Large deviation bounds for matrix Brownian motion

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ABSTRACT. We prove large deviation bounds for the convergence of Hermitian matrix valued Brownian motion towards free Brownian motion. As a consequence, we obtain upper and lower bounds on the microstates entropy introduced by Voiculescu [**22**].

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1. Introduction

Let $(H_t^N; t \in [0,1])$ be a Brownian motion taking values in the space $\mathcal{H}_N$ of Hermitian $N \times N$ matrices. A large N limit of this Brownian motion is provided by Voiculescu's free Brownian motion, namely let $(\mathcal{A}, \varphi)$ be a non-commutative, probability space and $(S_t; t \in [0,1])$ be a free Brownian motion in $(\mathcal{A}, \varphi)$, then for all choices of $t_1, \dots, t_n \in [0,1]$ one has

$$\frac{1}{N} Tr(H_{t_1}^N \dots H_{t_n}^N) \to_{N\to\infty} \varphi(S_{t_1} \dots S_{t_n}) \tag{1}$$

in probability, provided the covariance of H^N is suitably normalized. This convergence extends readily to families $\{H^{N,i}, 1 \le i \le m\}$ of independent copies of H^N, namely for all choices of $t_1, \dots, t_n \in [0,1]$ and $i_1, \dots, i_l \in \{1, \dots, m\}^n$, one has

$$\frac{1}{N} Tr(H_{t_1}^{N,i_1} \dots H_{t_n}^{N,i_n}) \to_{N\to\infty} \varphi(S_{t_1}^{i_1} \dots S_{t_n}^{i_n}) \tag{2}$$

for a family of m free Brownian motions $\{S^1, \dots, S^m\}$.

We will study the large deviations to this typical behaviour. It turns out that the Hermitian Brownian motion can deviate towards non bounded processes, and for this reason it will be convenient for us to replace the monomials in (2) by other "test-functions". Indeed let us define, for $l \in \{1, \dots, m\}$, unitary matrices

$$U_t^{N,l} := (H_t^{N,l} + 4i)(H_t^{N,l} - 4i)^{-1}$$

and unitary elements in $\mathcal{A}$, by

$$\Omega_t^l := (S_t^l + 4i)(S_t^l - 4i)^{-1}$$

then one can check that the following convergence holds in probability

$$\frac{1}{N} Tr((U_{t_1}^{N,i_1})^{\varepsilon_1} \dots (U_{t_n}^{N,i_n})^{\varepsilon_n}) \to_{N\to\infty} \varphi((\Omega_{t_1}^{i_1})^{\varepsilon_1} \dots (\Omega_{t_n}^{i_n})^{\varepsilon_n})$$

for all $t_1, \dots, t_n \in [0,1]$, $i_1, \dots, i_n \in \{1 \dots, m\}$ and $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$.

For any family of $N \times N$ unitary matrices $(V_t^i; t \in [0,1], i \in \{1, \dots, m\})$, (or more generally any family of unitaries in some tracial non-commutative probability space) the numbers $\frac{1}{N} Tr((V_{t_1}^{i_1})^{\varepsilon_1} \dots (V_{t_n}^{i_n})^{\varepsilon_n})$ define a tracial state on the $*$-algebra of the free group with generators $(u_t^i; t \in [0,1], i \in \{1 \dots, m\})$. In particular we shall denote $\hat{\sigma}^N$ the (random) tracial state defined by the matrices $(U_t^{N,i}; t \in [0,1], i \in \{1, \dots m\})$. It is in this space of tracial states that we shall study large deviations. More precisely, we shall restrict to a subspace of tracial states τ such that for all n, any $i_1, \dots, i_n \in \{1, \dots, m\}^n$ and any $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}^n$ the quantity $\tau((u_{t_1}^{i_1})^{\varepsilon_1} \dots (u_{t_n}^{i_n})^{\varepsilon_n})$ is a continuous function of $t_1, \dots, t_n$.

We shall prove the existence of a good rate function I on this space of states, given by a variational formula, such that for any closed subset F, one has

$$\limsup_{N \to \infty} N^{-2} \log \mathbb{P}\left(\hat{\sigma}^N \in F\right) \leq - \inf_{\tau \in F} I(\tau).$$

Furthermore we shall also give a lower bound for the probability that the empirical state is in a neighbourhood of some state satisfying a suitable stochastic differential equation. This lower bound is one of the main improvements with respect to earlier results in this direction, especially [**7**], [**8**]. We shall relate our results with Voiculescu's entropies and the results of [**7**] and [**8**] in section 7. In particular we prove that the microstate-free free entropy is always larger than the microstate entropy (see Corollary 7.5), this is one of the main outcomes of our approach to the free entropy.

This paper is organized as follows. In section 2 and 3 we gather preliminary results on hermitian brownian motion and its free analogue. Section 4 is devoted to a technical result on the convergence of some conditional expectations to their free analogues. In section 5 we state and prove the large deviation upper bound, while in section 6 we prove the lower bound. We compare these large deviation results with Voiculescu's free entropies in section 7. Finally in section 8 we prove a technical result which was needed in section 5.

2. Hermitian Brownian motion and tracial states on path space.

2.1. Tracial states. Let $m \in \mathbb{N}$ and $\mathcal{F}_{[0,1]}^m$ be the group $*$-algebra of the free group with generators $(u_t^i; t \in [0,1], i \in \{1, \dots, m\})$, and $\mathcal{F}_{[0,1]}^{m,sa}$ its self-adjoint part. We denote by $\mathcal{M}(\mathcal{F}_{[0,1]}^m)$ the set of all tracial states on $\mathcal{F}_{[0,1]}^m$ and by $\mathcal{M}^c(\mathcal{F}_{[0,1]}^m)$ the subset of tracial states τ such that for all $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}^n$, and all $i_1, \dots, i_n \in \{1, \dots, m\}^n$, the quantity $\tau((u_{t_1}^{i_1})^{\varepsilon_1} \dots (u_{t_n}^{i_n})^{\varepsilon_n})$ depends continuously on $t_1, \dots, t_n$. If $U = (U_t^i; t \in [0,1], i \in \{1, \dots, m\})$ is a family of unitary operators on some Hilbert space, then one can define $P(U)$ for any $P \in \mathcal{F}_{[0,1]}^m$ by substitution. If φ is a tracial state on the von Neumann algebra generated by U, then there is a unique tracial state τ on $\mathcal{F}_{[0,1]}^m$, called the distribution of $(U_t^i, t \in [0,1], i \in \{1, \dots, m\})$, such that

$$\tau(P) = \varphi(P(U))$$

for all $P \in \mathcal{F}_{[0,1]}^m$. In particular if $S = (S^1, \dots, S^m)$ is a m-dimensional free Brownian motion then the family of unitary elements $(\frac{S_t^l + 4i}{S_t^l - 4i}; t \in [0,1], l \in \{1, \dots m\})$ determines a state $\sigma \in \mathcal{M}^c(\mathcal{F}_{[0,1]}^m)$.

Let $X = (X^i_t; t \in [0,1], i \in \{1, \dots, m\})$ be a family of self-ajoint operators in some tracial non-commutative probability space $(\mathcal{A}, \varphi)$, and let $\tau \in \mathcal{M}(\mathcal{F}^m_{[0,1]})$ denote the distribution of $(\frac{X^l_s+4i}{X^l_s-4i}; s \in [0,1], l \in \{1, \dots, m\})$. Let $S = (S^1, \dots, S^m)$ be a m-dimensional free Brownian motion, free with X, then we denote by $\widetilde{\tau}^t$ the distribution of $(\frac{Y^l_s+4i}{Y^l_s-4i}; s \in [0,1], l \in \{1, \dots, m\})$ where, for $l \in \{1, \dots, m\}$, one has $Y^l_s = X^l_s$ if $s \leq t$ and $Y^l_s = X^l_t + S^l_{s-t}$ for $s > t$. In particular, one has $\widetilde{\tau}^1 = \tau$ and $\widetilde{\tau}^0 = \sigma$ for all $\tau \in \mathcal{M}^c(\mathcal{F}^m_{[0,1]})$. Note that $\widetilde{\tau}^t$ depends only on τ and not on the particular realization of X.

We endow $\mathcal{M}^c(\mathcal{F}^m_{[0,1]})$ with the topology such that a sequence $(\tau_n)_{n\geq 1}$ converges to τ if for every k, every $i_1, \dots, i_k \in \{1, \dots, m\}$ and every $\varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\}$ the functions

$$t_1, \dots, t_k \mapsto \tau_n((u^{i_1}_{t_1})^{\varepsilon_1} \dots (u^{i_k}_{t_k})^{\varepsilon_k})$$

converge uniformly to

$$\tau((u^{i_1}_{t_1})^{\varepsilon_1} \dots (u^{i_k}_{t_k})^{\varepsilon_k}).$$

Then $\mathcal{M}^c(\mathcal{F}^m_{[0,1]})$ is a metric space, with distance

$$d(\tau_1, \tau_2) = \sum_{k=1}^{\infty} 2^{-k} \sup_{\substack{i_1, \dots, i_k \in \{1, \dots, m\}^k \\ \varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\}^k}} \sup_{0 \leq t_1, \dots, t_k \leq 1} |(\tau_1 - \tau_2)((u^{i_1}_{t_1})^{\varepsilon_1} \dots (u^{i_k}_{t_k})^{\varepsilon_k})|$$

Some compact subsets of $\mathcal{M}^c(\mathcal{F}^m_{[0,1]})$ are described in the following Lemma.

Lemma 2.1. *For any function* $g : [0,1] \to \mathbb{R}^+$ *such that* $\lim_{x \to 0} g(x) = 0$*, the set*

$$K_g := \left\{ \tau \in \mathcal{M}^c(\mathcal{F}^m_{[0,1]}) : \ \forall s \leq t \in [0,1], \max_{1 \leq i \leq m} \tau(\left|u^i_s - u^i_t\right|^2) \leq g(t-s) \right\}$$

is a compact subset of $\mathcal{M}^c(\mathcal{F}^m_{[0,1]})$.

Proof. Applying Cauchy-Schwarz inequality we find that for all $i_1, \dots i_k \in \{1, \dots, m\}^k$ and $\varepsilon_1 \dots \varepsilon_k \in \{\pm 1\}^k$, the functions $t_1, \dots, t_k \mapsto \tau((u^{i_1}_{t_1})^{\varepsilon_1} \dots (u^{i_k}_{t_k})^{\varepsilon_k})$ for $\tau \in K_g$, are equicontinuous. The result follows from Arzela's theorem. $\square$

We shall also consider the following sets

$$\Gamma_L := \left\{ \tau \in \mathcal{M}^c(\mathcal{F}^m_{[0,1]}) ; \max_{1 \leq i \leq m} \sup_{t \in [0,1]} \tau(|u^i_t - 1|^{-2}) \leq L \right\}.$$

Any $\tau \in \Gamma_L$ can be mapped via the inverse of the Möbius map $\psi(x) = (x-4i)^{-1}(x+4i)$ into the law of a self-adjoint operator whose variance is less than $2^6 L$.

For all $L > 0$, the set Γ_L is a closed subset of $\mathcal{M}^c(\mathcal{F}^m_{[0,1]})$.

We denote $\mathcal{M}^{c,\infty}(\mathcal{F}^m_{[0,1]}) = \cup_{L>0} \Gamma_L$.

Using the identity $\tau(S^2_t) = t$ for a free Brownian motion, one checks readily the

Lemma 2.2. *For any function* $g : [0,1] \to \mathbb{R}^+$ *such that* $\lim_{x \to 0} g(x) = 0$*, there exists a constant* $C > 0$ *such that for any* $s \in [0,1]$*, any* $L \in \mathbb{R}^+$*, and any* $\tau \in \Gamma_L \cap K_g$*, one has* $\widetilde{\tau}^s \in \Gamma_{CL} \cap K_h$*, with* $h(x) = C(g(x) + |x|)$.

2.2. Hermitian Brownian motion.

Let $H^N(t); t \in [0,1]$ be a Brownian motion with values in the space of Hermitian matrices, normalized so that, for any Hermitian matrix A, the process $\mathrm{tr}_N(AH_t^N)$ is a real Brownian motion with variance $E[\mathrm{tr}_N(AH_t^N)^2] = \mathrm{tr}_N(A^2)t$ (here and in the sequel we use the notation $\mathrm{tr}_N = \frac{1}{N}Tr$ for the normalized trace on $M_N(\mathbb{C})$). Let $(H^{N,1}, \dots, H^{N,m})$ be m independent copies of H^N. As in the introduction we denote by $U_t^{N,i}$ the image of the Hermitian Brownian motion $H_t^{N,i}$ by the Möbius transformation $\psi(x) = \frac{x+4i}{x-4i}$, and by $\hat{\sigma}^N \in \mathcal{M}^c(\mathcal{F}^m_{[0,1]})$ the distribution of $U^N = (U_t^{N,l}; t \in [0,1], l \in \{1, \dots, m\})$, i.e. one has, for all $P \in \mathcal{F}^m_{[0,1]}$,

$$\hat{\sigma}^N(P(u)) = \mathrm{tr}_N(P(U^N)).$$

3. Malliavin Calculus and Clark-Ocone formula

3.1. Classical case.

Let $(B_t; t \in [0,1])$ be a d-dimensional Brownian motion on $[0,1]$, with its canonical filtration of $\sigma-$fields $(\mathcal{F}_s)_{s\in[0,1]}$. Denote by D the Malliavin gradient on the L^2 space of B (see e.g. [**13**]), and by $\mathbb{D}^{1,2}$ its domain.

Theorem 3.1. *(Clark-Ocone formula) Let $F \in \mathbb{D}^{1,2}$, then*

$$F = \mathbb{E}[F] + \int_0^1 \langle \, \mathbb{E}[D_s F | \mathcal{F}_s] \, , dB_s \rangle_{\mathbb{R}^d}.$$

The proof of this formula can be found in [**13**] (Proposition 1.3.5 p. 42) in the one dimensional case and in [**11**] for any dimension.

3.2. Matrix Clark-Ocone formula.

In order to apply the Clark-Ocone formula to matrix-valued processes we need to introduce certain non-commutative differential operators. For $s \in [0,1]$ and $l \in \{1, \dots, m\}$, let ∇_s^l be the linear map defined on $\mathcal{F}^m_{[0,1]}$ by

$$\begin{aligned}\nabla_s^l((u_{t_1}^{i_1})^{\varepsilon_1} \dots (u_{t_n}^{i_n})^{\varepsilon_n}) = \; & \textstyle\sum_{p=1}^n ((u_{t_p}^{i_p})^{\varepsilon_p} - 1)(u_{t_{p+1}}^{i_{p+1}})^{\varepsilon_{p+1}} \dots (u_{t_n}^{i_n})^{\varepsilon_n} \times \\ & (u_{t_1}^{i_1})^{\varepsilon_1} \dots (u_{t_{p-1}}^{i_{p-1}})^{\varepsilon_{p-1}} ((u_{t_p}^{i_p})^{\varepsilon_p} - 1) \tfrac{\varepsilon_p}{8i} 1_{i_p = l} 1_{[0,t_p]}(s).\end{aligned}$$

Remark that this formula for ∇_s^l formally comes from an application of the cyclic gradient formula

$$\nabla_s^l(x_{t_1}^{i_1} \dots x_{t_n}^{i_n}) = -\sum_{p=1}^n x_{t_{p+1}}^{i_{p+1}} \dots x_{t_n}^{i_n} x_{t_1}^{i_1} \dots x_{t_{p-1}}^{i_{p-1}} 1_{i_p=l} 1_{[0,t_p]}(s)$$

to the (non convergent in general) series $u_t^j = (1/4i)(x_t^j + 4i) \sum_k (x_t^j/4i)^k$, $j \in \{1, \dots, m\}$. Using the classical Clark-Ocone formula we can now derive the following variant. We denote by $(\mathcal{H}_s; s \in [0,1])$ the filtration of σ-fields of the Hermitian Brownian motion $H^N = (H^{N,1}, \dots, H^{N,m})$. We set $\nabla_s = (\nabla_s^1, \dots, \nabla_s^m)$, $A.B = \sum_{i=1}^m A_i B_i$ for any $(A_1, \dots, A_m)$ and $(B_1, \dots, B_m)$ in $M_N(\mathbb{C})^m$, and $|A|^2 = A.A^* = \sum_{i=1}^m A_i A_i^*$.

Proposition 3.2. *For every $P \in \mathcal{F}^m_{[0,1]}$ and $N \in \mathbb{N}$, one has*

$$\mathrm{tr}_N P\left[U^N\right] = \mathbb{E}\left[\mathrm{tr}_N P\left(U^N\right)\right] + \int_0^1 \mathrm{tr}_N \left\{ \mathbb{E}\left[\nabla_s P\left(U^N\right) | \mathcal{H}_s\right] . dH^N(s)\right\}.$$

The proposition follows from Theorem 3.1 applied to $F = \mathrm{tr}_N(P(U^N))$. A simple computation shows that, for any smooth function φ, the gradient of the function $M \mapsto \mathrm{tr}_N \varphi(M)$ on the space of hermitian matrices is the cyclic derivative of $\varphi(M)$. The extension to a function of several matrices is straightforward and yields the cyclic gradient.

3.3. Exponential martingales.

From Proposition 3.2, we deduce

Proposition 3.3. *For any $P \in \mathcal{F}^{m,sa}_{[0,1]}$, any $t \in [0,1]$, one has*

$$\mathbb{E}\Big[\exp\{N^2\Big(\mathbb{E}\left[\mathrm{tr}_N P\left(U^N\right)|\mathcal{F}_t\right] - \mathbb{E}\left(\mathrm{tr}_N P\left(U^N\right)\right) \\ -\tfrac{1}{2}\int_0^t \mathrm{tr}_N\left\{\left| \mathbb{E}\left[\nabla_s P\left(U^N\right)|\mathcal{H}_s\right]\right|^2\right\} ds\Big)\}\Big] = 1.$$

Proof. Indeed, for $P \in \mathcal{F}^{m,sa}_{[0,1]}$ the processes $s \mapsto \mathbb{E}\left[\nabla^l_s P\left(U^N\right)|\mathcal{H}_s\right]$, for $l = 1, \ldots, m$, are bounded and adapted so that, by Proposition 3.2

$$\begin{aligned} M^P_t &= \mathrm{tr}_N\left[\, \mathbb{E}\left(P(U_N)|\mathcal{H}_t\right)\right] - \mathbb{E}\left[\mathrm{tr}_N P(U_N)\right] \\ &= \int_0^t \mathrm{tr}_N\left(\, \mathbb{E}\left[\nabla_s P\left(U^N\right)|\mathcal{H}_s\right].dH^N(s)\right) \end{aligned}$$

is a bounded martingale with angle bracket

$$\langle \int_0^{\cdot} tr_N\left(\, \mathbb{E}\left[\nabla_s P\left(U^N\right)|\mathcal{H}_s\right].dH^N(s)\right)\rangle_t = \\ \tfrac{1}{N^2}\sum_{l=1}^m \int_0^t \mathrm{tr}_N\left(\left| \mathbb{E}\left[\nabla^l_s P\left(U^N\right)|\mathcal{H}_s\right]\right|^2\right) ds.$$

and the result follows from exponential martingale theory. □

4. Convergence of conditional expectations

In the following we shall need results on the convergence of conditional expectations with respect to the Hermitian Brownian motion towards corresponding quantities for free Brownian motion. We state these results in the following Proposition.

If $(\mathcal{A}, \varphi)$ is a tracial non-commutative probability space, and $\mathcal{B} \subset \mathcal{A}$ a von Neumann subalgebra, we denote by $\varphi(.|\mathcal{B})$ the conditional expectation onto $\mathcal{B}$.

Proposition 4.1. *For $P \in \mathcal{F}^m_{[0,1]}$, $\tau \in \mathcal{M}^c(\mathcal{F}^m_{[0,1]})$, $\epsilon > 0$, $l \in \mathbb{N}$, $L \in \mathbb{R}^+$, define*

$$\begin{aligned} H &:= H(P, L, \epsilon, N, \tau, l) \\ &= \operatorname*{ess\ sup}_{\{d(\hat{\sigma}^N,\tau)<\epsilon;\hat{\sigma}^N \in K_{L\sqrt{\cdot}} \cap \Gamma_L\}} \left|\mathrm{tr}_N\left(\mathbb{E}[P(U^N)|\mathcal{H}_t]^l\right) - \tau\left(\tilde{\tau}^t(P|\mathcal{B}_t)^l\right)\right| \end{aligned}$$

then one has, for every $l \in \mathbb{N}$

$$\sup_{L>0} \limsup_{\epsilon\to 0} \limsup_{N\to\infty} \sup_{\substack{\tau \in K_{L\sqrt{\cdot}} \cap \Gamma_L \\ t \in [0,1]}} H = 0 \tag{3}$$

For the proof we shall use the following three lemmas.

Lemma 4.2. *Let $q, n_1, \ldots, n_r \in \mathbb{N}$, then there exists coefficients α_π, $c_\pi \in \mathbb{N}$, indexed by partitions $\pi = (\pi_1, \ldots, \pi_k)$ of the set*

$$\{1, \ldots, q\} \cup \{(1,1), \ldots, (1, n_1)\} \cup \ldots \cup \{(r,1), \ldots, (r, n_r)\}$$

such that, for all $N > 0$ *and all sets of* $N \times N$ *matrices* $A_1, \dots, A_q; A_{(i,j)}; 1 \le i \le r; 1 \le j \le n_i$ *one has*

$$\mathbb{E}[A_1 Z A_2 \dots A_q Z \prod_{i=1}^{r} \mathrm{tr}_N(A_{(i,1)} Z \dots A_{(i,n_i)} Z)] = \sum_{\pi} \alpha_\pi N^{-c_\pi} A_{\pi_1} \prod_{j=2}^{k} \mathrm{tr}_N(A_{\pi_j})$$

where Z *is a gaussian Hermitian matrix with covariance* $E[\mathrm{tr}_N(AZ)^2] = \mathrm{tr}_N(A^2)$, *the sum is over partitions such that* $\pi_1 \subset \{1, \dots, q\}$, *and* $A_{\{u_1, \dots, u_k\}} = A_{u_1} \dots A_{u_k}$ *if* $u_1 < u_2 \dots < u_k$.

Proof. This follows from Wick's formula.

LEMMA 4.3. *Let* S *be a semi-circular variable free with* $a_1, \dots, a_n$, *then one has, with* $\varphi(.|a_1, \dots, a_n)$ *denoting conditional expectation onto the algebra generated by* $a_1, \dots, a_n$,

$$\varphi(a_1 S a_2 S a_3 \dots S a_n S | a_1, \dots, a_n) = \sum_{\pi; c_\pi = 0} \alpha_\pi a_{\pi_1} \prod_{j=2}^{k} \varphi(a_{\pi_j})$$

where the sum is over partitions of $\{1, \dots, n\}$ *and* α_π *is as in Lemma 4.2*

Proof. The existence of such a formula follows, e.g., from Speicher's combinatorial theory of free cumulants [**18**]. The fact that random gaussian matrices become asymptotically free with constant matrices (see [**22**]) implies that the coefficients coincide with those of Lemma 4.2. □

A detailed proof of the preceding two lemmas can be found in [**9**].

LEMMA 4.4. *Let* $\delta > 0$, *then there exists a two variables non-commutative polynomial* $F(U, Z)$ *such that, for every tracial non-commutative probability space* $(\mathcal{A}, \varphi)$, *any selfadjoint* $x, z \in \mathcal{A}$ *such that* $\|z\| \le 3$ *one has* $\|F(u, z) - \widetilde{u}\| < \delta$, *where* $u = \frac{x+4i}{x-4i}$ *and* $\widetilde{u} = \frac{x+z+4i}{x+z-4i}$.

Proof. One has

$$\widetilde{u} = 1 + 8i(x + z - 4i)^{-1} = 1 + \left(1 + \frac{u-1}{8i} z\right)^{-1} (u - 1)$$

Since $\|\frac{u-1}{8i} z\| \le \frac{3}{4}$, one can expand $(1 + \frac{u-1}{8i} z)^{-1}$ as a convergent geometric series and truncate it to get F. □

Proof of Proposition 4.1.

Recall that, by definition, U^N is the image by $\psi(x) = (x + 4i)(x - 4i)^{-1}$ of the m-dimensional Hermitian matrix Brownian motion H^N, e.g $U^N = \Psi(H^N)$ where, for any $X = (X^1, \dots, X^m) \in \mathcal{C}([0,1], \mathcal{H}_N^m)$, $\Psi(X)$ is the element of $\mathcal{C}([0,1], \mathcal{U}_N^m)$ given, for $l \in \{1, \dots, m\}$ and $t \in [0,1]$, by

$$\Psi(X)_t^l = \psi(X_t^l).$$

Introducing an auxiliary m-tuple of matrix valued brownian motions $\widetilde{H}^N$, independent from U^N, then one has, for any $P \in \mathcal{F}_{[0,1]}^m$,

$$\mathbb{E}\left[P\left(U^N\right) | \mathcal{H}_s\right] = \mathbb{E}_{\widetilde{H}^N}\left[P\left(\Psi(\Psi^{-1}(U^N)_{.\wedge s} + \widetilde{H}^N_{.-s\vee 0})\right)\right]$$

where $\mathbb{E}_{\widetilde{H}^N}$ is the expectation with respect to $\widetilde{H}^N$. We set, for $s \in [0,1]$ and $U^N \in \Psi(\mathcal{C}([0,1], \mathcal{H}_N^m))$,

$$\widetilde{U}^{N,s} = \Psi(\Psi^{-1}(U^N)_{.\wedge s} + \widetilde{H}^N_{.-s\vee 0}).$$

By Lemma 4.4, for any time $t \in [0,1]$, we can approximate $\widetilde{U}^{N,s}_t$ by a polynomial function of $U^N_{t\wedge s}$ and $\widetilde{H}^N_{t-s\vee 0}$ uniformly on the set where $||\widetilde{H}^N_{t-s\vee 0}|| \le 3$.

Letting $P = (u^{i_1}_{t_1})^{\varepsilon_1}\dots(u^{i_k}_{t_k})^{\varepsilon_k}$ be a monomial, we conclude that for any $\delta > 0$, there exists a non-commutative polynomial $Q(u_t; t \le s, x_{u-s\vee 0})$ such that on $\cap_{p=1}^k\{||\widetilde{H}^N_{t_p-s\vee 0}|| \le 3\}$, one has

$$\|P(\widetilde{U}^{N,s}) - Q(U^N_v; v \le s, \widetilde{H}^N_{u-s\vee 0})\| \le \delta.$$

We can also do the same substitution in the computation of $\widetilde{\tau}^t(P|\mathcal{B}_t)$ to obtain that, since the free Brownian motion is uniformly bounded by 2,

$$\|P\left(\Psi(\Psi^{-1}(u)_{.\wedge s} + S_{.-s\vee 0})\right) - Q(u_u; u \le s, S_{u-s\vee 0})\| \le \delta \quad \text{in } (A, \widetilde{\tau}^t)$$

According to Lemma 4.2, the conditional expectation

$$\mathbb{E}[Q(U^N_u; u \le s, \widetilde{H}^N_{u-s\vee 0})|\mathcal{H}_s],$$

which is obtained by integrating the gaussian increments, has a finite polynomial expansion

$$\sum_{m=0}^{k} N^{-m} R_m(U^N_v; v \le s) \tag{4}$$

where the polynomials R_m depend only on the joint moments of $(U^N_s; s \le t)$. Moreover, according to Lemma 4.3, one has

$$\tau(Q(u_u; u \le s, x_{u-s}; u > s)|\mathcal{B}_s) = \widetilde{R}_0(u_u; u \le s)$$

for some polynomial $\widetilde{R}_0$ of degree $\le k$, depending on the moments of $u_v; v \le s$ under τ. As $\epsilon \to 0$ the coefficients of R_0 in (4) converge towards those of $\widetilde{R}_0$. Putting things together, it remains to bound the quantities

$$\mathbb{E}[|Q(U^N_u; u \le s, H^N_{u-s\vee 0})|1_{\|H^N_{t_j} - H^N_t\| > 3}|\mathcal{H}_t]$$

and

$$\mathbb{E}[P(U^N)1_{\|H^N_{t_j} - H^N_t\| > 3}|\mathcal{H}_t].$$

It follows from standard results in random matrix theory that for $s < t$ one has $\mathbb{P}(\|H^N_t - H^N_s\| > 2\sqrt{t-s}) \to 0$ as $N \to \infty$. On the other hand one can bound above $\mathbb{E}[|Q(U^N_s; s \le t, H^N_u - H^N_t; u > t)|^2|\mathcal{H}_t]$ using e.g. Lemma 4.2, so that the result follows by Cauchy-Schwarz inequality. ☐

5. Large deviation upper bound

5.1. Statement of the upper bound. We begin here our study of the large deviation properties of $\hat{\sigma}^N$. Observe that, since H^N is a Hermitian matrix valued process whose entries have almost surely continuous paths, then

$$\mathbb{P}\left(\hat{\sigma}^N \in \mathcal{M}^{c,\infty}_{[0,1]}\right) = 1.$$

Let us now define the rate function governing the deviations of the law of $\hat{\sigma}_N$.We set $I(\tau) = +\infty$ if $\tau \notin \mathcal{M}^c(\mathcal{F}^m_{[0,1]})$ and otherwise

$$I(\tau) = \sup_{t\in[0,1]} \sup_{P \in \widetilde{\mathcal{F}}^{m,sa}_{[0,1]}} \left\{ \widetilde{\tau}^t(P) - \widetilde{\tau}^0(P) - \frac{1}{2}\int_0^t \tau\left[|\widetilde{\tau}^s(\nabla_s P|\mathcal{B}_s)|^2\right] ds \right\}.$$

We note that the conditional expectation $\widetilde{\tau}^s\left(\,\cdot|\mathcal{B}_s\right)$ has its range in $\mathcal{A}$, therefore the expression

$$\tau\left[\left|\widetilde{\tau}^s\left(\nabla_s P|\mathcal{B}_s\right)\right|^2\right] = \widetilde{\tau}^s\left[\left|\widetilde{\tau}^s(\nabla_s P|\mathcal{B}_s)\right|^2\right]$$

is well defined.

THEOREM 5.1. *For every closed set* $F \subset \mathcal{M}^c(\mathcal{F}^m_{[0,1]})$ *one has*

$$\lim\sup_{N\to\infty} \frac{1}{N^2}\log I\!P(\hat{\sigma}^N \in F) \leq -\inf_{\tau\in F} I(\tau)$$

We shall prove Theorem 5.1 in section 5.4 but first we need to establish some facts about the rate function I.

5.2. Study of the rate function.

THEOREM 5.2. *The function I defined in section 5.1 satisfies the following properties*
1) I is a good rate function, i.e. $I \geq 0$ and the set $\{I \leq M\}$ is a compact subset of $\mathcal{M}^{c,\infty}(\mathcal{F}^m_{[0,1]})$ for every $M > 0$.
2) If $I(\tau) < \infty$, then there exists a map $s \mapsto K^\tau_s \in L^2(\mathcal{B}_s,\tau)^m$ such that $\int_0^1 \sum_{i=1}^m \tau((K^{\tau,i}_s)^2)ds < \infty$ and
a) $\inf_{P\in\mathcal{F}^{m,sa}_{[0,1]}} \int_0^1 \tau\left[\left|\widetilde{\tau}^s\left(\nabla_s P|\mathcal{B}_s\right) - K^\tau_s\right|^2\right] ds = 0$.
b) For any $P \in \mathcal{F}^m_{[0,1]}$, one has

$$\tau(P) = \widetilde{\tau}^0(P) + \int_0^1 \tau\left(\widetilde{\tau}^s\left(\nabla_s P|\mathcal{B}_s\right).K^\tau_s\right) ds$$

COROLLARY 5.3. *With the notations as above, if $I(\tau) < +\infty$ then one has*

$$I(\tau) = \frac{1}{2}\int_0^1 \tau(|K^\tau_s|^2)ds.$$

Since the proof of Theorem 5.2 is rather technical, we defer it to Section 8.

5.3. Exponential Tightness. In this section, we prove that the distributions of $\hat{\sigma}^N$ form an exponentially tight family on $\mathcal{M}^c(\mathcal{F}^m_{[0,1]})$.

LEMMA 5.4.

$$\limsup_{L\to\infty}\limsup_{N\to\infty}\frac{1}{N^2}\log I\!P(\hat{\sigma}^N \in (K_{L\sqrt{\cdot}} \cap \Gamma_L)^c) = -\infty,$$

or, equivalently,

$$\limsup_{L\to\infty}\limsup_{N\to\infty}\frac{1}{N^2}\log I\!P(\sup_{t\in[0,1]} \mathrm{tr}_N(|1-U^N_t|^{-2}) > L) = -\infty, \tag{5}$$

and

$$\limsup_{L\to\infty}\limsup_{N\to\infty}\frac{1}{N^2}\log I\!P(\sup_{\substack{t,s\in[0,1]\\ t\neq s}} \frac{\mathrm{tr}_N(|U^N_s - U^N_t|^2)}{\sqrt{|t-s|}} > L) = -\infty. \tag{6}$$

Proof. Observe that $\mathrm{tr}_N(|1-U^N_t|^{-2}) \leq 8 + 2\mathrm{tr}_N(|H^N_t|^2)$. Then (5) follows from standard estimates for suprema of Brownian motions. For the proof of (6), note that $|U^N_t - U^N_s| \leq 4^{-2}|H^N_t - H^N_s|$ and that for $p \in \{1,\ldots,m\}$,

$$
\begin{aligned}
\mathbb{P}&\left(\sup_{s\neq t, s,t\in[0,1]} \frac{\mathrm{tr}_N\left(H^{N,p}(t)-H^{N,p}(s)\right)^2}{\sqrt{t-s}} > L\right)\\
&\le \mathbb{P}\left(\sum_{l=1}^{N^2} \sup_{s\neq t, s,t\in[0,1]} \frac{\left(B_t^l - B_s^l\right)^2}{\sqrt{t-s}} > LN^2\right)\\
&\le \exp(-\alpha L N^2)\left(\mathbb{E}\left(\exp[\alpha\|B\|^2_{\frac{1}{4}}]\right)\right)^{N^2}
\end{aligned}
$$

where $B_1,\dots,B_{N^2}$ are independent Brownian motions, $\|.\|_{\frac{1}{4}}$ is the $\frac{1}{4}$-Hölder norm, and $\alpha>0$ is such that $E\left(\exp[\alpha\|B\|^2_{\frac{1}{4}}]\right) = C < +\infty$ (see th 4.1 in [**12**] for the existence of α). Thus, for any $N\in\mathbb{N}$, any $p\in\{1,\dots,m\}$,

$$
\frac{1}{N^2}\log\mathbb{P}\left(\sup_{s\neq t, s,t\in[0,1]} \frac{\mathrm{tr}_N\left(H_t^{N,p}-H_s^{N,p}\right)^2}{\sqrt{t-s}} > L\right) \le -\alpha L + \log C \to -\infty
$$

as $L\to\infty$. ☐

5.4. Weak large deviation upper bound. Since, for any $L\in\mathbb{R}^+$, the set $K_{L\sqrt{.}}\cap\Gamma_L$ is a compact subset of $\mathcal{M}^{c,\infty}(\mathcal{F}^m_{[0,1]})$, we see that Lemma 5.4 reduces the proof of Theorem 5.1 to that of the following weak large deviation upper bound.

Lemma 5.5. *For any $\tau\in\cup_{L\in\mathbb{N}}(K_{L\sqrt{.}}\cap\Gamma_L)$, one has*

$$
\limsup_{\epsilon\to 0}\limsup_{N\to\infty} \mathbb{P}(d(\hat{\sigma}^N,\tau)\le\epsilon) \le -I(\tau). \tag{7}
$$

Proof. Let $\tau\in\cup_{L\in\mathbb{N}}(K_{L\sqrt{.}}\cap\Gamma_L)$ and $P\in\mathcal{F}^{m,sa}_{[0,1]}$, and set

$$
\zeta_P^N(t) := \exp\{N^2 l_{P,t}^N\} \tag{8}
$$

where $l^N_{P,t}$ is the random variable

$$
l^N_{P,t} := \mathrm{tr}_N\ \mathbb{E}(P|\mathcal{F}_t) - \mathbb{E}(\mathrm{tr}_N P) - \frac{1}{2}\int_0^t \mathrm{tr}_N\left\{|\ \mathbb{E}(\nabla_s P|\mathcal{F}_s)|^2\right\} ds.
$$

For any $\epsilon>0$, $L>0$, $t\in[0,1]$,one has

$$
\begin{aligned}
\mathbb{P}(d(\hat{\sigma}^N,\tau)\le\epsilon) \quad &\le \quad \mathbb{P}(\hat{\sigma}^N\notin K_{L\sqrt{.}}\cap\Gamma_L) + \\
&\qquad \mathbb{E}\left[1_{\{d(\hat{\sigma}^N,\tau)\le\epsilon\}\cap\{\hat{\sigma}^N\in K_{L\sqrt{.}}\cap\Gamma_L\}}\frac{\zeta_P^N(t)}{\zeta_P^N(t)}\right] \qquad (9)\\
&\le \quad \mathbb{P}(\hat{\sigma}^N\notin K_{L\sqrt{.}}\cap\Gamma_L) + \exp\{-N^2 \operatorname*{ess\,inf}_{\substack{d(\hat{\sigma}^N,\tau)<\epsilon\\ \hat{\sigma}^N\in K_{L\sqrt{.}}\cap\Gamma_L}} l^N_{P,t}\} \qquad (10)
\end{aligned}
$$

where we have used $\mathbb{E}[\zeta_P^N(t)]=1$ (Proposition 3.3) in the last line. Now, according to Proposition 4.1 (with $l=1$ and 2) if we set

$$
l_{P,t}(\tau) = \tilde{\tau}^t(P) - \tilde{\tau}^0(P) - \frac{1}{2}\int_0^t \tau(\tilde{\tau}^s(\nabla_s P|\mathcal{B}_s)^2)ds,
$$

then one has

$$
\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\operatorname*{ess\,sup}_{\substack{d(\hat{\sigma}^N,\tau)<\epsilon\\ \hat{\sigma}^N\in K_{L\sqrt{.}}\cap\Gamma_L}} |l^N_{P,t} - l_{P,t}(\tau)| = 0.
$$

Therefore, for any $L \in \mathbb{R}^+$, one has

$$\limsup_{\epsilon \to 0} \limsup_{N \to \infty} N^{-2} \log \mathbb{P}(d(\hat{\sigma}^N, \tau) \leq \epsilon) \leq \\ \max\{-l_{P,t}(\tau), \limsup_{N \to \infty} N^{-2} \log \mathbb{P}(\hat{\sigma}^N \in (K_{L\sqrt{\cdot}} \cap \Gamma_L)^c)\}.$$

Letting L go to infinity gives, thanks to Lemma 5.4,

$$\limsup_{\epsilon \to 0} \limsup_{N \to \infty} N^{-2} \log \mathbb{P}(d(\hat{\sigma}^N, \tau) \leq \epsilon) \leq -l_{P,t}(\tau).$$

We can now take the supremum over $t \in [0,1]$ and $P \in \mathcal{F}^{m,sa}_{[0,1]}$ to conclude. ☐

6. Large deviation lower bound

6.1. Existence of processes with smooth K. We shall prove a large deviation lower bound for processes satisfying certain smoothness conditions. Before that we will prove that indeed there exist such processes. More precisely one has

THEOREM 6.1. *For any $K \in \mathcal{F}^{m,sa}_{[0,1]}$, there exists a unique $\tau \in \mathcal{M}^{c,\infty}(\mathcal{F}^m_{[0,1]})$ such that for all $P \in \mathcal{F}^m_{[0,1]}$, all $t \in [0,1]$, one has*

$$\tilde{\tau}^t(P) = \tilde{\tau}^0(P) + \int_0^t \tilde{\tau}^s \left(\nabla_s P . \tilde{\tau}^s(\nabla_s K | \mathcal{B}_s)\right) ds \tag{11}$$

Furthermore, one has

$$I(\tau) = \frac{1}{2} \int_0^1 \tau(|\tilde{\tau}^s(\nabla_s K|\mathcal{B}_s)|^2) ds < \infty$$

Proof. Let ζ^N_K be as in (8), and consider the probability distribution $\mathbb{P}^N_K = \zeta^N_K(1) d\mathbb{P}$. Since $K \in \mathcal{F}^{m,sa}_{[0,1]}$, we can find a finite constant $C(K)$ such that, for all $N > 0$, one has

$$\zeta^N_G(1) \leq e^{C(K)N^2} \tag{12}$$

By Lemma 5.4 we get

$$\limsup_{L \to \infty} \limsup_{N \to \infty} \frac{1}{N^2} \log \mathbb{P}^N_K(\hat{\sigma}^N \notin K_{L\sqrt{\cdot}} \cap \Gamma_L) = -\infty \tag{13}$$

therefore the distributions of $\hat{\sigma}^N$ under $\mathbb{P}^N_K$ form an exponentially tight family, and in particular $\hat{\sigma}^N$ has almost surely converging subsequences. Using Girsanov's theorem and Proposition 3.2 we get that for any $P \in \mathcal{F}^m_{[0,1]}$, any $t \in [0,1]$, one has

$$\begin{aligned} \mathrm{tr}_N(\ \mathbb{E}(P(U^N)|\mathcal{H}_t)) &= \mathbb{E}(\mathrm{tr}_N P) + \int_0^t \mathrm{tr}_N(\ \mathbb{E}(\nabla_s P|\mathcal{H}_s).\ \mathbb{E}(\nabla_s K|\mathcal{H}_s)) ds \\ &\quad + \int_0^t \mathrm{tr}_N \left[\ \mathbb{E}(\nabla_s P|\mathcal{H}_s).d\widetilde{H}_N(s)\right] \end{aligned}$$

where $\widetilde{H}_N$ is a $\mathbb{P}^N_K$- Hermitian Brownian motion. The process

$$t \mapsto \int_0^t \mathrm{tr}_N \left[\ \mathbb{E}(\nabla_s P|\mathcal{H}_s).d\widetilde{H}_N(s)\right]$$

is a martingale whose bracket is of order N^{-2}, therefore it converges to zero in probability as $N \to \infty$. Henceforth, by Proposition 4.1 any limit point μ of $\hat{\sigma}^N$ under $\mathbb{P}_K^N$ satisfies

$$\widetilde{\mu}^t(P) = \widetilde{\mu}^0(P) + \int_0^t \widetilde{\mu}^s(\nabla_s P.\widetilde{\mu}^s(\nabla_s K|\mathcal{B}_s))ds \tag{14}$$

for any $P \in \mathcal{F}_{[0,1]}^m$ and any $t \in [0,1]$. This proves the existence of τ and it remains to show uniqueness. We first prove that if τ satisfies (11), then it corresponds to the distribution of $\frac{X_t+4i}{X_t-4i}$ for some bounded process X. Let us apply, for $l \in \{1, \dots, m\}$, equation (11) to $P = |1+\varepsilon - u_t^l|^{-2m}$. Although P is not really a polynomial one can check easily by expanding the inverse as a geometric series that indeed one has $P = \lim P_n$ and $\nabla_s P = \lim \nabla_s P_n$ for some sequence of polynomials, where convergence holds in norm, so that we can apply formula (11). By an explicit computation we get

$$\widetilde{\tau}^v(\left|\nabla_s |1+\varepsilon - u_t^l|^{-2m}\right|) \le \widetilde{\tau}^v(2m|1+\varepsilon - u_t^l|^{-2m+1})$$

for $v, s \le t$, therefore since K is bounded, for any $v \in [0,1]$,

$$\widetilde{\tau}^v(|1+\varepsilon - u_t^l|^{-2m}) \le \widetilde{\tau}^0(|1+\varepsilon - u_t^l|^{-2m}) + 2mL_K \int_0^v \widetilde{\tau}^s(|1+\varepsilon - u_t^l|^{-2m+1})ds$$

for some constant $L_K := \sup_{s\in[0,1]} ||\nabla_s K||_\infty < \infty$, and we conclude by Gronwall's lemma that $\widetilde{\tau}^v(|1+\varepsilon - u_t^l|^{-2m}) \le D^{2m}$ for some constant D independent of v, t, l and ε. Therefore τ is the distribution of $\{\frac{X_t^l+4i}{X_t^l-4i}, t \in [0,1], l \in \{1, \dots, m\}\}$ for some bounded process X_t.

We shall now prove that the distribution of $Y_t := X_t - \int_0^t K_s^\tau ds$, where $K_s^\tau := \widetilde{\tau}^s(\nabla_s K|\mathcal{B}_s)$, is that of a free Brownian motion. We shall rely on the following result, which is a free version of Paul Lévy's well known theorem on the characterization of Brownian motion as the unique martingale with continuous paths and square bracket equal to t, and which may be of independent interest.

Theorem 6.2. *Let $(\mathcal{B}_s; s \in [0,1])$ be an increasing family of von Neumann subalgebras, in a non-commutative probability space $(\mathcal{A}, \tau)$, and let*

$$(Z_s = (Z_s^1, \dots, Z_s^m); s \in [0,1])$$

be an m-tuple of self-adjoint processes adapted to $(\mathcal{B}_s; s \in [0,1])$, such that Z is bounded, $Z_0 = 0$, and for all $s < t$ one has

1. $\tau(Z_t|\mathcal{B}_s) = Z_s$
2. $\tau(|Z_t - Z_s|^4) \le K(t-s)^2$ *for some constant K,*
3. *For any $l, p \in \{1, \dots, m\}$, and all $A, B \in \mathcal{B}_s$, one has*

$$\tau(AZ_t^l BZ_t^p) = \tau(AZ_s^l BZ_s^p) + 1_{p=l}(t-s)\tau(A)\tau(B) + o(t-s),$$

then Z is a free Brownian motion, i.e. for all $s < t$ the elements $Z_t^l - Z_s^l; l \in \{1, \dots, m\}$ are free with $\mathcal{B}_s$, and have a semi-circular distribution of covariance $(t-s)I_m$.

Proof. Because of the invariance of the conditions under time translation, it is enough to prove that $Z_t - Z_0$ is free with $\mathcal{B}_0$, and of semi-circular distribution with covariance tI_m. We can assume that $Z_0 = 0$, and one has for any $i_1, \dots i_n \in \{1, \dots, m\}$,

$$\tau(Z_t^{i_1} \dots Z_t^{i_n}) = \tau((Z_s^{i_1} + (Z_t^{i_1} - Z_s^{i_1})) \dots (Z_s^{i_n} + (Z_t^{i_n} - Z_s^{i_n}))).$$

From condition 1 we get $\tau(Z_t^l - Z_s^l|\mathcal{B}_s) = 0$, and expanding the above product using 2 and 3 gives

$$\begin{gathered}\tau(Z_t^{i_1}\dots Z_t^{i_n}) - \tau(Z_s^{i_1}\dots Z_s^{i_n}) = \\ (t-s)\textstyle\sum_{0\le k+p\le n-2}\sum_{i_k=i_p}\tau(Z_s^{i_1}\dots Z_s^{i_{k-1}}Z_s^{i_{k+p+1}}\dots Z_s^{i_n})\tau(Z_s^{i_{k+1}}\dots Z_s^{i_{k+p-1}}) \\ +o(t-s)\end{gathered}$$

where we have used non commutative Hölder's inequality in order to bound the terms containing at least three $(Z_t^l - Z_s^l)$ factors. It follows that the quantities $\tau(Z_t^{i_1}\dots Z_t^{i_n})$ satisfy a system of differential equations whose initial conditions are known. It is easy to see that this system has a unique solution, resulting with the observation that there exists at most one process (in distribution) satisfying 1, 2 and 3.

Since the free m-dimensional brownian motion satisfies also 1, 2, 3 we conclude that $Z_t - Z_0$ is a free brownian motion. For the freeness property with respect to $\mathcal{B}_0$, we consider a quantity of the form

$$\tau(A_1 Z_t^{i_1} A_2 Z_t^{i_2}\dots A_n Z_t^{i_n})$$

which again satisfies the same differential equation as when Z_t is a free Brownian motion free with $\mathcal{B}_0$. ☐

In order to apply Theorem 6.2 to the process Y we have to check the 3 conditions. First we apply (11) to $P = (X_t^l - X_s^l)Q_s$, where $Q_s \in \mathcal{B}_s \cap \mathcal{F}^m_{[0,1]}$. Although P does not belong to $\mathcal{F}^m_{[0,1]}$ one can again check that it is a limit of a sequence of P_n in $\mathcal{F}^m_{[0,1]}$, such that $\nabla_s P_n$ converges to $\nabla_s P$, so there is no problem in applying formula (11). One has $\nabla_u^k P = \delta_{kl}1_{u\in[s,t]}Q_s + W$ where $\tilde{\tau}^u(W|\mathcal{B}_s) = 0$. We thus find that for all $Q_s \in \mathcal{B}_s$, one has

$$\begin{aligned}\tau((X_t^l - X_s^l)Q_s) &= \tilde{\tau}^0((X_t^l - X_s^l)Q_s) + \int_0^1 \tau(\tilde{\tau}^u(\nabla_u[(X_t^l - X_s^l)Q_s]|\mathcal{B}_u).K_u)du \\ &= \tau(\int_s^t Q_s K_u^l du)\end{aligned}$$

from which we get that condition 1 is satisfied by $X_t - \int_0^t K_s ds$.

We now apply (11) to $P = (X_t^l - X_s^l)^4$ (the same remark as above applies). Since $\tilde{\tau}^0((X_t^l - X_s^l)^4) = 2(t-s)^2$, $\nabla_u^k[(X_t^l - X_s^l)^4] = 0$ for $u \notin [s,t]$ and $\nabla_u^k[(X_t^l - X_s^l)^4] = 4\delta_{kl}(X_t^l - X_s^l)^3$ for $u \in [s,t]$, one has

$$\tau((X_t^l - X_s^l)^4) = 2(t-s)^2 + \int_s^t \tau(\tilde{\tau}^u(4(X_t^l - X_s^l)^3|\mathcal{B}_u)K_u^{\tau,l})du$$

Since $K_u^{\tau,l}$ is uniformly bounded in norm, using Hölder's inequality and Gronwall lemma, we get the bound 2.

Condition 3 can be checked in a similar way as condition 1.

We conclude that X is solution to the stochastic differential equation

$$X_t = S_t + \int_0^t K_s ds$$

with $K_s = \tau(\nabla_s K|\mathcal{B}_s)$. Observing that for $K \in \mathcal{F}^m_{[0,1]}$, $X \to K_s(X)$ is uniformly Lipschitz, e.g there exists a finite constant C such that for all $s \in [0,1]$,

$$||K_s(X) - K_s(Y)|| \le C\sup_{u\le s}||X_u - Y_u||,$$

we can use the usual Gronwall argument to prove the uniqueness of the solution to this equation, establishing the uniqueness of τ. ☐

Remark that from the above proof we can extract the following result.

Proposition 6.3. *With notations as in the proof of Theorem 6.1, under the probability* $I\!P_K^N$, *the trace states* $\hat{\sigma}^N$ *converge in probability, as* $N \to \infty$, *towards* τ.

6.2. Statement of the lower bound. We can now state the large deviation lower bound for states coming from an element $K \in \mathcal{F}_{[0,1]}^{m,sa}$ in $\mathcal{M}_b^{c,\infty}(\mathcal{F}_{[0,1]}^m)$.

Theorem 6.4. *Let* $K \in \mathcal{F}_{[0,1]}^{m,sa}$ *and let* τ *be the associated tracial state as in Theorem 6.1, then one has*

$$\lim_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{N^2}\log I\!P(d(\hat{\sigma}^N,\tau)<\delta)\geq -I(\tau).$$

Proof. As in the upper bound proof we can write for any $L \in \mathbb{R}^+$, with ζ as in (8)

$$\begin{aligned}\mathbb{P}(d(\hat{\sigma}^N,\tau)<\delta) &\geq \mathbb{E}\left[1_{d(\hat{\sigma}^N,\tau)<\delta,\hat{\sigma}^N\in K_{L\sqrt{\cdot}}\cap\Gamma_L}\frac{\zeta_K^N(1)}{\zeta_K^N(1)}\right]\\ &\geq \mathbb{E}\left[1_{d(\hat{\sigma}^N,\tau)<\delta;\hat{\sigma}^N\in K_{L\sqrt{\cdot}}\cap\Gamma_L}\zeta_K^N(1)\right]\\ &\qquad\times\exp\{-N^2\operatorname*{ess\,sup}_{\substack{d(\hat{\sigma}^N,\tau)<\delta\\ \hat{\sigma}^N\in K_{L\sqrt{\cdot}}\cap\Gamma_L}} l_{K,1}^N\}\end{aligned}$$

Using Theorem 4.1 we have, for any $L \in \mathbb{R}^+$, that

$$\begin{aligned}&\liminf_{\delta\to 0}\liminf_{N\to\infty}\inf_{\substack{d(\hat{\sigma}^N,\tau)<\delta\\ \hat{\sigma}^N\in K_{L\sqrt{\cdot}}\cap\Gamma_L}} l_{K,1}^N\\ &\qquad= \tilde{\tau}^0(K)-\tilde{\tau}^1(K)+\frac{1}{2}\int_0^t\tau(|\tilde{\tau}^s(\nabla_s K|\mathcal{B}_s)|^2)ds\\ &\qquad= -I(\tau)\end{aligned}$$

Therefore we get, for any $L \in \mathbb{R}^+$,

$$\begin{aligned}&\lim_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{N^2}\log\mathbb{P}\left(d(\hat{\sigma}^N,\tau)<\delta\right)\geq\\ &\quad -I(\tau)+\lim_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{N^2}\log\ \mathbb{E}\left[1_{d(\hat{\sigma}^N,\tau)<\delta;\hat{\sigma}^N\in K_{L\sqrt{\cdot}}\cap\Gamma_L}\zeta_K^N(1)\right].\end{aligned}$$

In order to conclude, we need to show that the last term in the above r.h.s. vanishes. This follows from Proposition 6.3 and the observation that

$$\lim_{L\to\infty}\liminf_{N\to\infty}\mathbb{P}_K^N(\hat{\sigma}^N\in K_{L\sqrt{\cdot}}\cap\Gamma_L)=1$$

by (12) and Lemma 5.4. ☐

Observe for later purpose (see section 7) that we can also prove

Corollary 6.5. *Let* $K \in \mathcal{F}_{[0,1]}^{m,sa}$ *and let* τ *be the associated tracial state as in Theorem 6.1, then one has*

$$\lim_{R\to\infty}\lim_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{N^2}\log I\!P(d(\hat{\sigma}^N,\tau)<\delta;||H_1^{N,i}||\leq R,1\leq i\leq m)\geq -I(\tau).$$

Proof. In view of the previous proof, this amounts to showing that

$$\limsup_{R\to\infty}\limsup_{N\to\infty}\mathbb{P}_K^N(\exists i\in\{1,\dots,m\}, ||H_1^{N,i}||\geq R)<1$$

By Girsanov formula, under $\mathbb{P}_K^N$, the process $(H_t^{N,1},\dots,H_t^{N,m})_{t\geq 0}$ is a strong solution to the stochastic differential equation

$$dH_t^{N,i}=dW_t^{N,i}+E[\nabla_t^i K|\mathcal{H}_t](H^{N,1},\dots,H^{N,m})dt$$

where WN,i are hermitian brownian motions. In particular,

$$||H_1^{N,i}||\leq ||W_1^{N,i}||+\sup_{t\in[0,1]}||E[\nabla_t^i K|\mathcal{H}_t]||\leq C+||W_1^{N,i}||$$

for a finite constant C. The result thus follows from the well known fact that the spectral radius of Wigner's matrices converges almost surely to 2. □

7. Large deviation functional and non-commutative entropies

7.1. Comparison between Voiculescu's first definition of free entropy and the large deviation functional. Let us recall Voiculescu's first definition of free entropy [**21**]. Let $\mathbb{C}_H^m:=\mathbb{C}\langle X_1,\dots,X_n\rangle$ be the free *-algebra generated by m self-adjoint elements $(X_1,\dots,X_n)$. Let $\mathcal{M}(\mathbb{C}_H^m)$ be the set of tracial states on $\mathbb{C}\langle X_1,\dots,X_n\rangle$, for which $X_1,\dots,X_n$ are bounded in the GNS representation. For $n\in\mathbb{N}$, $\mu\in\mathcal{M}(\mathbb{C}_H^m)$, $R>0$, $N\in\mathbb{N}$, $\epsilon>0$, let $\Gamma_R(\mu,n,N,\epsilon)$ be the set of n-tuples of $N\times N$ Hermitian matrices $(A_1,..,A_n)$ such that

$$|\tau(X_{i_1}..X_{i_p})-\mathrm{tr}_N(A_{i_1}..A_{i_p})|<\epsilon$$

for all $1\leq p\leq n$, $i_1,..,i_p\in\{1,..,n\}^p$, and $||A_j||_\infty\leq R$. The free entropy of μ is defined as

$$\chi(\mu)=\sup_{R>0}\inf_{n\in\mathbb{N}}\inf_{\epsilon>0}\limsup_{N\to\infty}\left(N^{-2}\log\lambda(\Gamma_R(\mu,n,N,\epsilon))+\frac{n}{2}\log N\right).$$

Replacing Lebesgue measure λ by the Gaussian probability measure

$$P(dA_1^N,..,dA_m^N)=\frac{1}{Z_N^n}\exp\left\{-\frac{N^2}{2}\mathrm{tr}_N\big(\sum_{k=1}^n(A_k^N)^2\big)\right\}\lambda(dA_1^N,..,dA_n^N),$$

if we set

$$\underline{\chi}(\mu)=\sup_{R>0}\inf_{n\in\mathbb{N}}\inf_{\epsilon>0}\limsup_{N\to\infty}\frac{1}{N^2}\log P\big(\Gamma_R(\mu,n,N,\epsilon)\big),$$

then a simple Laplace's estimate yields

$$\chi(\mu)=\underline{\chi}(\mu)+\frac{1}{2}\sum_{k=1}^n\mu(X_k^2)+nC$$

for some well defined finite constant C.

Keeping in tune with our previous computations, we shall rather consider the following entropy, using functional calculus on unitary matrices. Let $\mathbb{C}_U^m=\mathbb{C}\langle U_1,\dots,U_n,U_1^{-1},\dots,U_n^{-1}\rangle$ be the $*$-algebra of the free group with n generators, and let $\mathcal{M}(\mathbb{C}_U^m)$ to be the set of tracial states on $\mathbb{C}_U^m$. Consider $\Gamma_R^U(\nu,n,N,\epsilon)$ the set of n-tuples of unitary $N\times N$ matrices $V_1,..,V_n$ satisfying

$$|\nu(U_{i_1}^{\varepsilon_1}..U_{i_p}^{\varepsilon_p})-\mathrm{tr}_N(V_{i_1}^{\varepsilon_1}..V_{i_p}^{\varepsilon_p})|<\epsilon$$

for all $1 \leq p \leq n$, $i_1, .., i_p \in \{1, .., n\}^p$, $\varepsilon_1, \cdots, \varepsilon_p \in \{-1, +1\}^p$, and

$$-1 \leq 2^{-1}(V_j + V_j^*) \leq 1 - 2(R^2+1)^{-1}$$

for all $j \in \{1, \cdots n\}$. Define

$$\widetilde{\chi}(\nu) := \sup_{R>0} \inf_{n \in \mathbb{N}} \inf_{\epsilon > 0} \limsup_{N \to \infty} \frac{1}{N^2} \log P\big((\psi(A_1^N), \cdots, \psi(A_n^N)) \in \Gamma_R^U(\mu, n, N, \epsilon)\big).$$

Denote Ψ the map from $\mathbb{C}\langle X_1, \dots, X_n \rangle$ into $\mathbb{C}\langle U_1, \dots, U_n, U_1^{-1}, \dots, U_n^{-1} \rangle$ such that

$$\Psi(X^1, \dots, X^m)_l = \psi(X^l) = \frac{X^l + 4i}{X^l - 4i}.$$

Then, we claim that

Lemma 7.1. *For any $\mu \in \mathcal{M}_H^m$ for which the GNS representation consists of bounded operators one has*

$$\chi(\mu) = \widetilde{\chi}(\mu \circ \Psi).$$

Proof. The only point is to notice that ψ can be approximated by polynomial functions uniformly on compact sets, as well as its inverse on $\Re(z) \in [-1, 1 - 2(R^2 + 1)^{-1}] \cap \{|z| = 1\}$.

$\square$

For any state $\tau \in \mathcal{M}^{c,\infty}(\mathcal{F}_{[0,1]}^m)$ one can consider the state $\tau \circ \pi_1$ on $\mathbb{C}_U^m$ induced by the evaluation map at time 1, namely for any $P \in \mathbb{C}_U^m$ one has $\tau \circ \pi_1(P) = \tau(P(U^1(1), \dots, U^m(1)))$. Now, by Theorems 5.1 and 6.4 and its corollary 6.5, we obtain the following result.

Theorem 7.2. *Denote by τ_K the state associated with $K \in \mathcal{F}_{[0,1]}^{m,sa}$, as in Theorem 6.1. For any $\mu \in \mathcal{M}_U^m$, one has*

$$-\lim_{\delta \to 0} \inf\{I(\tau_K); d(\tau_K \circ \pi_1, \mu) < \delta, K \in \mathcal{F}_{[0,1]}^{m,sa}\} \leq \widetilde{\chi}(\mu) \tag{15}$$

and

$$\begin{aligned} \widetilde{\chi}(\mu) &\leq -\lim_{\delta \to \infty} \inf\{I(\tau); d(\tau \circ \pi_1, \mu) < \delta, \tau \in \mathcal{M}^{c,\infty}(\mathcal{F}_{[0,1]}^m)\} \\ &= -\inf\{I(\tau), \tau \circ \pi_1 = \mu\}. \end{aligned} \tag{16}$$

If μ is of the form $\tau_K \circ \pi_1$ for some state $K \in \mathcal{F}_{[0,1]}^{m,sa}$, or if it is in the closure of the set of such states, then the upper and lower bounds for free entropy given by Theorem 7.2 coincide and we have a formula for Voiculescu's free entropy. This raises the problem of determining which states satisfy this condition. One could hope that any state indeed satisfies it, but this seems to be a very difficult question since the truth of this would imply in particular a positive solution to Connes' problem of embedding arbitrary II_1-factors into the ultraproduct of the hyperfinite factor.

In the much simpler case where one considers the spectral measure of a non-centered Gaussian Wigner's matrix, O. Zeitouni and A. Guionnet have shown that upper and lower bounds given by Theorem 7.2 coincide but the proof is already highly non trivial.

7.2. The infimum is reached at a brownian bridge. Following [**8**], we shall now prove that the infimum in the right hand side of (16) can be replaced by the value of I at the law of the Brownian bridge. Hereafter, for $\mu \in \mathcal{M}^{c,\infty}(\mathcal{F}^m_{\{1\}}) \subset \mathcal{M}^m_U$, we denote $\tau_\mu \in \mathcal{M}^{c,\infty}(\mathcal{F}^m_{[0,1]})$ the distribution of the process

$$\left\{\Psi\left(tX^l + (1-t)S^l_{\frac{t}{1-t}}, 1 \le l \le m\right), t \in [0,1]\right\}$$

where $\{X^1, \dots, X^m\}$ is distributed as $\mu \circ \Psi^{-1}$ and $\{S^1, \dots, S^m\}$ is a m-dimensional free Brownian motion, free with $\{X^1, \dots, X^m\}$.

THEOREM 7.3. *For any* $\mu \in \mathcal{M}(\mathcal{F}^m_{\{1\}})$, *one has*

$$\tilde{\chi}(\mu) \le -I(\tau_\mu). \tag{17}$$

Proof. For any $\mu \in \mathcal{M}^{c,\infty}(\mathcal{F}^m_{\{1\}})$, any $\delta, \epsilon > 0$, one has

$$\mathbb{P}\left(d(\hat{\sigma}^N \circ \pi_1, \mu) \le \epsilon\right) = \frac{\mathbb{P}\left(d(\hat{\sigma}^N \circ \pi_1, \mu) \le \epsilon; d(\hat{\sigma}^N, \tau_\mu) \le \delta\right)}{(1 - \mathbb{P}\left(d(\hat{\sigma}^N, \tau_\mu) > \delta | d(\hat{\sigma}^N \circ \pi_1, \mu) \le \epsilon)\right)\}} \tag{18}$$

For any n-tuple of matrices $X^1, \dots, X^m$, the conditional distribution of hermitian brownian motion knowing that $H^{N,1}(1) = X^1, \dots H^{N,m}(1) = X^m$ is that of $\left(tX^l + (1-t)\tilde{H}^{N,l}_{\frac{t}{1-t}}, 1 \le l \le m\right), t \in [0,1]$ for some hermitian brownian motion $(\tilde{H}^{N,1}, \dots, \tilde{H}^{N,m})$. For N large, $X^1, \dots, X^m$ and $(\tilde{H}^{N,1}, \dots, \tilde{H}^{N,m})$ are asymptotically free, the distribution of the hermitian brownian motion being close to that of free brownian motion. It follows that for any $\delta > 0$ one has

$$\limsup_{\epsilon \to 0} \limsup_{N \to \infty} \mathbb{P}\left(d(\hat{\sigma}^N, \tau_\mu) > \delta | d(\hat{\sigma}^N \circ \pi_1, \mu) \le \epsilon\right) = 0$$

We deduce that for any $\delta > 0$

$$\limsup_{\epsilon \to 0} \limsup_{N \to \infty} \frac{1}{N^2} \log \mathbb{P}\left(d(\hat{\sigma}^N \circ \pi_1, \mu) \le \epsilon\right)$$

$$= \limsup_{\epsilon \to 0} \limsup_{N \to \infty} \frac{1}{N^2} \log \mathbb{P}\left(\{d(\hat{\sigma}^N \circ \pi_1, \mu) \le \epsilon\} \cap \{d(\hat{\sigma}^N, \tau_\mu) \le \delta\}\right) \tag{19}$$

Using theorem 5.1 we get

$$\limsup_{\epsilon \to 0} \limsup_{N \to \infty} \frac{1}{N^2} \log \mathbb{P}\left(d(\hat{\sigma}^N \circ \pi_1, \mu) \le \epsilon\right) \le - \lim_{\delta \to 0} \inf_{d(\tau, \tau_\mu) < \delta} I(\tau) = -I(\tau_\mu)$$

where we have used in the last equality that I is a good rate function. □

We will now establish for the free brownian bridge an equation which is the analogue of a well known property of the usual brownian bridge.

PROPOSITION 7.4. *The state* τ_μ *satisfies the equation, for all* $P \in \mathcal{F}^m_{[0,1]}$, *all* $t \in [0,1]$,

$$\begin{aligned} \tilde{\tau_\mu}^t(P) &= \tilde{\tau_\mu}^0(P) + \int_0^t \tau_\mu\left(\tilde{\tau_\mu}^s(\nabla_s P | \mathcal{B}_s).\frac{X - X_s}{1-s}\right) ds \\ &= \tilde{\tau_\mu}^0(P) + \int_0^t \tilde{\tau_\mu}^s\left(\tilde{\tau_\mu}^s(\nabla_s P | \mathcal{B}_s).\tau_\mu\left(\frac{X - X_s}{1-s} | \mathcal{B}_s\right)\right) ds. \end{aligned} \tag{20}$$

Proof. First, we remark that one has

$$X_t = S_t + \int_0^t \frac{X - X_s}{1-s} ds$$

where $(S_t; 0 \leq t \leq 1)$ is a free Brownian motion, free with X. Now we can take a dual projection onto the subfiltration $\mathcal{B}_s$, generated by the process X (i.e. $\mathcal{B}_s$ is the von Neumann algebra generated by $X_u; u \leq s$), to get that

$$X_t = \hat{S}_t + \int_0^t \tau_\mu \left(\frac{X - X_s}{1-s} | \mathcal{B}_t \right) ds \tag{21}$$

where $(\hat{S}_t; 0 \leq t \leq 1)$ is a martingale in the filtration $\mathcal{B}_s$. In fact, is not difficult to check, using Theorem 6.2 that indeed $(\hat{S}_t; 0 \leq t \leq 1)$ as defined by (21) is a free Brownian motion. This implies formula (20), as in the proof of Theorem 6.1. □

Since the free Brownian bridge is a Markov process, one has

$$\tau_\mu \left(\frac{X - X_s}{1-s} | \mathcal{B}_s \right) = \tau_\mu \left(\frac{X - X_s}{1-s} | X_s \right) = \frac{X_s}{s} - \mathcal{J}^{\mu_s}$$

where $\mathcal{J}^\mu$ denote the non-commutative Hilbert transform of $\mu \in \mathcal{M}_U^m$ and $\mu_t = \tau_\mu \circ \pi_t$ is the time marginal of the free Brownian bridge. The last line equality a direct consequence of Corollary 3.9 in [**23**].

From this and Theorem 7.3 we deduce that for any $\mu \in \mathcal{M}(\mathcal{F}^m_{\{1\}})$ such that $\chi^*(\mu) > -\infty$, one has

$$\widetilde{\chi}(\mu) \leq \chi^*(\mu) + \frac{1}{2} \inf_{P \in \mathcal{F}^m_{[0,1]}} \int_0^1 \widetilde{\tau_\mu}^t \left[\left| \mathcal{J}^{\mu_t} - \widetilde{\tau_\mu}^t (\nabla_t P | \mathcal{F}_t) - \frac{X_t}{t} \right|^2 \right] dt \tag{22}$$

where

$$\chi^*(\mu) = -\frac{1}{2} \int_0^1 \widetilde{\tau_\mu}^t \left[\left| \mathcal{J}^{\mu_t} - \frac{X_t}{t} \right|^2 \right] dt = -\frac{1}{2} \int_0^1 \widetilde{\tau_\mu}^t \left[|\mathcal{J}^{\mu_t}|^2 \right] dt - \frac{1}{2} \sum_{i=1}^m \mu(X_i^2) + mC$$

is the microstates-free entropy, up to a Gaussian term.

We now show that the second term in the right hand side in (22) vanishes. It is enough to prove that functions of the form $t \mapsto \widetilde{\tau_\mu}^t (\nabla_t P | \mathcal{B}_t)$ form a total family in the Hilbert space of functions $t \mapsto Y_t$ with $Y_t \in \mathcal{B}_t$ (Y is adapted) and $\tau_\mu(\int_0^1 |Y_s|^2 ds) < \infty$. Indeed, for any $0 < t_0 < \cdots < t_n \leq 1$, any $(j_i)_{1 \leq i \leq n} \in \{1, \cdots, m\}$, any family $Q_i \in \mathcal{F}^{m,sa}_{[0,t_{i-1}]}$ for $1 \leq i \leq n$, if Q is defined by

$$Q = \sum_{i=1}^n \left((1 - u_{t_i}^{j_i})^{-1} - (1 - u_{t_{i-1}}^{j_i})^{-1} \right) Q_i \tag{23}$$

then it satisfies

$$\widetilde{\tau}^u (\nabla_u^j [Q] | \mathcal{B}_u) = \frac{1}{8i} \sum_{i=1}^n 1_{(t_{i-1}, t_i]}(u) Q_i(u). \tag{24}$$

Note that we have been cheating a little since $(1 - u_{t_i}^{j_i})^{-1}$ does not belong to $\mathcal{F}^m_{[0,1]}$, however as noted earlier, the Malliavin operator can be extended to this situation, and one has

$$\nabla_s^l (1 - u_t^j)^{-1} = \frac{1_{l=j}}{8i} 1_{[0,t]}(s).$$

Hence, the family $\{(\widetilde{\tau_\mu}^t(\nabla_t P|\mathcal{B}_t))_{t\in[0,1]}, P \in F^{m,sa}_{[0,1]}\}$ contains the adapted stepwise constant functions. We show that these processes form a dense subspace of square integrable adapted processes. Let H be the Hilbert space obtained by the GNS construction for τ. By considering coordinates, it is easy to see that $C_0([0,1],H)$ is dense in $L^2([0,1],H)$, therefore if δ is positive, continuous, has support in $[-1,0]$, and integral 1, then any L^2 adapted process f_t is the limit of $f*\delta_n$ with $\delta_n(x) = n\delta(nx)$, which is a continuous adapted process. Finally any continuous adapted process can be approximated in L^2 by stepwise adapted processes. This yields the required density.

Finally, observe that the above bounds can be generalized to tracial states for which $\chi^*(\mu) = -\infty$ by noticing that $I(\tau_\mu)$ can be approximated by its restriction to the time interval $[\delta, 1-\delta]$ for $\delta > 0$. For this retriction, the non-commutative Hilbert transform of τ_μ is uniformly bounded and therefore belong to L^2, so that the above arguments hold. Lettin δ going to zero shows that $\chi(\mu) \leq \chi^*(\mu)$ even when $\chi^*(\mu) = -\infty$.

Putting things together, we get the following inequality between the two entropies of Voiculescu.

COROLLARY 7.5. *For any tracial state τ,*

$$\chi(\mu) \leq \chi^*(\mu)$$

8. Proof of Theorem 5.2

Observe first that, by an elementary quadratic form argument,

$$I(\tau) = \frac{1}{2} \sup_{P\in\mathcal{F}^{m,sa}_{[0,1]}} \sup_{t\in[0,1]} \left\{ \frac{\left(\widetilde{\tau}^t(P) - \widetilde{\tau}^0(P)\right)^2}{\int_0^t \tau\left(|\widetilde{\tau}^s\left(\nabla_s P|\mathcal{B}_s\right)|^2\right) ds} \right\} \tag{25}$$

so that $I \geq 0$.

Let us now show that the level sets of I are compact in $\mathcal{M}^{c,\infty}(\mathcal{F}^m_{[0,1]})$. We shall prove that for any $M \in \mathbb{R}_+$, there exists $L(M) < \infty$ such that

$$\{I \leq M\} \subset K_{L(M)\sqrt{\cdot}} \cap \Gamma_{L(M)}. \tag{26}$$

which implies, by Lemma 2.1, that $\{I \leq M\}$ is relatively compact. Since $I|_{K_{L(M)\sqrt{\cdot}}\cap\Gamma_{L(M)}}$ is lower semicontinuous, as the supremum of the functions

$$l_{P,t}(\tau) = \widetilde{\tau}^t(P) - \widetilde{\tau}^0(P) - \frac{1}{2}\int_0^t \tau\left\{\left|\widetilde{\tau}^s\left(\nabla_s P|\mathcal{B}_s\right)\right|^2\right\} ds$$

which are continuous on $K_{L(M)\sqrt{\cdot}} \cap \Gamma_{L(M)}$, we obtain that $\{I \leq M\}$ is compact. We first show that for any $M \in \mathbb{R}_+$, there exists $L_1(M) < \infty$ such that

$$\{I \leq M\} \subset \Gamma_{L_1(M)}. \tag{27}$$

Let, for $n \in \mathbb{N}^*$,

$$f_n(z) = (1 + n^{-1} - \Re(z))^{-1}$$

then for any $t \in [0,1]$, any $l \in \{1, \dots, m\}$, any $\tau \in \{I \leq M\}$, any $s \in [0,1]$, one has

$$\widetilde{\tau}^s(f_n(u_t^l)) \leq \widetilde{\tau}^0(f_n(u_t^l)) + M + \frac{1}{2}\int_0^s \tau\left(\widetilde{\tau}^s(\nabla_u f_n(u_t^l)|\mathcal{B}_u)^2\right) du. \tag{28}$$

$$\nabla_u f_n(u_t^l) \quad = \quad -i 1_{u \le t} f_n(u_t^l)^2 \left(\frac{u_t^l + (u_t^l)^*}{2} - 1 \right) (u_t^l - (u_t^l)^*) \tag{29}$$

Furthermore, for any $n \in \mathbb{N}$, and any z with $|z| = 1$, one has

$$|(\Re(z) - 1) f_n(z)| \le 1, \tag{30}$$

and, if we let $a_n = 1 + n^{-1} + \sqrt{(1 + n^{-1})^2 - 1}$, so that

$$1 + n^{-1} - \Re(z) = (2a_n)^{-1} |z - a_n|^2$$

for $|z| = 1$, then

$$\begin{aligned} |f_n(z)(z - \bar{z})| &\le 2 |f_n(z)(z - a)| \\ &= 4a_n |z - a_n|^{-1} = 2\sqrt{2a_n f_n(z)} \end{aligned} \tag{31}$$

Equations (29), (30) and (31) show that for any $t \in [0, 1]$, one has

$$|\nabla_u f_n(u_t^l)|^2 \le 1_{u \le t} 8 a_n f_n(u_t^l). \tag{32}$$

Plugging this estimate into (28) shows that for any $t \in [0, 1]$, any $\tau \in \{I \le M\}$, any $s \in [0, 1]$,

$$\begin{aligned} \tilde{\tau}^s(f_n(u_t^l)) &\le \tilde{\tau}^0(f_n(u_t^l)) + M + \frac{1}{2} \int_0^s \tau \left(\tilde{\tau}^u (\nabla_u^l f_n(u_t^l) | \mathcal{B}_u)^2 \right) du \\ &\le \tilde{\tau}^0(f_n(u_t^l)) + M + \frac{1}{2} \int_0^s \tilde{\tau}^u \left(\nabla_u f_n(u_t^l))^2 \right) du \\ &\le \tilde{\tau}^0(f_n(u_t^l)) + M + 4a_n \int_0^{s \wedge t} \tilde{\tau}^u \left(f_n(u_t^l) \right) du. \end{aligned} \tag{33}$$

Therefore, by Gronwall's lemma (since $u \mapsto \tilde{\tau}^u \left(f_n(u_t^l) \right)$ is uniformly bounded when $n \in \mathbb{N}$) that

$$\tilde{\tau}^s(f_n(u_t^l)) \le \left(\tilde{\tau}^0(f_n(u_t^l)) + M \right) e^{4a_n s \wedge t}. \tag{34}$$

Since

$$\sup_{t \in [0,1]} \sup_{n \in \mathbb{N}} \tilde{\tau}^0(f_n(u_t^l)) := \frac{1}{2} \sup_{t \in [0,1]} \sup_{n \in \mathbb{N}} \sigma \left(\frac{S_t^2 + 1}{2 + n^{-1} + n^{-1} S_t^2} \right) \le 2,$$

we deduce from (34) that

$$\max_{l \in \{1, \dots, m\}} \sup_{t, s \in [0,1]} \sup_{n \in \mathbb{N}} \tilde{\tau}^s(f_n(u_t^l)) \le (2 + M) e^4. \tag{35}$$

Finally, observe that f_n are, on $|z| = 1$, non negative functions which grow pointwise towards $2|1 - z|^{-2}$ so that monotone convergence theorem and (35) result with

$$\max_{l \in \{1, \dots, m\}} \sup_{t, s \in [0,1]} \tilde{\tau}^s(|1 - U_t^l|^{-2}) \le (1 + M) e^1$$

and therefore (27) holds with $L_1(M) = (1 + M)e^4$. We now prove that for any $M \in \mathbb{N}$, there exists $L_2(M)$ such that

$$\{I \le M\} \subset K_{L_2(M)\sqrt{\cdot}}. \tag{36}$$

To this end, we take, for $s \le t \in [0, 1]$, $P = |u_t^l - u_s^l|^2 \in \mathcal{F}_{[0,1]}^m$ and observe that for any $u \in [0, 1]$,

$$|\nabla_u P|(U) \le 2^4 1_{s \le u \le t} + 2^2 1_{u \le s} |U_t^l - U_s^l|^2.$$

Therefore, using (25), we obtain that, for any $\tau \in \{I \le M\}$, any $v \in [0,1]$,

$$\begin{aligned}(37)\ \tilde{\tau}^s(|U_t^l - U_s^l|^2) \quad &\le \quad \tilde{\tau}^0(|U_t^l - U_s^l|^2) + \\ &\sqrt{2M}\left(2^9|t-s| + 2^5\int_0^{v\wedge s}\tau\left(\tilde{\tau}^s(|U_t^l - U_s^l|^2|\mathcal{B}_u)^2\right)du\right)^{\frac{1}{2}}.\end{aligned}$$

Now for $u \le s$, one has

$$\begin{aligned}(38)\qquad \tilde{\tau}^s(|U_t^l - U_s^l|^2|\mathcal{B}_u) \quad &= \quad 4\tau\Big(\Big|(\psi^{-1}(U_u^l) + S_{t-u}^l + i)^{-1} - \\ &\qquad (\psi^{-1}(U_u^l) + S_{s-u}^l + i)^{-1}\Big|^2|\mathcal{B}_u\Big) \\ &\le \quad 4\sigma\left(|S_{t-u}^l - S_{s-u}^l|^2\right) = 2^4|t-s|.\end{aligned}$$

and

$$\begin{aligned}(39)\ \ \tilde{\tau}^0(|U_t^l - U_s^l|^2) \quad &= \quad 4\sigma\left(\left|(S_t^l + i)^{-1} - (S_s^l + i)^{-1}\right|^2\right) \le 4\sigma\left(\left|S_t^l - S_s^l\right|^2\right) \\ &= \quad 2^4|t-s|.\end{aligned}$$

so that (37), (38) and (39) imply that

$$(40)\qquad \tilde{\tau}^s(|U_t^l - U_s^l|^2) \le 2^4|t-s| + 2^6\sqrt{M}|t-s|^{\frac{1}{2}},$$

from which we deduce (36) with $L_2(M) = 2^4 + 2^6\sqrt{M}$.Equations (27) and (36) imply (26) with $L(M) = \max\{L_1(M), L_2(M)\}$.

We now prove points a) and b) of the second claim. Let $\tau \in \mathcal{M}^{c,\infty}(\mathcal{F}_{[0,1]}^m)$ be such that $I(\tau) < +\infty$, and consider the Hilbert space of adapted processes $s \mapsto L_s \in \mathcal{B}_s$ such that $\int_0^t \tau(|L_s|^2)ds < \infty$, and the closed subspace generated by the processes $s \mapsto \tilde{\tau}^s(\nabla_s P|\mathcal{B}_s)$ where $P \in \mathcal{F}^{m,sa}{}_{[0,1]}$. According to (25), when $I(\tau) < +\infty$, one has

$$(41)\qquad (\tau(P) - \sigma(P))^2 \le 2I(\tau)\int_0^1 \tau\left(|\tilde{\tau}^s(\nabla_s P|\mathcal{B}_s)|^2\right)ds$$

therefore, by Riesz's Theorem there exists K_s such that for every $P \in \tilde{\mathcal{F}}_{[0,t]}^{m,sa}$ one has

$$(42)\qquad \tau(P) - \sigma(P) = \int_0^1 \tau\left(\tilde{\tau}^s(\nabla_s P|\mathcal{B}_s).K_s\right)ds.$$

which completes the proof of the theorem.

□

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