CAUCHY-RIEMANN 3-MANIFOLDS AND EINSTEIN FILLINGS

OLIVIER BQUARD

In 1985 Fefferman and Graham started a program to study conformal geometry by attaching to a conformal manifold in dimension $d$ an Einstein metric in dimension $d+1$, and reading conformal invariants from Riemannian invariants of the Einstein metric. The relation is by looking at asymptotically hyperbolic Einstein metrics, with the conformal structure obtained as a conformal limit at infinity. This program was extremely fruitful, especially with the recent insight of the physicists who understood that it is related to their AdS/CFT correspondence (Maldacena, Witten).

The goal of this article is an exposition of a circle of similar ideas in a different context: three dimensional strictly pseudoconvex CR structures and four dimensional Einstein metrics. The Einstein metrics are now modeled at infinity on the complex hyperbolic space (asymptotically complex hyperbolic, or ACH, metrics), and the CR structure again is obtained from the Einstein metric as a conformal limit at infinity. CR structures as boundaries at infinity of Kähler-Einstein metrics have been considered a long time ago (see for example [CY80,BE90]), but their a priori less natural study as boundaries at infinity of Einstein metrics began only in [Biq00]—after the first explicit $SU_2$-invariant examples of Hitchin [Hit95]. This leads to a rich geometry with new insight on the classical subject of three dimensional CR manifolds.

The topics covered in this article are: existence, asymptotic development and regularity at infinity of ACH Kähler-Einstein and Einstein metrics; selfdual Einstein ACH metrics and their CR boundaries; a new intrinsic invariant of three dimensional CR manifolds coming from the correspondence between CR manifolds and ACH Einstein metrics, and applications to filling CR manifolds. This is based on the articles [Biq00,BH05,Biq05,BHR].

1. CAUCHY-RIEMANN STRUCTURES

A CR (Cauchy-Riemann) structure on a manifold $X^3$ is a codimension 1 distribution $H \subset TX$ with a complex structure $J$ on $H$.

**Example 1.** If $M^4$ is a complex manifold and $X \subset M$ is a real hypersurface, then $X$ has a natural CR structure where $H$ is the maximal complex subspace in $TX$.

Suppose $M$ is oriented (for example, this is the case if $M$ is the boundary of a complex manifold). A CR structure on $M$ gives an orientation
on $H$ and therefore on $TM/H$, so it makes sense to look at a 1-form $\eta$ on $M$ with kernel $H$, and positive on $TM/H$. The Levi form of $M$ is the quadratic form $\gamma = d\eta(\cdot, J\cdot)$, it depends on the choice of $\eta$ only up to a positive function. The CR structure is strictly pseudoconvex if the Levi form is positive. In this case the underlying distribution $H$ is a contact distribution (compatible with the given orientation).

Example 2. Suppose $X$ is the boundary of a domain $M$ in $\mathbb{C}^2$, with defining function $\phi$, so that $M = \{\phi < 0\}$. Then one can choose $\eta = d^C\phi$ and the Levi form is $dd^C\phi = 2i\partial \bar{\partial} \phi$.

It is well-known that strictly pseudoconvex structures in dimension 3 are not all embeddable as hypersurfaces of $\mathbb{C}^2$. Actually, deep results of Epstein [Eps92], Lempert [Lem92] and Bland [Bla94] tell us when a CR deformation of the standard 3-sphere $S^3$ is embeddable in $\mathbb{C}^2$. Moreover the stability theorem of Lempert says that, if a small CR deformation of the 3-sphere bounds a small complex deformation of the 4-ball, then it is embeddable in $\mathbb{C}^2$ as a small deformation of $S^3$.

Example 3. The left-invariant CR structures on the 3-sphere are not embeddable (of course except the standard one).

Deformations. Given a strictly pseudoconvex CR manifold $X^3$, the deformations of $X$, as a CR manifold, can be parameterized as follows: first, up to diffeomorphisms, the contact structure does not change, so one can suppose that it is fixed. Therefore, one has only to parameterize a complex structure $J$ in the contact directions $H$: the tangent space is the sections of the bundle $\sigma^2_0 H$ of symmetric trace-free endomorphisms of $H$ (or, equivalently, $J$-anti-linear endomorphisms). From the point of view of complex geometry, one can parameterize a nearby CR structure by a tensor $\varphi \in \Omega^{0,1} \otimes T^{1,0}$ representing the $T^{0,1}$ space of the new CR structure as a graph $T^{0,1} \rightarrow T^{1,0}$; the corresponding section of $\sigma^2_0 H$ is $\varphi + \varphi^*$.

The group of contactomorphisms of $X$ acts on the space of contact structures. Fix a contact form $\eta$ and the corresponding Reeb vector field $R$. An infinitesimal contactomorphism can be parameterized by a real function $f$, corresponding to the vector field $f R + \xi df$, where $\xi : \Omega^1 X \rightarrow TX$ is the duality induced by the symplectic form $d\eta$. Then on $S^3$ the infinitesimal action of the contactomorphisms is by

\[ f \rightarrow \tilde{\xi} \tilde{\partial} f \in \Omega^{0,1} \otimes T^{1,0}, \]

and the kernel can be identified with the Lie algebra of the group $SU_{1,2}$ of CR-automorphisms of $S^3$.

Cartan curvature. Finally, there exists a fourth order tensor of any CR structure, the Cartan curvature $Q$, section of $\Omega^{0,1} \otimes T^{1,0}$, whose vanishing implies that the CR structure is locally isomorphic to the standard structure on $S^3$: such CR structures are called spherical CR structures. At
the standard CR structure on $S^3$, this fits into a deformation complex for CR structures:

$$
\Omega^0 \rightarrow \Omega^0 \otimes T^{1,0} \xrightarrow{dQ} \Omega^0 \otimes T^{1,0} \rightarrow \Omega^0.
$$

The first arrow is the infinitesimal action (i), the middle arrow the differential of the Cartan curvature, and the last arrow the linearization of a Bianchi identity always satisfied by the Cartan curvature: here $\Re(\bar{\partial} \bar{\partial})^*$. The fourth order operator $dQ$ is hypoelliptic transversely to the action of contactomorphisms, and its kernel reduces to the infinitesimal action of the contactomorphisms [CL90].

2. ACH METRICS

The ball in $\mathbb{C}^2$ has a natural complete Kähler-Einstein metric, the Bergman metric—a model for complex hyperbolic space. An explicit formula, with sectional curvature normalized between $-4$ and $-1$ (constant holomorphic sectional curvature $-4$), is, if $\rho$ denotes the radius in $\mathbb{C}^2$:

$$
g = d\rho^2 + \rho^2 \eta^2 + \gamma + \frac{2}{1 - \rho^2},
$$

where $\eta = J(d\rho/\rho)$ is the standard connection 1-form on the Hopf bundle $S^3 \rightarrow S^2$, and $\gamma$ is the restriction of the standard metric of $S^3$ to the contact distribution, actually $\gamma(\cdot, \cdot) = d\left(\frac{\eta}{2}\right)(\cdot, J\cdot)$. In polar coordinates, with $\rho = \tanh(r)$, one gets

$$(iii)\quad g = dr^2 + \sinh^2(2r)\left(\frac{\eta}{2}\right)^2 + \sinh^2(r)\gamma.
$$

One can recover the CR structure on the sphere at infinity from the metric by taking the limit on concentric spheres $S_\rho$:

$$
\gamma = \lim_{\rho \to 1} (1 - \rho^2)g|_{S_\rho}.
$$

The limit is finite only on the contact distribution $H$. The conformal factor $(1 - \rho^2)$ could be changed by multiplication by a positive function on $S^3$, so only the conformal class of $\gamma$ is well defined, that is the metric induces the CR structure $J$ on the boundary at infinity.

**Cheng-Yau metric.** More generally a strictly pseudoconvex domain in $\mathbb{C}^n$ carries a unique complete Kähler-Einstein metric [CY80]. Moreover, the asymptotic behavior of this Cheng-Yau metric is similar to the one of the Bergman metric, that is

$$(iv)\quad g \sim \frac{d\phi^2 + 4\eta^2}{4\phi^2} + \frac{\gamma}{2\phi},
$$

or, equivalently,

$$(v)\quad g \sim g_0 = dr^2 + \frac{1}{4}e^{4r}\eta^2 + \frac{1}{4}e^{2r}\gamma,$$

where $\phi = 2e^{-2r}$ is a defining function of the boundary, $\eta$ a contact form on the boundary, $\gamma(\cdot, \cdot) = d\eta(\cdot, J\cdot)$. 

In view of the asymptotics of the Bergman metric, or more generally of the Cheng-Yau metric, it is natural to make the following definition.

**Definition 4.** Let $X^3$ be a strictly pseudoconvex CR manifold, and $M^4$ a manifold with $\partial M = X$. Identify a neighborhood of $X$ in $M$ with $(0, \infty) \times X$, with coordinate $r$ on the first factor going to infinity. A metric $g$ on $M$ is Asymptotically Complex Hyperbolic (in short ACH) near the boundary $X$ if

$$g = g_0 + O(e^{-r}),$$

where $g_0$ is defined by formula (v), and $O(e^{-r})$ is with respect to $g_0$.

The CR structure is called the conformal infinity, or the boundary at infinity of the metric $g$. As for the Bergman metric, it can be recovered from the metric $g$ by taking a conformal limit of $g$ restricted to the slices $M_R = \{r = R\}$ when $R$ goes to infinity.

It is easy to check that an ACH metric has its curvature tensor close at order $O(e^{-r})$ of the curvature tensor of complex hyperbolic space. This fact might be seen as a justification for the terminology “Asymptotically Complex Hyperbolic”.

**Example 5.** Let $\Sigma$ be a hyperbolic Riemann surface, then the total space of the disk bundle $\sqrt{T\Sigma}$ carries a complete ACH complex hyperbolic metric (induced by a representation $\pi_1(\Sigma) \to SU_{1,1} \subset SU_{1,2}$).

**Asymptotically Real Hyperbolic metrics.** Of course the theory here is parallel to the theory in the real case. The real hyperbolic space, with curvature $-1$, can be written in the model of the ball as

$$g = \frac{4}{(1 - \rho^2)^2} \sum (dx^i)^2,$$

and there is a conformal limit on the sphere at infinity:

$$\gamma = \lim_{\rho \to 1} g|_{S_\rho}.$$

Asymptotically Hyperbolic (AH) metrics on $M^n$ with boundary a conformal metric $\gamma$ on $\partial M = \{\phi = 0\}$ are defined as metrics asymptotic to

$$\frac{d\phi^2 + \gamma}{\phi^2},$$

they have sectional curvature going to $-1$ at the boundary.

**3. Elliptic Operators and ACH Metrics**

In this section, we do not state any precise result, but try to give a rough idea of the analysis of elliptic operators on ACH metrics, mainly through the example of the scalar Laplacian.
So look at a CR 3-manifold $X^3$, and fix a contact structure $\eta$, and therefore a metric $\gamma$ on the contact distribution. We shall denote the Laplacian in horizontal directions by $\square$:

$$\square W = -\sum_{i=1}^{2} (\nabla^W_{e_i})^2,$$

where $e_i$ is an orthonormal basis of the contact distribution, and $\nabla^W$ is the Tanaka-Webster connection. This formula makes sense also on tensor bundles rather than on functions. We shall use also the Reeb vector field $R$ and the derivative $\nabla^W R$.

In complex hyperbolic space, with polar coordinates (iii), one has the following formula for the scalar Laplacian:

$$\Delta = -\partial^2 \partial r^2 - 2(\coth(2r) + \coth(r)) \frac{\partial}{\partial r} + \frac{1}{\sinh^2(2r)} (\nabla^W_R)^2 - \frac{1}{\sinh^2 r} \square W.$$

On a general ACH manifold $M^4$ with conformal boundary $X$, the leading order terms are the same. If one looks only at smooth functions, then the leading term is

$$-\partial^2 \partial r^2 - 4 \frac{\partial}{\partial r}.$$

This is the indicial operator associated to the Laplacian. Solutions for the indicial operator are 1 and $e^{-4r}$, leading to indicial roots 0 and 4. Actually the Laplacian is well-known to be an isomorphism between Sobolev spaces $H^2 \to L^2$, and the fact that the indicial roots are 0 and 4 implies that it is still an isomorphism between weighted spaces corresponding to a decay $e^{-\lambda r}$ for any $0 < \lambda < 4$, that is between spaces

(vi) $H^2_\delta = e^{\delta r} H^2 \to L^2_\delta = e^{\delta r} L^2$

for any $-2 < \delta < 2$ (the translation between $\lambda$ and $\delta$ comes from the fact that the limit decay rate for $L^2$ functions is $e^{-2r}$).

This result can be used to solve the Dirichlet problem on $M$: given a function $\phi$ on $X$, find a harmonic function $f$ in the interior with boundary value $\phi$.

**Formal solution.** First one can try to find the asymptotic development of $f$ near the boundary. Construct a function $f_0$ by extending $\phi$ along rays, then from the formula for the Laplacian it is clear that $\Delta f_0 = O(e^{-2r})$, actually

$$\Delta f_0 = \phi_2 e^{-2r} + O(e^{-3r}),$$

where $\phi_2$ is a function on $X$ with an expression in terms of $\phi$ and its two first derivatives. Solving the indicial problem $(-\partial^2 \partial r^2 - 4 \partial r) f_2 = \phi_2 e^{-2r}$ by $f_2 = \frac{1}{4} e^{-2r} \phi_2$ gives an approximate solution $f_0 + f_2$ satisfying

$$\Delta (f_0 + f_2) = \phi_3 e^{-3r} + O(e^{-4r}).$$
This is the beginning of an induction procedure which stops at the indicial root 4: the indicial problem\((-\frac{\partial^2}{\partial r^2} - 4\frac{\partial}{\partial r}) f_4 = \phi_4 e^{-4r}\) is solved by \(F_4 = -\frac{1}{4} r e^{-4r} \phi_4\), and then
\[
\Delta(f_0 + \cdots + f_3 + F_4) = O(r e^{-5r});
\]
but there is a possible undetermined term of type
\[
f_4 = e^{-4r} \phi.
\]
After that, the formal resolution by functions \(r^j e^{-kr}\) ("polyhomogeneous expansion") continues without further obstruction.

**Global solution.** The undetermined term can be fixed only after finding a global solution: one can look at a finite number of terms of the formal resolution, so one gets an approximate solution \(f\) with \(\Delta f = O(e^{-\lambda r})\). Then we apply the isomorphism theorem \([\text{vi}]\) to correct \(f\) by a \(g \in H^2_\delta\) such that \(\Delta(f + g) = 0\) (actually \(g \in H^\infty_\delta\) by elliptic regularity). There are two cases:

- if \(\lambda \leq 4\) one can take \(\delta = \lambda - 2 - \epsilon\);
- if \(\lambda > 4\) one can take only \(\delta = 4 - \epsilon\): this reflects the fact that \(g\) contains the undetermined term, necessarily at order \(O(e^{-4r})\).

An important fact is that the solution using the isomorphism theorem \([\text{vi}]\) is smooth, but with a very weak control in the directions of \(X\): indeed \(g \in H^\infty_\delta\) only means, for a horizontal vector \(h \in T X\), one has \(e^{-kr} (\nabla^W_h)^k g \in L^2\), and this is very far from saying that the coefficients of the development near \(X\) are smooth functions on \(X\). This means that the isomorphism theorem can be refined, so that extra regularity of \(\Delta g\) along \(X\) will give extra regularity of \(g\) along \(X\).

In particular, one can prove that if the data \(\phi\) on the boundary has only regularity \(F S^k\) (the Folland-Stein spaces of functions with \(k\) horizontal derivatives in \(L^2\)), then the undetermined term has regularity \(F S^{k-4}\) (roughly speaking, one more order in the development translates in one less derivative along \(X\), as is clear from the explicit terms written above).

For a general operator of Laplacian type on a tensor bundle, the indicial operator has the form
\[
-\frac{\partial^2}{\partial r^2} - 4\frac{\partial}{\partial r} + A,
\]
where \(A\) is some constant linear operator. There may be different indicial roots corresponding to the eigenspace decomposition under \(A\) of the restriction of the bundle on the boundary.

This kind of idea has been used in particular to prove a complete asymptotic expansion for the Cheng-Yau metric \([\text{LM82}]\): an undetermined term occurs at order 6. A full understanding of the scalar Laplacian and its resolvent is achieved by the \(\Theta\)-pseudodifferential calculus \([\text{EMM91}]\). For analysis on AH spaces, see \([\text{Maz91}]\).
4. Asymptotics for ACH Kähler-Einstein Metrics

Given a CR strictly pseudoconvex $X^3$, one can try to construct an ACH Kähler-Einstein metric in a neighborhood of $X$. This problem is reminiscent of the expansion of Fefferman [Fef76] for the Kähler-Einstein metric on a complex domain, except that there is no complex coordinate with respect to which the equation could be written.

Identify a neighborhood of $X$ with $M = (0, \infty) \times X$ with variable $r$ on the first factor.

A high order asymptotically Kähler-Einstein metric is constructed in the following steps [BH05, section 3]: fix a contact form $\eta$ and therefore a metric $\gamma$ on the contact distribution and the initial ACH metric $g_0$ by formula (v).

**Proposition 6.** There is a formal integrable complex structure on $M$, given by power series in the variable $e^{-2r}$. It is uniquely determined by the CR structure, up to diffeomorphisms equal to the identity on the boundary.

The terms can be calculated by an iterative process involving the Tanaka-Webster connection and torsion. (Remind that a defining function for $X$ in $M$ is $\phi = e^{-2r}$, so the formal complex structure is actually well defined on the manifold with boundary $\bar{M}$). The fact that the series might diverge is not important for us, since we shall only use a finite number of terms.

**Proposition 7.** One can construct an approximate ACH Kähler-Einstein metric on $M$, given by an expansion

$$g_{KE} = g_0 + e^{-r}g_1 + e^{-2r}g_2 + \cdots + e^{-4r}g_4 + O(e^{-5r}).$$

The metric is Kähler with respect to the formal complex structure defined above, and Einstein up to order 4:

$$\text{Ric}(g_{KE}) + 6g_{KE} = O(e^{-5r}).$$

The approximate Kähler-Einstein metric is, up to order 4, formally determined by the CR structure only, and all the terms $g_i$ have explicit expressions as polynomials in the Webster connection and torsion and their derivatives.

Let me explain quickly why the first terms of the Kähler-Einstein metric are determined by the CR structure of $X$ only, since this is fundamental in the sequel. Denote by $\omega$ the Kähler form of the Kähler-Einstein metric, and look at an infinitesimal deformation $\hat{\omega} = f\omega + \varphi$, where $f$ is a function and $\varphi$ an antiselfdual 2-form. Then the variation of $\text{Ric}(\omega) + 6\omega$ is

$$2i\partial\bar{\partial}f + 6(f\omega + \varphi),$$

and in particular the contraction with $\omega$ is $\Delta f + 12f$; the indicial operator of $\Delta + 12$, as described in the previous section, is

$$-\frac{\partial^2}{\partial r^2} - 4\frac{\partial}{\partial r} + 12,$$
with indicial roots $-2$ and $6$. Therefore all terms up to order $6$ are formally determined, in particular the term $g_5$ and the logarithmic term at order $6$. The first undetermined term arises at order $6$, as in Fefferman’s expansion; but for our purposes the expansion up to order $4$ is sufficient.

5. **Einstein metrics**

A CR structure on $S^3$, bounding a strictly pseudoconvex domain in $\mathbb{C}^2$, is the boundary at infinity of a Kähler-Einstein metric, the Cheng-Yau metric. What is happening for CR structures not bounding a complex domain?

**Example 8.** A left-invariant CR structure on $S^3$ bounds an ACH Einstein metric on the $4$-ball [Hit95]. The Einstein metric is $SU_2$-invariant and given by explicit formulas in terms of $\vartheta$-functions.

There is a general existence theorem [Biq00]:

**Theorem 9.** A CR structure on $S^3$, close enough to the standard one, is the boundary at infinity of a complete ACH Einstein metric on the $4$-ball.

The Einstein metric is locally unique (modulo diffeomorphisms). Of course, when the CR structure is induced from an embedding into $\mathbb{C}^2$, the Einstein metric coincides with the Kähler-Einstein metric of Cheng and Yau.

The theorem is proved by an implicit function theorem argument, using the analysis of elliptic operators for ACH manifolds described section 3.

**Example 10.** Certain resolutions of finite quotients of $\mathbb{C}^2$ by a finite subgroup of $U_2$ carry ACH Einstein metrics [CS04]. The boundary at infinity is the link of the singularity.

In the real case, AH Einstein metrics with given conformal boundary have an expansion—the Fefferman-Graham expansion [FG85]. The local theorem analogous to theorem 9 was proved in [GL91]. In dimension 4, Anderson has found fairly general conditions on a conformal metric to bound an AH Einstein metric, see [And05] for a nice survey on the subject.

It is expected that, similarly, a general class of CR structures should be fillable by ACH Einstein metrics, but this remains a completely open problem.

6. **Regularity**

One can give a high-order asymptotic expansion for an ACH Einstein metric [BH05, corollary 5.4].

**Theorem 11.** Let $(M, g)$ be an ACH Einstein manifold, with boundary at infinity the CR manifold $X^3$, with contact distribution $H$. Construct from
then there exists a section \( k \) of \( \sigma^2 H \), such that \( g \) has the following development (up to a diffeomorphism):

\[
g = g_{KE} + ke^{-2r} + O(e^{-5r}).
\]

(The term \( ke^{-2r} \) is actually an order 4 term).

It is important to note that because the ACH metric in the contact distributions has growth \( e^r \), the tensor \( k \) has growth \( e^{-2r} \), hence the fourth order decay.

The term \( ke^{-2r} \) is the first undetermined term of the expansion. This is because the first indicial root for the linearized Einstein equation is 4, and corresponds to the subbundle \( \sigma^2 H \) of symmetric 2-tensors. It is interesting to note that it is different of the first indicial root for the Kähler-Einstein problem, giving a first undetermined term at order 6. This difference is essential is the following sections.

Theorem 11 contains a regularity result: the tensor \( k \) is \( C^\infty \) on \( X \) if the CR structure is \( C^\infty \). This result can be refined when the CR structure on the boundary is not smooth. Suppose that the regularity of the CR structure is only \( FS^k \) for some large \( k \) (also it will be important to consider \( k \) to be a real number). Then one has the following improvement [Biq05]:

**Theorem 12.** Given a CR structure on \( S^3 \), close enough to the standard one, of regularity \( FS^{\ell+\delta} \) with \( \ell \in \mathbb{N} \) large enough and \( 0 < \delta < 1 \), then the filling Einstein metric is still locally smooth, and can be written

\[
g = g_{KE} + ke^{-2r} + O(e^{-(4+\delta)r}),
\]

with \( k \) of regularity \( FS^{\ell-4+\delta} \).

The map taking the CR structure to the \( k \) tensor is a kind of nonlinear “Dirichlet-to-Neumann” map, of order 4.

### 7. Weyl tensor

The Weyl tensor \( W \in \Omega^2 \otimes \Omega^2 \) of an Einstein metric is harmonic as a 2-form with values in \( \Omega^2 \). Its components \( W_\pm \in \sigma^2 \Omega^2_\pm \) satisfy the same equation. Therefore, the analysis the Laplacian acting on \( \sigma^2 \Omega^2_\pm \) gives immediately some information on \( W_\pm \).

In order to understand the result, we need to decompose the bundles \( \sigma^2 \Omega^2_\pm \) at the boundary. The bundles \( \Omega^2_\pm M \) at the boundary can be identified with \( \Omega^1 X = \mathbb{R} \oplus \mathbb{H}^* \) by

\[
\omega \in \Omega^2_\pm M \mapsto \frac{\partial}{\partial r} \omega.
\]

Therefore, we have an identification, still at the boundary;

\[
\sigma^2 \Omega^2_\pm M = \mathbb{R} \oplus \mathbb{H}^* \oplus \sigma^2 \mathbb{H}^* ,
\]

where also \( \sigma^2 \mathbb{H}^* = \sigma^2 H \).
Concerning $W_-$, it turns out that the first indicial roots are 0 and 4, and they occur on the sub-piece $\sigma^2_0 H$. From this follows that, if the CR structure is $FS^{\ell + \delta}$,

$$(vii) \quad W_-(g) = w_-(g)e^{-2r} + O(e^{-(4+\delta)r}), \quad w_-(g) \in FS^{\ell - 4+\delta} (\sigma^2_0 H).$$

where $w_-(g)e^{-2r}$ is again an order 4 term. Actually, for a Kähler-Einstein metric, the first undetermined term occurs only at order 6, so the top order term of $W_-$ is completely determined: it is a fourth order invariant of the CR structure, and there is no choice, but the Cartan curvature:

$$(viii) \quad w_-(g_{KE}) = c_- Q$$

for some (computable) constant $c_-$. 

Concerning $W_+$, the situation is more complicated, and it is better to proceed as follows: in the Kähler-Einstein case one has $\Omega^2_\omega = R \omega \oplus \omega^\perp$, where $\omega$ is the Kähler form, and $W_+$ is determined by the scalar curvature:

$$(ix) \quad W_+ = \begin{pmatrix} s/6 & -s/12 \\ -s/12 & s/6 \end{pmatrix}.$$ 

Now remind from theorem \[11\] that at order 4, an ACH Einstein metric differs from a Kähler-Einstein $g_{KE}$ only by an undetermined term $ke^{-2r}$, where again $k \in \sigma^2_0 H$. From this one deduces that for some constant $c_+$, and the same $FS^{\ell + \delta}$ regularity on the CR structure on the boundary:

$$(x) \quad W_+(g) = W_+(g_{KE}) + c_+ ke^{-2r} + O(e^{-(4+\delta)r}),$$

where $k$ is the undetermined term for $g$ as in theorem \[12\] For the precise value of the constants $c_-$ and $c_+$, see \[BH05\].

8. Selfduality

The complex hyperbolic metric, and the Einstein metrics of examples\[8\] and\[10\] are also selfdual, that is $W_-=0$ (actually it is the selfduality which enables to construct these explicit examples). So a natural question is: when is an ACH Einstein metric selfdual ? if $g$ is selfdual and Kähler-Einstein, then $W_-=0$ and $W_+$ is given by \[ix\], so the curvature tensor is parallel: $g$ is locally symmetric, therefore complex hyperbolic.

A more precise question is the following: for which small CR deformations of $S^3$ is the filling ACH Einstein metric on the ball provided by theorem \[9\] selfdual ? let denote by $\mathcal{C}$ the space of all CR structures on $S^3$, $\mathcal{K} \subset \mathcal{C}$ the space of CR structures fillable by a Kähler-Einstein metric, and $\mathcal{A} \subset \mathcal{C}$ the space of CR structures fillable by a selfdual Einstein metric. To be precise, we ask $FS^k$-regularity for some large $k \notin \mathbb{N}$. Then \[Biq05\]:

**Theorem 13.** Near the standard CR structure $J_0$ of $S^3$, $\mathcal{A}$ and $\mathcal{K}$ are two transverse submanifolds of $\mathcal{C}$, whose intersection is reduced to the orbit of $J_0$ under the contactomorphism group.
The submanifold $\mathcal{K}$ was studied by Bland [Bla94], so the result is only a result on $\mathcal{A}$: it says that a complementary subspace of $\mathcal{K}$ (of infinite dimension modulo contactomorphisms) can be filled by selfdual Einstein metrics. In particular, it proves the existence of an infinite dimensional family of selfdual ACH Einstein metrics. The tangent space to $\mathcal{A}$ and $\mathcal{K}$ will be described in section 9.

The real case. This result is of course analogous to the similar result for AH metrics. The story begins by a local result of LeBrun [LeB82]: a real analytic conformal metric on $X^3$ is always the conformal infinity of a unique selfdual AH Einstein metric defined in a small collar neighborhood. The global picture was conjectured in 1991 by LeBrun as the “Positive Frequency Conjecture” and proved in [Biq02]: inside the space $\mathcal{M}$ of conformal metrics on $S^3$, the spaces $\mathcal{M}_\pm$ of conformal metrics fillable by a selfdual (resp. antiselfdual) Einstein metric on the 4-ball are transverse infinite dimensional submanifolds at the standard metric, and their intersection is reduced to the orbit of the standard metric under the diffeomorphism group (remind that the real hyperbolic space is selfdual and antiselfdual).

The tangent spaces to $\mathcal{M}_\pm$ are described in [Biq02, théorème 10.1] using a harmonic decomposition, but it is easy to check that it is equivalent to the following description. Consider the (complex rank 2) spinor bundle $\Sigma$ on the sphere, and its symmetric powers $\Sigma^\ell$; the infinitesimal conformal metrics are trace free symmetric 2-tensors, these are sections of the rank 5 bundle $\Sigma^4$. Now there is a Dirac operator $D: \Sigma^4 \subset \Sigma \Sigma^3 \rightarrow \Sigma \Sigma^3 = \Sigma^2 \oplus \Sigma^4$, and we decompose accordingly $D = D_2 + D_4$. The projection $D_2$ on $\Sigma^2 = TS^3$ is just the divergence, so its kernel can be identified with the tangent space of $\mathcal{M}$ modulo diffeomorphisms. Now $D_4$ acts on $\ker(D_2)$ with eigenspaces $E_1$, and one has (modulo diffeomorphisms):

$$
T.\mathcal{M}_+ = \oplus_{\lambda > 0} E_1, \\
T.\mathcal{M}_- = \oplus_{\lambda < 0} E_1.
$$

(xi)

The proof in the real case is through the twistor construction. This seems difficult in the complex case and I used a different method, described in the following section. But it is useful to note that the method in the complex case just proves theorem [13], which is an infinitesimal statement, while the twistor method of [Biq02] gives a much stronger result—a criterion for a conformal metric on $S^3$ (close to the standard one) to be the conformal infinity of a selfdual AH Einstein metric.
9. Idea of Proof of Theorem \[13\]

For a four dimensional metric $g$ on $M$, denote by $\Sigma_+$ and $\Sigma_-$ the two half-spin representations, and $\Sigma^\ell_\pm$ their symmetric powers, so that

\[ \Omega^1 M \cong \Sigma_+ \Sigma_- \quad \Omega^2 M \cong \Sigma^2 \pm \quad \sigma^2_0 \Omega^2 M \cong \Sigma^4 \pm. \]

The two half Weyl tensors $W_\pm$ are sections of $\Sigma^4 \pm$, and, if $g$ is Einstein, they are harmonic as 2-forms with values in 2-forms. This implies that they belong to the kernel of the two coupled Dirac operators:

\[ D_+ : \Sigma^4_+ \subset \Sigma_+ \Sigma^3_+ \longrightarrow \Sigma_- \Sigma^3_+ \]
\[ D_- : \Sigma^4_- \subset \Sigma_- \Sigma^3_- \longrightarrow \Sigma_+ \Sigma^3_- \]

From (vii), the antiselfdual part $W_-$ is in $L^2$, therefore belongs to the space of $L^2$-harmonic spinors defined by

\[ \mathcal{H}^A_-(g) = \{ \omega \in L^2(\Sigma^4_-), D_- \omega = 0 \}. \]

As in (vii), a harmonic spinor $\omega \in \mathcal{H}^A_-(g)$ has an asymptotic behaviour with a fourth order leading term:

\[ \omega = \partial_\infty \omega e^{-2r} + O(e^{-2r}), \quad \omega_\infty \in \sigma^2_0 H. \]

The spaces $\mathcal{H}^A_-(g)$ are infinite dimensional, but they depend smoothly on $g$:

**Proposition 14.** Near the complex hyperbolic metric:

1. the spaces $\mathcal{H}^A_-(g)$ form a smooth Hilbert bundle over the space of ACH metrics on the 4-ball;
2. a $L^2$ harmonic spinor $\omega \in \mathcal{H}^A_-(g)$ is completely determined by its value at infinity $\partial_\infty \omega$; the space of boundary values of harmonic spinors shall be denoted by $\partial_\infty \mathcal{H}^A_-(g)$.

The proof of the proposition in [Biq05] relies on an explicit calculation of $\mathcal{H}^A_-(g_0)$ for the complex hyperbolic metric $g_0$, using harmonic decompositions.

Once this is done, the selfduality problem, for a CR structure $J$ on $S^3$ near the standard one, becomes the following: let $g$ be the ACH Einstein metric with conformal infinity $J$, and consider

\[ P(J) = \partial_\infty W_- (g) \in \partial_\infty \mathcal{H}^A_-(g). \]

Then the selfduality equation becomes $P(J) = 0$, and the differential at the standard CR structure $J_0$ is

\[ d_{J_0} P(J) = \partial_\infty W_- \in \partial_\infty \mathcal{H}^A_-(g_0) \]

Remember from (viii) that if $J$ is fillable by an ACH Kähler-Einstein metric $g$, then the value at infinity of $W_-$ for $g$ is actually the Cartan curvature:

\[ P(J) = c_- Q(J). \]
Therefore, for \( \dot{J} \) fillable by an infinitesimal Kähler-Einstein metric:

\[
d_{\dot{J}} P(\dot{J}) = c_{-} dQ(\dot{J}).
\]

From this description and a calculation in harmonic decomposition follows [Biq05 lemme 3.2]:

**Claim.** Any \( \omega_{\infty} \in \partial_{\infty} \mathcal{H}^{1}(g_{0}) \) is the infinitesimal Cartan curvature of a \( \dot{J} \) fillable by an infinitesimal Kähler-Einstein metric.

This means that the operator \( P \) (with values in the bundle \( \partial_{\infty} \mathcal{H}^{1}_{-} \)) is submersive, actually its restriction

(xii)

\[
P : \mathcal{K} \to \partial_{\infty} \mathcal{H}^{1}_{-}
\]

is already submersive at \( J_{0} \), therefore the zero set \( \mathcal{A} = \{ P(J) = 0 \} \) of CR structures fillable by a selfdual Einstein metric is a submanifold, transverse to \( \mathcal{K} \) at \( J_{0} \). This ends the proof of theorem 13.

*More explicit description of \( \mathcal{K} \).* The method above gives a description of the tangent spaces of \( \mathcal{A} \) and \( \mathcal{K} \) at the standard CR structure \( J_{0} \) in terms of the harmonic decomposition on \( S^{3} \). However going slightly further in the calculations enables to get the following.

If \( J \in \mathcal{K} \), that is \( J \) is fillable by a Kähler-Einstein metric \( g \), then from equation (viii) one has

\[
Q(J) \in \partial \mathcal{H}^{1}(g).
\]

Actually one can check that the differential of (xii) at \( J_{0} \) is an isomorphism, therefore:

**Proposition 15.** Let \( J \) be a CR structure near the standard CR structure \( J_{0} \) on \( S^{3} \), and \( g \) the filling ACH Einstein metric. Then \( g \) is Kähler-Einstein if and only if \( Q(J) \in \partial \mathcal{H}^{1}(g) \).

Moreover, the tangent space at \( J_{0} \) of \( \mathcal{K} \) is

\[
T_{J_{0}} \mathcal{K} = (d_{J_{0}} Q)^{-1} \partial \mathcal{H}^{1}(g_{0}).
\]

Reminding that from the hypoelliptic complex (ii) the kernel of \( d_{J_{0}} Q \) is the infinitesimal action of the contactomorphisms, the statement means that the tangent space to \( \mathcal{K} \) at \( J_{0} \) modulo contactomorphisms can be identified to \( \partial \mathcal{H}^{1}(g_{0}) \) via \( d_{J_{0}} Q \).

*More explicit description of \( \mathcal{A} \).* Again we go slightly further [Biq05 théorème 3.11] and describe more explicitly the tangent space of \( \mathcal{A} \) modulo contactomorphisms, leaving the calculations to the interested reader. This tangent space is described in [Biq05] in terms of harmonic decompositions, which can be interpreted as follows:

**Proposition 16.** The tangent space at \( J_{0} \) to \( \mathcal{A} \) is

\[
T_{J_{0}} \mathcal{A} = (d_{J_{0}} Q)^{-1} \ker ((\partial_{\bar{\partial}})_{-}).
\]
This description fits nicely with the last operator of complex (ii), since the (infinitesimal) Bianchi identity is just the real part of \((\bar{\partial}_\gamma \bar{\partial})^+\).

In particular, this statement together with proposition 15 implies that in sections of \(\Omega^{0,1} \otimes T^{1,0}\) on \(S^3\) one has:

\[
\ker(\Re(\bar{\partial}_\gamma \bar{\partial})^+) = \partial H_4 - (g_0) \oplus \ker((\bar{\partial}_\gamma \bar{\partial})^+).
\]

In the Kähler-Einstein case, proposition 15 gives a good interpretation why \(\partial H_4 - (g_0)\) is the tangent space to \(K\) (modulo contactomorphisms). I do not know such interpretation of \(\ker((\bar{\partial}_\gamma \bar{\partial})^+)\) in the selfdual case.

In the asymptotically real hyperbolic case, described in the previous section, the tangent spaces at the real hyperbolic metric \(g_0\) to the spaces \(\mathcal{M}_\pm\) of conformal metrics on \(S^3\) fillable by \(\pm\)-selfdual Einstein metrics have a similar interpretation, actually it is easy to check that \(\text{(xvii)}\) is equivalent to

\[
T\mathcal{M}_\pm = \partial \mathcal{H}^A(g_0).
\]

10. The \(v\)-invariant

We now pass to a different topic, the construction of a new invariant of strictly pseudoconvex CR 3-manifolds from the correspondence with Einstein metrics [BH05].

**Hitchin’s formula in the AH case.** Our paradigm is the following formula in the AH case [Hit97]: if \((M^4, g)\) is an AH manifold, with conformal infinity \((X^3, \gamma)\), then one has

\[
\frac{1}{12\pi^2} \int_M |W_+|^2 - |W_-|^2 = \tau(M) + \eta(X).
\]

For the definition of the \(\eta\)-invariant, see section 14. The equation \(\text{(xv)}\) is essentially the Atiyah-Patodi-Singer formula [APS75a] for the signature \(\tau(M)\) of \(M\): the LHS and \(\eta(X)\) are conformal invariants, and the (also conformally invariant) trace-free part of the second fundamental form of the boundary vanishes for an AH metric, so the formula is a consequence of the usual APS formula for the conformal compactification \(\phi^2 g\), where \(\phi\) is a defining function of the boundary.

Formula \(\text{(xv)}\) is an illustration of the Fefferman-Graham program: constructing conformal invariants of \(X\) from invariants of a filling AH Einstein metric on \(M\). Another important invariant is the renormalized volume of AH Einstein metrics, see [Gra00] for a nice presentation.

**The Riemannian invariant.** We now pass to the complex case. First look at the Riemannian invariant. We consider the integral of characteristic classes for \(\chi - 3\tau\):

**Theorem 17.** Let \((M^4, g)\) be an ACH Einstein manifold. Then the integral

\[
\frac{1}{8\pi^2} \int_M 3|W_+|^2 - |W_-|^2 + \frac{\text{scal}^2}{24}
\]
converges, the result shall be denoted by $(\chi - 3\tau)(g)$.

From development (vii), it is clear that $W_\tau \in L^2$. The proof of convergence for the term $|W_\tau|^2 - \frac{1}{24} \text{scal}^2$ relies on the fact that it vanishes for a Kähler-Einstein metric, and on the development (vii) for $W_\tau$.

If $g$ is Kähler-Einstein, the integral for $\chi - 3\tau$ is the same as the one for $3C_2 - (c_1)^2$ and convergence in this case has been proved by Burns and Epstein [BE90].

The invariant of the boundary. Choose a defining function $\phi$ of the boundary $X^3 = \partial M$. Denote by $g_\epsilon$ the restriction of $g$ to $X_\epsilon = \{\phi = \epsilon\}$. Then the integral for the characteristic class $\chi - 3\tau$ on a manifold with boundary is transformed by standard formulas into a topological term of the interior plus contributions of the boundary, in which appears the $\eta$-invariant, already seen in formula (xv):

$$\frac{1}{8\pi^2} \int_{\phi > \epsilon} 3|W_\tau|^2 - |W_+|^2 + \frac{\text{scal}^2}{24} = \chi(M) - 3\tau(M) - 3\eta(g_\epsilon) + \int_{X_\epsilon} \text{local terms},$$

(xvi)

where the “local terms” are combination of the second fundamental form of $X_\epsilon$ and of the curvature of $g$. Denote the quantity on the second line by

$$\nu_\epsilon := -3\eta(g_\epsilon) + \int_{X_\epsilon} \text{local terms},$$

(xvii)

From the convergence of the integral (theorem 17), it is clear that one has a limit

$$\nu = \lim_{\epsilon \to 0} \nu_\epsilon$$

(xviii)

which does not depend on the choice of the defining function $\phi$. The important point here is:

Claim. The limit $\nu$ depends only on the asymptotic development of $g$ at infinity of theorem [11] up to order 4. Moreover, it depends only on the formally determined terms.

This means that the limit $\nu$ depends only on the four first terms of the formal Kähler-Einstein metric $g_{KE}$ constructed in proposition 7 so ultimately only on the CR structure of $X^3$. This process can be reversed: from a CR manifold $X$, one can construct the formal Kähler-Einstein metric and take the limit (xviii): therefore, a global ACH Einstein filling is not needed, and this gives the following result.

Theorem 18. Given any compact strictly pseudoconvex CR manifold $X^3$, the limit in (xviii) calculated with the formal Kähler-Einstein metric of proposition 7 defines an intrinsic invariant $\nu(X) \in \mathbb{R}$. 
Moreover, if \( X \) is the boundary at infinity of an ACH Einstein manifold \((M, g)\), then \((\chi - 3\tau)(g)\) defined in theorem 17 is related to \(\nu\) by:

\[
(\chi - 3\tau)(g) = \chi(M) - 3\tau(M) + \nu(X).
\]

Note that in (xvii) both terms diverge, so the limit \(\nu\) can be understood as a renormalization of \(\eta(g_\epsilon)\) when \(\epsilon \to 0\). From (iv) the leading term of \(g_\epsilon\) is \(g_\epsilon \sim \eta^2/\epsilon^2 + \gamma/2\epsilon\); as the \(\eta\)-invariant is conformally invariant, one can calculate it instead for the family of metrics

\[
g_\epsilon := 2\epsilon^{-1}\eta^2 + \gamma,
\]

converging to the Carnot-Carathéodory metric \(\gamma\) on \(\ker \eta\) as \(\epsilon \to 0\). The actual metrics \(g_\epsilon\) differ from \(\gamma_\epsilon/2\epsilon\) only by local terms involving the Tanaka-Webster torsion and curvature and their derivatives, so one can say finally that \(\nu\) is a renormalization of \(\eta(\gamma_\epsilon)\) by adding local terms when \(\epsilon \to 0\).

**Example 19.** The sphere \(S^3\) with the standard CR structure is the boundary at infinity of the Bergman metric on the ball \(B^4\), for which the integral for \((\chi - 3\tau)(g)\) vanishes identically. Therefore, since \(\chi(B^4) = 1\) and \(\tau(B^4) = 0\), one deduces from theorem 18 the value

\[\nu(S^3) = -1.\]

**Remark.** Using the asymptotics of an ACH Einstein metric, it is possible to study other invariants. In particular, Marc Herzlich [Her] has studied the renormalized volume in this situation, a “conformal anomaly” appears as in the AH case with even dimensional boundary.

11. **Relation with the Burns-Epstein invariant**

**The Burns-Epstein invariant.** Burns and Epstein [BE88] defined an invariant \(\mu(J)\) of a strictly pseudoconvex CR 3-manifold \((X^3, J)\) with trivial holomorphic canonical bundle. This invariant is defined by a Chern-Simons type integral, using the Cartan connection of the CR structure. It can also be seen in the following way, shown to me by Michel Rumin.

First remind some formalism for pseudo-hermitian geometry. Choose a local coframe \((\eta, \theta^1, \bar{\theta}^1)\) such that \(d\eta = i\theta^1 \wedge \bar{\theta}^1\). The Tanaka-Webster connection form is a purely imaginary 1-form \(\omega_1^1\) and the Tanaka-Webster torsion a \((0,1)\)-form \(\tau^1 = \tau_1^1\bar{\theta}^1\), such that

\[
d\theta^1 = \bar{\theta}^1 \wedge \omega_1^1 + \eta \wedge \tau^1.
\]

The Tanaka-Webster curvature \(R\) is defined by

\[
d\omega_1^1 = R\theta^1 \wedge \bar{\theta}^1 + (\tau_1^1 - \tau_1^1) \wedge \eta.
\]

The canonical bundle \(K\) of the CR structure, generated by \(\theta^1 \wedge \eta\), carries a natural holomorphic structure, and therefore a canonical connection with curvature \(\Omega\) satisfying \(\Omega \wedge \eta = 0\). In the frame \(\theta^1 \wedge \eta\), the connection form is \(\omega_1^1 + iR\eta\).
Suppose now that one has a global holomorphic trivialisation of $K$ by a non vanishing holomorphic section: changing the contact form by a conformal factor, one can make the norm of the holomorphic section constant, so for this choice the bundle $K$ is flat, $d(\omega_1 + i R \eta) = 0$. Then the formula for the invariant $\mu$ in [BE88] is easily seen to reduce to:

$$\mu = \frac{1}{16\pi^2} \int_X (4|\tau|^2 - |R|^2) \eta \wedge d\eta.$$

The connection of $K$ remains flat when we transform the contact form to $e^f \eta$, with $f$ the real part of a CR holomorphic function on $X$: one can check that the integral above remains invariant under such change of $\eta$, therefore defining an intrinsic invariant of the CR structure. Actually the invariant is well defined as soon as the bundle $K$ is flat for some choice of $\eta$, so this includes the case where a power of $K$ is trivial.

**Variation of the $v$-invariant.** As the $v$-invariant is defined as a boundary term of characteristic classes, its variation can be expressed by an integral of local terms [BH05, theorem 8.1]:

**Theorem 20.** On the space of CR structures, the derivative of $v$ is

$$d_J v(J) = -\frac{3}{8\pi^2} \int_X \langle Q(J), \hat{J} \rangle \eta \wedge d\eta,$$

where $\langle \cdot, \cdot \rangle$ is the Hermitian scalar product on $\Omega^{0,1} \otimes T^{1,0}$.

**Comparison of the two invariants.** For a general $X^3$, the invariant $\mu$ is not defined, but Cheng and Lee [CL90] introduced a relative version $\mu(J, J')$ for two CR structures with the same underlying contact structure, such that $\mu(J, J') = \mu(J) - \mu(J')$ when $\mu$ is defined for $J$ and $J'$. The variation of $\mu(J, J')$ is the same as the variation of $v/3$, therefore:

$$v(J) - v(J') = 3\mu(J, J')$$

for two CR structures $J$ and $J'$ defined on the same contact structure.

In particular: if the contact structure $H$ on $X^3$ is fixed, then the difference $v(J) - 3\mu(J)$, when defined, is a constant:

$$v(J) - 3\mu(J) = \text{cst(contact structure)}$$

for $J$ with trivial holomorphic canonical bundle.

Since the $\mu$ invariant, or its relative version, are given by integral of local terms as in formula (xx), they lead more easily to explicit calculations. If the constant in (xxii) can be fixed, this leads to explicit calculations of $v$. For example, the standard CR structure on $S^3$ has $\mu = -1$, so one deduces from example [19] that for boundaries of strictly pseudoconvex domains in $C^2$, one has

$$v = 3\mu + 2.$$
that \( d\alpha_3 = \alpha_1 \wedge \alpha_2 \), etc. Take \( \alpha_3 \) as the contact form, then the metrics corresponding to the CR structures are \( \lambda^{-1}\alpha_1^2 + \lambda\alpha_2^2 \), with the \( \mu \) invariant calculated by Burns and Epstein \( [BE88 \text{ 4.1.A}] \), from which one deduces via (xxiii) the result:

\[
(\text{xxiv}) \quad \nu(\lambda^{-1}\alpha_1^2 + \lambda\alpha_2^2) = -1 + \frac{9}{4} \frac{(1 - \lambda^2)^2}{\lambda^2}.
\]

**Characteristic numbers of complex domains.** If \((M^4, g)\) is an ACH manifold with Kähler-Einstein metric, and \(\partial M\) has trivial holomorphic canonical bundle, Burns and Epstein \([BE90]\) prove:

\[
(\text{xxv}) \quad \int_M c_2 - \frac{1}{3} c_1^2 = \chi(M) - \frac{1}{3} \tilde{c}_1(M)^2 + \mu(\partial M),
\]

where \( \tilde{c}_1(M) \) is a lifting of \( c_1(M) \) in \( H^2(M, \partial M) \), and the square \( \tilde{c}_1(M)^2 \) does not depend on the choice of \( \tilde{c}_1(M) \).

Of course, for a Kähler-Einstein metric, the integral in the LHS of (xxv) is just one third of the integral \( (\chi - 3\tau)(g) \). Comparing with theorem 18 gives immediately

\[
(\text{xxvi}) \quad \nu(\partial M) - 3\mu(\partial M) = 2\chi(M) + 3\tau(M) - \tilde{c}_1(M)^2
\]

for an ACH Kähler-Einstein \((M, g)\) with holomorphic tangent bundle of \( \partial M \) trivial. So in this case, the constant of (xxii) can be explicitly calculated in terms of the topology of the filling complex manifold. Note that for a closed complex manifold one has \( c_1^2 = 2\chi + 3\tau \) so this term arises only when one looks at manifolds with boundary.

By the way, formula (xxvi) gives also an intrinsic explanation why the variation of \( \nu \) and \( 3\mu \) must be the same.

### 12. MiyaoKa-Yau inequality

If \( g \) is Kähler-Einstein, then \( |W_+|^2 = \frac{\text{scal}^2}{24} \) so that

\[
(\chi - 3\tau)(g) = \frac{1}{8\pi^2} \int_M |W_-|^2 \geq 0.
\]

The vanishing of \( (\chi - 3\tau)(g) \) implies \( W_- = 0 \) and therefore \( g \) is complex hyperbolic. This leads to an inequality on \( \nu(\partial M) \).

Actually it is a nontrivial fact that this inequality extends in the Einstein case. This was first observed by LeBrun \([LeB95]\), in the case of a compact manifold: if \( g \) is an Einstein metric on a closed compact \( M^4 \), and some Seiberg-Witten invariant of \( M \) is non trivial, then \( \chi(M) \geq 3\tau(M) \), with equality if and only if the metric is complex hyperbolic. A generalization of the result in the finite volume case was given in \([Biq97]\), and the ACH (infinite volume) case was done by Rollin \([Rol04]\), using the version of the Seiberg-Witten invariants constructed by Kronheimer and Mrowka \([KM97]\) for a 4-manifold with contact boundary.
Theorem 21 (Rollin). Suppose that $M^4$ carries an ACH Einstein metric $g$ (therefore $\partial M$ inherits a CR structure with some underlying contact structure), and a Kronheimer-Mrowka invariant of $(M, \partial M)$ does not vanish. Then $(\chi - 3\tau)(g) \geq 0$, that is

$$\chi(M) - 3\tau(M) \geq -\nu(\partial M),$$

with equality if and only if $g$ is complex hyperbolic.

The hypothesis on the Kronheimer-Mrowka invariants is true in particular in the two following cases:

- $(M, \partial M)$ carries a Kähler ACH metric;
- more generally, there exists on $M$ a symplectic form compatible with the contact structure on $\partial M$.

The theorem can be seen as giving obstructions on the topology of an ACH Einstein or Kähler-Einstein metric filling a given CR 3-manifold. It leads also to the following rigidity result:

Corollary 22. If $M^4$ has an ACH complex-hyperbolic metric $g$, then any other ACH Einstein metric on $M$ inducing the same CR structure on the boundary at infinity is equal to $g$ (up to a diffeomorphism).

The proof is clear from the theorem: for the complex hyperbolic metric, one has $\chi(M) - 3\tau(M) = \nu(\partial M)$. For another ACH metric inducing the same CR structure at infinity, $\nu(\partial M)$ does not change since it is an invariant of the CR structure only, so we are still in the equality case and the metric is complex hyperbolic.

Actually the corollary was stated in [Rol04] only for the complex hyperbolic space itself, but the same proof works more generally for quotients. It is interesting to note that the similar result in the real case is known only for the hyperbolic space itself, not for the quotients.

13. Explicit calculations for $\nu$

In a few cases, the definition of the invariant $\nu$ by the direct formula (xviii) is possible: this requires to calculate the formal Kähler-Einstein metric and the $\eta$-invariant of the slices. In [BHR] we have been able to do that in the case of $S^1$-invariant spherical CR structures (with $S^1$-action transverse to the contact distribution). The geometric setting is the following.

We consider a holomorphic line bundle $L \to \Sigma^2$ of degree $d < 0$ over a Riemann surface $\Sigma$ (the sign condition on $d$ is to ensure strict pseudoconvexity of the CR structure, for example $S^3$ is the bundle $\mathcal{O}(-1)$ over $S^2$). Choose a constant curvature metric on $\Sigma$ and on $L$, then the circle bundle $X^3 \to \Sigma^2$, with the connection 1-form as contact form, has vanishing torsion and constant Webster curvature, so is spherical. For this metric, the formal Kähler-Einstein metric is especially simple, and relying on Komuro’s formula for the $\eta$-invariant of circle bundles [Kom84].
one gets the formula:

\[(xxvii) \quad \nu(X) = -d - 3 - \frac{\chi(\Sigma)^2}{4d} \]

In view of theorem 20, the variation of \(\nu\) on the space of spherical CR structures vanishes, so \(\nu\) is constant on each component, therefore it is not surprising that formula (xxvii) contains only topological terms.

A more general situation can be dealt with, allowing both \(\Sigma\) and the circle bundle to have orbifold singularities: at each orbifold point, there is a local fundamental group \(\mathbb{Z}/\alpha\mathbb{Z}\), and a generator acts on a local chart on \(\Sigma\) as \(\exp(i \frac{2\pi \beta}{\alpha})\) and on the fiber as \(\exp(i \frac{2\pi \gamma}{\alpha})\), with \(\beta\) and \(\gamma\) prime to \(\alpha\). Then the formula (xxvii) has to be modified in the following way: first \(d\) and \(\chi(\Sigma)\) are now orbifold characteristic numbers (therefore, rational numbers); second there are additional contributions from the singularities:

\[(xxviii) \quad \nu(X) = -d - 3 - \frac{\chi(\Sigma)^2}{4d} - 12 \sum \frac{1}{4a} \sum_{j} s(\alpha_j, \beta_j, \gamma_j), \]

where \(s(\alpha, \beta, \gamma)\) is the Dedekind sum \(\frac{1}{4a} \sum \frac{1}{a-1} \cot \left( \frac{k \beta}{a} \pi \right) \cot \left( \frac{k \gamma}{a} \pi \right)\). The contributions at the singularities are already present in the formula for the \(\eta\)-invariant of Ouyang [Ouy94] which is used to prove (xxviii).

As an application of this formula, one can calculate the \(\nu\)-invariant of the lens space \(L(p, q)\) obtained by quotient of \(S^3\) by \(\mathbb{Z}/p\mathbb{Z}\) with generator acting by \(\exp(\frac{2\pi i}{p})\), \(\exp(\frac{2\pi i}{q})\), where \(q\) is prime with \(p\):

\[(xxix) \quad \nu(L(p, q)) = -\frac{1}{p} + 12 s(p, q, 1). \]

This can be compared with \(\eta(L(p, q)) = -4s(p, q, 1)\) from [APS75b].

An application of such calculations is to make explicit the obstruction from theorem 21 to the existence of an ACH Einstein filling. But spherical CR manifolds are locally the boundary at infinity of a complex hyperbolic metric, and it is therefore natural to ask whether a spherical CR manifold \(X^3\) is the boundary at infinity of a (smooth) complex hyperbolic manifold \(M^4\). This is a question one can look at using the theory of group acting on complex hyperbolic space, but the invariant \(\nu\) gives also some result: here we are in the equality case of the Miyaoka-Yau inequality, therefore one must have

\[(xxx) \quad \chi(M) - 3\tau(M) = -\nu(X)\]

and in particular:

\[(xxxi) \quad \nu(X) \in \mathbb{Z}. \]

In view of formula (xxviii), it is clear at once that a lot of CR spherical manifolds do not satisfy this obstruction.
Remark. For a circle bundle over a hyperbolic Riemann surface $\Sigma$, equality (xxx) applied to the corresponding disk bundle gives $\chi(\Sigma) + 3 = d + 3 + \frac{\chi(\Sigma)}{4d}$, and the only solution is $d = \chi(\Sigma)/2$. This gives a simple proof of the fact that the only disk bundles over $\Sigma$ admitting a complex hyperbolic metric are square roots of the tangent bundle (example 5).

In general, one expects that the complex hyperbolic metric filling a spherical CR structure may have orbifold singularities and cusps. In that case formula (xxx) still makes sense, but the left hand side must be interpreted in the appropriate orbifold sense; there are also corrections for cusp ends, see [Biq97]. In particular, $\nu(X)$ is only a rational number.

14. SPECTRAL ASPECTS

The $\nu$-invariant is a CR analogue of the (conformally invariant) $\eta$-invariant. But there exists a version of the $\eta$-invariant in CR geometry, due to Rumin.

First remind that the $\eta$-invariant is defined by considering the De Rham complex

$$\Omega^0 \to \Omega^1 \to \Omega^2 \to \Omega^3.$$  

The $\eta$ function is constructed from the eigenvalues $\lambda$ of the operator $d^*$ acting on closed two forms by the formula $\eta(s) = \sum (\text{sign} \lambda) |\lambda|^{-s}$. One can prove that $\eta$ is holomorphic for $\Re s > -\frac{1}{2}$, and the $\eta$-invariant is defined as $\eta(0)$.

In contact geometry, the De Rham complex is replaced by the Rumin complex [Rum90],

$$\Omega^0 \to \Omega^1_H \to \Omega^2_V \to \Omega^3,$$

which uses only derivation along the contact directions: here, $\Omega^1_H = H^*$ denotes horizontal one forms, and $\Omega^2_V$ vertical two forms (that is, two forms $\omega$ satisfying $\eta \wedge \omega = 0$). The middle operator is an order two hypoelliptic operator, and therefore the $\eta$-invariant for the operator $D^*$ acting on vertical two forms is well defined.

Remind that the $\nu$-invariant is defined by a “renormalization” of the invariants $\eta(\gamma, \epsilon)$ of the family of metrics defined by (xxix) when $\epsilon$ goes to zero. In this limit, $d^*$ has both bounded and unbounded eigenvalues: the bounded ones converge to the eigenvalues of $D^*$, but the transition at $\epsilon = 0$ from elliptic to hypoelliptic is very difficult to understand [Rum00]. Nevertheless there is some hope to use this limit to relate the $\nu$-invariant to the invariant $\eta(D^*)$ of Rumin, and this would give some insight on the still mysterious invariant $\eta(D^*)$, which, by the way, is probably not a CR invariant, but only a pseudohermitian invariant. A beginning of answer to this question is obtained in [BHR].
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IRMA, Université Louis Pasteur et CNRS, 7 rue René Descartes, F-67084 STRASBOURG CEDEX

E-mail address: olivier.biquard@math.u-strasbg.fr