# EINSTEIN METRICS WITH ANISOTROPIC BOUNDARY BEHAVIOUR

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In recent years the relation between complete, infinite volume, Einstein metrics and the geometry of their boundary at infinity has been intensively studied, especially since the advent of the physical AdS/CFT correspondence.

In all the previous examples of this correspondence, the Einstein metrics at infinity are supposed to be asymptotic to some fixed model—a symmetric space of noncompact type $G/K$. Here we shall restrict to the rank one case, where the examples are asymptotically real, complex or quaternionic hyperbolic metrics. The corresponding geometries at infinity ("parabolic geometries" modelled on $G/P$, where $P$ is a minimal parabolic subgroup of $G$) are conformal metrics, CR structures or quaternionic-contact structures. In this article, we introduce a new class of examples, which are no more asymptotic to a symmetric space. Actually the model at infinity is still given by a homogeneous Einstein space, which may vary from point to point on the boundary at infinity.

This phenomenon cannot occur in the most classical examples (real or complex hyperbolic spaces), because the algebraic structure at infinity (abelian group or Heisenberg group) has no deformation. But such deformations exist for the quaternionic Heisenberg group (except in dimension 7), and even in the 15-dimensional octonionic case. So these are the two cases on which this article shall focus. In the parabolic geometry language, these are the two cases where non regular examples exist.

More concretely, the basic quaternionic example is the sphere $S^{4m-1}$, with its $(4m-4)$-dimensional distribution $\mathscr{D}$, and the octonionic example is the sphere $S^{15}$ with a 8-dimensional distribution $\mathscr{D}$. At each point $x$ of the sphere, there is an induced nilpotent Lie algebra structure on $\mathfrak{n}_x = \mathscr{D}_x \oplus T_xS/\mathscr{D}_x$, given by the projection on $T_xS/\mathscr{D}_x$ of the bracket of two vector fields $X, Y \in \mathscr{D}_x$. It was proved in [2] that small deformations of $\mathscr{D}$, such that $\mathfrak{n}_x$ remains the quaternionic Heisenberg algebra for all $x$, are boundaries at infinity of complete Einstein metrics

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on the ball. This regularity assumption (that is, keeping the isomorphism type of the algebra $\mathfrak{n}_x$ fixed) is a strong differential system on $\mathscr{D}$: it was shown in [2] that such quaternionic-contact structures exist in abundance, but there is no octonionic example [11].

In this article, we relax the regularity assumption in these two cases. There is a beautiful family of examples, already known in the literature: the homogeneous Einstein metrics of Heber [8]. In the upper space model, each hyperbolic space is identified with the solvable group $S = AN$, with boundary at infinity the Heisenberg group $N$ (where $G = KAN$ is the Iwasawa decomposition). Then Heber proved that every deformation of $S$ carries a unique homogeneous Einstein metric. In particular, we can associate to a deformation of the nilpotent Lie algebra $\mathfrak{n}$ the homogeneous Einstein metric on the corresponding solvable group $S = AN$.

**Theorem 0.1.** Let $n = 4m - 1 \geq 11$ in the quaternionic case, or $n = 15$ in the octonionic case. Any small deformation of the $(4m-4)$-dimensional (in the quaternionic case) or 8-dimensional (in the octonionic case) distribution of $S^n$ is the boundary at infinity of a complete Einstein metric on the ball $B^{n+1}$.

At each point $x \in S^n$, the Einstein metric is asymptotic to Heber's homogeneous metric on the solvable group associated to the nilpotent algebra $\mathfrak{n}_x$.

Let us precise immediately the meaning of the statement on the asymptotic behaviour of the Einstein metric. Fix a defining function $t$ of the boundary $S^n$ of $B^{n+1}$ (for example $t = 1 - r^2$), then there exists a metric $\gamma$ on the distribution $\mathscr{D}$ and a metric $\eta$ on $TS/\mathscr{D}$, such that near the boundary the Einstein metric $g$ is asymptotic to

$$\frac{dt^2 + \eta}{t^2} + \frac{\gamma}{t}, \tag{1}$$

where we identify a neighborhood of $S$ in $B$ with $(0, \epsilon) \times S$. Then the 2-tensor $t^2 g$ extends continuously to the boundary $S$, and one recovers the distribution $\mathscr{D}$ from the metric as the kernel of $(t^2 g)|_S$. In the formula (1), the metric $\eta$ is extended to a 2-tensor on $TS$ by choosing a supplementary subspace to $\mathscr{D} \subset TS$, but the asymptotics of $g$ does not depend on this choice. As will be explained in remark 1.3, at each point $x \in S^n$, the meaning of (1) is that when one approaches the boundary point $x$, the Einstein metric approaches a homogeneous metric on a solvable group associated to the nilpotent Lie algebra $\mathfrak{n}_x$. This metric is Heber's metric. As Heber's metric on a given solvable group is unique, this implies that the choice of $(\eta_x, \gamma_x)$ is unique up to dilations $(\eta_x, \gamma_x) \to (\lambda^2 \eta_x, \lambda \gamma_x)$ for a positive real number $\lambda$.

Compared to previous results, the meaning of the theorem is that all deformations of the distributions on the boundary of the rank one symmetric spaces can be interpreted as boundaries at infinity of Einstein metrics, but maybe with an anisotropic behaviour (the asymptotics depends on the point at infinity). This gives new examples in the quaternionic case, for dimension at least 11, and in the octonionic case.

The relation between the regular examples and the new examples is perhaps best understood by remembering that the Einstein metrics associated to quaternionic-contact structures in dimension at least 11 are actually quaternionic-Kähler [3], so they keep the holonomy $Sp_m Sp_1$ of the hyperbolic space. This condition distinguishes exactly the regular case:

**Corollary 0.2.** In the quaternionic case, for $m \geq 3$, the Einstein metric constructed by the previous theorem is quaternionic-Kähler if and only if the distribution on $S^{4m-1}$ is regular (that is, is a quaternionic-contact structure).

The corollary follows from the fact that the boundary at infinity of a quaternionic-Kähler metric must be a quaternionic-contact structure [2].

There is a similar, but obvious, story in the octonionic case. The Cayley plane has holonomy $Spin_9$. If the Einstein metric keeps the $Spin_9$ condition, it is well-known that it is the hyperbolic metric ($Spin_9$ metrics are locally symmetric). On the other hand, a regular distribution of dimension 8 on $S^{15}$ must be standard. So we have a (trivial) example of the equivalence of the holonomy condition on the Einstein metric with the regularity condition on the boundary.

The article has two parts. The first part is algebraic, and consists in the construction of an approximate Einstein metric near the boundary at infinity. The new point here is that the model is not explicit: it is the solution of algebraic equations giving the metrics $\gamma$ and $\eta$ on the distribution $\mathscr{D}$ and on the quotient $TS/\mathscr{D}$. These equations have a nice interpretation in terms of a stronger geometric structure, a quaternionic (or octonionic) structure on $\mathscr{D}$, on which we add a gauge condition which enables to find a unique solution. This additional structure should be useful in future applications, in particular if one wishes to work out a Fefferman-Graham type development of the Einstein metric.

The second part is analytic, and consists in deforming an approximate Einstein metric into a solution of the equations. This relies basically on a deformation argument, which requires to understand the analytic properties of the deformation operator. If one has a good understanding of the analysis for the models (Einstein metrics on solvable groups), then one can probably use microlocal analysis to glue together the inverses of the deformation operator into the required parametrix. However here we prefer to avoid the analysis on these solvable groups, since more direct methods give the required result. Nevertheless, it is clear that the more sophisticated microlocal analysis may be required in further developments of the theory.

## 1. Algebraic considerations

Let $V_1$ and $V_2$ be vector spaces of dimensions $4m-4$ and 3 (in the quaternionic case) or of dimensions 8 and 7 (in the octonionic case). A formal Levi bracket is an element $\ell$ of $W = \wedge^2 V_1^* \otimes V_2$. This bracket makes

$$\mathfrak{n} = V_1 \oplus V_2 \tag{2}$$

into a two-graded nilpotent Lie algebra (as the Jacobi identity is trivially satisfied). The corresponding Lie group $N$ will then carry an invariant distribution $\mathscr{D}$ of same dimension as $V_1$. Consider the Lie bracket $[,]$ on sections of $\mathscr{D}$. This is a differential bracket, but the differential part of it only maps into $\mathscr{D}$. Hence the map

$$\begin{aligned} \mathscr{L} : \wedge^2 \Gamma(\mathscr{D}) &\rightarrow \Gamma(TN/\mathscr{D}) \\ \mathscr{L}(X,Y) &= [X,Y]/\mathscr{D} \end{aligned}$$

is an algebraic map, i.e. a section of $\wedge^2 \mathscr{D}^* \otimes (TN/\mathscr{D})$. If we designate by $L$ the group $GL(V_1) \oplus GL(V_2)$, then $\mathscr{D}$ and $TN/\mathscr{D}$ are bundles associated to an $L$-principal bundle $E \rightarrow N$.

In this set-up, $\mathscr{L}$ corresponds to an $L$-equivariant map $f_{\mathscr{L}}$ from $E$ to $W$. Designate by $E_p$ the fibre of $E$ at $p \in N$. By construction, $\ell$ is in the image $f_{\mathscr{L}}(E_p)$ for all $p$, and this image consists precisely of the $L$ orbit of $\ell$ in $W$.

Under the identifications $V_1 = \mathbb{H}^{m-1}$ and $V_2 = \operatorname{im}\mathbb{H}$, the quaternionic standard Levi bracket $\kappa$ is given by the choice of a Hermitian metric $h$ on $V_1$; in this case, $\kappa$ is simply the imaginary part of $h$.

Similarly, the standard octonionic bracket (also designated $\kappa$) is also defined by identifications $V_1 = \mathbb{O}$, $V_2 = \operatorname{im}\mathbb{O}$ and a choice of $h$.

In general, $\kappa$ is only defined up to $L$-action; but as $\ell$ is only defined up to $L$-action, we will assume our choice of $\kappa$ is fixed.

We identify $G_0$ as the stabiliser of $\kappa$ in $L$; this can be seen as the group that stabilises the quaternionic or octonionic structure. In the quaternionic case,

$$G_0 = \mathbb{R}_+^* Sp(1)Sp(m-1),$$

while in the octonionic case,

$$G_0 = \mathbb{R}_+^* Spin(7).$$

In general, we consider a manifold $X^n$ of dimension $n = 4m-1$ in the quaternionic case, or $n = 15$ in the octonionic case, and a distribution $\mathscr{D} \subset TX$ of dimension equal that of $V_1$. At each point $x$ of $X$, the image in $TX/\mathscr{D}$ of the bracket $[X_1, X_2]$ of two vector fields in $\mathscr{D}$ is an algebraic map, $\mathscr{L}_x \in \Lambda^2\mathscr{D}_x^* \otimes (T_xX/\mathscr{D}_x)$. If we choose a linear identification $e : V_1 \oplus V_2 \xrightarrow{\sim} \mathscr{D}_x \oplus T_xX/\mathscr{D}_x$, sending $V_1$ to $\mathscr{D}_x$ and $V_2$ to $T_xX/\mathscr{D}_x$, then the algebraic bracket $\mathscr{L}_x$ can be identified with the element $e^*\mathscr{L}_x$ in $W$, but there is an ambiguity, since we can compose $e$ by the action of $L$ on $V_1 \oplus V_2$. Therefore $\mathscr{L}_x$ defines only a $L$-orbit in $W$, which will be denoted by $\mathscr{O}(\mathscr{L}_x)$. In more intrinsic terms: $e$ is an element of the product $E$ of the frame bundles of $\mathscr{D}$ and $TX/\mathscr{D}$ (this is a $L$-principal bundle), and the formula $f_{\mathscr{L}}(e) = e^*\mathscr{L}_x$ defines a $L$-equivariant map $f_{\mathscr{L}} : E_x \to W$. Then the orbit $\mathscr{O}(\mathscr{L}_x) \subset W$ is exactly the image $f_{\mathscr{L}}(E_x)$.

The main result of this section is:

**Proposition 1.1.** There exists a $L$-invariant open set $U \subset W$ (that is an open set of $L$-orbits), containing $\kappa$, with the following property. If $X^n$ has a distribution $\mathscr{D}$ such that for every $x \in X$ the induced bracket $\mathscr{L}_x$ satisfies $\mathscr{O}(\mathscr{L}_x) \subset U$, then there exist metrics $\eta$ and $\gamma$ on $TX/\mathscr{D}$ and $\mathscr{D}$, such that, choosing any splitting $TX = \mathscr{D} \oplus V$, the metric

$$g = \frac{dt^2 + \eta}{t^2} + \frac{\gamma}{t} \tag{3}$$

on $\mathbb{R}_+^* \times X$ is asymptotically Einstein when $t \to 0$:

$$\mathrm{Ric}(g) = \lambda g + O(t^{\frac{1}{2}}), \tag{4}$$

where $\lambda = -m-2$ in the quaternionic case, $\lambda = -9$ in the octonionic case. Moreover, this choice of $\eta$ and $\gamma$ is unique, up to the conformal transformation:

$$(\eta, \gamma) \to (f^2\eta, f\gamma)$$

for $f$ a strictly positive function $X \to \mathbb{R}$.

In this statement, it is important to note that the asymptotic behaviour (4) does not depend on the choice of splitting $TX = \mathscr{D} \oplus V$.

Note the square root $t^{\frac{1}{2}}$ in (4): this is natural with the form (3) of the metric, e.g. in the asymptotically complex case the expansion of the Einstein metric involves powers of $t^{\frac{1}{2}}$ rather than $t$. It might be interesting to study whether a good choice of the splitting $TX = \mathscr{D} \oplus V$ might lead to a metric $g$ which is Einstein already up to the higher order $O(t)$.

**Remark 1.2.** In the special case where $X$ is the nilpotent group $N$ associated to an algebraic bracket $\ell \in \Lambda^2 V_1^* \otimes V_2$, and the distribution $\mathscr{D}$ is the associated distribution, then the splitting (2) gives a canonical choice for $V$. Then the metric (3) is an invariant metric on the solvable group $S = \mathbb{R}_+^* \ltimes N$. In particular, $\mathrm{Ric}(g) - \lambda g$ is homogeneous and has constant norm, so by (4) it must be zero. Therefore the metric is Einstein. This is the metric constructed by Heber [8] on $S$ (by uniqueness of Heber's metric). The proof of the proposition will give another construction of this metric, at least for small deformations of the distribution. Conversely, it also follows from our proof that the open set $U$ can be taken equal to the set of brackets $\ell$ such that an Einstein metric exists on the associated solvable group.

**Remark 1.3.** In general, at each point $x \in X$ the bracket $\mathscr{L}_x$ gives a nilpotent Lie algebra structure on $\mathfrak{n}_x = \mathscr{D}_x \oplus T_x X / \mathscr{D}_x$. Let $N_x$ be the corresponding nilpotent group and $S_x$ the associated solvable group $S_x = \mathbb{R}_+^* \ltimes N_x$. We are going to see the relation between the asymptotically Einstein metric (3) and $S_x$. To simplify notation, let us consider only the quaternionic case (but the octonionic case is similar). Near $x$ we choose coordinates $(x_1, \dots, x_n)$ on $X$ such that $\mathscr{D}_x$ is generated by the vector fields $(\frac{\partial}{\partial x_4}, \dots, \frac{\partial}{\partial x_n})$. The distribution $\mathscr{D}$ is given by the kernel of three 1-forms, $\alpha_1$, $\alpha_2$ and $\alpha_3$, and we can suppose that near the point $x$ one has $\eta = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$. Moreover we can arrange the coordinates so that at the point $x$ one has $\alpha_i = dx_i$. Then we consider the homothety

$$h_r(t, x_1, \dots, x_n) = (rt, rx_1, rx_2, rx_3, \sqrt{r}x_4, \dots, \sqrt{r}x_n). \tag{5}$$

Note $\alpha_i = \alpha_i^j dx_j$, with $\alpha_i^j(0) = \delta_i^j$. Then one has

$$\bar{\alpha}_i := \lim_{r \to 0} \frac{1}{\sqrt{r}} h_r^* \alpha_i = dx_i + \sum_{j,k=4}^{n} x_k \frac{\partial \alpha_i^j}{\partial x_k}(0) dx_j.$$

The three forms $\bar{\alpha}_1$, $\bar{\alpha}_2$ and $\bar{\alpha}_3$ are homogeneous, and define exactly the horizontal distribution of the nilpotent group $N_x$. Denote $\bar{\gamma} := \gamma(0)$

and $\bar\eta = \bar\alpha_1^2 + \bar\alpha_2^2 + \bar\alpha_3^2$, then, when $r \to 0$, one obtains the limit

$$h_r^* g \longrightarrow \bar g = \frac{dt^2 + \bar\eta}{t^2} + \frac{\bar\gamma}{t}.$$

This metric is now defined globally in the coordinates $(t, x_1, \ldots, x_n)$, and is actually an invariant metric on the solvable group $S_x$. (This is a nice way to see the solvable group $S_x$ directly from the distribution as the limit of the inhomogeneous rescaling (5) when $r \to 0$). From (4) one has $\mathrm{Ric}(h_r^* g) + \lambda h_r^* g = O(r^{\frac12} t^{\frac12})$ and therefore at the limit $\mathrm{Ric}(\bar g) + \lambda \bar g = 0$, that is $\bar g$ is Einstein. Therefore $\bar g$ must be the unique homogeneous Einstein metric on $S_x$ mentioned in the previous remark. This justifies the statement in Theorem 0.1 that at each point the constructed metric is asymptotic to the corresponding Heber's metric.

*Proof of proposition 1.1.* The uniqueness comes from remark 1.3 and the uniqueness of the homogeneous Einstein metric on $S_x$ proved by Heber. Later in the paper, a weaker uniqueness will also be proved.

We will calculate the Ricci tensor of the metric (3) as a function of $\gamma$ and $\eta$. The calculation is local, so we can choose orthonormal frames $\{\check X_i\}$ and $\{\check Y_i\}$ of $(TX/\mathscr{D})$ and $\mathscr{D}$, respectively.

On $M = \mathbb{R}_+^* \times X$, we can define an orthonormal frame via:

$$\begin{aligned} X_0 &= t\frac{\partial}{\partial t}, \\ X_i &= t\check X_i, \\ Y_i &= \sqrt{t}\check Y_i. \end{aligned}$$

Let $O(a)$ denote sections of $TM$ whose norm (under $g$) tends to zero at least as last as $t^a$. Then we may calculate the Lie brackets of the above frame elements:

$$\begin{aligned} [X_0, X_i] &= X_i, \\ [X_0, Y_i] &= \frac12 Y_i, \\ [X_i, X_j] &= O(1), \\ [X_i, Y_j] &= O(1/2), \\ [Y_i, Y_j] &= \mathscr{L}_{ij} + O(1/2), \end{aligned}$$

where $\mathscr{L}_{ij} = \mathscr{L}_{Y_i Y_j}$. In future, if $\upsilon$ is a section of $TX/\mathscr{D}$, we will denote by $\mathscr{L}^\upsilon$ the $\upsilon$ component of $\mathscr{L}$ – i.e.

$$\mathscr{L}^\upsilon_{\xi\zeta} = g(\mathscr{L}_{\xi\zeta}, \upsilon),$$

which we will abbreviate as $\mathscr{L}^k$ for $\upsilon = X_k$. Similarly, for $\xi$ a section of $\mathscr{D}$ we denote by $\mathscr{L}^{\upsilon}_{\xi}$ the section of $\mathscr{D}$ defined by:

$$g(\mathscr{L}^{\upsilon}_{\xi}, Y_j) = \mathscr{L}^{\upsilon}_{\xi Y_j},$$

and use the short forms $\mathscr{L}^k_i = \mathscr{L}^k_{Y_i}$, $\mathscr{L}^k_{i\xi} = \mathscr{L}^k_{Y_i\xi}$ and $\mathscr{L}^k_{ij} = \mathscr{L}^k_{Y_iY_j}$.

Now let $\nabla$ be the Levi-Civita connection of $g$. We can calculate $\nabla$ by using the Koszul formula:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X \cdot g(Y,Z) + Y \cdot g(X,Z) - Z \cdot g(X,Y) \\ &\quad + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X). \end{aligned}$$

Since our frame elements are orthonormal, the formula reduces to

$$2g(\nabla_X Y, Z) = g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X),$$

giving:

$$\begin{aligned} \nabla_{X_0} X_0 &= 0, \\ \nabla_{X_0} X_i = \nabla_{X_0} Y_i &= 0, \\ \nabla_{X_i} X_0 &= -X_i, \\ \nabla_{Y_i} X_0 &= -\frac{1}{2} Y_i, \\ \nabla_{X_i} X_j &= \delta_{ij} X_0 + O(1/2), \\ \nabla_{X_i} Y_j = \nabla_{Y_j} X_i &= -\frac{1}{2}\mathscr{L}^i_j + O(1/2), \\ \nabla_{Y_i} Y_j = \nabla_{Y_j} Y_i &= \frac{1}{2}\mathscr{L}_{ij} + \frac{1}{2}\delta_{ij} X_0 + O(1/2). \end{aligned}$$

So in this frame, $\nabla = d + A + O(1/2)$ where $A \in \Gamma(T^*X \otimes \operatorname{End} TX)$ is independent of $t$. In detail:

$$\begin{aligned} A(X_0) &= 0, \\ A(X_i) &: \begin{cases} X_0 \to -X_i \\ X_j \to \delta_{ij} X_0 \\ Y_j \to -\frac{1}{2}\mathscr{L}^i_j, \end{cases} \\ A(Y_i) &: \begin{cases} X_0 \to -\frac{1}{2} Y_i \\ X_j \to -\frac{1}{2}\mathscr{L}^j_i \\ Y_j \to \frac{1}{2}\mathscr{L}_{ij} + \frac{1}{2}\delta_{ij} X_0. \end{cases} \end{aligned}$$

One has $dA(X,Y) = X \cdot A(Y) - Y \cdot A(X) - A([X,Y])$. Note that differentiating $A$ in the $X_0$ direction is zero, while differentiating $A$ in the direction of $X_i$ or $Y_i$ picks up a $t$ or $\sqrt{t}$ term, and hence become $O(1/2)$. Thus $dA(X,Y) = -A([X,Y]) + O(1/2)$.

The curvature $R$ of $\nabla$ is $dA + [A, A] + O(1/2)$, which immediately implies that

$$
\begin{aligned}
R_{X_0,X_i} &= -A(X_i) + O(1/2), \\
R_{X_0,Y_i} &= -\frac{1}{2}A(Y_i) + O(1/2), \\
R_{X_i,X_j} &= [A(X_i), A(X_j)] + O(1/2), \\
R_{X_i,Y_j} &= [A(X_i), A(Y_j)] + O(1/2), \\
R_{Y_i,Y_j} &= -A(\mathscr{L}_{ij}) + [A(X_i), A(X_j)] + O(1/2).
\end{aligned}
$$

The commutator terms are given by:

$$
[A(X_i), A(X_j)] \quad : \quad \begin{cases} X_0 & \rightarrow \quad 0 \\ X_k & \rightarrow \quad \delta_{ik}X_j - \delta_{jk}X_i \\ Y_k & \rightarrow \quad \frac{1}{4}\left(\mathscr{L}^i_{\mathscr{L}^j_k} - \mathscr{L}^j_{\mathscr{L}^i_k}\right), \end{cases}
$$

$$
[A(X_i), A(Y_j)] \quad : \quad \begin{cases} X_0 & \rightarrow \quad -\frac{1}{4}\mathscr{L}^i_j \\ X_k & \rightarrow \quad \frac{1}{4}\mathscr{L}^i_{\mathscr{L}^k_j} + \frac{1}{2}\delta_{ik}Y_j \\ Y_k & \rightarrow \quad -\frac{1}{2}\delta_{jk}X_i + \frac{1}{4}\mathscr{L}_{j\mathscr{L}^i_k}, \end{cases}
$$

$$
[A(Y_i), A(Y_j)] \quad : \quad \begin{cases} X_0 & \rightarrow \quad \frac{1}{2}\mathscr{L}_{ji} \\ X_k & \rightarrow \quad \frac{1}{4}\left(\mathscr{L}_{j\mathscr{L}^k_i} - \mathscr{L}_{i\mathscr{L}^k_j}\right) + \frac{1}{2}\mathscr{L}^k_{ji}X_0 \\ Y_k & \rightarrow \quad \frac{1}{4}\left(\mathscr{L}_j^{\mathscr{L}_{ik}} - \mathscr{L}_i^{\mathscr{L}_{jk}} + \delta_{ik}Y_j - \delta_{jk}Y_i\right). \end{cases}
$$

Now we need to take the Ricci-trace of this expression:

$$
\begin{aligned}
\mathrm{Ric}_{X_0,X_0} &= \sum_i g(X_i, R_{X_i,X_0}X_0) + \sum_i g(Y_i, R_{Y_i,X_0}X_0) \\
&= \sum_i g(X_i, -X_i) + \sum_i g(Y_i, -\frac{1}{4}Y_i) + O(1/2) \\
&= \lambda + O(1/2).
\end{aligned}
$$

Here $\lambda$ is equal to $-3-(4m-4)/4 = -m-2$ in the quaternionic case, and $-7-8/4 = -9$ in the octonionic case. The cross-terms of the Ricci curvature all vanish:

$$\begin{aligned}
\mathrm{Ric}_{X_0,X_i} &= \sum_j g(X_j, R_{X_j,X_0}X_i) + \sum_i g(Y_j, R_{Y_j,X_0}X_i) \\
&= \sum_j g(X_j, \delta_{ij}X_0) + \sum_j \frac{1}{4}\mathscr{L}_{jj} + O(1/2) \\
&= O(1/2), \\
\mathrm{Ric}_{X_0,Y_i} &= \sum_j g(X_j, R_{X_j,X_0}Y_i) + \sum_i g(Y_j, R_{Y_j,X_0}Y_i) \\
&= O(1/2), \\
\mathrm{Ric}_{X_i,Y_j} &= g(X_0, R_{X_0,X_i}Y_j) + \sum_j g(X_j, R_{X_j,X_i}Y_j) + \sum_i g(Y_j, R_{Y_j,X_i}Y_j) \\
&= O(1/2),
\end{aligned}$$

the last two expressions vanishing because they are sums of terms of type $g(X, Y)$ with $X \perp Y$. Next, the $\mathscr{D} \times \mathscr{D}$ term is:

$$\begin{aligned}
\mathrm{Ric}_{X_i,X_j} &= g(X_0, R_{X_0,X_i}X_j) + \sum_k g(X_k, R_{X_k,X_i}X_j) + \sum_k g(Y_k, R_{Y_k,X_i}X_j) \\
&= -\delta_{ij} + \delta_{ij} - \sum_k \delta_{ij} g(X_k, X_k) - \sum_k \frac{1}{2}\delta_{ij} g(Y_k, Y_k) \\
&\quad -\sum_k \frac{1}{4} g(Y_k, \mathscr{L}^i_{\mathscr{L}^j_k}) + O(1/2).
\end{aligned}$$

In the quaternionic case, this is

$$\mathrm{Ric}_{X_i,X_j} = \lambda\delta_{ij} + (1-m)\delta_{ij} + \sum_{k=1}^{4m-4} \frac{1}{4}\mathscr{L}^i_{k\mathscr{L}^j_k} + O(1/2). \tag{6}$$

In the octonionic case, this is

$$\mathrm{Ric}_{X_i,X_j} = \lambda\delta_{ij} - 2\delta_{ij} + \sum_{k=1}^{8} \frac{1}{4}\mathscr{L}^i_{k\mathscr{L}^j_k} + O(1/2). \tag{7}$$

Finally the $(TX/\mathscr{D}) \times (TX/\mathscr{D})$ term is

$$\begin{aligned}
\mathrm{Ric}_{Y_i,Y_j} &= g(X_0, R_{X_0,Y_i}Y_j) + \sum_k g(X_k, R_{X_k,Y_i}Y_j) + \sum_k g(Y_k, R_{Y_k,Y_i}Y_j) \\
&= -\frac{1}{4}\delta_{ij} + \sum_k \left(\frac{1}{4}\mathscr{L}^k_{i\mathscr{L}^k_j} - \frac{1}{2}\delta_{ij}\right) + \frac{1}{4}\left(\delta_{ij} + \sum_k 3\mathscr{L}^{\mathscr{L}_{kj}}_{ik} - \delta_{ij}\right) \\
&\quad + O(1/2) \\
&= -\frac{1}{2}\sum_k \delta_{ij} + \frac{1}{4}\sum_k (2\mathscr{L}^{\mathscr{L}_{kj}}_{ik} - \delta_{ij}) + O(1/2).
\end{aligned}$$

since

$$\sum_k \mathscr{L}^{\mathscr{L}_{kj}}_{ik} = \sum_{kp} \mathscr{L}^p_{ik}\mathscr{L}^p_{kj} = \sum_{pk} \mathscr{L}^k_{ip}\mathscr{L}^k_{pj} = \sum_k -\mathscr{L}^k_{i\mathscr{L}^k_j}.$$

In the quaternionic case, the curvature is

$$\mathrm{Ric}_{Y_i,Y_j} = \lambda\delta_{ij} + \frac{3}{2}\delta_{ij} + \frac{1}{2}\sum_{k=1}^{4m-4} \mathscr{L}_{ik}^{\mathscr{L}_{kj}} + O(1/2). \tag{8}$$

In the octonionic case, it is:

$$\mathrm{Ric}_{Y_i,Y_j} = \lambda\delta_{ij} + \frac{7}{2}\delta_{ij} + \frac{1}{2}\sum_{k=1}^{8} \mathscr{L}_{ik}^{\mathscr{L}_{kj}} + O(1/2). \tag{9}$$

Now $(M, g)$ is asymptotically Einstein if $\mathrm{Ric}_{X_i,X_j} = \lambda\delta_{ij} + O(1/2)$ and $\mathrm{Ric}_{Y_i,Y_j} = \lambda\delta_{ij} + O(1/2)$. From now on, we will use the Einstein summation convention, where any repeated index is summed over. Then the equations (6) and (8) imply that in the quaternionic case, we must have:

$$\begin{array}{rcl} \mathscr{L}_{ij}^k \mathscr{L}_{op}^q \gamma^{io}\gamma^{jp} & = & 4(m-1)\eta^{kq}, \\ \mathscr{L}_{ij}^k \mathscr{L}_{op}^q \gamma^{io}\eta_{kq} & = & 3\gamma_{jp}, \end{array} \tag{10}$$

while equations (7) and (9) imply that in the octonionic case, we must have:

$$\begin{array}{rcl} \mathscr{L}_{ij}^k \mathscr{L}_{op}^q \gamma^{io}\gamma^{jp} & = & 8\eta^{kq}, \\ \mathscr{L}_{ij}^k \mathscr{L}_{op}^q \gamma^{io}\eta_{kq} & = & 7\gamma_{jp}. \end{array} \tag{11}$$

For an $\ell$ sufficiently close to $\kappa$, these equations can be solved (see Theorem 1.7), and the solution is unique up to conformal transformations. □

A natural question is whether, as in the quaternionic-contact case, the conformal class $(\eta, \gamma)$ comes with a quaternionic structure on $\mathscr{D}$ and $TX/\mathscr{D}$. The same applies for the octonionic-contact structures, of course. We propose here a construction, where instead of looking only for a conformal class, one constructs directly a quaternionic or octonionic structure. As a byproduct, the system (10) or (11) is interpreted in a natural way, see (15), and existence of a solution is provided.

The automorphism group of these structures is $G_0$, which is contained in the conformal automorphism group

$$G' = \mathbb{R}_+^* \times SO(\eta) \times SO(\gamma)$$

of $(\eta, \gamma)$. Thus it seems that to get the quaternionic/octonionic structures on the manifold, we need to impose extra equations beyond (10) and (11).

These can best be understood by looking at the normality $\partial^*$ operator described in [6], [5] and [7]. It is an algebraic Lie algebra codifferential, which extends naturally to a bundle operator on associated bundles. If $X$ is a quaternionic- or octonionic-contact manifold, $\mathscr{K}$ the corresponding Levi-bracket, and $\mathscr{M}$ is any section of $\wedge^2\mathscr{D}^* \otimes (TX/\mathscr{D})$. Then $\partial^*\mathscr{M} = \alpha\mathscr{M} \oplus \beta\mathscr{M}$ where

$$(\alpha\mathscr{M})^r_q = (\gamma^{jr}\gamma^{ip}\eta_{ko})(\mathscr{M}^k_{ij}\mathscr{K}^o_{pq}),$$
$$(\beta\mathscr{M})^k_r = -\tfrac{1}{2}(\eta_{or}\gamma^{ip}\gamma^{jq})(\mathscr{M}^k_{ij}\mathscr{K}^o_{pq}), \tag{12}$$

Einstein summation over repeated indexes being assumed. Note that these expressions are invariant under conformal transformations $(\eta, \gamma) \to (f^2\eta, f\gamma)$. If we apply $\alpha$ and $\beta$ to $\mathscr{K}$ itself, we get:

**Lemma 1.4.** In the quaternionic-contact case:

$$\begin{aligned}(\alpha\mathscr{K}) &= 3Id_{\mathscr{D}}\\ -2(\beta\mathscr{K}) &= 4(m-1)Id_{TX/\mathscr{D}},\end{aligned}$$

while in the octonionic-contact case:

$$\begin{aligned}(\alpha\mathscr{K}) &= 7Id_{\mathscr{D}}\\ -2(\beta\mathscr{K}) &= 8Id_{TX/\mathscr{D}},\end{aligned}$$

the same numbers as in equations (10) and (11).

*Proof.* Fix $\eta$ and $\gamma$, and pick local orthonormal sections $\{I_1, \cdots, I_p\}$ of $TX/\mathscr{D}$, where $p = 3$ in the quaternionic case, and $p = 7$ in the octonionic case. These all correspond to complex structures on $\mathscr{D}$. Then for $X, Y \in \Gamma(\mathscr{D})$, $\mathscr{K}(X, Y)$ can be written as:

$$\mathscr{K}(X,Y) = \sum_{i=1}^{p} \gamma(I_i(X), Y)I_i.$$

By extension, define $I_0$ to be the identity transformation of $\mathscr{D}$. Now pick local orthonormal sections $\{Y_1, \cdots Y_q\}$ of $\mathscr{D}$, chosen so that $I_iY_j$ is orthogonal to all $I_kY_l$ whenever $i \neq k$ or $j \neq k$. This is possible, as $\gamma$ must be hermitian with respect to these complex structures. Here, $q = m-1$ for quaternionic structures, and $q = 1$ for octonionic structures.

Again, we may rewrite $\mathscr{K}$ as:

$$\mathscr{K} = \sum_{i=0,j,k=1}^{i,j=p,k=q} -(I_jI_iY_k)^* \otimes (I_iY_k)^* \otimes I_j.$$

If we raise and lower all indexes with $\eta$ and $\gamma$, we get $\mathscr{K}^*$, which is

$$\mathscr{K} = \sum_{i=0,j,k=1}^{i,j=p,k=q} -(I_jI_iY_k) \otimes (I_iY_k) \otimes (I_j)^*.$$

Now $\alpha\mathscr{K}$ involves taking the trace of $\mathscr{K}$ and $\mathscr{K}^*$ over one of the $\mathscr{D}$ components and over the $TX/\mathscr{D}$ components. The trace over the

$TX/\mathscr{D}$ component is trivial; and if $\llcorner$ denotes contraction between a space and its dual,

$$\begin{aligned}
\alpha\mathscr{K} &= \sum_{k,o=1}^{q}\sum_{j,r=1}^{p}\sum_{i,l=0}^{p}\left((I_jI_iY_k)^*\llcorner(I_rI_lY_o)\right)(I_j\llcorner I_r^*)\,(I_iY_k)^*\otimes(I_lY_o) \\
&= \sum_{k,o=1}^{q}\sum_{j,r=1}^{p}\sum_{i,l=0}^{p}\left((I_jI_iY_k)^*\llcorner(I_rI_lY_o)\right)\delta_{jr}(I_iY_k)^*\otimes(I_lY_o) \\
&= \sum_{k,o=1}^{q}\sum_{j=1}^{p}\sum_{i,l=0}^{p}\left((I_jI_iY_k)^*\llcorner(I_jI_lY_o)\right)(I_iY_k)^*\otimes(I_lY_o) \\
&= \sum_{k,o=1}^{q}\sum_{j=1}^{p}\sum_{i,l=0}^{p}\delta_{il}\delta_{ko}(I_iY_k)^*\otimes(I_lY_o) \\
&= \sum_{j=1}^{p}\sum_{i=0}^{p}\sum_{k=0}^{q}(I_iY_k)^*\otimes(I_iY_k) \\
&= pId_{\mathscr{D}}.
\end{aligned}$$

The $-2\beta\mathscr{K}$ term is the contraction of $\mathscr{K}$ and $\mathscr{K}^*$ over both their $\mathscr{D}$ components; it is

$$\begin{aligned}
-2\beta\mathscr{K} &= \sum_{k,o=1}^{q}\sum_{j,r=1}^{p}\sum_{i,l=0}^{p}\left((I_jI_iY_k)^*\llcorner(I_rI_lY_o)\right)\left((I_iY_k)^*\llcorner(I_lY_o)\right)I_j\otimes I_r^* \\
&= \sum_{k,o=1}^{q}\sum_{j,r=1}^{p}\sum_{i,l=0}^{p}\left((I_jI_iY_k)^*\llcorner(I_rI_lY_o)\right)(\delta_{ko}\delta_{il})I_j\otimes I_r^* \\
&= \sum_{k=1}^{q}\sum_{i=0}^{p}\sum_{j,r=1}^{p}\left((I_jI_iY_k)^*\llcorner(I_rI_iY_k)\right)I_j\otimes I_r^* \\
&= \sum_{k=1}^{q}\sum_{i=0}^{p}\sum_{j,r=1}^{p}\delta_{jr}I_j\otimes I_r^* \\
&= \sum_{k=1}^{q}\sum_{i=0}^{p}\sum_{j=1}^{p}I_j\otimes I_j^* \\
&= q(p+1)Id_{TX/\mathscr{D}}.
\end{aligned}$$

Then substituting in the values for $p$ and $q$ gives the result. $\square$

Now if $\mathscr{N}$ is a section of $\wedge^2\mathscr{D}^*\otimes(TX/\mathscr{D})$, we may use it in equations (12) instead of $\mathscr{K}$; in that case, define

$$\begin{aligned}
(\alpha_{\mathscr{N}}\mathscr{M})_q^r &= (\gamma^{jr}\gamma^{ip}\eta_{ko})(\mathscr{M}_{ij}^k\mathscr{N}_{pq}^o), \\
(\beta_{\mathscr{N}}\mathscr{M})_r^k &= -\frac{1}{2}(\eta_{or}\gamma^{ip}\gamma^{jq})(\mathscr{M}_{ij}^k\mathscr{N}_{pq}^o).
\end{aligned}$$

Similarly, though $\mathscr{K}$ defines the conformal class of $(\eta,\gamma)$, (through the reduction to structure group $G_0\subset G'$), there is no reason to require that $\mathscr{K}$ be the Levi-bracket of the distribution $\mathscr{D}$. Given $(\eta,\gamma)$ on a general manifold with distribution $\mathscr{D}$ of correct dimension and co-dimension, they define a (local) class of compatible brackets $\mathscr{K}$ of quaternionic-contact or octonionic-contact type. Then the equations (10) and (11) can be rewritten as saying that we must find $(\eta, \gamma)$ such that for any $\mathscr{K}$ compatible with them,

$$\alpha_{\mathscr{K}} \mathscr{K} = \alpha_{\mathscr{L}} \mathscr{L}, \tag{13}$$

$$\beta_{\mathscr{K}} \mathscr{K} = \beta_{\mathscr{L}} \mathscr{L}, \tag{14}$$

or, more compactly,

$$\partial^*_{\mathscr{K}} \mathscr{K} = \partial^*_{\mathscr{L}} \mathscr{L}. \tag{15}$$

It is easy to see that these equations are conformally invariant.

**Remark.** It is useful to compare these equations with those defining a 'Damek-Ricci' space (this is a subclass of Heber's metrics, see [10]). For any section $Z$ of $\mathscr{D}$, we may define an endomorphism $J_Z$ of $\mathscr{D}$ by

$$\gamma(J_Z X, Y) = \eta(Z, \mathscr{L}(X, Y)),$$

for sections $X$ and $Y$ of $\mathscr{D}$. Then $X$ is asymptotically Damek-Ricci if $J_Z^2 = -1\eta(Z, Z)$. Now if $\{Z_j\}$ is a local orthonormal frame for $TX/\mathscr{D}$, then we may rewrite $\alpha_{\mathscr{L}} \mathscr{L}$ once more as

$$\begin{aligned} \alpha_{\mathscr{L}} \mathscr{L} &= \mathrm{Tr}_\gamma \Big( \sum_{jk} \eta(Z_j, Z_k) \eta(Z_j, \mathscr{L}) \otimes \eta(Z_k, \mathscr{L}) \Big) \\ &= \mathrm{Tr}_\gamma \Big( \sum_j J_{Z_j} \otimes J_{Z_j} \Big) \\ &= -\sum_j J_{Z_j}^2. \end{aligned}$$

Since $Z_j$ is normal, $J_{Z_j}^2 = -Id_{\mathscr{D}}$, and Damek-Ricci spaces must solve equation (13). Similarly, for $Z$ and $Z'$ sections of $TX/\mathscr{D}$

$$-2\eta((\beta_{\mathscr{L}} \mathscr{L})(Z), Z') = \mathrm{Tr}_\gamma \ \mathrm{Tr}_\gamma \ J_{Z'} \otimes J_Z.$$

Since this must be symmetric, it values are determined by taking $Z = Z'$; in which case it is $4(m-1)\eta(Z, Z)$ in the quaternionic case, and $8\eta(Z, Z)$ in the octonionic one. Consequently Damek-Ricci spaces are special solutions of equation (15), as are any spaces that are asymptotically Damek-Ricci (i.e. spaces with $\mathscr{L}$, $\gamma$ and $\eta$ such that the relation $J_Z^2 = -1\eta(Z, Z)$ holds).

It is still somewhat unsatisfactory that there is a large class of $\mathscr{K}$ compatible with a given $\mathscr{L}$. It would be better to have a procedure that fixes $\mathscr{K}$ uniquely (and hence the quaternionic/octonionic structure, as well as $(\eta, \gamma)$).

In the quaternionic case, the dimension of $L$ (the full graded automorphism group) is $(4m-4)^2+3^2 = 16m^2-32m+25$, while the group $G'$ is of dimension $(4m-4)(4m-5)/2+3+1 = 8m^2-18m+14$ and $G_0$ is of dimension $(2m-2)(2m-1)/2+3+1 = 2m^2-3m+1$.

Looking at equation (15), one can see that $\alpha_{\mathscr{L}}\mathscr{L}$ takes values in the $\gamma$-symmetric component of $\mathscr{D}\otimes\mathscr{D}^*$, while $\beta_{\mathscr{L}}\mathscr{L}$ takes values in the $\eta$ symmetric component of $(TX/\mathscr{D})\otimes(TX/\mathscr{D})^*$. These values are not completely independent, however: the $\gamma$ trace of $\alpha_{\mathscr{L}}\mathscr{L}$ is the complete trace of $\mathscr{L}$ with itself, as is the $\eta$ trace of $-2\beta_{\mathscr{L}}\mathscr{L}$. Hence there is one extra relation, giving a total of $(4m-4)(4m-3)/2+6-1 = 8m^2-14m+11$ independent equations – just the right amount to reduce the structure group from $L$ to $G'$.

Now let us consider a slight deformation of a quaternionic-contact structure, $\mathscr{L} = \mathscr{K} + \epsilon\mathscr{M}$. Re-writing equation (15):

$$\begin{aligned} 0 &= \partial^*_{\mathscr{L}}\mathscr{L} - \partial^*_{\mathscr{K}}\mathscr{K} \\ &= \epsilon\,(\partial^*_{\mathscr{M}}\mathscr{K} + \partial^*_{\mathscr{K}}\mathscr{M}) + O(\epsilon^2). \end{aligned}$$

The $\epsilon$ term is the symmetric part of $\partial^*_{\mathscr{K}}\mathscr{M}$; so, to first order, the requirement is that $\partial^*_{\mathscr{K}}\mathscr{L}$ be completely anti-symmetric. A method for fixing $\mathscr{K}$ is suggested by the following lemma:

**Lemma 1.5.** The equation $\partial^*_{\mathscr{K}}\mathscr{M} = 0$ consists of $16m^2-32m+24$ independent equations, which is exactly enough to restrict the structure group from $L$ to $G_0$.

*Proof.* The operator $\partial^*$ takes values in $\mathscr{D}\otimes\mathscr{D}^*\oplus(TX/\mathscr{D})\otimes(TX/\mathscr{D})^*$. This bundle may be identified with $E(\mathfrak{l})$, where $\mathfrak{l}$ is the Lie algebra of $L$. The bracket $\mathscr{K}$ defines a reduction to the structure group $G_0$ and hence $E_0$, a $G_0$-principal bundle. This defines the vector bundle $E_0(\mathfrak{g}_0)$, with $\mathfrak{g}_0$ the Lie algebra of $G_0$. The inclusion $E_0 \subset E$ defines an inclusion of this bundle into $E(\mathfrak{l})$. Then the book [7] implies that the image of $\partial^*$ is transverse to $E_0(\mathfrak{g}_0)$, giving us our dimensionality result. □

So the natural candidate for fixing $\mathscr{K}$ would be one whose derivative close to a quaternionic-contact structure is one where the anti-symmetric part of $\partial^*_{\mathscr{K}}\mathscr{M}$ vanishes.

The simplest such condition is to simply require that the anti-symmetric part of $\partial^*_{\mathscr{K}}\mathscr{L}$ vanishes. Thus:

**Definition 1.6** (Compatibility)**.** The algebraic bracket $\mathscr{K}$, a section of $\wedge^2\mathscr{D}^*\otimes(TX/\mathscr{D})$, is compatible with the Levi bracket $\mathscr{L}$ if:

(1) $\mathscr{K}$ is of quaternionic-contact or octonionic-contact type – hence the dimension and co-dimension of $\mathscr{D}$ is correct, and $\mathscr{K}$ defines a pair of metric $(\eta, \gamma)$ up to conformal transformations,
(2) $\partial^*_{\mathscr{K}}\mathscr{K} = \partial^*_{\mathscr{L}}\mathscr{L}$,
(3) $\partial^*_{\mathscr{K}}\mathscr{L}$ is symmetric.

Now, this definition is similar, but not identical, with the condition for non-regular two-graded geometries laid out in [1]; indeed, the condition there (that $\partial^*_{\mathscr{K}}\mathscr{L} = 0$) is precisely the infinitesimal version of the above.

We now phrase our general result. The setting is the same as for proposition 1.1: we consider a manifold $X^n$ of dimension $n = 4m - 1$ in the quaternionic case, or $n = 15$ in the octonionic case, and a distribution $\mathscr{D} \subset TX$ of codimension 3 in the quaternionic case, and 7 in the octonionic case. Remind that the vector field bracket is an algebraic map, $\mathscr{L}_x \in \Lambda^2\mathscr{D}^*_x \otimes (T_xX/\mathscr{D}_x)$. The isomorphism type of $\mathscr{L}_x$ defines in $W$ a $L$ orbit, $\mathscr{O}(\mathscr{L}_x)$, which does not depend on any choice.

**Theorem 1.7.** There is an open set $U \subset W$, containing the standard bracket $\kappa$, such that for all points $x \in X$ where $\mathscr{O}(\mathscr{L}_x)$ intersects $U$, there exists a locally unique, continuously defined, choice of compatible algebraic bracket $\mathscr{K}_x$.

*Proof.* We need to prove the existence and uniqueness of compatible $\mathscr{K}$. This is a purely algebraic construction, so we may work at a point. If we choose the natural bracket $\kappa$ to be fixed in $\wedge^2V_1^* \otimes V_2$, define $\theta$ as the map $\wedge^2V_1^* \otimes V_2 \to \mathfrak{s}$,

$$\theta = \frac{1}{2}\left(\partial^*_\ell\ell + \partial^*_\kappa\ell - \partial^*_\ell\kappa\right) - \partial^*_\kappa\kappa. \tag{16}$$

Note that the first $\ell$ term must by symmetric, while the other two $\ell$ terms together are anti-symmetric, so there is no overlap between them. Another important fact is that the Lie algebra $\mathfrak{g}_0$ has one symmetric part (the grading element) and the rest is anti-symmetric. We already know that the image of $\partial^*_\ell\ell$ is of co-dimension one in the symmetric part of $\mathfrak{s}$; its image it precisely the part transverse to the grading element $(2Id, Id)$. Now $\partial^*_\kappa\ell - \partial^*_\ell\kappa$ is simply the anti-symmetric part of $2\partial^*_\kappa\ell$. We know that $2\partial^*_\kappa\ell$ must be transverse to $\mathfrak{g}_0$, and hence so is its anti-symmetric part. Consequently $\theta$ maps into $\mathfrak{l}/\mathfrak{g}_0$.

Now if $f_{\mathscr{L}}(E_x)$ intersects the zero set of $\theta$, then there is a point $p \in E_x$ such that $\theta(f_{\mathscr{L}}(p)) = 0$. Then if we define $\mathscr{K}_x$ by the property that $f_{\mathscr{K}}(p) = \kappa$, we will get the vanishing of the bundle version of equation (16). Hence this $\mathscr{K}$ will be compatible.

So what we need to show is that the $L$-orbit of the zero set of $\theta$ contains an open set $U$ around $\kappa$. Now consider the map $\Theta : L \times W \to \mathfrak{s}$,

$$\Theta(s, \ell) = \theta(s \cdot \ell),$$

where $s \cdot l$ denotes the action of $s \in L$ on $\ell$. We wish to calculate the derivative of this map in the $L$ directions around the point $(\kappa, Id)$. Let $s \in \mathfrak{l}$; then a little bit of calculations demonstrate that this derivative is

$$D_\Theta(s)(\kappa, Id) = \partial^*_\kappa(\partial_\kappa s)$$

where

$$(\partial_\kappa s)(x, y) = \kappa(s(x), y) + \kappa(x, s(y)) - s(\kappa(x, y))$$

(see [1] for more details of how this is derived). The book [7] then demonstrates that $\partial^*_\kappa \partial_\kappa$ is an invertible map from the image of $\partial^*$ to itself, with kernel equal to $\mathfrak{g}_0$. An extra subtlety is needed to demonstrate that result, namely the vanishing of the first cohomology groups $H^{(1)}(\mathfrak{g}^+, \mathfrak{g})$ in homogeneity zero, see [7] and [1]. But Kostant's proof of the Bott-Borel-Weil theorem ([9]) show that this is indeed the case in our situation.

Hence, under the action of $L$, $\theta(s \cdot \kappa)$ must trace out an open neighbourhood of zero in $\mathfrak{l}/\mathfrak{g}_0$. This property must extend to points $\ell$ close to $\kappa$ by the implicit function theorem, defining our set $U$.

Now let $\ell$ be in $U$ intersected with the zero set of $\theta$. If $\ell$ is close enough to $\kappa$ (possibly restricting $U$ to a smaller open subset), we know that if $B_\ell \subset S$ is defined such that $\theta(b \cdot \ell) = 0$ for all $b \in B_\ell$, then $B$ must be of same dimension as $\mathfrak{g}_0$ (at least around the identity in $L$). However, if $g \in G_0$, then

$$\begin{aligned}
\theta(g \cdot \ell) &= \frac{1}{2}\left(\partial^*_{g \cdot l} g \cdot \ell + \partial^*_\kappa (g \cdot \ell) - \partial^*_{(g \cdot \ell)} \kappa\right) - \partial^*_\kappa \kappa \\
&= \frac{1}{2}\left(\partial^*_{g \cdot \ell} g \cdot \ell + g \cdot (\partial^*_\kappa \ell - \partial^*_\ell \kappa)\right) - \partial^*_\kappa \kappa \\
&= \frac{1}{2}(\partial^*_\ell \ell) - \partial^*_\kappa \kappa = 0,
\end{aligned}$$

since $g$ is a conformal transformation, commutes with $\partial^*$, and $\partial^*_\kappa \ell - \partial^*_\ell \kappa = 0$ by the assumption $\theta(\ell) = 0$.

Hence around the identity, dimension count implies that $B_\ell$ is precisely the group $G_0$. Action by $G_0$ preserves $\kappa$, so does not affect the value of $\mathscr{K}_x$. Consequently, the choice of $\mathscr{K}_x$ is locally unique for $\ell \in U$. □

**Remark.** As noted before, the condition that $\partial^*_{\mathscr{K}}\mathscr{L}$ be symmetric can be replaced with any other condition that approximates the one above to first order. There are more natural candidates for that – involving, for instance, the decomposition of the partial trace $\mathscr{L}^k_{ij}\mathscr{L}^r_{lo}\gamma^{il}$ into irreducible $G_0$ components, and the vanishing of one of these components. But since we're been unable to find a direct use of such a result (it affect the curvature of the asymptotically Einstein metric, but it's not clear exactly how), we've stuck with the simpler condition in this paper.

## 2. Construction of the Einstein metrics

In this section, we prove theorem 0.1, along the lines of [2]. Because we restrict to the case of small deformations of the model hyperbolic metric, we are able to give a short direct proof, in which the main step is a uniform estimate for the norm of the inverse of the linearization.

We start with the quaternionic or octonionic hyperbolic space $M$, whose metric in polar coordinates is expressed in both cases by

$$g_0 = dr^2 + \sinh^2(\tfrac{r}{2})\gamma_0 + \sinh^2(r)\eta_0. \tag{17}$$

Here $\eta_0 = \alpha_0^2$, where $\alpha_0$ is a 1-form on $S^{4m-1}$ (resp. $S^{15}$) with values in $\mathbb{R}^3$ (resp. $\mathbb{R}^7$), and $\gamma_0$ is the induced metric on the $4(m-1)$-dimensional (resp. 8-dimensional) distribution $\mathscr{D}_0$ of $S^n$.

We will need the mean curvature $H_0(r) = \partial_r \log v$ of the spheres $r = \mathrm{cst}$, where $v$ is the volume element. It is given by $H_0(r) = 2(m-1)\coth(\frac{r}{2}) + 3\coth(r)$ in the quaternionic case, or $H_0(r) = 4\coth(\frac{r}{2}) + 7\coth(r)$ in the octonionic case. Also we note

$$\mathscr{H} = \lim_{r\to\infty} H_0(r)$$

the limit at infinity, so that $\mathscr{H} = 2m+1$ in the quaternionic case and $\mathscr{H} = 11$ in the octonionic case.

Suppose that we have now a small perturbation $\mathscr{D}$ of the distribution $\mathscr{D}_0$. From proposition 1.1 we have constructed $\gamma$ and $\eta$ on $\mathscr{D}$ and $TS/\mathscr{D}$ such that

$$g_{\mathscr{D}} = dr^2 + \sinh^2(\tfrac{r}{2})\gamma + \sinh^2(r)\eta \tag{18}$$

is asymptotically Einstein :

$$\mathrm{Ric}(g_{\mathscr{D}}) - \lambda g_{\mathscr{D}} = O(e^{-\frac{r}{2}}), \tag{19}$$

with $\lambda = -m-2$ (resp. $\lambda = -9$). Here the norms are with respect to $g_{\mathscr{D}}$. Actually, in the proof of proposition 1.1, we proved more, that is there is a development for the curvature,

$$R = R_0 + e^{-\frac{r}{2}}R_1 + e^{-r}R_2 + \cdots , \tag{20}$$

where the terms $R_i$ do not depend on $r$, the term $R_1$ depends on one derivative of the bracket $\mathscr{L}$ on the boundary, and the other terms depend on two derivatives of $\mathscr{L}$. This immediately implies

$$|\nabla^k(\operatorname{Ric}(g_{\mathscr{D}}) - \lambda g_{\mathscr{D}})| \leq c_k e^{-\frac{r}{2}} \text{ for all } k, \tag{21}$$

where $c_k$ can be made small if $\mathscr{L}$ is $C^{k+2}$ close to the standard bracket.

Of course, the formula (18) does not give a smooth metric at the origin. To remedy this, we choose a cutoff function $\chi(r)$, such that $\chi(r) = 1$ for $r \geq R+1$ and $\chi(r) = 0$ or $r \leq R-1$. Then we define

$$g = \chi g_{\mathscr{D}} + (1-\chi) g_0. \tag{22}$$

The metric $g$ is a global filling of $(\eta, \gamma)$ in the ball.

The first observation is that the metrics $g_{\mathscr{D}}$ have uniform geometry:

**Lemma 2.1.** Suppose $k \geq 2$. For $\mathscr{D}$ varying in a fixed $C^{k+1}$ neighbourhood of $\mathscr{D}_0$, the sectional curvature of $g$ is negative, the curvatures of $g$ and their $(k-2)$ covariant derivatives are uniformly bounded.

*Proof.* A $C^{k+1}$ control of $\mathscr{D}$ gives a $C^k$ control of the conformal metric $(\eta, \gamma)$, since one derivative is needed to calculate the Levi bracket and $(\eta, \gamma)$ is then obtained as the solution of algebraic equations. Therefore we have a $C^k$ control on the coefficients of $g$. The lemma then follows from the form (20) of the curvature. $\square$

This implies that balls for the metrics $g_{\mathscr{D}}$ are uniformly comparable with Euclidean balls. Then the Hölder norm of a function $f$ is defined as the supremum of the Hölder norms of $f$ on each ball of radius 1.

The analysis of the Einstein equation requires the use of weighted Hölder spaces. Our weight function will be

$$w(r) = \cosh(r)^{\delta} \tag{23}$$

and we then define the weighted Hölder space $C^{k,\alpha}_{\delta} = w^{-\delta} C^{k,\alpha}$. Of course, from the initial estimate (19), the weight we are interested in is $\delta = \frac{1}{2}$.

As in [2, chapter I], the Einstein metric will be constructed as a solution $h$ of the equation

$$\Phi^g(h) := \operatorname{Ric}(h) - \lambda h + \delta^*_h(\delta_g h + \tfrac{1}{2} d \operatorname{Tr}_g h) = 0, \tag{24}$$

and we require that $h$ is asymptotic to $g$ in the sense that

$$h - g \in C^{2,\alpha}_{1/2}. \tag{25}$$

Indeed, by [2, lemma I.1.4], a solution $h$ of $\Phi^g(h) = 0$ then satisfies $\delta_g h + \frac{1}{2} d \operatorname{Tr}_g h = 0$ and $\operatorname{Ric}(h) = \lambda h$. Given lemma 2.1 (in particular, the negative curvature of $g$ implies that the linearization of $\Phi^g$ has no

$L^2$ kernel), the proof in [2] applies and proves that if the data $(\eta, \gamma)$ is sufficiently close to $(\eta_0, \gamma_0)$ in $C^{2,\alpha}$ norm, that is if $\mathscr{D}$ is sufficiently close to $\mathscr{D}_0$ in $C^{3,\alpha}$ norm, then one can find a solution $h$ of (24), if one has a uniform bound on the inverse of the linearization of $\Phi^g$. This is provided by:

**Lemma 2.2.** Suppose that $\frac{1}{2}(\mathscr{H} - \sqrt{\mathscr{H}^2 - 8}) < \delta < \frac{1}{2}(\mathscr{H} + \sqrt{\mathscr{H}^2 - 8})$. For $\mathscr{D}$ sufficiently close to $\mathscr{D}_0$ in $C^{3,\alpha}$ norm, the linearization $P_g = d_g \Phi^g : C^{2,\alpha}_\delta(\operatorname{Sym}^2 T^*M) \to C^\alpha_\delta(\operatorname{Sym}^2 T^*M)$ is invertible and the norm of the inverse is uniformly bounded.

From the value of $\mathscr{H}$, we check that the weight $\delta = \frac{1}{2}$ indeed satisfies the hypothesis, so theorem 0.1 follows from the lemma.

So we now concentrate on the proof of the lemma. One has

$$P_g = \tfrac{1}{2}\nabla^*\nabla - \mathring{R}_g.$$

The property of the curvature term $\mathring{R}_g$ we need is the following [2, lemmas I.4.1 and I.4.2]: for the hyperbolic metric $g_0$, the largest eigenvalue of $\mathring{R}_{g_0}$ is equal to 1 (instead of 4 in [2], because here we normalize here the sectional curvature of $g_0$ in $[-1, -\frac{1}{4}]$ instead of $[-4, -1]$). This immediately implies that, for $\mathscr{D}$ close enough to $\mathscr{D}_0$ in $C^3$ norm, one has

$$\mathring{R}_g \leq 1 + \epsilon. \tag{26}$$

For the function $w$ depending on $r$ only, one has

$$\Delta w = -\partial_r^2 w - H(r)\partial_r w, \tag{27}$$

where $H(r) = \partial_r \log v$ is the mean curvature. For the metric $g$ given by (22), the mean curvature $H(r)$ coincides with $H_0(r)$ for $r \geq R+1$ or $r \leq R-1$, and for $R-1 \leq r \leq R+1$ we get $|H(r) - H_0(r)| \leq \epsilon$ if we suppose $(\eta, \gamma)$ close enough to $(\eta_0, \gamma_0)$.

An easy calculation gives, for the hyperbolic metric,

$$-\frac{\Delta w}{w} - 2\frac{|dw|^2}{w^2} = \delta\left(\mathscr{H} - \delta + \frac{\dim \mathscr{D}}{2\cosh r} + \frac{\delta + 1}{\cosh^2 r}\right). \tag{28}$$

It follows that, for the metric $g$, if $\mathscr{D}$ is sufficiently close to $\mathscr{D}_0$,

$$-\frac{\Delta w}{w} - 2\frac{|dw|^2}{w^2} \geq \delta(\mathscr{H} - \delta - \epsilon). \tag{29}$$

Using this property of the weight function $w$, we can now establish lemma 2.2 using the maximum principle. From Kato's inequality,

$$\langle u, \nabla^*\nabla u\rangle = |u|\Delta|u| + |\nabla u|^2 - |d|u||^2 \geq |u|\Delta|u|.$$

Using the formula

$$\begin{aligned} w\Delta|u| &= \Delta(w|u|) - w|u|\Big(\frac{\Delta w}{w} + 2\frac{|dw|^2}{|w|^2}\Big) + 2\Big\langle \frac{dw}{w}, d(w|u|)\Big\rangle \\ &\geq \Delta(w|u|) + \delta(\mathscr{H} - \delta - \epsilon)w|u| + 2\Big\langle \frac{dw}{w}, d(w|u|)\Big\rangle \end{aligned}$$

it follows from (26) that

$$w|Pu| \geq \tfrac{1}{2}\Delta(w|u|) + \Big(\tfrac{1}{2}\delta(\mathscr{H}-\delta-\epsilon)-1-\epsilon\Big)w|u| + \Big\langle \frac{dw}{w}, d(w|u|)\Big\rangle. \tag{30}$$

Let $A = \frac{1}{2}\delta(\mathscr{H} - \delta - \epsilon) - 1 - \epsilon$. If $\delta$ satisfies the hypothesis of lemma 2.2, then one can choose $\epsilon$ sufficiently small so that $A > 0$. Then by the maximum principle applied to $w|u|$, it follows that

$$\sup(w|u|) \leq A^{-1} \sup(w|Pu|). \tag{31}$$

(A priori we cannot apply the maximum principle to $w|u|$ since it has not to go to zero at infinity, but we can apply it for $w = (\cosh r)^{\delta'}$ for any $\delta' < \delta$; then taking $\delta' \to \delta$ gives the estimate).

From this estimate, it is immediate that if $v \in C^\alpha_\delta$, then one can solve $Pu = v$ with $u \in C^0_\delta$ and $\|u\|_{C^0_\delta} \leq A^{-1}\|v\|_{C^0_\delta}$. It remains to obtain a bound on higher derivatives, but from the uniform geometry lemma 2.1, applying the usual elliptic estimate in each ball, one obtains a constant $C$ such that

$$\|u\|_{C^{2,\alpha}_\delta} \leq C\Big(\|Pu\|_{C^\alpha_\delta} + \|u\|_{C^0_\delta}\Big) \leq C(1 + A^{-1})\|Pu\|_{C^\alpha_\delta}$$

which is the required estimate.

**Remark.** The previous lemma does not give an optimal interval of weights for the isomorphism. In [2] the optimal interval for $g_0$ is calculated; using microlocal analysis, it is proved in [4] that the same interval holds if the distribution $\mathscr{D}$ is quaternionic-contact (the regular case). In general, the optimal interval may depend on the supremum of the eigenvalues of the curvatures $\mathring{R}_x$, where $\mathring{R}_x$ is the curvature of the homogeneous Einstein model attached to the point $x$ of the boundary.

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