A small hole close to the boundary for the two-dimensional Laplace Dirichlet problem

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Abstract

We study a Dirichlet problem in a planar domain with a small hole close to the boundary. For each pair $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ of positive parameters, we define a perforated domain $\Omega_{\boldsymbol{\varepsilon}}$ obtained by making a small perforation of size $\varepsilon_1\varepsilon_2$ in an open set. The distance of the cavity from the boundary is instead controlled by ε_1. As $\varepsilon_1 \to 0$, the perforation shrinks to a point and at the same time approaches the boundary. We consider separately two cases: the case when $\boldsymbol{\varepsilon}$ tends to $(0,0)$ and the case when ε_1 tends to 0 and ε_2 is fixed. In the first case we show that the solution of a Dirichlet problem defined in $\Omega_{\boldsymbol{\varepsilon}}$ displays a logarithmic behavior when $\boldsymbol{\varepsilon} \to 0$. In the second case instead, the asymptotic behavior of the solution can be described in terms of real analytic functions of ε_1. We will also show that the energy integral and the total flux on the exterior boundary have a different limiting behavior in the two cases.

Keywords: Dirichlet problem; singularly perturbed perforated domain; planar domain; Laplace operator; real analytic continuation in Banach space; asymptotic expansion

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1 Introduction

This paper deals with the asymptotic behavior of the solution of a Dirichlet problem for the Laplace equation in a domain with a small hole which is approaching to the outer boundary. In [2] we study the case when the dimension of the ambient space is greater than or equal to three, while here we focus on the two-dimensional case, which presents some peculiarities and needs a separate treatment. We will consider two cases: the case when the hole is 'moderately close' to the boundary, *i.e.*, its size tends to zero 'faster' than the distance from the outer boundary, and the case when the size of the hole and the distance from the boundary

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are comparable. We will see that the asymptotic behavior in the two cases is different: a logarithmic behavior arises in the first case while we have a real analytic continuation result in the second case. In addition, the energy integral and the total flux of the solution on the outer boundary may have a different limit value.

Let us present the geometry of the problem. We introduce a notation for the upper half plane by setting

$$\mathbb{R}^2_+ \equiv \{\mathsf{x} = (x_1, x_2) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \ : \ x_2 > 0\}\,.$$

Then we consider a domain Ω which represents the 'unperturbed' domain and which is such that

$$\Omega \text{ is an open bounded connected subset of } \mathbb{R}^2_+ \text{ of class } \mathscr{C}^{1,\alpha}, \tag{H_1}$$

where $\alpha \in]0,1[$ is a regularity parameter. For the definition of functions and sets of the usual Schauder classes $\mathscr{C}^{k,\alpha}$ $(k = 0,1)$, we refer to Gilbarg and Trudinger [15, §6.2]. We denote by $\partial\Omega$ the boundary of Ω and we set

$$\partial_0\Omega \equiv \partial\Omega \cap \partial\mathbb{R}^2_+, \qquad \partial_+\Omega \equiv \partial\Omega \cap \mathbb{R}^2_+.$$

On the flat boundary, we consider the following assumption

$$\partial_0\Omega \text{ is an open neighborhood of } 0 \text{ in } \partial\mathbb{R}^2_+. \tag{H_2}$$

Since we want to make a perforation in Ω of a given shape, we fix another set ω satisfying the following assumption:

$$\omega \text{ is a bounded open connected subset of } \mathbb{R}^2 \text{ of class } \mathscr{C}^{1,\alpha} \text{ such that } 0 \in \omega.$$

Then we fix a point

$$\mathsf{p} = (p_1, p_2) \in \mathbb{R}^2_+,$$

and we define the inclusion $\omega_{\boldsymbol{\varepsilon}}$ by

$$\omega_{\boldsymbol{\varepsilon}} \equiv \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\omega\,, \qquad \forall\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2\,.$$

Let $\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}'' \in \mathbb{R}^2$. By convention, we write $\boldsymbol{\varepsilon}' \le \boldsymbol{\varepsilon}''$ (resp. $\boldsymbol{\varepsilon}' < \boldsymbol{\varepsilon}''$) if and only if $\varepsilon'_j \le \varepsilon''_j$ (resp. $\varepsilon'_j < \varepsilon''_j$), for $j = 1,2$. Then we denote $]\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}''[$ the open rectangular domain of the $\boldsymbol{\varepsilon} \in \mathbb{R}^2$ such that $\boldsymbol{\varepsilon}' < \boldsymbol{\varepsilon} < \boldsymbol{\varepsilon}''$. With this notation, there is $\boldsymbol{\varepsilon}^{\mathrm{ad}} \equiv (\varepsilon_1^{\mathrm{ad}}, \varepsilon_2^{\mathrm{ad}}) > \mathbf{0}$ such that

$$\overline{\omega_{\boldsymbol{\varepsilon}}} \subseteq \Omega, \qquad \forall\boldsymbol{\varepsilon} \in]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[.$$

In addition, since we are interested in the case when $\boldsymbol{\varepsilon}$ is close to $\mathbf{0}$, we can assume without loss of generality that

$$\varepsilon_1^{\mathrm{ad}} < 1 \text{ and } 1 < \varepsilon_2^{\mathrm{ad}} < 1/\varepsilon_1^{\mathrm{ad}}.$$

So that, $\varepsilon_1\varepsilon_2 < 1$ for all $\boldsymbol{\varepsilon} \in]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$. This technical condition will allow us to deal with the function $1/\log(\varepsilon_1\varepsilon_2)$, as we do in Section 3, and also to consider the case when $\varepsilon_2 = 1$ in Section 4.

The rectangular domain $]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$ plays the role of the set of 'admissible' coefficients $\boldsymbol{\varepsilon}$ for which we define the perforated domain

$$\Omega_{\boldsymbol{\varepsilon}} \equiv \Omega \setminus \overline{\omega_{\boldsymbol{\varepsilon}}}\,.$$

One verifies that $\Omega_{\boldsymbol{\varepsilon}}$ is a bounded connected open subset of class $\mathscr{C}^{1,\alpha}$ with boundary $\partial\Omega_{\boldsymbol{\varepsilon}}$ consisting of two connected components: $\partial\Omega$ and $\partial\omega_{\boldsymbol{\varepsilon}} = \varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2\partial\omega$. The size of the hole $\omega_{\boldsymbol{\varepsilon}}$ is controlled by the product $\varepsilon_1\varepsilon_2$, whereas the distance of the hole $\omega_{\boldsymbol{\varepsilon}}$ from the boundary of Ω is controlled by the parameter ε_1. Clearly, as the pair $\boldsymbol{\varepsilon} \in]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$ approaches a degenerate value $(0, \varepsilon_2^\star)$ both the size of the cavity and its distance from the boundary $\partial\Omega$ tend to 0. If $\varepsilon_2^\star = 0$, then the ratio of the size of the hole and the distance from the boundary tends to 0 and we can say that the size tends to zero 'faster' than the distance. If instead $\varepsilon_2^\star > 0$, then the size and the distance from the boundary tend to zero at the same rate. Figure 1 illustrates our geometrical setting.

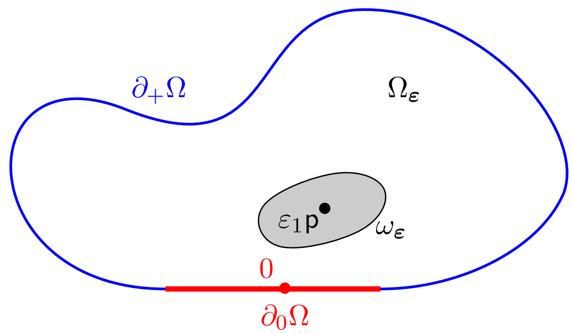


Figure 1: Geometrical settings.

We now take a function $g^{\mathrm{o}} \in \mathscr{C}^{1,\alpha}(\partial\Omega)$ and a function $g^{\mathrm{i}} \in \mathscr{C}^{1,\alpha}(\partial\omega)$ and for $\boldsymbol{\varepsilon} \in]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$ we consider the following Dirichlet problem in $\Omega_{\boldsymbol{\varepsilon}}$:

$$\begin{cases} \Delta u(\mathsf{x}) = 0, & \forall x \in \Omega_{\boldsymbol{\varepsilon}}\,, \\ u(\mathsf{x}) = g^{\mathrm{o}}(\mathsf{x}), & \forall \mathsf{x} \in \partial\Omega\,, \\ u(\mathsf{x}) = g^{\mathrm{i}}\left(\frac{\mathsf{x}-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right), & \forall \mathsf{x} \in \partial\omega_{\boldsymbol{\varepsilon}}\,. \end{cases} \tag{1.1}$$

As is well known problem (1.1) has a unique solution in $\mathscr{C}^{1,\alpha}(\overline{\Omega_{\boldsymbol{\varepsilon}}})$. We will denote such a solution by $u_{\boldsymbol{\varepsilon}}$. We wish to investigate the asymptotic behavior of the solution $u_{\boldsymbol{\varepsilon}}$ when $\boldsymbol{\varepsilon} \to (0,0)$ and when $\varepsilon_1 \to 0$ and $\varepsilon_2 > 0$ is fixed. In other words, we want to answer to the following questions.

$$\begin{array}{c}\text{Let } \mathsf{x} \in \Omega \text{ be fixed. What can be said on the map } \boldsymbol{\varepsilon} \mapsto u_{\boldsymbol{\varepsilon}}(\mathsf{x}) \\ \text{when } \varepsilon_1 \to 0 \text{ and either } \varepsilon_2 \to 0 \text{ or } \varepsilon_2 \text{ is fixed?}\end{array} \tag{1.2}$$

We observe that we don't consider in this paper the case when ε_2 is close to 0 and ε_1 remains positive. Such a case corresponds to a boundary value problem in a domain with a hole which collapses to a point in its interior and it has received the attention of many authors.

1.1 Explicit computations on a toy problem

To explain the results that we are going to prove in this work, we first consider a toy problem where we can make the solution explicit. We denote by $\mathcal{B}(\mathsf{x}, \rho)$ the ball of $\mathbb{R}^2$ centered at x

and of radius ρ, we take a function $g^{\mathrm{i}} \in \mathscr{C}^{1,\alpha}(\partial\mathcal{B}(0,1))$, and for $\boldsymbol{\varepsilon} \in]\mathbf{0},(1,1)[$ we introduce the following Dirichlet problem in the perforated half space $\mathbb{R}^2_+ \setminus \mathcal{B}((0,\varepsilon_1),\varepsilon_1\varepsilon_2)$:

$$\begin{cases} \Delta u_{\boldsymbol{\varepsilon}}(\mathsf{x}) = 0, & \forall \mathsf{x} \in \mathbb{R}^2_+ \setminus \mathcal{B}((0,\varepsilon_1),\varepsilon_1\varepsilon_2) \\ u_{\boldsymbol{\varepsilon}}(\mathsf{x}) = 0, & \forall \mathsf{x} \in \partial\mathbb{R}^2_+, \\ u_{\boldsymbol{\varepsilon}}(\mathsf{x}) = g^{\mathrm{i}}\left(\frac{\mathsf{x}-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right), & \forall \mathsf{x} \in \partial\mathcal{B}((0,\varepsilon_1),\varepsilon_1\varepsilon_2), \\ \lim_{\mathsf{x}\to\infty} u_{\boldsymbol{\varepsilon}}(\mathsf{x}) = 0\,. \end{cases} \tag{1.3}$$

where $\mathsf{p} = (0,1)$. Then we consider the conformal map

$$\varphi_a :\ z \mapsto \frac{z - ia}{z + ia}$$

with inverse

$$\varphi_a^{-1} :\ z \mapsto -ia\ \frac{z+1}{z-1}.$$

When a is real, φ_a maps the imaginary axis onto the unit circle. Moreover, if

$$a(\boldsymbol{\varepsilon}) = a(\varepsilon_1,\varepsilon_2) = \varepsilon_1\sqrt{1-\varepsilon_2^2},$$

then $\varphi_{a(\boldsymbol{\varepsilon})}$ maps the circle of center $(0,\varepsilon_1)$ and radius $\varepsilon_1\varepsilon_2$ to the circle of center the origin and radius

$$\rho(\varepsilon_2) = \sqrt{\frac{1-\sqrt{1-\varepsilon_2^2}}{1+\sqrt{1+\varepsilon_2^2}}}.$$

We observe that the maps $a :\ (\varepsilon_1,\varepsilon_2) \mapsto a(\boldsymbol{\varepsilon})$ and $\rho :\ \varepsilon_2 \mapsto \rho(\varepsilon_2)$ are analytic.

Since harmonic functions are transformed into harmonic functions by a conformal map, we can now transport problem (1.3) on the annular domain $\mathcal{B}(0,1) \setminus \mathcal{B}(0,\rho(\varepsilon_2))$ by means of the map $\varphi_{a(\boldsymbol{\varepsilon})}$. We obtain new problem for the unknown function $\underline{u}_{\boldsymbol{\varepsilon}} = u_{\boldsymbol{\varepsilon}} \circ \varphi^{-1}_{a(\boldsymbol{\varepsilon})}$:

$$\begin{cases} \Delta \underline{u}_{\boldsymbol{\varepsilon}} = 0, & \text{in } \mathcal{B}(0,1) \setminus \mathcal{B}(0,\rho(\varepsilon_2)) \\ \underline{u}_{\boldsymbol{\varepsilon}} = 0, & \text{on } \partial\mathcal{B}(0,1), \\ \underline{u}_{\boldsymbol{\varepsilon}}(z) = \underline{g}^{\mathrm{i}}_{\boldsymbol{\varepsilon}}(\arg z), & \text{for all } z \in \partial\mathcal{B}(0,\rho(\varepsilon_2)), \end{cases} \tag{1.4}$$

where the new boundary condition is given by

$$\underline{g}^{\mathrm{i}}_{\boldsymbol{\varepsilon}}(\theta) = g^{\mathrm{i}}\left(-\frac{i}{\varepsilon_2}\left(\sqrt{1-\varepsilon_2^2}\ \frac{\rho(\varepsilon_2)e^{i\theta}+1}{\rho(\varepsilon_2)e^{i\theta}-1}\ +1\right)\right), \qquad \forall\theta \in [0,2\pi[.$$

To obtain an analytic expression of its solution, we decompose $\underline{g}^{\mathrm{i}}_{\boldsymbol{\varepsilon}}$ in Fourier series

$$\underline{g}^{\mathrm{i}}_{\boldsymbol{\varepsilon}}(\theta) = a_0(\underline{g}^{\mathrm{i}}_{\boldsymbol{\varepsilon}}) + \sum_{k\geq 1} a_k(\underline{g}^{\mathrm{i}}_{\boldsymbol{\varepsilon}}) \cos k\theta + b_k(\underline{g}^{\mathrm{i}}_{\boldsymbol{\varepsilon}}) \sin k\theta,$$

so that, in polar coordinates,

$$\underline{u}_{\boldsymbol{\varepsilon}}(r,\theta) = a_0(\underline{g}^{\mathrm{i}}_{\boldsymbol{\varepsilon}})\frac{\log r}{\log\rho(\varepsilon_2)} + \sum_{k\geq 1}\left(a_k(\underline{g}^{\mathrm{i}}_{\boldsymbol{\varepsilon}})\cos k\theta + b_k(\underline{g}^{\mathrm{i}}_{\boldsymbol{\varepsilon}})\sin k\theta\right)\frac{r^k - r^{-k}}{\rho(\varepsilon_2)^k - \rho(\varepsilon_2)^{-k}}.$$

Then, we can recover $u_{\boldsymbol{\varepsilon}}$ by the pull back $u_{\boldsymbol{\varepsilon}} = \underline{u}_{\boldsymbol{\varepsilon}} \circ \varphi_{a(\boldsymbol{\varepsilon})}$. To that end, we observe that in polar coordinates we have $\varphi_{a(\boldsymbol{\varepsilon})}(\mathsf{x}) = r_{\boldsymbol{\varepsilon}}(\mathsf{x})e^{i\theta_{\boldsymbol{\varepsilon}}(\mathsf{x})}$ with

$$r_{\boldsymbol{\varepsilon}}(\mathsf{x}) = \left|\frac{x_1 + ix_2 - ia(\boldsymbol{\varepsilon})}{x_1 + ix_2 + ia(\boldsymbol{\varepsilon})}\right|, \qquad \theta_{\boldsymbol{\varepsilon}}(\mathsf{x}) = \arg\left(\frac{x_1 + ix_2 - ia(\boldsymbol{\varepsilon})}{x_1 + ix_2 + ia(\boldsymbol{\varepsilon})}\right).$$

As an example, if we assume that $g^{\mathrm{i}} = 1$, then the solution of (1.3) is

$$u_{\boldsymbol{\varepsilon}}(\mathsf{x}) = \frac{\log r_{\boldsymbol{\varepsilon}}(\mathsf{x})}{\log \rho(\varepsilon_2)} = \frac{\log\left(x_1^2 + \left(x_2 - \varepsilon_1\sqrt{1-\varepsilon_2^2}\right)^2\right) - \log\left(x_1^2 + \left(x_2 + \varepsilon_1\sqrt{1-\varepsilon_2^2}\right)^2\right)}{\log\left(1 - \sqrt{1-\varepsilon_2^2}\right) - \log\left(1 + \sqrt{1+\varepsilon_2^2}\right)}.$$

We observe that, *for any fixed* $\mathsf{x} \in \mathbb{R}^2_+$ *and for* ε_1, ε_2 *positive and sufficiently small, the map* $\boldsymbol{\varepsilon} \mapsto u_{\boldsymbol{\varepsilon}}(\mathsf{x})$ *is analytic. When* $\boldsymbol{\varepsilon} \to \mathbf{0}$, *the function* $u_{\boldsymbol{\varepsilon}}$ *tends to* 0 *with a main term of order* $\varepsilon_1(\log\varepsilon_2)^{-1}$. *In addition, for* $\varepsilon_2 > 0$ *fixed the map* $\varepsilon_1 \mapsto u_{\boldsymbol{\varepsilon}}(\mathsf{x})$ *has an analytic continuation around* $\varepsilon_1 = 0$. We aim, in this work, to prove similar results also for problem (1.1), and thus to answer to question (1.2) by investigating the analyticity properties of the function $\boldsymbol{\varepsilon} \mapsto u_{\boldsymbol{\varepsilon}}(\mathsf{x})$. In addition, instead of the evaluation of $u_{\boldsymbol{\varepsilon}}$ to a point x, we will consider its restriction to suitable subsets of Ω and the restriction of the rescaled function $u_{\boldsymbol{\varepsilon}}(\varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\cdot)$ to suitable open subsets of $\mathbb{R}^2 \setminus \omega$, then we will study functionals related to $u_{\boldsymbol{\varepsilon}}$, such as the energy integral and the total flux on $\partial\Omega$. In the next subsection we describe our main results.

1.2 Main results

There are many ways to answer to question (1.2) depending on the approach that one applies to study (1.1). In literature, most of the papers dedicated to the analysis of problems with small holes employ expansion methods to provide asymptotic approximations of the solution. As an example, we mention the method of matching outer and inner asymptotic expansions proposed by Il'in (see, *e.g.*, [16]), the compound asymptotic expansion method of Maz'ya, Nazarov, and Plamenevskij [24] and of Kozlov, Maz'ya, and Movchan [17] and the mesoscale asymptotic approximations presented by Maz'ya, Movchan, and Nieves [23] to study Green's kernels in domains with small cavities. We also mention the works of Bonnaillie-Noël, Lacave, and Masmoudi [7], Chesnel and Claeys [8], and Dauge, Tordeux, and Vial [12]. Boundary value problems in domains with moderately close holes have been analyzed with the method of multiscale asymptotic expansions by Bonnaillie-Noël, Dambrine, Tordeux, and Vial [5, 6], Bonnaillie-Noël and Dambrine [3], and Bonnaillie-Noël, Dambrine, and Lacave [4]. We also observe that problems in domains with small holes and inclusions have a large number of applications. They appear, for example, in the study of inverse problems (cf.,*e.g.*, the monograph of Ammari and Kang [1]) and of shape and topological optimization (as it is described in the monograph by Novotny and Sokołowsky [26]).

In this paper, we answer to question (1.2) by applying an approach which is in a sense alternative to the expansion methods. We will use the 'functional analytic approach' which was proposed by Lanza de Cristoforis [18, 19] and then developed together with Dalla Riva and Musolino for the analysis of problems in domains with small holes (cf., *e.g.*, [9, 10, 11]). Our aim is to represent the map which takes $\boldsymbol{\varepsilon}$ to (suitable restrictions of) the solution $u_{\boldsymbol{\varepsilon}}$ in terms of real analytic maps with values in convenient Banach spaces of functions and of known elementary functions of ε_1 and ε_2 (for the definition of real analytic maps in Banach spaces, see Deimling [13, p. 150]). We observe that then we can recover asymptotic approximations like the one that one can get from the expansion methods. For example, if we know that the function in (1.2) equals for ε_1 and ε_2 small and positive a real analytic function defined in a whole neighborhood of $(0,0)$, then we know that such a map can be expanded in power series for ε_1 and ε_2 small and the truncated series provides an approximation of the solution.

To perform our analysis, in addition to assumptions (H_1)–(H_2), we assume that Ω satisfies the following condition:

$$\overline{\partial_0\Omega} \text{ is a finite union of closed disjoint intervals in } \partial\mathbb{R}^2_+. \tag{H_3}$$

In particular we note that assumption (H_3) implies the existence of linear and continuous extension operators $E^{k,\alpha}$ from $\mathscr{C}^{k,\alpha}(\overline{\partial_+\Omega})$ to $\mathscr{C}^{k,\alpha}(\partial\Omega)$, for $k=0,1$. This will allow us to pass from functions defined on $\partial_+\Omega$ to $\partial\Omega$ (and viceversa), preserving their regularity.

For proving our analyticity result, we shall consider a regularity condition on the Dirichlet datum around the origin, namely:

$$\text{there exists } r_0>0 \text{ such that the restriction } g^{\mathrm{o}}_{|\mathcal{B}(0,r_0)\cap\partial_0\Omega} \text{ is real analytic.} \tag{H_4}$$

As happens for the solution of a Dirichlet problem in a domain with a small hole 'far' from the boundary, we show that $u_{\boldsymbol{\varepsilon}}$ converges as $\varepsilon_1 \to 0$ to the function u_0 which is the unique solution in $\mathscr{C}^{1,\alpha}(\overline{\Omega})$ of the following Dirichlet problem in the unperturbed domain Ω:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega\,, \\ u = g^{\mathrm{o}} & \text{on } \partial\Omega\,. \end{cases} \tag{1.5}$$

We observe that u_0 is harmonic, and therefore analytic, in the interior of Ω. This fact can be useful to study the Dirichlet problem in a domain with a hole which shrinks to an interior point of Ω. If instead the hole shrinks to a point on the boundary, as it does in this paper, then we have to introduce condition (H_4) in oder to ensure that u_0 has an analytic (actually harmonic) extension around the limit point. Indeed, by (H_4) and by a classical argument based on the Cauchy-Kovalevskaya Theorem (cf. [2]), we can prove the following result.

Proposition 1.1. *There exists $r_1 \in]0,r_0]$ and a function U_0 from $\overline{\mathcal{B}(0,r_1)}$ to $\mathbb{R}$ such that $\mathcal{B}^+(0,r_1)\subseteq\Omega$ and*

$$\begin{cases} \Delta U_0 = 0 & \text{in } \mathcal{B}(0,r_1), \\ U_0 = u_0 & \text{in } \overline{\mathcal{B}^+(0,r_1)}. \end{cases}$$

Here, we denote $\mathcal{B}^+(0,r) = \mathcal{B}(0,r)\cap\mathbb{R}^2_+$.

Then, possibly shrinking $\varepsilon_1^{\mathrm{ad}}$ we can assume that

$$\varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\overline{\omega} \subseteq \mathcal{B}(0,r_1), \qquad \forall\boldsymbol{\varepsilon}\in\,]-\boldsymbol{\varepsilon}^{\mathrm{ad}},\boldsymbol{\varepsilon}^{\mathrm{ad}}[\,. \tag{1.6}$$

We now show our answers to question (1.2) in the cases when $\boldsymbol{\varepsilon}\to\mathbf{0}$ and when $\varepsilon_1\to 0$ and $\varepsilon_2>0$ is fixed.

1.2.1 What happens when $\varepsilon \to 0$?

To answer to question (1.2) in the case when $\boldsymbol{\varepsilon} \to \mathbf{0}$ we need to introduce a curve $\eta \mapsto \boldsymbol{\varepsilon}(\eta) \equiv (\varepsilon_1(\eta), \varepsilon_2(\eta))$ which describes the values attained by the parameter $\boldsymbol{\varepsilon}$ in a certain specific way. The reason is that the quotient

$$\frac{\log \varepsilon_1}{\log(\varepsilon_1 \varepsilon_2)} \tag{1.7}$$

plays an important role in the description of $u_{\boldsymbol{\varepsilon}}$ for $\boldsymbol{\varepsilon}$ small. However the limit of (1.7) as $\boldsymbol{\varepsilon} \to \mathbf{0}$ is not defined. Then, we take a function $\boldsymbol{\varepsilon}(\cdot)$ from $]0,1[$ to $]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$ such that

$$\lim_{\eta \to 0^+} \boldsymbol{\varepsilon}(\eta) = \mathbf{0}, \tag{1.8}$$

and such that the limit

$$\lim_{\eta \to 0^+} \frac{\log \varepsilon_1(\eta)}{\log(\varepsilon_1(\eta)\varepsilon_2(\eta))} \quad \text{exists and equals} \quad \lambda \in [0,1[. \tag{1.9}$$

It will also be convenient to denote by $\boldsymbol{\delta}$ the function

$$\begin{aligned} \boldsymbol{\delta} : \quad]0,1[\quad &\to \quad \mathbb{R}^2, \\ \eta \quad &\mapsto \quad \boldsymbol{\delta}(\eta) \equiv \big(\delta_1(\eta), \delta_2(\eta)\big) \equiv \left(\frac{1}{\log\big(\varepsilon_1(\eta)\varepsilon_2(\eta)\big)}, \frac{\log \varepsilon_1(\eta)}{\log\big(\varepsilon_1(\eta)\varepsilon_2(\eta)\big)} \right). \end{aligned} \tag{1.10}$$

So that

$$\lim_{\eta \to 0^+} \boldsymbol{\delta}(\eta) = (0, \lambda) \,.$$

In Section 3 we will prove the following Theorem 1.2 where we describe $u_{\boldsymbol{\varepsilon}(\eta)}$ in terms of a real analytic function of four real variables evaluated at $(\boldsymbol{\varepsilon}(\eta), \boldsymbol{\delta}(\eta))$.

Theorem 1.2. *Let $\lambda \in [0,1[$. Let Ω' be an open subset of Ω with $0 \notin \overline{\Omega'}$. Then there are $\boldsymbol{\varepsilon}' \in]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$, an open neighborhood $\mathcal{V}_\lambda$ of $(0,\lambda)$ in $\mathbb{R}^2$, and a real analytic map*

$$\mathfrak{U}_{\Omega'} :]-\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}'[\times \mathcal{V}_\lambda \to \mathscr{C}^{1,\alpha}(\overline{\Omega'})$$

such that

$$\overline{\omega_{\boldsymbol{\varepsilon}}} \cap \overline{\Omega'} = \emptyset \,, \qquad \forall \boldsymbol{\varepsilon} \in]-\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}'[\,, \tag{1.11}$$

and

$$u_{\boldsymbol{\varepsilon}(\eta)|\overline{\Omega'}} = \mathfrak{U}_{\Omega'}\big[\boldsymbol{\varepsilon}(\eta), \boldsymbol{\delta}(\eta)\big] \,, \qquad \forall \eta \in]0, \eta'[\,, \tag{1.12}$$

where the latter equality holds for all functions $\boldsymbol{\varepsilon}(\cdot)$ from $]0,1[$ to $]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$ which satisfy (1.8) – (1.9) and with $\boldsymbol{\delta}(\cdot)$ as in (1.10), and for all $\eta' \in]0,1[$ small enough so that

$$(\boldsymbol{\varepsilon}(\eta), \boldsymbol{\delta}(\eta)) \in]0, \boldsymbol{\varepsilon}'[\times \mathcal{V}_\lambda \,, \qquad \forall \eta \in]0, \eta'[\,.$$

In addition,

$$\mathfrak{U}_{\Omega'}[\mathbf{0}, (0,\lambda)] = u_{0|\overline{\Omega'}} \,. \tag{1.13}$$

We stress here that the map $\mathfrak{U}_{\Omega'}$ depends on the limit value λ, but not on the specific curve $\boldsymbol{\varepsilon}(\cdot)$ which satisfies (1.9). A similar result is then given also for the behavior of the solution of problem (1.1) close to the hole and of the energy integral. In particular, we show that the limiting value of the energy integral is given by

$$\lim_{\eta\to 0}\int_{\Omega_\varepsilon}|\nabla u_{\boldsymbol{\varepsilon}(\eta)}|^2\,d\mathsf{x}=\int_{\Omega}|\nabla u_0|^2\,d\mathsf{x}+\int_{\mathbb{R}^2\setminus\omega}|\nabla v_0|^2\,d\mathsf{x}, \tag{1.14}$$

where $v_0\in\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2\setminus\omega)$ is the unique solution of

$$\begin{cases}\Delta v_0=0 & \text{in } \mathbb{R}^2\setminus\omega\,,\\ v_0=g^{\mathrm{i}} & \text{on } \partial\omega\,,\\ \sup_{\mathbb{R}^2\setminus\omega}|v_0|<+\infty\,. \end{cases} \tag{1.15}$$

In addition, for the flux on $\partial\Omega$ we will show that

$$\lim_{\eta\to 0}\int_{\partial\Omega}\nu_\Omega\cdot\nabla u_{\boldsymbol{\varepsilon}(\eta)}\,d\sigma=0.$$

1.2.2 What happens when $\varepsilon_1\to 0$ and $\varepsilon_2>0$ is fixed?

We observe that we can confine ourself to consider $\varepsilon_2=1$ fixed. Then the case of $\varepsilon_2=\varepsilon_2^*\in\,]0,\varepsilon_{0,2}[$ fixed is obtained by rescaling the reference domain ω with the factor ε_2^*. We also observe that in this case we are dealing with a one parameter problem. Accordingly, we find convenient to define $\varepsilon^{\mathrm{ad}}\equiv\varepsilon_1^{\mathrm{ad}}$, $\omega_\varepsilon\equiv\omega_{\varepsilon_1,1}$, $\Omega_\varepsilon\equiv\Omega_{\varepsilon_1,1}$, and $u_\varepsilon\equiv u_{\varepsilon_1,1}$ for all $\varepsilon\in\,]-\varepsilon^{\mathrm{ad}},\varepsilon^{\mathrm{ad}}[$. In Section 4 we will prove the following Theorem 1.3 holds.

Theorem 1.3. *Let Ω' be an open subset of Ω such that $0\notin\overline{\Omega'}$. Then there are $\varepsilon'\in]0,\varepsilon_1^{\mathrm{ad}}[$ and a real analytic map*

$$\mathfrak{U}_{\Omega'}:]-\varepsilon',\varepsilon'[\to\mathscr{C}^{1,\alpha}(\overline{\Omega'})$$

such that

$$\overline{\omega_\varepsilon}\cap\overline{\Omega'}=\emptyset\qquad\forall\varepsilon\in]-\varepsilon',\varepsilon'[\,, \tag{1.16}$$

and

$$u_{\varepsilon|\overline{\Omega'}}=\mathfrak{U}_{\Omega'}[\varepsilon],\qquad\forall\varepsilon\in]0,\varepsilon'[\,. \tag{1.17}$$

Moreover we have

$$\mathfrak{U}_{\Omega'}[0]=u_{0|\overline{\Omega'}}\,. \tag{1.18}$$

In addition, we shall prove a similar result for the behavior of u_ε near the boundary of the hole (cf. Theorem 4.7), for the energy integral $\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|\,d\mathsf{x}$ (cf. Theorem 4.9), and for the total flux through the outer boundary $\int_{\partial\Omega}\nu_\Omega\cdot\nabla u_\varepsilon\,d\sigma$ (cf. Theorem 4.11). In particular we will show that the limiting value of the energy integral is

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^2\,d\mathsf{x}=\int_{\Omega}|\nabla u_0|^2\,d\mathsf{x}+\int_{\mathbb{R}^2_+\setminus(\mathsf{p}+\omega)}|\nabla w_*|^2\,d\mathsf{x}\,. \tag{1.19}$$

and the limiting value of the total flux is

$$\int_{\mathsf{p}+\partial\omega}\nu_{\mathsf{p}+\omega}\cdot\nabla w_*\,d\sigma \tag{1.20}$$

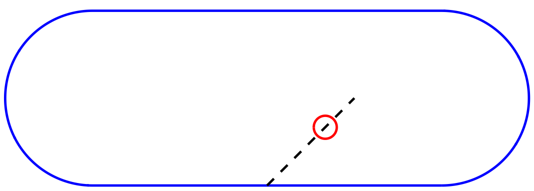
where w_* is the unique solution in $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^2_+} \setminus (\mathsf{p}+\omega))$ of

$$\begin{cases} \Delta w_* = 0 & \text{in } \mathbb{R}^2_+ \setminus (\mathsf{p}+\overline{\omega})\,, \\ w_* = g^{\mathrm{i}} & \text{on } \mathsf{p}+\partial\omega\,, \\ w_* = g^{\mathrm{o}}(0) & \text{on } \partial\mathbb{R}^2_+\,, \\ \lim_{\mathsf{X}\to\infty} w_*(\mathsf{X}) = g^{\mathrm{o}}(0)\,. \end{cases} \tag{1.21}$$

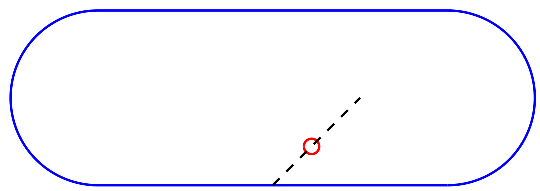
We observe that for suitable choices of g^{o} and g^{i} the limiting value of the energy integral differs from the one in (1.14). This emphasizes the difference between the two regimes.

1.3 Numerical illustration of the results.

In order to illustrate the main results stated before, we present some numerical simulations. The domain Ω is a stadium that is the union of the rectangle $[-2,2]\times[0,2]$ and two half-disks. The origin $(0,0)$ is in the middle of a segment of the boundary. We choose $\mathsf{p}=(1,1)$ and the inclusion is a small disk as described in Figure 2. The small parameter $\boldsymbol{\varepsilon}$ is chosen as $\varepsilon_1 = \left(\frac{2}{3}\right)^{n_1}$ and $\varepsilon_2 = \left(\frac{2}{3}\right)^{n_2}$ for integers $1 \le n_1 \le 16$ and $1 \le n_2 \le 20$.



(a) $n_1 = 1$ and $n_2 = 4$, 4312 triangles.



(b) $n_1 = 2$ and $n_2 = 4$, 4364 triangles.

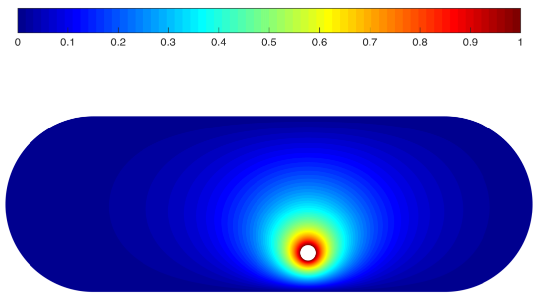
Figure 2: Different computational domains.

In order to approximate the solution $u_{\boldsymbol{\varepsilon}}$ of the boundary value problem, we use a $\mathbb{P}^4$ finite element method on an adapted triangular mesh thanks to the Finite Element Library MÉLINA (see [22]). Figures 3–5 exhibit the computed $\mathscr{H}^1$ energy $\|\nabla u_{\boldsymbol{\varepsilon}}\|^2_{\mathscr{L}^2(\Omega_{\boldsymbol{\varepsilon}})}$ in the previously defined configurations. In Figure 3, we take $g^{\mathrm{o}}=0$ and $g^{\mathrm{i}}=1$, then the sum in (1.14) is 0 while the limiting energy (1.19) is strictly positive. Besides the limit value of the total flux (1.20) equals 0 only for special choices of g^{o}, g^{i}. To illustrate this fact we consider now $g^{\mathrm{o}}=x_2$ and either $g^{\mathrm{i}}=0=g^{\mathrm{o}}(0)$ (see Figure 4), either $g_i = 1 \neq g^{\mathrm{o}}(0)$ (see Figure 5).

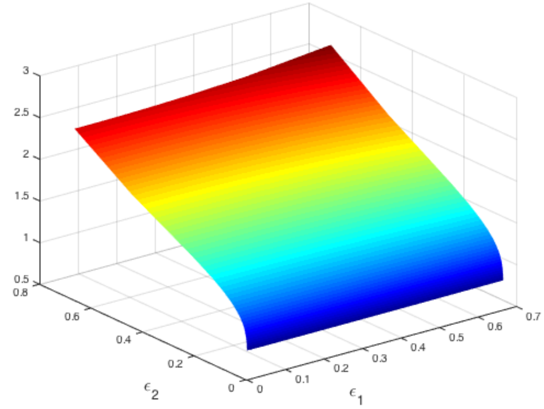
In the numerical results for $g^{\mathrm{i}}=1$ that is with $g^{\mathrm{i}} \neq g^{\mathrm{o}}(0)$, the energy has a different limit value whether both ε_1 and ε_2 tend to 0 or ε_1 tends to 0 and ε_2 is fixed. In the first case the limit is the same that we have in the very well known case when $\varepsilon_2 \to 0$ and ε_1 is fixed.

1.4 Structure of the paper

The paper is organised as follows. In Section 2, we recall some facts of classical potential theory and we present some results about layer potentials derived by the Green function of the half plane. Then in Section 3 we study the behavior of the solution of (1.1) for $\boldsymbol{\varepsilon} \to \mathbf{0}$ and prove Theorem 1.2. In Section 4 we consider the case when $\varepsilon_1 \to 0$ and $\varepsilon_2 = 1$ is fixed and prove Theorem 1.3.

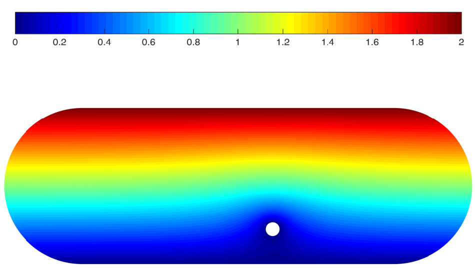


(a) u_ε for $n_1 = 2$ and $n_2 = 4$.

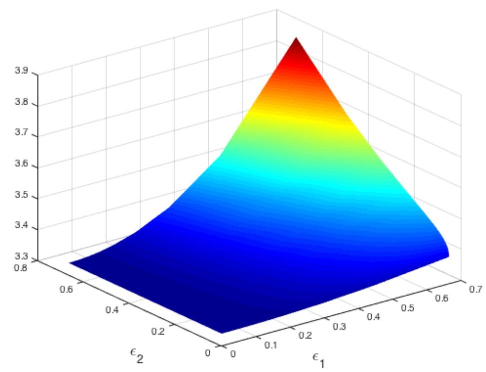


(b) Energy of u_ε.

Figure 3: Case where $g^{\mathrm{o}} = 0$ and $g^{\mathrm{i}} = 1$.

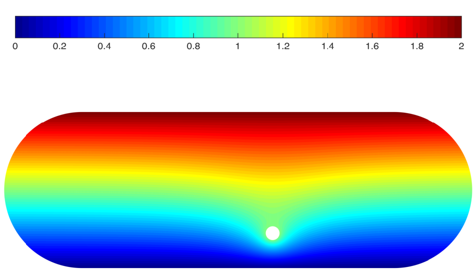


(a) u_ε for $n_1 = 2$ and $n_2 = 4$.

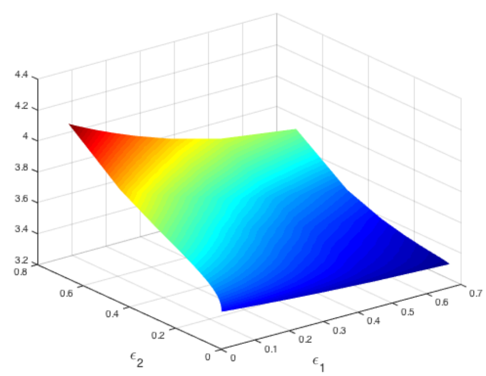


(b) Energy of u_ε

Figure 4: Case $g^{\mathrm{i}} = 0$ and $g^{\mathrm{o}} = x_2$.



(a) u_ε for $n_1 = 2$ and $n_2 = 4$.



(b) Energy of u_ε.

Figure 5: Case $g^{\mathrm{i}} = 1$ and $g^{\mathrm{o}} = x_2$.

2 Preliminaries and notations

As in [2], our approach is based on potential theory, and in particular on the use of layer potentials derived by the Dirichlet Green function for the upper-half plane. In this section, we recall some technical results and notation that we use in the sequel.

2.1 Single and double layer potentials

We denote by S the fundamental solution of Δ defined by

$$S(x) \equiv \frac{1}{2\pi} \log |\mathsf{x}| \qquad \forall \mathsf{x} \in \mathbb{R}^2 \setminus \{0\}\,.$$

Then we introduce the single and double layer potentials for a generic domain $\mathcal{D}$ which we assume to be an open bounded connected subset of $\mathbb{R}^2$ of class $\mathscr{C}^{1,\alpha}$.

Definition 2.1 (Definition of the layer potentials)**.** *For all $\phi \in \mathscr{C}^{0,\alpha}(\partial\mathcal{D})$, we define*

$$v_S[\partial\mathcal{D}, \phi](\mathsf{x}) \equiv \int_{\partial\mathcal{D}} \phi(y) S(\mathsf{x} - \mathsf{y})\, d\sigma_{\mathsf{y}}, \qquad \forall \mathsf{x} \in \mathbb{R}^2\,,$$

where $d\sigma$ denotes the area element on $\partial\mathcal{D}$. For all $\psi \in \mathscr{C}^{1,\alpha}(\partial\mathcal{D})$, we define

$$w_S[\partial\mathcal{D}, \psi](\mathsf{x}) \equiv -\int_{\partial\mathcal{D}} \psi(\mathsf{y})\, \nu_{\mathcal{D}}(\mathsf{y}) \cdot \nabla S(\mathsf{x} - \mathsf{y})\, d\sigma_{\mathsf{y}}, \qquad \forall \mathsf{x} \in \mathbb{R}^2\,,$$

where $\nu_{\mathcal{D}}$ denotes the outer unit normal to $\partial\mathcal{D}$ and the symbol '$\cdot$' denotes the scalar product in $\mathbb{R}^2$.

To describe the regularity properties of the layer potentials, we introduce the following notation.

Definition 2.2. *We denote by $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \mathcal{D})$ the space of functions on $\mathbb{R}^2 \setminus \mathcal{D}$ whose restrictions to $\overline{\mathcal{O}}$ belong to $\mathscr{C}^{1,\alpha}(\overline{\mathcal{O}})$ for all open bounded subsets $\mathcal{O}$ of $\mathbb{R}^2 \setminus \mathcal{D}$. Moreover, we denote by $\mathscr{C}^{0,\alpha}_{\#}(\partial\mathcal{D})$ the subspace of $\mathscr{C}^{0,\alpha}(\partial\mathcal{D})$ consisting of the functions ϕ with $\int_{\partial\mathcal{D}} \phi\, d\sigma = 0$.*

Then we have the following well known regularity properties of the single and double layer potentials.

Proposition 2.3 (Regularity of layer potentials)**.** *The function $v_S[\partial\mathcal{D}, \phi]$ is continuous from $\mathbb{R}^2$ to $\mathbb{R}$. The restrictions $v^i_S[\partial\mathcal{D}, \phi] \equiv v_S[\partial\mathcal{D}, \phi]_{|\overline{\mathcal{D}}}$ and $v^e_S[\partial\mathcal{D}, \phi] \equiv v_S[\partial\mathcal{D}, \phi]_{|\mathbb{R}^2 \setminus \mathcal{D}}$ belong to $\mathscr{C}^{1,\alpha}(\overline{\mathcal{D}})$ and to $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \mathcal{D})$, respectively. The restriction $w_S[\partial\mathcal{D}, \psi]_{|\mathcal{D}}$ extends to a function $w^i_S[\partial\mathcal{D}, \psi]$ of $\mathscr{C}^{1,\alpha}(\overline{\mathcal{D}})$ and the restriction $w_S[\partial\mathcal{D}, \psi]_{|\mathbb{R}^2 \setminus \overline{\mathcal{D}}}$ extends to a function $w^e_S[\partial\mathcal{D}, \psi]$ of $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \mathcal{D})$.*

We also recall the jump relations of the layer potentials (see, *e.g.*, Folland [14, Chap. 3]).

Proposition 2.4 (Jump relations of the layer potentials)**.** *For any $\mathsf{x} \in \partial\mathcal{D}$, $\psi \in \mathscr{C}^{1,\alpha}(\partial\mathcal{D})$, and $\phi \in \mathscr{C}^{0,\alpha}(\partial\mathcal{D})$, we have*

$$w^\sharp_S[\partial\mathcal{D}, \psi](\mathsf{x}) = \frac{\mathsf{s}_\sharp}{2}\psi(x) + w_S[\partial\mathcal{D}, \psi](\mathsf{x})\,,$$

$$\nu_\Omega(x) \cdot \nabla v^\sharp_S[\partial\mathcal{D}, \phi](x) = -\frac{\mathsf{s}_\sharp}{2}\phi(x) + \int_{\partial\mathcal{D}} \phi(y)\nu_\Omega(x) \cdot \nabla S(x - y)\, d\sigma_y\,,$$

where $\sharp = i, e$ and $\mathsf{s}_i = 1$, $\mathsf{s}_e = -1$.

In the sequel, we will exploit the following classical result of potential theory.

Lemma 2.5. *The map* $\begin{array}{rcl} \mathscr{C}^{0,\alpha}_{\#}(\partial\mathcal{D})\times\mathbb{R} & \to & \mathscr{C}^{1,\alpha}(\partial\mathcal{D}) \\ (\phi,\xi) & \mapsto & v_S[\partial\mathcal{D},\phi]_{|\partial\mathcal{D}}+\xi \end{array}$ *is an isomorphism.*

2.2 Green function for the upper-half plane and associated layer potentials

As in [2], an effective tool to analyze problem (1.1) is represented by layer potentials derived by the Dirichlet Green function for the upper-half plane instead of the classical fundamental solution S. Indeed, by exploiting them, we will be able to get rid of the integral equation on $\partial_0\Omega$, which is the part of the bounday of $\partial\Omega$ where the inclusion $\omega_{\boldsymbol{\varepsilon}}$ collapses for $\varepsilon_1 = 0$. In this subsection, we recall some notation, definitions and results from [2].

We denote by ς the reflexion with respect to the axis $(\mathbf{0},\mathbf{e_1})$, so that $\varsigma((x_1,x_2)) \equiv (x_1,-x_2)$ for all $\mathsf{x}=(x_1,x_2)\in\mathbb{R}^2$, and we define $\varsigma(\mathcal{D})\equiv\{\mathsf{x}\in\mathbb{R}^2 \mid \varsigma(\mathsf{x})\in\mathcal{D}\}$, for all subsets $\mathcal{D}$ of $\mathbb{R}^2$. Then we denote by G the Green function defined by

$$G(\mathsf{x},\mathsf{y})\equiv S(\mathsf{x}-\mathsf{y})-S(\varsigma(\mathsf{x})-\mathsf{y}),\quad \forall\mathsf{x}\in\mathbb{R}^2 \text{ and } \mathsf{y}\in\mathbb{R}^2\setminus\{\mathsf{x},\varsigma(\mathsf{x})\}.$$

Clearly,

$$G(\mathsf{x},\mathsf{y})=G(\mathsf{y},\mathsf{x}),\quad \forall\mathsf{x}\in\mathbb{R}^2 \text{ and } \mathsf{y}\in\mathbb{R}^2\setminus\{\mathsf{x},\varsigma(\mathsf{x})\}\,, \tag{2.1}$$

and

$$G(\mathsf{x},\mathsf{y})=0,\quad \forall\mathsf{x}\in\partial\mathbb{R}^2_+ \text{ and } \mathsf{y}\in\mathbb{R}^2\setminus\{\mathsf{x},\varsigma(\mathsf{x})\}\,. \tag{2.2}$$

We are now ready to introduce analogs of the classical layer potentials obtained by replacing S by the Green function G. In the sequel, $\mathcal{D}_+$ is an open bounded connected subset of $\mathbb{R}^2_+$ of class $\mathscr{C}^{1,\alpha}$.

Definition 2.6 (Definition of layer potentials derived by G)**.** *For any $\phi\in\mathscr{C}^{0,\alpha}(\partial\mathcal{D}_+)$, we define*

$$v_G[\partial\mathcal{D}_+,\phi](\mathsf{x})\equiv\int_{\partial\mathcal{D}_+}\phi(\mathsf{y})G(\mathsf{x},\mathsf{y})\,d\sigma_{\mathsf{y}},\qquad \forall\mathsf{x}\in\mathbb{R}^2\,.$$

For any subset of the boundary $\Gamma\subseteq\partial\mathcal{D}_+$ and for any $\psi\in\mathscr{C}^{1,\alpha}(\partial\mathcal{D}_+)$, we define

$$w_G[\Gamma,\psi](\mathsf{x})\equiv\int_{\Gamma}\psi(\mathsf{y})\;\nu_{\mathcal{D}_+}(\mathsf{y})\cdot\nabla_{\mathsf{y}}\,G(\mathsf{x},\mathsf{y})\,d\sigma_{\mathsf{y}},\qquad \forall\mathsf{x}\in\mathbb{R}^2\,.$$

By exploiting regularity properties and jump formulas for the classical layer potentials, we can deduce the corresponding results for $v_G[\partial\mathcal{D}_+,\phi]$ and $w_G[\partial\mathcal{D}_+,\psi]$. Here we set $\partial_0\mathcal{D}_+\equiv\partial\mathcal{D}_+\cap\partial\mathbb{R}^2_+$ and $\partial_+\mathcal{D}_+\equiv\partial\mathcal{D}_+\cap\mathbb{R}^2_+$.

Proposition 2.7 (Regularity and jump relations for layer potentials derived by G)**.** *Let $\phi\in\mathscr{C}^{0,\alpha}(\partial\mathcal{D}_+)$ and $\psi\in\mathscr{C}^{1,\alpha}(\partial\mathcal{D}_+)$. Then*

- *the functions $v_G[\partial\mathcal{D}_+,\phi]$ and $w_G[\partial\mathcal{D}_+,\psi]$ are harmonic in $\mathcal{D}_+$, $\varsigma(\mathcal{D}_+)$, and $\mathbb{R}^2\setminus(\mathcal{D}_+\cup\varsigma(\mathcal{D}_+))$;*
- *the function $v_G[\partial\mathcal{D}_+,\phi]$ is continuous from $\mathbb{R}^2$ to $\mathbb{R}$;*

- *the restrictions* $v_G^i[\partial\mathcal{D}_+,\phi] \equiv v_G[\partial\mathcal{D}_+,\phi]_{|\overline{\mathcal{D}_+}}$ *and* $v_G^e[\partial\mathcal{D}_+,\phi] \equiv v_G[\partial\mathcal{D}_+,\phi]_{|\overline{\mathbb{R}^2_+\setminus\mathcal{D}_+}}$ *belong to* $\mathscr{C}^{1,\alpha}(\overline{\mathcal{D}_+})$ *and to* $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^2_+\setminus\mathcal{D}_+})$*, respectively;*
- *the restriction* $w_G[\partial\mathcal{D}_+,\psi]_{|\mathcal{D}_+}$ *extends to a function* $w_G^i[\partial\mathcal{D}_+,\psi]$ *of* $\mathscr{C}^{1,\alpha}(\overline{\mathcal{D}_+})$ *and the restriction* $w_G[\partial\mathcal{D}_+,\psi]_{|\mathbb{R}^2_+\setminus\overline{\mathcal{D}_+}}$ *extends to a function* $w_G^e[\partial\mathcal{D}_+,\psi]$ *of* $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^2_+\setminus\mathcal{D}_+})$*.*

In particular, the jump relations for the double layer potential are

$$\begin{aligned} w_G^\sharp[\partial\mathcal{D}_+,\psi](\mathsf{x}) &= \frac{\mathsf{s}_\sharp}{2}\psi(\mathsf{x}) + w_G[\partial\mathcal{D}_+,\psi](\mathsf{x}), && \forall \mathsf{x}\in\partial_+\mathcal{D}_+\,, \\ w_G^i[\partial\mathcal{D}_+,\psi](\mathsf{x}) &= \psi(\mathsf{x}), && \forall \mathsf{x}\in\partial_0\mathcal{D}_+\,. \end{aligned}$$

where $\sharp = i, e$*,* $\mathsf{s}_i = 1$*, and* $\mathsf{s}_e = -1$*. In addition, it holds*

$$\begin{aligned} v_G[\partial\mathcal{D}_+,\phi](\mathsf{x}) &= 0, && \forall \mathsf{x}\in\partial\mathbb{R}^2_+\,, \\ w_G^e[\partial\mathcal{D}_+,\psi](\mathsf{x}) &= 0, && \forall \mathsf{x}\in\partial\mathbb{R}^2_+\setminus\partial_0\mathcal{D}_+\,. \end{aligned}$$

An analog of the Green representation formula can then be proved by means of the layer potentials with kernel G introduced in Definition 2.6 (see [2]).

Lemma 2.8 (Representation formulas with layer potentials derived by G)**.** *Let* $u^i \in \mathscr{C}^{1,\alpha}(\overline{\mathcal{D}_+})$ *be harmonic in* $\mathcal{D}_+$*, then*

$$w_G[\partial\mathcal{D}_+,u^i_{|\partial\mathcal{D}_+}] - v_G[\partial\mathcal{D}_+,\nu_{\mathcal{D}_+}\cdot\nabla u^i_{|\partial\mathcal{D}_+}] = \begin{cases} u^i & \text{in } \mathcal{D}_+, \\ 0 & \text{in } \mathbb{R}^2\setminus\overline{\mathcal{D}_+\cup\varsigma(\mathcal{D}_+)}\,. \end{cases}$$

2.3 The single layer potential operators derived by G on $\partial\Omega$

In this subsection, we present some results of [2] on the single layer potential associated to the set Ω which satisfies the assumptions (H_1), (H_2), and (H_3).

First of all, we consider the behavior at infinity of the Green function G and of $v_G[\partial\Omega,\phi]$.

Lemma 2.9 (Behavior at infinity of G)**.** *The following statements hold.*

(i) The function $(\mathsf{x},\mathsf{y})\mapsto|\mathsf{x}|G(\mathsf{x},\mathsf{y})$ *is bounded in* $(\mathbb{R}^2\setminus\mathcal{B}(0,d))\times\overline{\Omega}$*, where* $d\equiv 2\sup_{y\in\Omega}|y|$*.*

(ii) Let $\phi\in\mathscr{C}^{0,\alpha}(\partial\Omega)$*. Then the function* $\mathsf{x}\mapsto|\mathsf{x}|\,v_G[\partial\Omega,\phi](\mathsf{x})$ *is bounded in* $\mathbb{R}^2\setminus(\Omega\cup\varsigma(\Omega))$*. In particular,* $v_G[\partial\Omega,\phi]$ *is harmonic at infinity.*

As one can easily see, the single layer potential $v_G[\partial\Omega,\phi]$ does not depend on the values of the density ϕ on $\partial_0\Omega$, *i.e.*, it takes into account only $\phi_{|\partial_+\Omega}$. Thus, it is natural to introduce a quotient Banach space.

Definition 2.10. *We denote by* $\mathscr{C}^{0,\alpha}_+(\partial\Omega)$ *the quotient Banach space*

$$\mathscr{C}^{0,\alpha}(\partial\Omega)/\{\phi\in\mathscr{C}^{0,\alpha}(\partial\Omega)\mid\phi_{|\partial_+\Omega}=0\}\,.$$

Then we have that the single layer potential map

$$\begin{aligned} \mathscr{C}^{0,\alpha}_+(\partial\Omega) &\to \mathscr{C}^{1,\alpha}(\partial_+\Omega) \\ \phi &\mapsto v_G[\partial\Omega,\phi]_{|\partial_+\Omega} \end{aligned}$$

is well defined and one-to-one. In Proposition 2.11 here below we introduce the image space $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$ of $v_G[\partial\Omega,\cdot]_{|\partial_+\Omega}$.

Proposition 2.11 (Image of the single layer potential derived by G)**.** *Let $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$ denote the vector space*

$$\mathscr{V}^{1,\alpha}(\partial_+\Omega) = \left\{ v_G[\partial\Omega, \phi]_{|\partial_+\Omega},\ \forall \phi \in \mathscr{C}^{0,\alpha}_+(\partial\Omega) \right\}.$$

Let $\|\cdot\|_{\mathscr{V}^{1,\alpha}(\partial_+\Omega)}$ be the norm on $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$ defined by

$$\|f\|_{\mathscr{V}^{1,\alpha}(\partial_+\Omega)} \equiv \|\phi\|_{\mathscr{C}^{0,\alpha}_+(\partial\Omega)}$$

for all $(f,\phi) \in \mathscr{V}^{1,\alpha}(\partial_+\Omega) \times \mathscr{C}^{0,\alpha}_+(\partial\Omega)$ such that $f = v_G[\partial\Omega,\phi]_{|\partial_+\Omega}$. Then:

(i) $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$ endowed with the norm $\|\cdot\|_{\mathscr{V}^{1,\alpha}(\partial_+\Omega)}$ is a Banach space.

(ii) The operator $v_G[\partial\Omega,\cdot]_{|\partial\Omega}$ is an homeomorphism from $\mathscr{C}^{0,\alpha}_+(\partial\Omega)$ to $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$.

In the following proposition we characterize the functions of $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$.

Proposition 2.12 (Characterization of $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$)**.** *A function f belongs to $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$ if and only if $f = u^e_{|\partial_+\Omega}$, where $u^e \in \mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^2_+ \setminus \Omega})$ satisfies the following conditions:*

$$\Delta u^e = 0 \ \text{in}\ \mathbb{R}^2_+ \setminus \overline{\Omega}; \quad u^e_{|\partial\mathbb{R}^2_+ \setminus \partial_0\Omega} = 0; \quad \lim_{\mathsf{x}\to\infty} |\mathsf{x}|^{-1} u^e(\mathsf{x}) = 0; \quad \lim_{\mathsf{x}\to\infty} \frac{\mathsf{x}}{|\mathsf{x}|} \cdot \nabla u^e(\mathsf{x}) = 0.$$

Then we have a Green representation formula in the exterior domain $\mathbb{R}^2_+ \setminus \overline{\Omega}$.

Lemma 2.13 (Green representation formula in $\mathbb{R}^2_+ \setminus \overline{\Omega}$ with layer potentials derived by G)**.** *Let $u^e \in \mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^2_+ \setminus \Omega})$ be such that*

$$\Delta u^e = 0 \ \text{in}\ \mathbb{R}^2_+ \setminus \overline{\Omega}, \qquad \lim_{|\mathsf{x}|\to\infty} |\mathsf{x}|^{-1} u^e(\mathsf{x}) = 0 \qquad \text{and} \qquad \lim_{|\mathsf{x}|\to\infty} \frac{\mathsf{x}}{|\mathsf{x}|} \cdot \nabla u^e(\mathsf{x}) = 0.$$

Then we have

$$- w_G[\partial_+\Omega, u^e_{|\partial_+\Omega}](\mathsf{x}) + v_G[\partial\Omega, \nu_\Omega \cdot \nabla u^e_{|\partial\Omega}](\mathsf{x}) + \frac{2x_2}{2\pi} \int_{\partial\mathbb{R}^2_+ \setminus \partial_0\Omega} \frac{u^e(\mathsf{y})}{|\mathsf{x}-\mathsf{y}|^2}\, d\sigma_{\mathsf{y}} = \begin{cases} u^e(\mathsf{x}) & \forall \mathsf{x} \in \mathbb{R}^2_+ \setminus \overline{\Omega}, \\ 0 & \forall \mathsf{x} \in \Omega. \end{cases}$$

In the sequel we will also need the following technical Lemma 2.14, which can be proved by the properties of integral operators with harmonic kernel (and no singularity).

Lemma 2.14. *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ such that $\overline{\mathcal{O}} \cap \mathbb{R}^n_+$ is contained in Ω. Then $w_G[\partial_+\Omega, \psi]$ is harmonic on $\mathcal{O}$ for all $\psi \in \mathscr{C}^{1,\alpha}(\partial\Omega)$.*

2.4 Extending functions from $\mathscr{C}^{k,\alpha}(\overline{\partial_+\Omega})$ to $\mathscr{C}^{k,\alpha}(\partial\Omega)$.

We will use the following extension result which allows us to identify $\mathscr{C}^{0,\alpha}_+(\partial\Omega)$ and $\mathscr{C}^{0,\alpha}(\overline{\partial_+\Omega})$. For a proof we refer to Troianiello [27, proof of Lem. 1.5, p. 16].

Lemma 2.15 (Extension lemma)**.** *There exist linear and continuous extension operators $E^{k,\alpha}$ from $\mathscr{C}^{k,\alpha}(\overline{\partial_+\Omega})$ to $\mathscr{C}^{k,\alpha}(\partial\Omega)$, for $k = 0, 1$.*

3 Behavior of the solution of (1.1) for $\boldsymbol{\varepsilon}$ close to 0

When studying singular perturbation problems in perforated domains in the two-dimensional plane, it is well-known to see some logarithmic terms enter in the description of the effect of the perturbation. Such logarithmic terms do not appear in dimension higher than or equal to three and are generated by the specific behavior of the fundamental solution upon rescaling. Indeed, for $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$ we have

$$G(\varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2 \mathsf{X}, \varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2 \mathsf{Y}) = S(\mathsf{X} - \mathsf{Y}) + \frac{1}{2\pi}\log \varepsilon_1\varepsilon_2 - S(-2p_2\mathsf{e}_2 + \varepsilon_2(\varsigma(\mathsf{X}) - \mathsf{Y})) - \frac{1}{2\pi}\log\varepsilon_1 \,. \tag{3.1}$$

To handle the logarithimic terms, we need a representation formula for harmonic functions in $\Omega_{\boldsymbol{\varepsilon}}$ which is different from the one that we have exploit in [2] for the case of dimension ≥ 3.

First of all we note that, if $\boldsymbol{\varepsilon} \in]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$, then the sets $\Omega_{\boldsymbol{\varepsilon}}$ and $\omega_{\boldsymbol{\varepsilon}}$ satisfies the same assumption (H_1), (H_2), and (H_3) as Ω. Accordingly, we can apply the results of Subsection 2.3 with Ω replaced by $\Omega_{\boldsymbol{\varepsilon}}$ or $\omega_{\boldsymbol{\varepsilon}}$.

In that spirit, we denote by $v_G[\partial\omega_{\boldsymbol{\varepsilon}}, 1]$ the single layer potential with density function identically equal to 1 on $\partial\omega_{\boldsymbol{\varepsilon}}$:

$$v_G[\partial\omega_{\boldsymbol{\varepsilon}}, 1](\mathsf{x}) \equiv \int_{\partial\omega_{\boldsymbol{\varepsilon}}} G(\mathsf{x}, \mathsf{y})\, d\sigma_{\mathsf{y}}, \qquad \forall \mathsf{x} \in \mathbb{R}^2 \,.$$

We set $\mathscr{B}_{\boldsymbol{\varepsilon}} \equiv \mathscr{C}^{0,\alpha}_{+}(\partial\Omega) \times \mathscr{C}^{0,\alpha}_{\#}(\partial\omega_{\boldsymbol{\varepsilon}})$ (cf. Definitions 2.2 and 2.10). Then we have the following proposition.

Proposition 3.1. *Let $\boldsymbol{\varepsilon} \in]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$ and $\rho \in \mathbb{R} \setminus \{0\}$. Then the map*

$$\begin{array}{rcl} \mathscr{B}_{\boldsymbol{\varepsilon}} \times \mathbb{R} & \to & \mathscr{V}^{1,\alpha}(\partial_+\Omega_{\boldsymbol{\varepsilon}}) \\ (\boldsymbol{\phi}, \xi) & \mapsto & v_G[\partial\Omega, \phi_1]_{|\partial_+\Omega_{\boldsymbol{\varepsilon}}} + v_G[\partial\omega_{\boldsymbol{\varepsilon}}, \phi_2]_{|\partial_+\Omega_{\boldsymbol{\varepsilon}}} + \ \xi \ (\rho v_G[\partial\omega_{\boldsymbol{\varepsilon}}, 1]_{|\partial_+\Omega_{\boldsymbol{\varepsilon}}}) \end{array}$$

is an isomorphism.

Proof. We have

$$v_G[\partial\Omega, \phi_1]_{|\partial_+\Omega_{\boldsymbol{\varepsilon}}} + v_G[\partial\omega_{\boldsymbol{\varepsilon}}, \phi_2]_{|\partial_+\Omega_{\boldsymbol{\varepsilon}}} + (\rho v_G[\partial\omega_{\boldsymbol{\varepsilon}}, 1]_{|\partial_+\Omega_{\boldsymbol{\varepsilon}}})\xi = v_G[\partial\Omega_{\boldsymbol{\varepsilon}}, \phi]_{|\partial_+\Omega_{\boldsymbol{\varepsilon}}}$$

with

$$\phi(\mathsf{x}) \equiv \begin{cases} \phi_1(\mathsf{x}) & \forall \mathsf{x} \in \partial\Omega \,, \\ \phi_2(\mathsf{x}) + \rho\xi & \forall \mathsf{x} \in \partial\omega_{\boldsymbol{\varepsilon}} \,. \end{cases}$$

Then the statement follows by the definition of $\mathscr{V}^{1,\alpha}$ as the image of the single layer potential derived by G (cf. Proposition 2.11). □

Now, by Proposition 3.1 and by the representation formula stated in Lemma 2.8 we have the following Proposition 3.2 where we show a suitable way to write a function of $\mathscr{C}^{1,\alpha}(\partial\Omega_{\boldsymbol{\varepsilon}})$ as a sum of layer potentials derived by G.

Proposition 3.2. *Let $\boldsymbol{\varepsilon} \in]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$. Let $f \in \mathscr{C}^{1,\alpha}(\partial\Omega_{\boldsymbol{\varepsilon}})$. Let $\rho \in \mathbb{R} \setminus \{0\}$. Then there exists a unique pair $(\boldsymbol{\phi}, \xi) = ((\phi_1, \phi_2), \xi) \in \mathscr{B}_{\varepsilon} \times \mathbb{R}$ such that*

$$f = w^i_G[\partial\Omega_{\boldsymbol{\varepsilon}}, f]_{|\partial\Omega_{\boldsymbol{\varepsilon}}} + v_G[\partial\Omega, \phi_1]_{|\partial\Omega_{\boldsymbol{\varepsilon}}} + v_G[\partial\omega_{\boldsymbol{\varepsilon}}, \phi_2]_{|\partial\Omega_{\boldsymbol{\varepsilon}}} + (\rho v_G[\partial\omega_{\boldsymbol{\varepsilon}}, 1]_{|\partial\Omega_{\boldsymbol{\varepsilon}}})\xi \,.$$

3.1 Defining the operator $\mathfrak{L}$

Let $\boldsymbol{\varepsilon} \in]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$. By the previous Proposition 3.2, we can look for solutions of problem (1.1) in the form

$$w_G^i[\partial\Omega_{\boldsymbol{\varepsilon}}, u_{\boldsymbol{\varepsilon}|\partial\Omega_{\boldsymbol{\varepsilon}}}] + v_G[\partial\Omega, \phi_1] + v_G[\partial\omega_{\boldsymbol{\varepsilon}}, \phi_2] + (\rho v_G[\partial\omega_{\boldsymbol{\varepsilon}}, 1])\xi$$

for a suitable $(\boldsymbol{\phi}, \xi) \in \mathscr{B}_\varepsilon \times \mathbb{R}$. We split the integral on $\partial\Omega_{\boldsymbol{\varepsilon}}$ as the sum of integrals on $\partial\Omega$ and on $\partial\omega_{\boldsymbol{\varepsilon}}$, we add and subtract $v_G^i[\partial\Omega, \nu_\Omega \cdot \nabla u_{0|\partial\Omega}]$, and we obtain

$$\begin{aligned} w_G^i[\partial\Omega, g^{\mathrm{o}}] - v_G^i[\partial\Omega, \nu_\Omega \cdot \nabla u_{0|\partial\Omega}] - w_G^e\Big[\partial\omega_{\boldsymbol{\varepsilon}}, g^{\mathrm{i}}\big(\frac{\cdot - \varepsilon_1 \mathsf{p}}{\varepsilon_1\varepsilon_2}\big)\Big] \\ + v_G[\partial\Omega, \phi_1 + \nu_\Omega \cdot \nabla u_{0|\partial\Omega}] + v_G[\partial\omega_{\boldsymbol{\varepsilon}}, \phi_2] + (\rho v_G[\partial\omega_{\boldsymbol{\varepsilon}}, 1])\xi\,. \end{aligned}$$

Then we note that

$$u_0 = w_G^i[\partial\Omega, g^{\mathrm{o}}] - v_G^i[\partial\Omega, \nu_\Omega \cdot \nabla u_{0|\partial\Omega}]$$

and, by taking $\rho = (\varepsilon_1\varepsilon_2 \log(\varepsilon_1\varepsilon_2))^{-1}$ and by performing change of variable in the integrals over $\partial\omega_{\boldsymbol{\varepsilon}}$, we deduce that the solutions of (1.1) can be written in the form

$$\begin{aligned} u_0(\mathsf{x}) - \varepsilon_1\varepsilon_2 \int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot (\nabla_{\mathsf{y}} G)(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y})\, g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}} + v_G[\partial\Omega, \mu_1](\mathsf{x}) \\ + \int_{\partial\omega} G(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y})\, \mu_2(\mathsf{Y})\, d\sigma_{\mathsf{Y}} + \frac{\xi}{\log(\varepsilon_1\varepsilon_2)} \int_{\partial\omega} G(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y})\, d\sigma_{\mathsf{Y}}, \qquad \forall \mathsf{x} \in \Omega_{\boldsymbol{\varepsilon}} \end{aligned} \tag{3.2}$$

provided that $(\mu_1, \mu_2, \xi) \in \mathscr{C}_+^{0,\alpha}(\partial\Omega) \times \mathscr{C}_\#^{0,\alpha}(\partial\omega) \times \mathbb{R}$ is choosen in such a way that the boundary conditions of (1.1) are satisfied.

Now define $\mathscr{B} \equiv \mathscr{C}_+^{0,\alpha}(\partial\Omega) \times \mathscr{C}_\#^{0,\alpha}(\partial\omega)$. We can verify that the (extension to $\overline{\Omega_{\boldsymbol{\varepsilon}}}$ of the) harmonic function in (3.2) solves problem (1.1) if and only if the pair $(\boldsymbol{\mu}, \xi) \in \mathscr{B} \times \mathbb{R}$ solves

$$\mathfrak{L}[\boldsymbol{\varepsilon}, \tfrac{1}{\log(\varepsilon_1\varepsilon_2)}, \tfrac{\log\varepsilon_1}{\log(\varepsilon_1\varepsilon_2)}, \boldsymbol{\mu}, \xi] = 0\,, \tag{3.3}$$

where $\mathfrak{L}[\boldsymbol{\varepsilon}, \boldsymbol{\delta}, \boldsymbol{\mu}, \xi] \equiv (\mathfrak{L}_1[\boldsymbol{\varepsilon}, \boldsymbol{\delta}, \boldsymbol{\mu}, \xi], \mathfrak{L}_2[\boldsymbol{\varepsilon}, \boldsymbol{\delta}, \boldsymbol{\mu}, \xi])$ is defined for all $(\boldsymbol{\varepsilon}, \boldsymbol{\delta}, \boldsymbol{\mu}, \xi) \in] - \boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathbb{R}^2 \times \mathscr{B} \times \mathbb{R}$ by

$$\begin{aligned} \mathfrak{L}_1[\boldsymbol{\varepsilon}, \boldsymbol{\delta}, \boldsymbol{\mu}, \xi](\mathsf{x}) \equiv\; & v_G[\partial\Omega, \mu_1](\mathsf{x}) \\ & + \int_{\partial\omega} G(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y})\, \mu_2(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \\ & + \delta_1\, \xi \int_{\partial\omega} G(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y})\, d\sigma_{\mathsf{Y}} \\ & - \varepsilon_1\varepsilon_2 \int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot (\nabla_{\mathsf{y}} G)(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y})\, g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}}, \qquad \forall \mathsf{x} \in \partial_+\Omega\,, \end{aligned}$$

$$
\begin{aligned}
\mathfrak{L}_2[\boldsymbol{\varepsilon},\boldsymbol{\delta},\boldsymbol{\mu},\xi](\mathsf{X}) \equiv\; & v_S[\partial\omega,\mu_2](\mathsf{X}) + \rho_\omega(1-\delta_2)\,\xi \\
& - \int_{\partial\omega} S(-2p_2\mathsf{e}_2 + \varepsilon_2(\varsigma(\mathsf{X})-\mathsf{Y}))\,\mu_2(\mathsf{Y})\,d\sigma_{\mathsf{Y}} \\
& + \delta_1\xi \int_{\partial\omega} (S(\mathsf{X}-\mathsf{Y}) - S(-2p_2\mathsf{e}_2+\varepsilon_2(\varsigma(\mathsf{X})-\mathsf{Y})))\,d\sigma_{\mathsf{Y}} \\
& + \int_{\partial_+\Omega} G(\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{X},\mathsf{y})\,\mu_1(\mathsf{y})\,d\sigma_{\mathsf{y}} \\
& - \varepsilon_2 \int_{\partial\omega} \nu_\omega(\mathsf{Y})\cdot\nabla S(-2p_2\mathsf{e}_2+\varepsilon_2(\varsigma(\mathsf{X})-\mathsf{Y}))\,g^{\mathrm{i}}(\mathsf{Y})\,d\sigma_{\mathsf{Y}} \\
& + U_0(\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{X}) - w_S[\partial\omega,g^{\mathrm{i}}](\mathsf{X}) - \frac{g^{\mathrm{i}}(\mathsf{X})}{2}, \qquad \forall \mathsf{X}\in\partial\omega\,,
\end{aligned}
$$

with

$$
\rho_\omega \equiv \frac{1}{2\pi}\int_{\partial\omega} d\sigma\,.
$$

Thus, in order to find the solution of problem (1.1) it suffices to find a solution of (3.3). Hence, in order to study the asymptotic behavior of $u_{\boldsymbol{\varepsilon}}$, we are now reduced to analyze the behavior of the solutions of the system of integral equations (3.3).

3.2 Real analyticity of the operator $\mathfrak{L}$

We are going to apply the implicit function theorem for real analytic maps to equation (3.3) (see Deimling [13, Thm. 15.3]). As a first step, we prove that $\mathfrak{L}$ defines a real analytic nonlinear operator between suitable Banach spaces.

Proposition 3.3 (Real analyticity of $\mathfrak{L}$)**.** *The map $\mathfrak{L}$ defined by*

$$
\begin{array}{rcl}
]-\boldsymbol{\varepsilon}^{\mathrm{ad}},\boldsymbol{\varepsilon}^{\mathrm{ad}}[\times\mathbb{R}^2\times\mathscr{B}\times\mathbb{R} & \to & \mathscr{V}^{1,\alpha}(\partial_+\Omega)\times\mathscr{C}^{1,\alpha}(\partial\omega) \\
(\boldsymbol{\varepsilon},\boldsymbol{\delta},\boldsymbol{\mu},\xi) & \mapsto & \mathfrak{L}[\boldsymbol{\varepsilon},\boldsymbol{\delta},\boldsymbol{\mu},\xi]
\end{array}
$$

is real analytic.

Proof. We split the proof component by component.

Study of $\mathfrak{L}_1$. First we prove that $\mathfrak{L}_1$ is real analytic.

First step: the range of $\mathfrak{L}_1$ is a subset of $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$. Let $(\boldsymbol{\varepsilon},\boldsymbol{\delta},\boldsymbol{\mu},\xi)\in\,]-\boldsymbol{\varepsilon}^{\mathrm{ad}},\boldsymbol{\varepsilon}^{\mathrm{ad}}[\times\mathbb{R}^2\times\mathscr{B}\times\mathbb{R}$. Let $U^e[\boldsymbol{\varepsilon},\boldsymbol{\delta},\boldsymbol{\mu},\xi]$ denote the function defined by

$$
\begin{aligned}
U^e[\boldsymbol{\varepsilon},\boldsymbol{\delta},\boldsymbol{\mu},\xi](\mathsf{x}) \equiv\; & v_G^e[\partial\Omega,\mu_1](\mathsf{x}) \\
& + \int_{\partial\omega} G(\mathsf{x},\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{Y})\,\mu_2(\mathsf{Y})\,d\sigma_{\mathsf{Y}} \\
& + \delta_1\,\xi\int_{\partial\omega} G(\mathsf{x},\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{Y})\,d\sigma_{\mathsf{Y}} \\
& - \varepsilon_1\varepsilon_2\int_{\partial\omega} \nu_\omega(\mathsf{Y})\cdot(\nabla_{\mathsf{y}}G)(\mathsf{x},\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{Y})\,g^{\mathrm{i}}(\mathsf{Y})\,d\sigma_{\mathsf{Y}}, \quad \forall\mathsf{x}\in\overline{\mathbb{R}^2_+\setminus\Omega}\,.
\end{aligned}
$$

The function $U^e[\boldsymbol{\varepsilon},\boldsymbol{\delta},\boldsymbol{\mu},\xi]$ belongs to $\in\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^2_+\setminus\Omega})$ by the properties of the (Green) single layer potential and by the properties of integral operators with real analytic kernel and no singularity. In addition, one verifies that

$$\begin{cases} \Delta U^e[\boldsymbol{\varepsilon}, \boldsymbol{\delta}, \boldsymbol{\mu}, \xi] = 0 & \text{in } \mathbb{R}^2_+ \setminus \overline{\Omega}, \\ U^e[\boldsymbol{\varepsilon}, \boldsymbol{\delta}, \boldsymbol{\mu}, \xi] = 0 & \text{on } \partial\mathbb{R}^2_+ \setminus \partial_0\Omega, \\ \lim_{\mathsf{x}\to\infty} U^e[\boldsymbol{\varepsilon}, \boldsymbol{\delta}, \boldsymbol{\mu}, \xi](\mathsf{x}) = 0, \\ \lim_{\mathsf{x}\to\infty} \frac{\mathsf{x}}{|\mathsf{x}|} \cdot \nabla U^e[\boldsymbol{\varepsilon}, \boldsymbol{\delta}, \boldsymbol{\mu}, \xi](\mathsf{x}) = 0 \end{cases}$$

(see also Lemma 2.9). Then, by the characterization of $\mathscr{V}^{1,\alpha}$ in Proposition 2.12, we conclude that $\mathfrak{L}_1[\boldsymbol{\varepsilon}, \boldsymbol{\delta}, \boldsymbol{\mu}, \xi] = U^e[\boldsymbol{\varepsilon}, \boldsymbol{\delta}, \boldsymbol{\mu}, \xi]_{|\partial_+\Omega} \in \mathscr{V}^{1,\alpha}(\partial_+\Omega)$.

Second step: $\mathfrak{L}_1$ is real analytic. We observe that

$$\mathfrak{L}_1[\boldsymbol{\varepsilon}, \boldsymbol{\delta}, \boldsymbol{\mu}, \xi] = v_G[\partial\Omega, \mu_1]_{|\partial_+\Omega} + \mathfrak{f}[\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi]$$

where

$$\begin{aligned} \mathfrak{f}[\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi](\mathsf{x}) \equiv & \int_{\partial\omega} G(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y})\, \mu_2(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \\ & + \delta_1\, \xi \int_{\partial\omega} G(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y})\, d\sigma_{\mathsf{Y}} \\ & - \varepsilon_1\varepsilon_2 \int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot (\nabla_{\mathsf{y}} G)(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y})\, g^{\mathsf{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}}, \qquad \forall \mathsf{x} \in \overline{\partial_+\Omega}\,. \end{aligned}$$

Since that the map which takes μ_1 to $v_G[\partial\Omega, \mu_1]_{|\partial_+\Omega}$ is linear and continuous from $\mathscr{C}^{0,\alpha}_+(\partial\Omega)$ to $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$, it is real analytic. Then, to prove that $\mathfrak{L}_1$ is real analytic we have to show that the map which takes $(\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi)$ to $\mathfrak{f}[\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi](\mathsf{x})$ is real analytic from $]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\, \times \mathbb{R} \times \mathscr{C}^{0,\alpha}_\#(\partial\omega) \times \mathbb{R}$ to $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$. To that end, we will show that there is a real analytic map

$$\phi :]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\, \times \mathbb{R} \times \mathscr{C}^{0,\alpha}_\#(\partial\omega) \times \mathbb{R} \to \mathscr{C}^{0,\alpha}_+(\partial\Omega)$$

such that

$$\mathfrak{f}[\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi] = v_G[\partial\Omega, \phi[\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi]]_{|\partial_+\Omega} \tag{3.4}$$

for all $(\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi) \in\,]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\, \times \mathbb{R} \times \mathscr{C}^{0,\alpha}_\#(\partial\omega) \times \mathbb{R}$. Then the real analyticity of $\mathfrak{f}$ will follow from the definition of the Banach space $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$ in Proposition 2.11.

We will obtain such map ϕ as the sum of two real analytc terms. To construct the first one, we begin by observing that $\mathfrak{f}$ is real analytic from $]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\, \times \mathbb{R} \times \mathscr{C}^{0,\alpha}_\#(\partial\omega) \times \mathbb{R}$ to $\mathscr{C}^{1,\alpha}(\overline{\partial_+\Omega})$ by the properties of integral operators with real analytic kernel and no singularities (see Lanza de Cristoforis and Musolino [20, Prop. 4.1 (ii)]). Then, by the extension Lemma 2.15, the composed map

$$\begin{aligned}]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\, \times \mathbb{R} \times \mathscr{C}^{0,\alpha}_\#(\partial\omega) \times \mathbb{R} &\to \mathscr{C}^{1,\alpha}(\partial\Omega) \\ (\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi) &\mapsto E^{1,\alpha} \circ \mathfrak{f}[\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi] \end{aligned}$$

is real analytic. Then we denote by $u^i[\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi]$ the unique solution of the Dirichlet problem in Ω with boundary datum $E^{1,\alpha} \circ \mathfrak{f}[\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi]$. Since the map from $\mathscr{C}^{1,\alpha}(\partial\Omega)$ to $\mathscr{C}^{1,\alpha}(\overline{\Omega})$ which takes a function ψ to the unique solution of the Dirichlet problem in Ω with boundary datum ψ is linear and continuous, the map from $]-\varepsilon^{\mathrm{ad}},\varepsilon^{\mathrm{ad}}[\times\mathbb{R}\times\mathscr{C}^{0,\alpha}_{\#}(\partial\omega)\times\mathbb{R}$ to $\mathscr{C}^{1,\alpha}(\overline{\Omega})$ which takes $(\varepsilon,\delta_1,\mu_2,\xi)$ to $u^i[\varepsilon,\delta_1,\mu_2,\xi]$ is real analytic. In particular we have that

$$\text{the map}\quad \begin{array}{rcl}]-\varepsilon^{\mathrm{ad}},\varepsilon^{\mathrm{ad}}[\times\mathbb{R}\times\mathscr{C}^{0,\alpha}_{\#}(\partial\omega)\times\mathbb{R} & \to & \mathscr{C}^{0,\alpha}_{+}(\partial\Omega) \\ (\varepsilon,\delta_1,\mu_2,\xi) & \mapsto & \nu_\Omega\cdot\nabla u^i[\varepsilon,\delta_1,\mu_2,\xi]_{|\partial\Omega} \end{array}\quad \text{is real analytic.} \tag{3.5}$$

The map in (3.5) will be the first term in the sum which gives ϕ. To obtain the second term, we begin by taking

$$\begin{aligned} u^e[\varepsilon,\delta_1,\mu_2,\xi](\mathsf{x}) \equiv & \int_{\partial\omega} G(\mathsf{x},\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{Y})\,\mu_2(\mathsf{Y})\,d\sigma_{\mathsf{Y}} \\ & + \delta_1\,\xi\int_{\partial\omega} G(\mathsf{x},\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{Y})\,d\sigma_{\mathsf{Y}} \\ & - \varepsilon_1\varepsilon_2\int_{\partial\omega}\nu_\omega(\mathsf{Y})\cdot(\nabla_{\mathsf{y}}G)(\mathsf{x},\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{Y})\,g^{\mathrm{i}}(\mathsf{Y})\,d\sigma_{\mathsf{Y}}, \qquad \forall \mathsf{x}\in\overline{\mathbb{R}^2_+\setminus\Omega}\,. \end{aligned}$$

By standard properties of integral operators with real analytic kernels and no singularity (see Lanza de Cristoforis and Musolino [20, Prop. 4.1 (ii)]), we have that the map from $]-\varepsilon^{\mathrm{ad}},\varepsilon^{\mathrm{ad}}[\times\mathbb{R}\times\mathscr{C}^{0,\alpha}_{\#}(\partial\omega)\times\mathbb{R}$ to $\mathscr{C}^{0,\alpha}(\overline{\partial_+\Omega})$ which takes $(\varepsilon,\delta_1,\mu_2,\xi)$ to

$$\begin{aligned} &\nu_\Omega\cdot\nabla u^e[\varepsilon,\delta_1,\mu_2,\xi](\mathsf{x}) \\ &\quad = \nu_\Omega(\mathsf{x})\cdot\int_{\partial\omega}\nabla_{\mathsf{x}}G(\mathsf{x},\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{Y})\,\mu_2(\mathsf{Y})\,d\sigma_{\mathsf{Y}} \\ &\qquad + \delta_1\,\xi\,\nu_\Omega(\mathsf{x})\cdot\int_{\partial\omega}\nabla_{\mathsf{x}}G(\mathsf{x},\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{Y})\,d\sigma_{\mathsf{Y}} \\ &\qquad - \varepsilon_1\varepsilon_2\sum_{j,k=1}^{2}(\nu_\Omega(\mathsf{x}))_j\int_{\partial\omega}(\nu_\omega(\mathsf{Y}))_k(\partial_{x_j}\partial_{y_k}G)(\mathsf{x},\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{Y})\,g^{\mathrm{i}}(\mathsf{Y})\,d\sigma_{\mathsf{Y}}, \qquad \forall\mathsf{x}\in\overline{\partial_+\Omega} \end{aligned}$$

is real analytic. Since by the extension Lemma 2.15 we can identify $\mathscr{C}^{0,\alpha}_{+}(\partial\Omega)$ with $\mathscr{C}^{0,\alpha}(\overline{\partial_+\Omega})$, we deduce that

$$\text{the map}\quad \begin{array}{rcl}]-\varepsilon^{\mathrm{ad}},\varepsilon^{\mathrm{ad}}[\times\mathbb{R}\times\mathscr{C}^{0,\alpha}_{\#}(\partial\omega)\times\mathbb{R} & \to & \mathscr{C}^{0,\alpha}_{+}(\partial\Omega) \\ (\varepsilon,\delta_1,\mu_2,\xi) & \mapsto & \nu_\Omega\cdot\nabla u^e[\varepsilon,\delta_1,\mu_2,\xi]_{|\partial\Omega} \end{array}\quad \text{is real analytic.} \tag{3.6}$$

We now show that the maps in (3.5) and (3.6) provide the two terms for the construction of ϕ. First, we observe that $u^i[\varepsilon,\delta_1,\mu_2,\xi]_{|\partial_+\Omega}=\mathfrak{f}[\varepsilon,\delta_1,\mu_2,\xi]$, and thus by the representation formula in Lemma 2.8 we have

$$0 = w_G[\partial_+\Omega,\mathfrak{f}[\varepsilon,\delta_1,\mu_2,\xi]](\mathsf{x}) - v_G[\partial_+\Omega,\nu_\Omega\cdot\nabla u^i[\varepsilon,\delta_1,\mu_2,\xi]_{|\partial_+\Omega}](\mathsf{x}) \tag{3.7}$$

for all $\mathsf{x}\in\mathbb{R}^2_+\setminus\overline{\Omega}$. In addition, one verifies that $u^e[\varepsilon,\delta_1,\mu_2,\xi]\in\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2_+\setminus\Omega)$ and that

$$\begin{cases} \Delta u^e[\varepsilon,\delta_1,\mu_2,\xi]=0 & \text{in } \mathbb{R}^2_+\setminus\overline{\Omega}, \\ u^e[\varepsilon,\delta_1,\mu_2,\xi](\mathsf{x})=0 & \text{on } \partial\mathbb{R}^2_+\setminus\partial_0\Omega, \\ \lim_{\mathsf{x}\to\infty}u^e[\varepsilon,\delta_1,\mu_2,\xi](\mathsf{x})=0\,, & \\ \lim_{\mathsf{x}\to\infty}\frac{\mathsf{x}}{|\mathsf{x}|}\cdot\nabla u^e[\varepsilon,\delta_1,\mu_2,\xi](\mathsf{x})=0 & \end{cases} \tag{3.8}$$

(see also Lemma 2.9). Then, by (3.8), by equality $u^e[\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi]_{|\partial_+\Omega} = \mathfrak{f}[\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi]$, and by the exterior representation formula in Lemma 2.13 we have

$$u^e[\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi](\mathsf{x}) = -w_G[\partial_+\Omega, \mathfrak{f}[\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi]](\mathsf{x}) + v_G[\partial_+\Omega, \nu_\Omega \cdot \nabla u^e[\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi]_{|\partial_+\Omega}](\mathsf{x}) \quad (3.9)$$

for all $\mathsf{x} \in \mathbb{R}^2_+ \setminus \overline{\Omega}$. Then, by taking the sum of (3.7) and (3.9) and by the continuity properties of the (Green) single layer potential we obtain that (3.4) holds with

$$\phi[\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi] = \nu_\Omega \cdot \nabla u^e[\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi]_{|\partial_+\Omega} - \nu_\Omega \cdot \nabla u^i[\boldsymbol{\varepsilon}, \delta_1, \mu_2, \xi]_{|\partial_+\Omega} .$$

In addition, by (3.5) and (3.6), ϕ is real analytic from $]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathbb{R} \times \mathscr{C}^{0,\alpha}_{\#}(\partial\omega) \times \mathbb{R}$ to $\mathscr{C}^{0,\alpha}_{+}(\partial\Omega)$. The analyticity of $\mathfrak{L}_1$ is now proved.

Study of $\mathfrak{L}_2$. The analyticity of the map $\mathfrak{L}_2$ from $]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathbb{R}^2 \times \mathscr{B} \times \mathbb{R}$ to $\mathscr{C}^{1,\alpha}(\partial\omega)$ is a consequence of:

- the real analyticity of U_0 (see also assumption (1.6))
- the mapping properties of the single layer potential (see Lanza de Cristoforis and Rossi [21, Thm. 3.1] and Miranda [25]) and of the integral operators with real analytic kernels and no singularity (see Lanza de Cristoforis and Musolino [20, Prop. 4.1 (ii)]).

□

3.3 Functional analytic representation theorems

3.3.1 Analysis of (3.3) via the implicit function theorem

In this subsection, we study equation (3.3) around a degenerate pair $(\boldsymbol{\varepsilon}, \boldsymbol{\delta}) = (\mathbf{0}, (0, \lambda))$, with $\lambda \in [0, 1[$. As a first step, we investigate equation (3.3) for $(\boldsymbol{\varepsilon}, \boldsymbol{\delta}) = (\mathbf{0}, (0, \lambda))$.

Proposition 3.4. *Let $\lambda \in [0, 1[$. There exists a unique $(\boldsymbol{\mu}^*, \xi^*) \in \mathscr{B} \times \mathbb{R}$ such that*

$$\mathfrak{L}[\mathbf{0}, (0, \lambda), \boldsymbol{\mu}^*, \xi^*] = 0$$

and we have

$$\mu_1^* = 0$$

and

$$v_S[\partial\Omega, \mu_2^*]_{|\partial\omega} + \rho_\omega (1 - \lambda)\, \xi^* = -g^{\mathrm{o}}(0) + w_S[\partial\omega, g^{\mathrm{i}}]_{|\partial\omega} + \frac{g^{\mathrm{i}}}{2} .$$

Proof. First of all, we observe that for all $(\boldsymbol{\mu}, \xi) \in \mathscr{B} \times \mathbb{R}$, we have

$$\begin{cases} \mathfrak{L}_1[\mathbf{0}, (0, \lambda), \boldsymbol{\mu}, \xi](\mathsf{x}) = & v_G[\partial\Omega, \mu_1](\mathsf{x}), \qquad \forall \mathsf{x} \in \partial_+\Omega\,, \\ \mathfrak{L}_2[\mathbf{0}, (0, \lambda), \boldsymbol{\mu}, \xi](\mathsf{X}) = & v_S[\partial\omega, \mu_2](\mathsf{X}) + \rho_\omega (1 - \lambda)\, \xi \\ & \quad + v_G[\partial\Omega, \mu_1](0) + g^{\mathrm{o}}(0) - w_S[\partial\omega, g^{\mathrm{i}}](\mathsf{X}) - \frac{g^{\mathrm{i}}(\mathsf{X})}{2}, \qquad \forall \mathsf{X} \in \partial\omega\,. \end{cases}$$

By Proposition 2.11 (ii), the unique function in $\mathscr{C}^{0,\alpha}_{+}(\partial\Omega)$ such that $v_G[\partial\Omega,\mu_1]=0$ on $\partial_+\Omega$ is $\mu_1=0$. On the other hand, by classical potential theory, there exists a unique pair $(\mu_2,\xi)\in\mathscr{C}^{0,\alpha}_{\#}(\partial\omega)\times\mathbb{R}$ such that (cf. Lemma 2.5)

$$v_S[\partial\omega,\mu_2](\mathsf{X})+\rho_\omega(1-\lambda)\,\xi=-g^{\mathrm{o}}(0)+w_S[\partial\omega,g^{\mathrm{i}}](\mathsf{X})+\frac{g^{\mathrm{i}}(\mathsf{X})}{2},\qquad\forall\mathsf{X}\in\partial\omega.$$

Now the validity of the proposition is proved. □

Then, by the implicit function theorem for real analytic maps (see Deimling [13, Thm. 15.3]) we deduce the following theorem.

Theorem 3.5. *Let $\lambda\in[0,1[$. Let $(\boldsymbol{\mu}^*,\xi^*)$ be as in Proposition 3.4. Then there exist $\boldsymbol{\varepsilon}^*\in]\mathbf{0},\boldsymbol{\varepsilon}^{\mathrm{ad}}[$, an open neighborhood $\mathcal{V}_\lambda$ of $(0,\lambda)$ in $\mathbb{R}^2$, an open neighborhood $\mathcal{U}^*$ of $(\boldsymbol{\mu}^*,\xi^*)$ in $\mathscr{B}\times\mathbb{R}$, and a real analytic map $\Phi\equiv(\Phi_1,\Phi_2,\Phi_3)$ from $]-\boldsymbol{\varepsilon}^*,\boldsymbol{\varepsilon}^*[\times\mathcal{V}_\lambda$ to $\mathcal{U}^*$ such that the set of zeros of $\mathfrak{L}$ in $]-\boldsymbol{\varepsilon}^*,\boldsymbol{\varepsilon}^*[\times\mathcal{V}_\lambda\times\mathcal{U}^*$ coincides with the graph of Φ.*

Proof. The partial differential of $\mathfrak{L}$ with respect to $(\boldsymbol{\mu},\xi)$ evaluated at $(\mathbf{0},(0,\lambda),\boldsymbol{\mu}^*,\xi^*)$ is delivered by

$$\begin{aligned}
&\partial_{(\boldsymbol{\mu},\xi)}\mathfrak{L}_1[\mathbf{0},(0,\lambda),\boldsymbol{\mu}^*,\xi^*](\boldsymbol{\phi},\zeta)=v_G[\partial\Omega,\phi_1]_{|\partial_+\Omega}\,,\\
&\partial_{(\boldsymbol{\mu},\xi)}\mathfrak{L}_2[\mathbf{0},(0,\lambda),\boldsymbol{\mu}^*,\xi^*](\boldsymbol{\phi},\zeta)=v_S[\partial\omega,\phi_2]_{|\partial\omega}+\rho_\omega(1-\lambda)\,\zeta\,,
\end{aligned}$$

for all $(\boldsymbol{\phi},\zeta)\in\mathscr{B}\times\mathbb{R}$. Then by Proposition 2.11 and by the properties of the single layer potential we deduce that $\partial_{(\boldsymbol{\mu},\xi)}\mathfrak{L}[\mathbf{0},(0,\lambda),\boldsymbol{\mu}^*,\xi^*]$ is an isomorphism from $\mathscr{B}\times\mathbb{R}$ to $\mathscr{V}^{1,\alpha}(\partial_+\Omega)\times\mathscr{C}^{1,\alpha}(\partial\omega)$. Then the theorem follows by the implicit function theorem (see Deimling [13, Thm. 15.3]) and by Proposition 3.3. □

3.3.2 Macroscopic behavior

Since $\log\varepsilon_1/\log(\varepsilon_1\varepsilon_2)$ has no limit when $\boldsymbol{\varepsilon}\in]\mathbf{0},\boldsymbol{\varepsilon}^{\mathrm{ad}}[$ tends to $\mathbf{0}$, we have to introduce a specific curve of parameters $\boldsymbol{\varepsilon}$. Then, we take a function $\boldsymbol{\varepsilon}(\cdot)$ from $]0,1[$ to $]\mathbf{0},\boldsymbol{\varepsilon}^{\mathrm{ad}}[$ such that assumptions (1.8) and (1.9) hold (cf. Theorem 1.2). In the following Lemma 3.6, we provide a convenient representation for the solution $u_{\boldsymbol{\varepsilon}(\eta)}$.

Lemma 3.6 (Representation formula at macroscopic scale)**.** *Let the assumptions of Theorem 3.5 hold. Let $\boldsymbol{\varepsilon}(\cdot)$ be a function from $]0,1[$ to $]\mathbf{0},\boldsymbol{\varepsilon}^{\mathrm{ad}}[$ such that assumptions* (1.8) *and* (1.9) *hold. Let $\boldsymbol{\delta}(\cdot)$ be as in* (1.10)*. Then*

$$\begin{aligned}
u_{\boldsymbol{\varepsilon}(\eta)}(\mathsf{x})&=u_0(\mathsf{x})-\varepsilon_1(\eta)\varepsilon_2(\eta)\int_{\partial\omega}\nu_\omega(\mathsf{Y})\cdot(\nabla_{\mathsf{y}}G)(\mathsf{x},\varepsilon_1(\eta)\mathsf{p}+\varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{Y})\,g^{\mathrm{i}}(\mathsf{Y})\,d\sigma_{\mathsf{Y}}\\
&+v_G\big[\partial\Omega,\Phi_1\big[\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta)\big]\big](\mathsf{x})\\
&+\int_{\partial\omega}G(\mathsf{x},\varepsilon_1(\eta)\mathsf{p}+\varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{Y})\,\Phi_2\big[\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta)\big](\mathsf{Y})\,d\sigma_{\mathsf{Y}}\\
&+\delta_1(\eta)\Phi_3\big[\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta)\big]\int_{\partial\omega}G(\mathsf{x},\varepsilon_1(\eta)\mathsf{p}+\varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{Y})\,d\sigma_{\mathsf{Y}}
\end{aligned}$$

for all $\mathsf{x}\in\Omega_{\boldsymbol{\varepsilon}(\eta)}$ and for all $\eta\in]0,1[$ such that $(\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta))\in]\mathbf{0},\boldsymbol{\varepsilon}^*[\times\mathcal{V}_\lambda$.

As a consequence of this representation formula, $u_{\boldsymbol{\varepsilon}(\eta)}(\mathsf{x})$ can be written as a converging power series of four real variables evaluated at $\big(\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta)\big)$ for η positive and small. A similar result holds for the restrictions $u_{\boldsymbol{\varepsilon}(\eta)|\overline{\Omega'}}$ to any open subset Ω' of Ω such that $0\notin\overline{\Omega'}$. Namely, we are now in the position to prove Theorem 1.2.

Proof of Theorem 1.2. Let $\boldsymbol{\varepsilon}^*$ and $\mathcal{V}_\lambda$ be as in Theorem 3.5. We take $\boldsymbol{\varepsilon}'\in]\mathbf{0},\boldsymbol{\varepsilon}^*[$ such that (1.11) holds true. Then, we define

$$\begin{aligned}\mathfrak{U}_{\Omega'}[\boldsymbol{\varepsilon},\boldsymbol{\delta}](\mathsf{x})\equiv{}& u_0(\mathsf{x})-\varepsilon_1\varepsilon_2\int_{\partial\omega}\nu_\omega(\mathsf{Y})\cdot(\nabla_{\mathsf{y}}G)(\mathsf{x},\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{Y})\,g^{\mathrm{i}}(\mathsf{Y})\,d\sigma_{\mathsf{Y}}\\&+v_G[\partial\Omega,\Phi_1[\boldsymbol{\varepsilon},\boldsymbol{\delta}]](\mathsf{x})\\&+\int_{\partial\omega}G(\mathsf{x},\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{Y})\,\Phi_2[\boldsymbol{\varepsilon},\boldsymbol{\delta}](\mathsf{Y})\,d\sigma_{\mathsf{Y}}\\&+\delta_1\Phi_3[\boldsymbol{\varepsilon},\boldsymbol{\delta}]\int_{\partial\omega}G(\mathsf{x},\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{Y})\,d\sigma_{\mathsf{Y}}\end{aligned}$$

for all $\mathsf{x}\in\overline{\Omega'}$ and for all $(\boldsymbol{\varepsilon},\boldsymbol{\delta})\in]-\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}'[\times\mathcal{V}_\lambda$. By Theorem 3.5 and by a standard argument (see in the proof of Proposition 3.3 the argument used to study $\mathcal{L}_2$), we can show that $\mathfrak{U}_{\Omega'}$ is real analytic from $]-\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}'[\times\mathcal{V}_\lambda$ to $\mathscr{C}^{1,\alpha}(\overline{\Omega'})$. The validity of (1.12) follows by Remark 3.6 and the validity of (1.13) is deduced by Proposition 3.4, by Theorem 3.5 and a straightforward computation. □

3.3.3 Microscopic behavior

We now present a representation formula for $u_{\boldsymbol{\varepsilon}(\eta)}(\varepsilon_1(\eta)\mathsf{p}+\varepsilon_1(\eta)\varepsilon_2(\eta)\cdot)$, *i.e.*, for what we call the microscopic behavior of $u_{\boldsymbol{\varepsilon}(\eta)}$.

Lemma 3.7 (Representation formula at microscopic scale)**.** *Let the assumptions of Theorem 3.5 hold. Let $\boldsymbol{\varepsilon}(\cdot)$ be a function from $]0,1[$ to $]\mathbf{0},\boldsymbol{\varepsilon}^{\mathrm{ad}}[$ such that assumptions (1.8) and (1.9) hold. Let $\boldsymbol{\delta}(\cdot)$ be as in (1.10). Then*

$$\begin{aligned}&u_{\boldsymbol{\varepsilon}(\eta)}(\varepsilon_1(\eta)\mathsf{p}+\varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{X})=u_0(\varepsilon_1(\eta)\mathsf{p}+\varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{X})-w_S^e[\partial\omega,g^{\mathrm{i}}](\mathsf{X})\\&\quad-\varepsilon_2(\eta)\int_{\partial\omega}\nu_\omega(\mathsf{Y})\cdot\nabla S\Big(-2p_2\mathsf{e}_2+\varepsilon_2(\eta)(\varsigma(\mathsf{X})-\mathsf{Y})\Big)\,g^{\mathrm{i}}(\mathsf{Y})\,d\sigma_{\mathsf{Y}}\\&\quad+\int_{\partial_+\Omega}G(\varepsilon_1(\eta)\mathsf{p}+\varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{X},\mathsf{y})\,\Phi_1\big[\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta)\big](\mathsf{y})\,d\sigma_{\mathsf{y}}\\&\quad+v_S\big[\partial\omega,\Phi_2\big[\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta)\big]\big](\mathsf{X})\\&\quad-\int_{\partial\omega}S\Big(-2p_2\mathsf{e}_2+\varepsilon_2(\eta)(\varsigma(\mathsf{X})-\mathsf{Y})\Big)\,\Phi_2\big[\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta)\big](\mathsf{Y})\,d\sigma_{\mathsf{Y}}\\&\quad+\rho_\omega\big(1-\delta_2(\eta)\big)\Phi_3\big[\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta)\big]\\&\quad+\delta_1(\eta)\int_{\partial\omega}\Big(S(\mathsf{X}-\mathsf{Y})-S\Big(-2p_2\mathsf{e}_2+\varepsilon_2(\eta)(\varsigma(\mathsf{X})-\mathsf{Y})\Big)\Big)\,d\sigma_{\mathsf{Y}}\Phi_3\big[\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta)\big]\end{aligned}$$

for all $\mathsf{X}\in\mathbb{R}^2\setminus\omega$ and for all $\eta\in]0,1[$ such that $(\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta))\in]\mathbf{0},\boldsymbol{\varepsilon}^*[\times\mathcal{V}_\lambda$ and such that $\varepsilon_1(\eta)\mathsf{p}+\varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{X}\in\overline{\Omega_{\boldsymbol{\varepsilon}(\eta)}}$.

We now show that $u_{\boldsymbol{\varepsilon}(\eta)}(\varepsilon_1(\eta)\mathsf{p}+\varepsilon_1(\eta)\varepsilon_2(\eta)\cdot)$ for η close to 0 can be expressed as a real analytic map evaluated at $\big(\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta)\big)$.

Theorem 3.8. *Let the assumptions of Theorem 3.5 hold. Let ω' be an open bounded subset of $\mathbb{R}^2\setminus\overline{\omega}$ and let $\boldsymbol{\varepsilon}''\in]\mathbf{0},\boldsymbol{\varepsilon}^*[$ be such that*

$$(\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\overline{\omega'})\subseteq\mathcal{B}(0,r_1)\qquad\forall\boldsymbol{\varepsilon}\in]-\boldsymbol{\varepsilon}'',\boldsymbol{\varepsilon}''[\,.$$

Then there is a real analytic map

$$\mathfrak{V}_{\omega'}:]-\boldsymbol{\varepsilon}'',\boldsymbol{\varepsilon}''[\times\mathcal{V}_\lambda\to\mathscr{C}^{1,\alpha}(\overline{\Omega'})\,.$$

such that

$$u_{\boldsymbol{\varepsilon}(\eta)}(\varepsilon_1(\eta)\mathsf{p}+\varepsilon_1(\eta)\varepsilon_2(\eta)\cdot)_{|\overline{\omega'}}=\mathfrak{V}_{\omega'}\left[\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta)\right],\qquad\forall\eta\in]0,\eta''[\,,\tag{3.10}$$

where the latter equality holds for all functions $\boldsymbol{\varepsilon}(\cdot)$ from $]0,1[$ to $]\mathbf{0},\boldsymbol{\varepsilon}^{\mathrm{ad}}[$ which satisfy (1.8) – (1.9) and with $\boldsymbol{\delta}(\cdot)$ as in (1.10), and for all $\eta''\in]0,1[$ small enough so that

$$\big(\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta)\big)\in]\mathbf{0},\boldsymbol{\varepsilon}''[\times\mathcal{V}_\lambda\qquad\forall\eta\in]0,\eta''[\,.$$

Moreover we have

$$\mathfrak{V}_{\omega'}[\mathbf{0},(0,\lambda)]=v_{0|\overline{\omega'}}\tag{3.11}$$

where $v_0\in\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2\setminus\omega)$ is the unique solution of (1.15).

Proof. We define

$$\begin{aligned}\mathfrak{V}_{\omega'}[\boldsymbol{\varepsilon},\boldsymbol{\delta}](\mathsf{X})\equiv{}&U_0(\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{X})-w_S^{\mathrm{e}}[\partial\omega,g^{\mathrm{i}}](\mathsf{X})\\&-\varepsilon_2\int_{\partial\omega}\nu_\omega(\mathsf{Y})\cdot\nabla S(-2p_2\mathsf{e}_2+\varepsilon_2(\varsigma(\mathsf{X})-\mathsf{Y}))\,g^{\mathrm{i}}(\mathsf{Y})\,d\sigma_{\mathsf{Y}}\\&+\int_{\partial_+\Omega}G(\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{X},\mathsf{y})\,\Phi_1[\boldsymbol{\varepsilon},\boldsymbol{\delta}](\mathsf{y})\,d\sigma_{\mathsf{y}}\\&+v_S[\partial\omega,\Phi_2[\boldsymbol{\varepsilon},\boldsymbol{\delta}]](\mathsf{X})\\&-\int_{\partial\omega}S(-2p_2\mathsf{e}_2+\varepsilon_2(\varsigma(\mathsf{X})-\mathsf{Y}))\,\Phi_2[\boldsymbol{\varepsilon},\boldsymbol{\delta}](\mathsf{Y})\,d\sigma_{\mathsf{Y}}\\&+\rho_\omega(1-\delta_2)\Phi_3[\boldsymbol{\varepsilon},\boldsymbol{\delta}]\\&+\delta_1\int_{\partial\omega}(S(\mathsf{X}-\mathsf{Y})-S(-2p_2\mathsf{e}_2+\varepsilon_2(\varsigma(\mathsf{X})-\mathsf{Y})))\,d\sigma_{\mathsf{Y}}\,\Phi_3[\boldsymbol{\varepsilon},\boldsymbol{\delta}]\end{aligned}$$

for all $\mathsf{X}\in\overline{\omega'}$ and for all $(\boldsymbol{\varepsilon},\boldsymbol{\delta})\in]-\boldsymbol{\varepsilon}'',\boldsymbol{\varepsilon}''[\times\mathcal{V}_\lambda$. Then, by Proposition 1.1 and by a standard argument (see the study of $\mathcal{L}_2$ in the proof of Proposition 3.3) we verify that $\mathfrak{V}_{\omega'}$ is real analytic from $]-\boldsymbol{\varepsilon}'',\boldsymbol{\varepsilon}''[\times\mathcal{V}_\lambda$ to $\mathscr{C}^{1,\alpha}(\overline{\omega'})$. The validity of (3.10) follows by Lemma 3.7. By a straightforward computation and by Proposition 3.4 one verifies that

$$\mathfrak{V}_{\omega'}[\mathbf{0},(0,\lambda)](\mathsf{X})=g^{\mathrm{o}}(0)-w_S^{\mathrm{e}}[\partial\omega,g^{\mathrm{i}}](\mathsf{X})+v_S[\partial\omega,\Phi_2[\mathbf{0},(0,\lambda)]](\mathsf{X})+\rho_\omega(1-\lambda)\,\Phi_3[\mathbf{0},(0,\lambda)]\tag{3.12}$$

for all $\mathsf{X}\in\overline{\omega'}$. Then, we deduce that the right hand side of (3.12) equals g^{i} on $\partial\omega$ by Proposition 3.4 and by the jump properties of the double layer potential. Hence, by the decaying properties at ∞ of the single and double layer potentials and by the uniqueness of the solution of the exterior Dirichlet problem, we deduce the validity of (3.11). □

3.3.4 Energy integral

We turn to consider the behavior of the energy integral $\int_{\Omega_{\varepsilon(\eta)}} \left|\nabla u_{\varepsilon(\eta)}\right|^2 \, d\mathsf{x}$ for η close to 0.

Theorem 3.9. *Let the assumptions of Theorem 3.5 hold. Then there exist $\varepsilon^{\mathfrak{E}} \in]\mathbf{0}, \varepsilon^*[$ and a real analytic function*

$$\mathfrak{E} :] - \varepsilon^{\mathfrak{E}}, \varepsilon^{\mathfrak{E}}[\times \mathcal{V}_\lambda \to \mathbb{R}$$

such that

$$\int_{\Omega_{\varepsilon(\eta)}} \left|\nabla u_{\varepsilon(\eta)}\right|^2 \, d\mathsf{x} = \mathfrak{E}\big(\varepsilon(\eta), \boldsymbol{\delta}(\eta)\big) \,, \qquad \forall \eta \in]0, \eta^{\mathfrak{E}}[\,, \tag{3.13}$$

where the latter equality holds for all functions $\varepsilon(\cdot)$ from $]0,1[$ to $]\mathbf{0}, \varepsilon^{\mathrm{ad}}[$ which satisfy (1.8) – (1.9) and with $\boldsymbol{\delta}(\cdot)$ as in (1.10), and for all $\eta^{\mathfrak{E}} \in]0,1[$ such that

$$\big(\varepsilon(\eta), \boldsymbol{\delta}(\eta)\big) \in]\mathbf{0}, \varepsilon^{\mathfrak{E}}[\times \mathcal{V}_\lambda \,, \qquad \forall \eta \in]0, \eta^{\mathfrak{E}}[\,.$$

In addition,

$$\mathfrak{E}(\mathbf{0}, (0,\lambda)) = \int_\Omega |\nabla u_0|^2 \, d\mathsf{x} + \int_{\mathbb{R}^2 \setminus \omega} |\nabla v_0|^2 \, d\mathsf{x} \,. \tag{3.14}$$

Proof. By the divergence theorem and by (1.1) we have

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, d\mathsf{x} &= \int_{\partial\Omega} u_\varepsilon \, \nu_\Omega \cdot \nabla u_\varepsilon \, d\sigma - \int_{\partial\omega_\varepsilon} u_\varepsilon \, \nu_{\omega_\varepsilon} \cdot \nabla u_\varepsilon \, d\sigma \\ &= \int_{\partial\Omega} g^{\mathrm{o}} \, \nu_\Omega \cdot \nabla u_\varepsilon \, d\sigma - \int_{\partial\omega_\varepsilon} g^{\mathrm{i}}\Big(\frac{\mathsf{x} - \varepsilon_1 \mathsf{p}}{\varepsilon_1 \varepsilon_2}\Big) \nu_{\omega_\varepsilon}(\mathsf{x}) \cdot \nabla u_\varepsilon(\mathsf{x}) \, d\sigma_{\mathsf{x}} \end{aligned} \tag{3.15}$$

for all $\varepsilon \in]\mathbf{0}, \varepsilon^{\mathrm{ad}}[$. Then we take a function $\varepsilon(\cdot)$ from $]0,1[$ to $]\mathbf{0}, \varepsilon^{\mathrm{ad}}[$ and a function $\boldsymbol{\delta}(\cdot)$ from $]0,1[$ to $\mathbb{R}^2$ which satisfy (1.8) – (1.10). By Lemma 3.6 we have

$$\int_{\partial\Omega} g^{\mathrm{o}} \, \nu_\Omega \cdot \nabla u_{\varepsilon(\eta)} \, d\sigma = I_{1,\eta} + \varepsilon_1(\eta)\varepsilon_2(\eta) I_{2,\eta} + I_{3,\eta} + \delta_1(\eta) I_{4,\eta} \tag{3.16}$$

for all $\eta \in]0,1[$ such that $(\varepsilon(\eta), \boldsymbol{\delta}(\eta)) \in]\mathbf{0}, \varepsilon^*[\times \mathcal{V}_\lambda$, where

$$\begin{aligned} I_{1,\eta} &= \int_{\partial\Omega} g^{\mathrm{o}}(\mathsf{x}) \, \nu_\Omega \cdot \nabla u_0 \, d\sigma + \int_{\partial\Omega} g^{\mathrm{o}}(\mathsf{x}) \, \nu_\Omega \cdot \nabla v_G\big[\partial\Omega, \Phi_1\big[\varepsilon(\eta), \boldsymbol{\delta}(\eta)\big]\big] \, d\sigma \,, \qquad (3.17) \\ I_{2,\eta} &= -\int_{\partial\Omega} g^{\mathrm{o}}(\mathsf{x}) \, \nu_\Omega(\mathsf{x}) \cdot \nabla_{\mathsf{x}} \int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot (\nabla_{\mathsf{y}} G)(\mathsf{x}, \varepsilon_1(\eta)\mathsf{p} + \varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{Y}) \, g^{\mathrm{i}}(\mathsf{Y}) \, d\sigma_{\mathsf{Y}} \, d\sigma_{\mathsf{x}} \,, \\ I_{3,\eta} &= \int_{\partial\Omega} g^{\mathrm{o}}(\mathsf{x}) \, \nu_\Omega(\mathsf{x}) \cdot \nabla_{\mathsf{x}} \int_{\partial\omega} G(\mathsf{x}, \varepsilon_1(\eta)\mathsf{p} + \varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{Y}) \, \Phi_2\big[\varepsilon(\eta), \boldsymbol{\delta}(\eta)\big](\mathsf{Y}) \, d\sigma_{\mathsf{Y}} \, d\sigma_{\mathsf{x}} \,, \\ I_{4,\eta} &= \Phi_3\big[\varepsilon(\eta), \boldsymbol{\delta}(\eta)\big] \int_{\partial\Omega} g^{\mathrm{o}}(\mathsf{x}) \, \nu_\Omega(\mathsf{x}) \cdot \nabla_{\mathsf{x}} \int_{\partial\omega} G(\mathsf{x}, \varepsilon_1(\eta)\mathsf{p} + \varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{Y}) \, d\sigma_{\mathsf{Y}} \, d\sigma_{\mathsf{x}} \,. \end{aligned}$$

By the Fubini theorem and by (2.1) it follows that

$$\begin{aligned} I_{2,\eta} &= -\int_{\partial\omega} g^{\mathrm{i}}(\mathsf{Y}) \, \nu_\omega(\mathsf{Y}) \cdot \nabla_{\mathsf{y}} \left(\int_{\partial\Omega} g^{\mathrm{o}}(\mathsf{x}) \, \nu_\Omega(\mathsf{x}) \cdot \nabla_{\mathsf{x}} G(\mathsf{y}, \mathsf{x}) \, d\sigma_{\mathsf{x}} \right)_{\mathsf{y} = \varepsilon_1(\eta)\mathsf{p} + \varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{Y}} d\sigma_{\mathsf{Y}} \\ I_{3,\eta} &= \int_{\partial\omega} \Phi_2\big[\varepsilon(\eta), \boldsymbol{\delta}(\eta)\big](\mathsf{Y}) \int_{\partial\Omega} g^{\mathrm{o}}(\mathsf{x}) \nu_\Omega(\mathsf{x}) \cdot \nabla_{\mathsf{x}} G(\varepsilon_1(\eta)\mathsf{p} + \varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{Y}, \mathsf{x}) \, d\sigma_{\mathsf{x}} \, d\sigma_{\mathsf{Y}} \\ I_{4,\eta} &= \delta_1(\eta)\Phi_3\big[\varepsilon(\eta), \boldsymbol{\delta}(\eta)\big] \int_{\partial\omega} \int_{\partial\Omega} g^{\mathrm{o}}(\mathsf{x}) \, \nu_\Omega(\mathsf{x}) \cdot \nabla_{\mathsf{x}} G(\mathsf{x}, \varepsilon_1(\eta)\mathsf{p} + \varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{Y}) \, d\sigma_{\mathsf{x}} \, d\sigma_{\mathsf{Y}} \end{aligned}$$

and, by the definition of the double layer potential derived by G (cf. Definition 2.6) and by (2.2), we deduce that

$$
\begin{aligned}
I_{2,\eta} &= -\int_{\partial\omega} g^{\mathrm{i}}(\mathsf{Y})\,\nu_\omega(\mathsf{Y})\cdot\nabla w_G[\partial_+\Omega, g^{\mathrm{o}}](\varepsilon_1(\eta)\mathsf{p}+\varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{Y})\,d\sigma_{\mathsf{Y}}\\
I_{3,\eta} &= \int_{\partial\omega}\Phi_2\big[\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta)\big](\mathsf{Y})\,w_G[\partial_+\Omega, g^{\mathrm{o}}](\varepsilon_1(\eta)\mathsf{p}+\varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{Y})\,d\sigma_{\mathsf{Y}} \qquad (3.18)\\
I_{4,\eta} &= \Phi_3\big[\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta)\big]\int_{\partial\omega} w_G[\partial_+\Omega, g^{\mathrm{o}}](\varepsilon_1(\eta)\mathsf{p}+\varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{Y})\,d\sigma_{\mathsf{Y}}
\end{aligned}
$$

for all $\eta\in]0,1[$ such that $(\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta))\in]\mathbf{0},\boldsymbol{\varepsilon}^*[\times\mathcal{V}_\lambda$.

Now we choose a specific domain ω' which satisfies the conditions in Theorem 3.8 and which in addition contains the boundary of ω in its closure, namely such that $\partial\omega\subseteq\overline{\omega'}$. Then, for such ω', we take $\boldsymbol{\varepsilon}^{\mathfrak{E}}\equiv\boldsymbol{\varepsilon}''$ with $\boldsymbol{\varepsilon}''$ as in Theorem 3.8. By (4.3) and by a change of variable in the integral, we have

$$
\int_{\partial\omega_{\boldsymbol{\varepsilon}}} g^{\mathrm{i}}\Big(\frac{\mathsf{x}-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\Big)\nu_{\omega_{\boldsymbol{\varepsilon}}}(\mathsf{x})\cdot\nabla u_{\boldsymbol{\varepsilon}}(\mathsf{x})\,d\sigma_{\mathsf{x}} = \int_{\partial\omega} g^{\mathrm{i}}\,\nu_\omega\cdot\nabla\mathfrak{V}_{\omega'}[\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta)]\,d\sigma \qquad (3.19)
$$

for all $\eta\in]0,1[$ such that $(\boldsymbol{\varepsilon}(\eta),\boldsymbol{\delta}(\eta))\in]\mathbf{0},\boldsymbol{\varepsilon}^{\mathfrak{E}}[\times\mathcal{V}_\lambda$.

Then we define

$$
\begin{aligned}
\mathfrak{E}_1(\boldsymbol{\varepsilon},\boldsymbol{\delta}) &\equiv \int_{\partial\Omega} g^{\mathrm{o}}\,\nu_\Omega\cdot\nabla(u_0+v_G[\partial_+\Omega,\Phi_1[\boldsymbol{\varepsilon},\boldsymbol{\delta}]])\,d\sigma\,,\\
\mathfrak{E}_2(\boldsymbol{\varepsilon},\boldsymbol{\delta}) &\equiv -\int_{\partial\omega} g^{\mathrm{i}}(\mathsf{Y})\,\nu_\omega(\mathsf{Y})\cdot\nabla w_G[\partial_+\Omega,g^{\mathrm{o}}](\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{Y})\,d\sigma_{\mathsf{Y}}\,,\\
\mathfrak{E}_3(\boldsymbol{\varepsilon},\boldsymbol{\delta}) &\equiv \int_{\partial\omega}\Phi_2[\boldsymbol{\varepsilon},\boldsymbol{\delta}](\mathsf{Y})\,w_G[\partial_+\Omega,g^{\mathrm{o}}](\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{Y})\,d\sigma_{\mathsf{Y}}\,,\\
\mathfrak{E}_4(\boldsymbol{\varepsilon},\boldsymbol{\delta}) &\equiv \Phi_3[\boldsymbol{\varepsilon},\boldsymbol{\delta}]\int_{\partial\omega} w_G[\partial_+\Omega,g^{\mathrm{o}}](\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\mathsf{Y})\,d\sigma_{\mathsf{Y}}\,,\\
\mathfrak{E}_5(\boldsymbol{\varepsilon},\boldsymbol{\delta}) &\equiv -\int_{\partial\omega} g^{\mathrm{i}}\,\nu_\omega\cdot\nabla\mathfrak{V}_{\omega'}[\boldsymbol{\varepsilon},\boldsymbol{\delta}]\,d\sigma
\end{aligned}
$$

and

$$
\mathfrak{E}(\boldsymbol{\varepsilon},\boldsymbol{\delta})\equiv\mathfrak{E}_1(\boldsymbol{\varepsilon},\boldsymbol{\delta})+\varepsilon_1\varepsilon_2\mathfrak{E}_2(\boldsymbol{\varepsilon},\boldsymbol{\delta})+\mathfrak{E}_3(\boldsymbol{\varepsilon},\boldsymbol{\delta})+\delta_1\mathfrak{E}_4(\boldsymbol{\varepsilon},\boldsymbol{\delta})+\mathfrak{E}_5(\boldsymbol{\varepsilon},\boldsymbol{\delta}) \qquad (3.20)
$$

for all $(\boldsymbol{\varepsilon},\boldsymbol{\delta})\in]-\boldsymbol{\varepsilon}^{\mathfrak{E}},\boldsymbol{\varepsilon}^{\mathfrak{E}}[\times\mathcal{V}_\lambda$. Then the validity (3.13) follows by (3.15)–(3.19). In addition, by Theorems 3.5 and 3.8, by Lemma 2.14, and by a standard argument (see in the proof of Theorem 3.3 the study of $\mathfrak{L}_2$), we can prove that the $\mathfrak{E}_i$'s are real analytic from $]-\boldsymbol{\varepsilon}^{\mathfrak{E}},\boldsymbol{\varepsilon}^{\mathfrak{E}}[$ to $\mathbb{R}$. Hence $\mathfrak{E}$ is real analytic from $]-\boldsymbol{\varepsilon}^{\mathfrak{E}},\boldsymbol{\varepsilon}^{\mathfrak{E}}[$ to $\mathbb{R}$.

To complete the proof we have to verify (3.14). We begin by observing that $\Phi_1[\mathbf{0},(0,\lambda)]=0$ by Proposition 3.4 and Theorem 3.5. Thus

$$
\mathfrak{E}_1(\mathbf{0},(0,\lambda))=\int_{\partial\Omega} g^{\mathrm{o}}\,\nu_\Omega\cdot\nabla u_0\,d\sigma=\int_\Omega|\nabla u_0|^2\,d\mathsf{x}\,. \qquad (3.21)
$$

By Lemma 2.7, we have $w_G[\partial_+\Omega,g^{\mathrm{o}}](0)=g^{\mathrm{o}}(0)$. Since $\Phi_2[\mathbf{0},(0,\lambda)]$ belongs to $\mathscr{C}^{1,\alpha}_{\#}(\partial\omega)$, we compute

$$
\mathfrak{E}_3(\mathbf{0},(0,\lambda))=g^{\mathrm{o}}(0)\int_{\partial\omega}\Phi_2[\mathbf{0},(0,\lambda)]\,d\sigma=0\,. \qquad (3.22)
$$

Then, by (3.11) and by the divergence theorem, we have

$$\mathfrak{E}_4(\mathbf{0},(0,\lambda)) = -\int_{\partial\omega} g^{\mathrm{i}}\, \nu_\omega \cdot \nabla v_0 \, d\sigma = \int_{\mathbb{R}^2\setminus\omega} |\nabla v_0|^2 \, d\mathsf{x}\,. \tag{3.23}$$

We conclude by (3.20) – (3.23). □

Finally, in the following Theorem 3.10 where we consider the total flux on $\partial\Omega$.

Theorem 3.10. *Let the assumptions of Theorem 3.5 hold. Then there exist* $\varepsilon^{\mathfrak{F}} \in]\mathbf{0}, \varepsilon^*[$ *and a real analytic function*

$$\mathfrak{F} :]-\varepsilon^{\mathfrak{F}}, \varepsilon^{\mathfrak{F}}[\times \mathcal{V}_\lambda \to \mathbb{R}$$

such that

$$\int_{\partial\Omega} \nu_\Omega \cdot \nabla u_{\varepsilon(\eta)} \, d\sigma = \mathfrak{F}\big(\varepsilon(\eta), \boldsymbol{\delta}(\eta)\big)\,, \qquad \forall \eta \in]0, \eta^{\mathfrak{F}}[$$

where the latter equality holds for all functions $\varepsilon(\cdot)$ *from* $]0,1[$ *to* $]\mathbf{0}, \varepsilon^{\mathrm{ad}}[$ *which satisfy* (1.8) – (1.9) *and with* $\boldsymbol{\delta}(\cdot)$ *as in* (1.10)*, and for all* $\eta^{\mathfrak{F}} \in]0,1[$ *such that*

$$\big(\varepsilon(\eta), \boldsymbol{\delta}(\eta)\big) \in]\mathbf{0}, \varepsilon^{\mathfrak{F}}[\times \mathcal{V}_\lambda\,, \qquad \forall \eta \in]0, \eta^{\mathfrak{F}}[\,. \tag{3.24}$$

In addition,

$$\mathfrak{F}(\mathbf{0},(0,\lambda)) = 0\,.$$

Proof. Let $\varepsilon(\cdot)$ from $]0,1[$ to $]\mathbf{0}, \varepsilon^{\mathrm{ad}}[$ and $\boldsymbol{\delta}(\cdot)$ from $]0,1[$ to $\mathbb{R}^2$ which satisfy (1.8) – (1.10). Then by the divergence theorem we have

$$\int_{\partial\Omega} \nu_\Omega \cdot \nabla u_{\varepsilon(\eta)} \, d\sigma = \int_{\partial\omega_\varepsilon} \nu_{\omega_\varepsilon} \cdot \nabla u_{\varepsilon(\eta)} \, d\sigma$$

for all $\eta \in]0,1[$ such that $(\varepsilon(\eta), \boldsymbol{\delta}(\eta)) \in]\mathbf{0}, \varepsilon^*[\times \mathcal{V}_\lambda$. Then we take ω' which satisfies the conditions in Theorem 3.8 and such that $\partial\omega \subseteq \overline{\omega'}$. Then, for such ω', we take $\varepsilon^{\mathfrak{F}} \equiv \varepsilon''$ with ε'' as in Theorem 3.8 and we deduce that

$$\int_{\partial\Omega} \nu_\Omega \cdot \nabla u_{\varepsilon(\eta)} \, d\sigma = \int_{\partial\omega} \nu_\omega \cdot \nabla \mathfrak{V}_{\omega'}[\varepsilon(\eta), \boldsymbol{\delta}(\eta)] \, d\sigma$$

for all $\eta \in]0,1[$ such that $(\varepsilon(\eta), \boldsymbol{\delta}(\eta)) \in]\mathbf{0}, \varepsilon^{\mathfrak{F}}[\times \mathcal{V}_\lambda$. Accordingly, we define

$$\mathfrak{F}(\varepsilon, \boldsymbol{\delta}) \equiv \int_{\partial\omega} \nu_\omega \cdot \nabla \mathfrak{V}_{\omega'}[\varepsilon, \boldsymbol{\delta}] \, d\sigma\,, \qquad \forall (\varepsilon, \boldsymbol{\delta}) \in]-\varepsilon^{\mathfrak{F}}, \varepsilon^{\mathfrak{F}}[\times \mathcal{V}_\lambda\,.$$

Then the equality (3.24) holds true. By Theorem 3.8, one deduces that $\mathfrak{F}$ is real analytic from $]-\varepsilon^{\mathfrak{F}}, \varepsilon^{\mathfrak{F}}[\times \mathcal{V}_\lambda$ to $\mathbb{R}$. Finally, by (3.11) we have

$$\mathfrak{F}(\mathbf{0},(0,\lambda)) = \int_{\partial\omega} \nu_\omega \cdot \nabla \mathfrak{V}_{\omega'}[\mathbf{0},(0,\lambda)] \, d\sigma = \int_{\partial\omega} \nu_\omega \cdot \nabla v_0 \, d\sigma$$

and the latter integral vanishes because of v_0 is harmonic at infinity (see (1.15)). □

4 Behavior of the solution of (1.1) for ε_1 close to 0 and $\varepsilon_2 = 1$

As noticed in the beginning of Section 3, when studying singular perturbation problems in perforated domains in the two-dimensional plane one would expect to have some logarithmic terms in the asymptotic formulas. Such logarithmic terms are generated by the specific behavior of the fundamental solution upon rescaling (cf. equality (3.1)). However, for our problem there will be no logarithmic term when $\varepsilon_2 = 1$ is fixed and we just consider the dependence upon ε_1. Indeed, for $\varepsilon_2 = 1$, we have

$$S(\varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2 \mathsf{X}) = S(\mathsf{p} - \mathsf{X}) + \frac{\log \varepsilon_1}{2\pi}$$

and thus

$$G(\varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2 \mathsf{X}, \varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2 \mathsf{Y}) = S(\mathsf{X} - \mathsf{Y}) - S(-2p_2 \mathsf{e}_2 + (\varsigma(\mathsf{X}) - \mathsf{Y}))$$

for all $\varepsilon_1 > 0$. Accordingly the rescaling of G gives rise to no logarithmic term.

Since we are dealing here with a one parameter problem, we find convenient to take $\varepsilon \equiv \varepsilon_1$, $\varepsilon^{\mathrm{ad}} \equiv \varepsilon_1^{\mathrm{ad}}$, $\Omega_\varepsilon \equiv \Omega_{\varepsilon_1,1}$, $\omega_\varepsilon \equiv \omega_{\varepsilon_1,1}$, and $u_\varepsilon \equiv u_{\varepsilon_1,1}$ for all $\varepsilon \in]-\varepsilon^{\mathrm{ad}}, \varepsilon^{\mathrm{ad}}[$.

4.1 Defining the operator Φ

Let $\varepsilon \in]-\varepsilon^{\mathrm{ad}}, \varepsilon^{\mathrm{ad}}[$. By Proposition 3.2 we can look for solutions of problem (1.1) under the form

$$w_G^i[\partial\Omega_\varepsilon, u_{\varepsilon|\partial\Omega_\varepsilon}] + v_G[\partial\Omega, \phi_1] + v_G[\partial\omega_\varepsilon, \phi_2] + v_G[\partial\omega_\varepsilon, 1]\,\xi$$

for suitable $(\phi_1, \phi_2, \xi) \in \mathscr{C}_+^{0,\alpha}(\partial\Omega) \times \mathscr{C}_\#^{0,\alpha}(\partial\omega_\varepsilon) \times \mathbb{R}$. We split the integral on $\partial\Omega_\varepsilon$ as the sum of integrals on $\partial\Omega$ and on $\partial\omega_\varepsilon$, we add and subtract $v_G^i[\partial\Omega, \nu_\Omega \cdot \nabla u_{0|\partial\Omega}]$ to obtain the new form

$$\begin{aligned} w_G^i[\partial\Omega, g^{\mathrm{o}}] - v_G^i[\partial\Omega, \nu_\Omega \cdot \nabla u_{0|\partial\Omega}] - w_G^e\Big[\partial\omega_\varepsilon, g^{\mathrm{i}}\big(\frac{\cdot - \varepsilon \mathsf{p}}{\varepsilon}\big)\Big] \\ + v_G[\partial\Omega, \phi_1 + \nu_\Omega \cdot \nabla u_{0|\partial\Omega}] + v_G[\partial\omega_\varepsilon, \phi_2] + v_G[\partial\omega_\varepsilon, 1]\,\xi\,. \end{aligned}$$

Since

$$u_0 = w_G^i[\partial\Omega, g^{\mathrm{o}}] - v_G^i[\partial\Omega, \nu_\Omega \cdot \nabla u_{0|\partial\Omega}]\,,$$

we finally look for solutions of (1.1) in the form

$$\begin{aligned} u_0(\mathsf{x}) - \varepsilon \int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot (\nabla_{\mathsf{y}} G)(\mathsf{x}, \varepsilon\mathsf{p} + \varepsilon\mathsf{Y})\, g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}} + v_G[\partial\Omega, \mu_1](\mathsf{x}) \\ + \int_{\partial\omega} G(\mathsf{x}, \varepsilon\mathsf{p} + \varepsilon\mathsf{Y})\, \mu_2(\mathsf{Y})\, d\sigma_{\mathsf{Y}} + \xi \int_{\partial\omega} G(\mathsf{x}, \varepsilon\mathsf{p} + \varepsilon\mathsf{Y})\, d\sigma_{\mathsf{Y}}, \qquad \forall \mathsf{x} \in \Omega_\varepsilon \end{aligned} \tag{4.1}$$

for suitable $(\boldsymbol{\mu}, \xi) \in \mathscr{B} \times \mathbb{R}$ ensuring that the boundary conditions of (1.1) are satisfied (here as in Section 3 we take $\mathscr{B} \equiv \mathscr{C}_+^{0,\alpha}(\partial\Omega) \times \mathscr{C}_\#^{0,\alpha}(\partial\omega)$).

The (extension to $\overline{\Omega_\varepsilon}$ of the) harmonic function in (4.1) solves problem (1.1) if and only if the pair $(\boldsymbol{\mu}, \xi)$ solves

$$\Phi[\varepsilon, \boldsymbol{\mu}, \xi] = 0\,, \tag{4.2}$$

with $\Phi[\varepsilon,\boldsymbol{\mu},\xi] \equiv (\Phi_1[\varepsilon,\boldsymbol{\mu},\xi], \Phi_2[\varepsilon,\boldsymbol{\mu},\xi])$ defined by

$$
\begin{aligned}
\Phi_1[\varepsilon,\boldsymbol{\mu},\xi](\mathsf{x}) \equiv\; & v_G[\partial\Omega,\mu_1](\mathsf{x}) \\
& + \int_{\partial\omega} G(\mathsf{x},\varepsilon\mathsf{p}+\varepsilon\mathsf{Y})\,\mu_2(s)\,d\sigma_{\mathsf{Y}} \\
& + \xi\int_{\partial\omega} G(\mathsf{x},\varepsilon\mathsf{p}+\varepsilon\mathsf{Y})\,d\sigma_{\mathsf{Y}} \\
& - \varepsilon\int_{\partial\omega} \nu_\omega(\mathsf{Y})\cdot(\nabla_{\mathsf{y}}G)(\mathsf{x},\varepsilon\mathsf{p}+\varepsilon\mathsf{Y})\,g^{\mathrm{i}}(\mathsf{Y})\,d\sigma_{\mathsf{Y}}, \qquad \forall \mathsf{x}\in\partial_+\Omega\,, \\
\Phi_2[\varepsilon,\boldsymbol{\mu},\xi](\mathsf{X}) \equiv\; & v_S[\partial\omega,\mu_2](\mathsf{X}) \\
& - \int_{\partial\omega} S(-2p_2\mathsf{e}_2+\varsigma(\mathsf{X})-\mathsf{Y})\,\mu_2(\mathsf{Y})\,d\sigma_{\mathsf{Y}} \\
& + \xi\int_{\partial\omega} (S(\mathsf{X}-\mathsf{Y})-S(-2p_2\mathsf{e}_2+\varsigma(\mathsf{X})-\mathsf{Y}))\,d\sigma_{\mathsf{Y}} \\
& + \int_{\partial_+\Omega} G(\varepsilon\mathsf{p}+\varepsilon\mathsf{X},\mathsf{y})\,\mu_1(\mathsf{y})\,d\sigma_{\mathsf{y}} \\
& - \int_{\partial\omega} \nu_\omega(\mathsf{Y})\cdot\nabla S(-2p_2\mathsf{e}_2+\varsigma(\mathsf{X})-\mathsf{Y})\,g^{\mathrm{i}}(\mathsf{Y})\,d\sigma_{\mathsf{Y}} \\
& + U_0(\varepsilon\mathsf{p}+\varepsilon\mathsf{X}) - w_S[\partial\omega,g^{\mathrm{i}}](\mathsf{X}) - \frac{g^{\mathrm{i}}(\mathsf{X})}{2}, \qquad \forall \mathsf{X}\in\partial\omega\,.
\end{aligned}
$$

Thus, it suffices to find a solution of (4.2) to solve problem (1.1). Therefore, we now analyze the behavior of the solutions of the system of integral equations (4.2).

4.2 Real analyticity of Φ

In the following Proposition 4.1 we state the real analyticity of Φ. We omit the proof, which is a straightforward modification of the proof of Proposition 3.3.

Proposition 4.1 (Real analyticity of Φ)**.** *The map*

$$
\begin{aligned}
]-\varepsilon_0,\varepsilon_0[\times\mathscr{B}\times\mathbb{R} &\to \mathscr{V}^{1,\alpha}(\partial_+\Omega)\times\mathscr{C}^{1,\alpha}(\partial\omega) \\
(\varepsilon,\boldsymbol{\mu},\xi) &\mapsto \Phi[\varepsilon,\boldsymbol{\mu},\xi]
\end{aligned}
$$

is real analytic.

In the sequel we set

$$
\tilde{\omega} \equiv \omega\cup(\varsigma(\omega)-2p_2\mathsf{e}_2)\,.
$$

Then $\tilde{\omega}$ is an open subset of $\mathbb{R}^2$ of class $\mathscr{C}^{1,\alpha}$ with two connected components, ω and $\varsigma(\omega)-2p_2\mathsf{e}_2$, and with boundary $\partial\tilde{\omega}$ consisting of two connected components, $\partial\omega$ and $\partial\varsigma(\omega)-2p_2\mathsf{e}_2$. One can also observe that $\tilde{\omega}$ is symmetric with respect to the horizontal axis $\mathbb{R}\times\{-p_2\}$. Then, for all functions ϕ from $\partial\omega$ to $\mathbb{R}$, we denote by $\tilde{\phi}$ the extension of ϕ to $\partial\tilde{\omega}$ defined by

$$
\tilde{\phi}(\mathsf{X}) \equiv \begin{cases} \phi(\mathsf{X}) & \text{if } \mathsf{X}\in\partial\omega\,, \\ -\phi(\varsigma(\mathsf{X})-2p_2\mathsf{e}_2) & \text{if } \mathsf{X}\in\partial(\varsigma(\omega)-2p_2\mathsf{e}_2)\,. \end{cases}
$$

In particular, the symbol $\tilde{1}$ will denote the function from $\partial\tilde{\omega}$ to $\mathbb{R}$ defined by

$$\tilde{1}(\mathsf{X}) \equiv \begin{cases} 1 & \text{if } \mathsf{X} \in \partial\omega\,, \\ -1 & \text{if } \mathsf{X} \in \partial(\varsigma(\omega) - 2p_2\mathsf{e}_2)\,. \end{cases}$$

If $k \in \mathbb{N}$, then we denote by $\mathscr{C}^{k,\alpha}_{\mathrm{odd}}(\partial\tilde{\omega})$ the subspace of $\mathscr{C}^{k,\alpha}(\partial\tilde{\omega})$ consisting of the functions ψ such that $\psi(\mathsf{X}) = -\psi(\varsigma(\mathsf{X}) - 2p_2\mathsf{e}_2)$ for all $\mathsf{X} \in \partial\tilde{\omega}$. The extensions $\tilde{\phi}$ belongs to $\mathscr{C}^{k,\alpha}_{\mathrm{odd}}(\partial\tilde{\omega})$ for all $\phi \in \mathscr{C}^{k,\alpha}(\partial\omega)$, in particular $\tilde{1} \in \mathscr{C}^{k,\alpha}_{\mathrm{odd}}(\partial\tilde{\omega})$. One can also prove that $v_S[\partial\tilde{\omega},\psi]_{|\partial\tilde{\omega}}$ and $w_S[\partial\tilde{\omega},\theta]_{|\partial\tilde{\omega}}$ belong to $\mathscr{C}^{1,\alpha}_{\mathrm{odd}}(\partial\tilde{\omega})$ for all $\psi \in \mathscr{C}^{0,\alpha}_{\mathrm{odd}}(\partial\tilde{\omega})$ and $\theta \in \mathscr{C}^{1,\alpha}_{\mathrm{odd}}(\partial\tilde{\omega})$.

Then, by classical potential theory we have the following Lemma 4.2.

Lemma 4.2. *The map from $\mathscr{C}^{0,\alpha}_{\#}(\partial\omega) \times \mathbb{R}$ to $\mathscr{C}^{1,\alpha}_{\mathrm{odd}}(\partial\tilde{\omega})$ which takes (μ,ξ) to*

$$v_S[\partial\tilde{\omega},\tilde{\mu}]_{|\partial\tilde{\omega}} + \xi\, v_S[\partial\tilde{\omega},\tilde{1}]_{|\partial\tilde{\omega}}$$

is an isomorphism.

Proof. By Lemma 2.5 the map which takes (μ,ξ) to $v_S[\partial\tilde{\omega},\mu]_{|\partial\tilde{\omega}} + \xi$ is an isomorphism from $\mathscr{C}^{0,\alpha}_{\#}(\partial\tilde{\omega}) \times \mathbb{R}$ to $\mathscr{C}^{1,\alpha}(\partial\tilde{\omega})$. Then the map from $\mathscr{C}^{0,\alpha}_{\mathrm{odd}}(\partial\tilde{\omega})$ to $\mathscr{C}^{1,\alpha}_{\mathrm{odd}}(\partial\tilde{\omega})$ which takes μ to $v_S[\partial\tilde{\omega},\tilde{\mu}]_{|\partial\tilde{\omega}}$ is an isomorphism. One concludes by observing that the map from $\mathscr{C}^{0,\alpha}_{\#}(\partial\omega) \times \mathbb{R}$ to $\mathscr{C}^{0,\alpha}_{\mathrm{odd}}(\partial\tilde{\omega})$ which takes (μ,ξ) to $\tilde{\mu} + \xi\,\tilde{1}$ is an isomorphism. □

4.3 Functional analytic representation theorems

As intermediate step for studying the dependence of (4.2) around the degenerate value, we now analyze equation (4.2) for the degenerate value $\varepsilon = 0$.

Proposition 4.3. *There exists a unique $(\boldsymbol{\mu}^*,\xi^*) \in \mathscr{B} \times \mathbb{R}$ such that*

$$\Phi[0,\boldsymbol{\mu}^*,\xi^*] = 0$$

and we have

$$\mu_1^* = 0$$

and

$$v_S[\partial\tilde{\omega},\tilde{\mu}_2^*](\mathsf{X}) + \xi^*\, v_S[\partial\tilde{\omega},\tilde{1}](\mathsf{X}) = -g^{\mathrm{o}}(0)\tilde{1}(\mathsf{X}) + w_S[\partial\tilde{\omega},\tilde{g}^{\mathrm{i}}](\mathsf{X}) + \frac{\tilde{g}^{\mathrm{i}}(\mathsf{X})}{2} \quad \forall \mathsf{X} \in \partial\tilde{\omega}\,.$$

Proof. First of all, we observe that for all $(\boldsymbol{\mu},\xi) \in \mathscr{B} \times \mathbb{R}$, we have

$$\begin{cases} \Phi_1[0,\boldsymbol{\mu},\xi](\mathsf{x}) & = v_G[\partial\Omega,\mu_1](\mathsf{x}), \qquad \forall \mathsf{x} \in \partial_+\Omega\,, \\ \Phi_2[0,\boldsymbol{\mu},\xi](\mathsf{X}) & = v_S[\partial\omega,\mu_2](\mathsf{X}) \\ & \quad - \displaystyle\int_{\partial\omega} S(-2p_2\mathsf{e}_2 + \varsigma(\mathsf{X}) - \mathsf{Y})\,\mu_2(\mathsf{Y})\,d\sigma_{\mathsf{Y}} \\ & \quad + \xi \displaystyle\int_{\partial\omega} (S(\mathsf{X}-\mathsf{Y}) - S(-2p_2\mathsf{e}_2 + \varsigma(\mathsf{X}) - \mathsf{Y}))\,d\sigma_{\mathsf{Y}} \\ & \quad - \displaystyle\int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot \nabla S(-2p_2\mathsf{e}_2 + \varsigma(\mathsf{X}) - \mathsf{Y})\,g^{\mathrm{i}}(\mathsf{Y})\,d\sigma_{\mathsf{Y}} \\ & \quad + g^{\mathrm{o}}(0) - w_S[\partial\omega,g^{\mathrm{i}}](\mathsf{X}) - \dfrac{g^{\mathrm{i}}(\mathsf{X})}{2}, \qquad \forall \mathsf{X} \in \partial\omega\,. \end{cases}$$

By Theorem 2.11 (ii), the unique function in $\mathscr{C}^{0,\alpha}_{+}(\partial\Omega)$ such that $v_G[\partial\Omega,\mu_1] = 0$ on $\partial_+\Omega$ is $\mu_1 = 0$. On the other hand, by a change of variable in integrals, one verifies that

$$\Phi_2[0,\boldsymbol{\mu},\xi](\mathsf{X}) = v_S[\partial\tilde\omega,\tilde\mu_2](\mathsf{X}) + \xi\, v_S[\partial\tilde\omega,\tilde 1](\mathsf{X}) + g^{\mathrm{o}}(0)\tilde 1(\mathsf{X}) - w_S[\partial\tilde\omega,\tilde g^{\mathrm{i}}](\mathsf{X}) - \frac{g^{\mathrm{i}}(\mathsf{X})}{2} \quad \forall \mathsf{X}\in\partial\omega\,.$$

Then, by Lemma 4.2, there exists a unique pair $(\mu_2,\xi)\in\mathscr{C}^{0,\alpha}_{\#}(\partial\omega)\times\mathbb{R}$ such that

$$v_S[\partial\tilde\omega,\tilde\mu_2](\mathsf{X}) + \xi\, v_S[\partial\tilde\omega,\tilde 1](\mathsf{X}) = -g^{\mathrm{o}}(0)\tilde 1(\mathsf{X}) + w_S[\partial\tilde\omega,\tilde g^{\mathrm{i}}](\mathsf{X}) + \frac{\tilde g^{\mathrm{i}}(\mathsf{X})}{2} \quad \forall \mathsf{X}\in\partial\tilde\omega\,.$$

Now the statement is proved. □

The main result of this section is obtained by exploiting the implicit function theorem for real analytic maps (see Deimling [13, Thm. 15.3]).

Theorem 4.4. *Let $(\boldsymbol{\mu}^*,\xi^*)$ be as in Proposition 4.3. Then there exist $0<\varepsilon^*<\varepsilon^{\mathrm{ad}}$, an open neighborhood $\mathcal{U}^*$ of $(\boldsymbol{\mu}^*,\xi^*)$ in $\mathscr{B}\times\mathbb{R}$, and a real analytic map $\Psi\equiv(\Psi_1,\Psi_2,\Psi_3)$ from $]-\varepsilon^*,\varepsilon^*[$ to $\mathcal{U}^*$ such that the set of zeros of Φ in $]-\varepsilon^*,\varepsilon^*[\,\times\,\mathcal{U}^*$ coincides with the graph of Ψ.*

Proof. The partial differential of M with respect to $(\boldsymbol{\mu},\xi)$ evaluated at $(0,\boldsymbol{\mu}^*,\xi^*)$ is delivered by

$$\begin{aligned}
&\partial_{(\boldsymbol{\mu},\xi)}\Phi_1[0,\boldsymbol{\mu}^*,\xi^*](\boldsymbol{\phi},\zeta) = v_G[\partial\Omega,\phi_1]_{|\partial_+\Omega}\,,\\
&\partial_{(\boldsymbol{\mu},\xi)}\Phi_2[0,\boldsymbol{\mu}^*,\xi^*](\boldsymbol{\phi},\zeta) = v_S[\partial\tilde\omega,\tilde\phi_2]_{|\partial\omega} + \zeta\, v_S[\partial\tilde\omega,\tilde 1]_{|\partial\omega}
\end{aligned}$$

for all $(\boldsymbol{\phi},\zeta)\in\mathscr{B}\times\mathbb{R}$. Then $\partial_{(\boldsymbol{\mu},\xi)}\Phi[0,\boldsymbol{\mu}^*,\xi^*]$ is an isomorphism from $\mathscr{B}\times\mathbb{R}$ to $\mathscr{V}^{1,\alpha}(\partial_+\Omega)\times\mathscr{C}^{1,\alpha}(\partial\omega)$ thanks to Proposition 2.11 and Lemma 4.2. The conclusion is reached by the implicit function theorem (see Deimling [13, Thm. 15.3]) and by Proposition 4.1. □

4.3.1 Macroscopic behavior

We first provide a representation for the solution u_ε.

Lemma 4.5. *Let the assumptions of Theorem 4.4 hold. Then*

$$\begin{aligned}
u_\varepsilon(\mathsf{x}) = u_0(\mathsf{x}) &- \varepsilon\int_{\partial\omega}\nu_\omega(\mathsf{Y})\cdot(\nabla_{\mathsf{y}}G)(\mathsf{x},\varepsilon\mathsf{p}+\varepsilon\mathsf{Y})\,g^{\mathrm{i}}(\mathsf{Y})\,d\sigma_{\mathsf{Y}} + v_G\big[\partial\Omega,\Psi_1[\varepsilon]\big](\mathsf{x})\\
&+\int_{\partial\omega}G(\mathsf{x},\varepsilon\mathsf{p}+\varepsilon\mathsf{Y})\,\Psi_2[\varepsilon]\,d\sigma_{\mathsf{Y}} + \Psi_3[\varepsilon]\int_{\partial\omega}G(\mathsf{x},\varepsilon\mathsf{p}+\varepsilon\mathsf{Y})\,d\sigma_{\mathsf{Y}}
\end{aligned}$$

for all $\mathsf{x}\in\Omega_\varepsilon$ and for all $\varepsilon\in\,]0,\varepsilon^*[$.

As a consequence of Lemma 4.5, $u_\varepsilon(\mathsf{x})$ can be written in terms of a converging power series of ε for ε positive and small. A similar result holds for the restrictions $u_{\varepsilon|\overline{\Omega'}}$ where Ω' is an open subset of Ω such that $0\notin\overline{\Omega'}$. Namely, we are now in the position to prove Theorem 1.3.

Proof of Theorem 1.3. Let ε_* be as in Theorem 4.4. Let $\varepsilon' \in]0, \varepsilon_*]$ be such that (1.16) holds true. We define

$$\mathfrak{U}_{\Omega'}[\varepsilon](\mathsf{x}) \equiv U_0(\mathsf{x}) - \varepsilon \int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot (\nabla_{\mathsf{y}} G)(\mathsf{x}, \varepsilon\mathsf{p} + \varepsilon\mathsf{Y})\, g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}} + v_G\big[\partial\Omega, \Psi_1[\varepsilon]\big](\mathsf{x}) \\ + \int_{\partial\omega} G(\mathsf{x}, \varepsilon\mathsf{p} + \varepsilon\mathsf{Y})\, \Psi_2[\varepsilon]\, d\sigma_{\mathsf{Y}} + \Psi_3[\varepsilon] \int_{\partial\omega} G(\mathsf{x}, \varepsilon\mathsf{p} + \varepsilon\mathsf{Y})\, d\sigma_{\mathsf{Y}}$$

for all $\mathsf{x} \in \overline{\Omega'}$ and for all $\varepsilon \in]-\varepsilon', \varepsilon'[$. Then, by Theorem 3.5 and by a standard argument (see the study of $\mathfrak{L}_2$ in the proof of Proposition 3.3) one verifies that $\mathfrak{U}_{\Omega'}$ is real analytic from $]-\varepsilon', \varepsilon'[$ to $\mathscr{C}^{1,\alpha}(\overline{\Omega'})$. The validity of (1.17) follows by Remark 4.5 and the validity of (1.18) can be deduced by Proposition 4.3, by Theorem 4.4, and by a straightforward computation. □

4.3.2 Microscopic behavior

As we have done in Remark 3.7 for $\boldsymbol{\varepsilon}$ small, we now present a representation formula for $u_\varepsilon(\varepsilon\mathsf{p} + \varepsilon\,\cdot)$.

Remark 4.6. *Let the assumptions of Theorem 4.4 hold. Then*

$$\begin{aligned} u_\varepsilon(\varepsilon\mathsf{p} + \varepsilon\mathsf{X}) = u_0(\varepsilon\mathsf{p} + \varepsilon\mathsf{X}) - w_S^e[\partial\omega, g^{\mathrm{i}}](\mathsf{X}) - \int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot \nabla S(-2p_2\mathsf{e}_2 + \varsigma(\mathsf{X}) - \mathsf{Y})\, g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \\ + \int_{\partial_+\Omega} G(\varepsilon\mathsf{p} + \varepsilon\mathsf{X}, \mathsf{y})\, \Psi_1[\varepsilon](\mathsf{y})\, d\sigma_{\mathsf{y}} \\ + v_S[\partial\omega, \Psi_2[\varepsilon]](\mathsf{X}) - \int_{\partial\omega} S(-2p_2\mathsf{e}_2 + \varsigma(\mathsf{X}) - \mathsf{Y})\, \Psi_2[\varepsilon]\, d\sigma_{\mathsf{Y}} \\ + \Psi_3[\varepsilon] \int_{\partial\omega} (S(\mathsf{X} - \mathsf{Y}) - S(-2p_2\mathsf{e}_2 + \varsigma(\mathsf{X}) - \mathsf{Y}))\, d\sigma_{\mathsf{Y}} \end{aligned}$$

for all $\mathsf{X} \in \mathbb{R}^2 \setminus \omega$ and for all $\varepsilon \in]0, \varepsilon^*[$ such that $\varepsilon\mathsf{p} + \varepsilon\mathsf{X} \in \overline{\Omega_\varepsilon}$.

We now show that $u_\varepsilon(\varepsilon\mathsf{p} + \varepsilon\,\cdot\,)$ can be expressed as a real analytic map of ε for ε small.

Theorem 4.7. *Let the assumptions of Theorem 4.4 hold. Let ω' be an open bounded subset of $\mathbb{R}^2 \setminus \overline{\omega}$. Let $\varepsilon' \in]0, \varepsilon^*[$ be such that*

$$(\varepsilon\mathsf{p} + \varepsilon\overline{\omega'}) \subseteq \mathcal{B}(0, r_1)\,, \qquad \forall \varepsilon \in]-\varepsilon', \varepsilon'[\,.$$

Then there exists a real analytic map $\mathfrak{V}_{\omega'}$ from $]-\varepsilon', \varepsilon'[$ to $\mathscr{C}^{1,\alpha}(\overline{\Omega'})$ such that

$$u_\varepsilon(\varepsilon\mathsf{p} + \varepsilon\,\cdot\,)_{|\overline{\omega'}} = \mathfrak{V}_{\omega'}[\varepsilon], \qquad \forall \varepsilon \in]0, \varepsilon'[\,. \tag{4.3}$$

Moreover we have

$$\mathfrak{V}_{\omega'}[0] = v_{*|\overline{\omega'}} + g^{\mathrm{o}}(0) \tag{4.4}$$

where $v_* \in \mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \tilde{\omega})$ is the unique solution of

$$\begin{cases} \Delta v_* = 0 & \textit{in } \mathbb{R}^2 \setminus \tilde{\omega}\,, \\ v_* = \tilde{g}^{\mathrm{i}} - \tilde{g}^{\mathrm{o}}(0) & \textit{on } \partial\tilde{\omega}\,, \\ \lim_{\mathsf{X}\to\infty} v_*(\mathsf{X}) = 0\,. \end{cases}$$

Proof. We define

$$\begin{aligned}\mathfrak{V}_{\omega'}[\varepsilon](\mathsf{X}) = {} & U_0(\varepsilon\mathsf{p}+\varepsilon\mathsf{X}) - w_S^e[\partial\omega, g^{\mathrm{i}}](\mathsf{X}) - \int_{\partial\omega} \nu_\omega(\mathsf{Y})\cdot\nabla S(-2p_2\mathsf{e}_2+\varsigma(\mathsf{X})-\mathsf{Y})\, g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \\ & + \int_{\partial_+\Omega} G(\varepsilon\mathsf{p}+\varepsilon\mathsf{X},\mathsf{y})\,\Psi_1[\varepsilon](\mathsf{y})\, d\sigma_{\mathsf{y}} \\ & + v_S[\partial\omega,\Psi_2[\varepsilon]](\mathsf{X}) - \int_{\partial\omega} S(-2p_2\mathsf{e}_2+\varsigma(\mathsf{X})-\mathsf{Y})\,\Psi_2[\varepsilon]\, d\sigma_{\mathsf{Y}} \\ & + \Psi_3[\varepsilon]\int_{\partial\omega}(S(\mathsf{X}-\mathsf{Y}) - S(-2p_2\mathsf{e}_2+\varsigma(\mathsf{X})-\mathsf{Y}))\, d\sigma_{\mathsf{Y}}\end{aligned}$$

for all $\mathsf{X}\in\overline{\omega'}$ and for all $\varepsilon\in\,]-\varepsilon',\varepsilon'[$. Then, one verifies that $\mathfrak{V}_{\omega'}$ is real analytic from $]-\varepsilon',\varepsilon'[$ to $\mathscr{C}^{1,\alpha}(\overline{\omega'})$ by Proposition 1.1, by Theorem 4.4, and by a standard argument (see in the proof of Proposition 3.3 the argument used to study $\mathcal{L}_2$). Relation (4.3) follows by Lemma 4.6.

Now, by a change of variables in integrals and by Proposition 4.3, one verifies that

$$\mathfrak{V}_{\omega'}[0](\mathsf{X}) \equiv g^{\mathrm{o}}(0) - w_S^e[\partial\tilde\omega,\tilde g^{\mathrm{i}}](\mathsf{X}) + v_S[\partial\tilde\omega,\tilde\Psi_2[0]](\mathsf{X}) + \Psi_3[0]v_S[\partial\tilde\omega,\tilde 1](\mathsf{X}) \tag{4.5}$$

for all $\mathsf{X}\in\overline{\omega'}$. The right hand side of (4.5) equals g^{i} on $\partial\omega$ by Proposition 4.3 and by the jump properties of the double layer potential. Then, the (harmonic) function

$$v_*(\mathsf{X}) \equiv -w_S^e[\partial\tilde\omega,\tilde g^{\mathrm{i}}](\mathsf{X}) + v_S[\partial\tilde\omega,\tilde\Psi_2[0]](\mathsf{X}) + \Psi_3[0]v_S[\partial\tilde\omega,\tilde 1](\mathsf{X}), \qquad \forall\mathsf{X}\in\mathbb{R}^2\setminus\tilde\omega \tag{4.6}$$

equals $\tilde g^{\mathrm{i}} - \tilde g^{\mathrm{o}}(0)$ on $\partial\tilde\omega$. By the decaying properties at ∞ of the single and double layer potentials, $\lim_{\mathsf{X}\to\infty} v_*(\mathsf{X})$ exists and is finite. Since $v_*(\mathsf{X}) = -v_*(\varsigma(\mathsf{X})-2p_2\mathsf{e}_2)$, $\lim_{\mathsf{X}\to\infty} v_*(\mathsf{X}) = 0$. Now, (4.4) holds by the uniqueness of the solution of the exterior Dirichlet problem. □

Remark 4.8. *If we take* $w_*(\mathsf{X}) \equiv v_*(\mathsf{X}-\mathsf{p}) + g^{\mathrm{o}}(0)$ *for all* $\mathsf{X}\in\mathbb{R}^2_+\setminus(\mathsf{p}+\omega)$, *then*

$$\mathfrak{V}_{\omega'}[0](\mathsf{X}-\mathsf{p}) = w_*(\mathsf{X}), \qquad \forall\mathsf{X}\in\mathsf{p}+\overline{\omega}'$$

and w_* *is the unique solution in* $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^2_+}\setminus(\mathsf{p}+\omega))$ *of* (1.21).

4.3.3 Energy integral

In Theorem 4.9 here below we turn to consider the energy integral $\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^2\, d\mathsf{x}$ for ε close to 0.

Theorem 4.9. *Let the assumptions of Theorem 4.4 hold. Then there exist* $0<\varepsilon^{\mathfrak{E}}<\varepsilon^*$ *and a real analytic map*

$$\mathfrak{E} :]-\varepsilon^{\mathfrak{E}},\varepsilon^{\mathfrak{E}}[\,\to\mathbb{R}$$

such that

$$\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^2\, d\mathsf{x} = \mathfrak{E}(\varepsilon), \qquad \forall\varepsilon\in\,]0,\varepsilon^{\mathfrak{E}}[\,. \tag{4.7}$$

In addition,

$$\mathfrak{E}(0) = \int_\Omega|\nabla u_0|^2\, d\mathsf{x} + \frac{1}{2}\int_{\mathbb{R}^2\setminus\tilde\omega}|\nabla v_*|^2\, d\mathsf{x}\,. \tag{4.8}$$

Proof. We take ω' as in Theorem 4.7 which in addition satisfies the condition $\partial\omega \subseteq \overline{\omega'}$. Then we set $\varepsilon^{\mathfrak{E}} \equiv \varepsilon''$ with ε'' as in Theorem 4.7 and we define

$$\begin{aligned}
\mathfrak{E}_1(\varepsilon) &\equiv \int_{\partial\Omega} g^{\mathrm{o}}\, \nu_\Omega \cdot \nabla(u_0 + v_G[\partial_+\Omega, \Psi_1[\varepsilon]])\, d\sigma\,, \\
\mathfrak{E}_2(\varepsilon) &\equiv -\int_{\partial\omega} g^{\mathrm{i}}(\mathsf{Y})\, \nu_\omega(\mathsf{Y}) \cdot \nabla w_G[\partial_+\Omega, g^{\mathrm{o}}](\varepsilon\mathsf{p} + \varepsilon\mathsf{Y})\, d\sigma_{\mathsf{Y}}\,, \\
\mathfrak{E}_3(\varepsilon) &\equiv \int_{\partial\omega} (\Psi_2[\varepsilon](\mathsf{Y}) + \Psi_3[\varepsilon])\; w_G[\partial_+\Omega, g^{\mathrm{o}}](\varepsilon\mathsf{p} + \varepsilon\mathsf{Y})\, d\sigma_{\mathsf{Y}}\,, \\
\mathfrak{E}_4(\varepsilon) &\equiv -\int_{\partial\omega} g^{\mathrm{i}}\, \nu_\omega \cdot \nabla\mathfrak{V}_{\omega'}[\varepsilon]\, d\sigma
\end{aligned}$$

and

$$\mathfrak{E}(\varepsilon) \equiv \mathfrak{E}_1(\varepsilon) + \varepsilon\mathfrak{E}_2(\varepsilon) + \mathfrak{E}_3(\varepsilon) + \mathfrak{E}_4(\varepsilon), \qquad \forall \varepsilon \in \left]-\varepsilon^{\mathfrak{E}}, \varepsilon^{\mathfrak{E}}\right[\,. \tag{4.9}$$

By Theorems 4.4 and 4.7, by Lemma 2.14, and by a standard argument (see in the proof of Proposition 3.3 the study of $\mathfrak{L}_2$), one verifies that the functions $\mathfrak{E}_i$'s and $\mathfrak{E}$ are real analytic from $]-\varepsilon^{\mathfrak{E}}, \varepsilon^{\mathfrak{E}}[$ to $\mathbb{R}$. Using the definition of $w_G[\partial_+\Omega, g^{\mathrm{o}}]$ and by the Fubini theorem, one gets

$$\mathfrak{E}_2(\varepsilon) = -\int_{\partial\Omega} g^{\mathrm{o}}(\mathsf{x})\, \nu_\Omega(\mathsf{x}) \cdot \nabla_{\mathsf{x}} \left(\int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot (\nabla_{\mathsf{y}} G)(\mathsf{x}, \varepsilon\mathsf{p} + \varepsilon\mathsf{Y}) g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \right) d\sigma_{\mathsf{x}}$$

and

$$\begin{aligned}
\mathfrak{E}_3(\varepsilon) = &\int_{\partial\Omega} g^{\mathrm{o}}(\mathsf{x})\, \nu_\Omega(\mathsf{x}) \cdot \nabla_{\mathsf{x}} \left(\int_{\partial\omega} G(\mathsf{x}, \varepsilon\mathsf{p} + \varepsilon\mathsf{Y}) \Psi_2[\varepsilon](\mathsf{Y})\, d\sigma_{\mathsf{Y}} \right) d\sigma_{\mathsf{x}} \\
&+ \int_{\partial\Omega} g^{\mathrm{o}}(\mathsf{x})\, \nu_\Omega(\mathsf{x}) \cdot \nabla_{\mathsf{x}} \left(\Psi_3[\varepsilon] \int_{\partial\omega} G(\mathsf{x}, \varepsilon\mathsf{p} + \varepsilon\mathsf{Y})\, d\sigma_{\mathsf{Y}} \right) d\sigma_{\mathsf{x}}
\end{aligned}$$

for all $\varepsilon \in]0, \varepsilon^{\mathfrak{E}}[$. Then, (4.7) follows by the divergence theorem, by Lemma 4.5, and by Theorem 4.7 (see also the proof of Theorem 3.9, where an analog argument is presented in full details).

To prove (4.8), we observe that $\Psi_1[0] = 0$ by Proposition 4.3 and Theorem 4.4. Thus

$$\mathfrak{E}_1(0) = \int_{\partial\Omega} g^{\mathrm{o}}\, \nu_\Omega \cdot \nabla u_0\, d\sigma = \int_\Omega |\nabla u_0|^2\, d\mathsf{x}\,. \tag{4.10}$$

By Lemma 2.7, $w_G[\partial_+\Omega, g^{\mathrm{o}}](0) = g^{\mathrm{o}}(0)$. Since $\Psi_2[0] \in \mathscr{C}^{1,\alpha}_{\#}(\partial\omega)$, we compute

$$\mathfrak{E}_3(0) = g^{\mathrm{o}}(0)\, \Psi_3[0] \int_{\partial\omega} d\sigma\,. \tag{4.11}$$

Then, we have

$$\begin{aligned}
\int_{\partial\omega} \nu_\omega \cdot \nabla v_*\, d\sigma &= -\int_{\partial\omega} \nu_\omega \cdot \nabla w^e_S[\partial\tilde{\omega}, \tilde{g}^{\mathrm{i}}]\, d\sigma + \int_{\partial\omega} \nu_\omega \cdot \nabla v^e_S[\partial\tilde{\omega}, \tilde{\Psi}_2[0] + \Psi_3[0]]\, d\sigma \\
&= -\int_{\partial\omega} \nu_\omega \cdot \nabla w^i_S[\partial\tilde{\omega}, \tilde{g}^{\mathrm{i}}]\, d\sigma + \int_{\partial\omega} \left(\tilde{\Psi}_2[0] + \Psi_3[0]\right) d\sigma = \Psi_3[0] \int_{\partial\omega} d\sigma\,.
\end{aligned}$$

where we have used successively (4.6), the jump properties of the (classical) single and double layer potentials, the divergence theorem, and $\Psi_2[0] \in \mathscr{C}^{1,\alpha}_{\#}(\partial\omega)$. Using (4.4) and the equality $v_*(\mathsf{X}) = -v_*(\varsigma(\mathsf{X}) - 2p_2\mathsf{e}_2)$ which holds for all $\mathsf{X} \in \mathbb{R}^2 \setminus \tilde{\omega}$, we have

$$\begin{aligned}
\mathfrak{E}_4(0) &= -\int_{\partial\omega} (g^{\mathrm{i}} - g^{\mathrm{o}}(0))\, \nu_\omega \cdot \nabla v_*\, d\sigma - g^{\mathrm{o}}(0) \int_{\partial\omega} \nu_\omega \cdot \nabla v_*\, d\sigma \\
&= -\frac{1}{2} \int_{\partial\tilde{\omega}} v_*\, \nu_{\tilde{\omega}} \cdot \nabla v_*\, d\sigma - g^{\mathrm{o}}(0) \int_{\partial\omega} \nu_\omega \cdot \nabla v_*\, d\sigma \\
&= \frac{1}{2} \int_{\mathbb{R}^2 \setminus \tilde{\omega}} |\nabla v_*|^2\, d\mathsf{x} - g^{\mathrm{o}}(0)\, \Psi_3[0] \int_{\partial\omega} d\sigma\,.
\end{aligned} \tag{4.12}$$

thanks to the divergence theorem. Relation (4.8) follows by (4.9) – (4.12). □

Remark 4.10. *If we take w_* as in Remark 4.8, then*

$$\mathfrak{E}(0) = \int_\Omega |\nabla u_0|^2\, d\mathsf{x} + \int_{\mathbb{R}^2_+ \setminus (\mathsf{p}+\omega)} |\nabla w_*|^2\, d\mathsf{x}\,.$$

Finally, we consider in the following Theorem 4.11 the total flux on $\partial\Omega$. The proof of Theorem 4.11 can be deduced by a straightforward modification of the proof of Theorem 3.10 and it is accordingly omitted.

Theorem 4.11. *Let the assumptions of Theorem 4.4 hold. Then there exist $\varepsilon^{\mathfrak{F}} \in]0, \varepsilon^*[$ and a real analytic function*

$$\mathfrak{F} :]-\varepsilon^{\mathfrak{F}}, \varepsilon^{\mathfrak{F}}[\to \mathbb{R}$$

such that

$$\int_{\partial\Omega} \nu_\Omega \cdot \nabla u_\varepsilon\, d\sigma = \mathfrak{F}(\varepsilon), \qquad \forall \varepsilon \in]0, \varepsilon^{\mathfrak{F}}[\,.$$

In addition,

$$\mathfrak{F}(0) = \int_{\partial\omega} \nu_\omega \cdot \nabla v_*\, d\sigma = \int_{\mathsf{p}+\partial\omega} \nu_{\mathsf{p}+\omega} \cdot \nabla w_*\, d\sigma\,.$$

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