A Dirichlet problem for the Laplace operator in a domain with a small hole close to the boundary

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Abstract

We take an open regular domain Ω in $\mathbb{R}^n$, $n \geq 3$. We introduce a pair of positive parameters ε_1 and ε_2 and we set $\boldsymbol{\varepsilon} \equiv (\varepsilon_1, \varepsilon_2)$. Then we define the perforated domain $\Omega_{\boldsymbol{\varepsilon}}$ by making in Ω a small hole of size $\varepsilon_1\varepsilon_2$ at distance ε_1 from the boundary. When $\boldsymbol{\varepsilon} \to (0,0)$, the hole approaches the boundary while its size shrinks at a faster rate. In $\Omega_{\boldsymbol{\varepsilon}}$ we consider a Dirichlet problem for the Laplace equation and we denote its solution by $u_{\boldsymbol{\varepsilon}}$. By an approach based on functional analysis and on the introduction of special layer potentials we show that the map which takes $\boldsymbol{\varepsilon}$ to (a restriction of) $u_{\boldsymbol{\varepsilon}}$ has a real analytic continuation in a neighbourhood of $(0,0)$. Then we compute some asymptotic expansions.

Keywords: Dirichlet problem; singularly perturbed perforated domain; Laplace operator; real analytic continuation in Banach space; asymptotic expansion

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1 Introduction

Elliptic boundary value problems in domains where a small part has been removed arise in the study of mathematical models for bodies with small perforations or inclusions and are of interest not only for the mathematical aspects, but also for the applications to elasticity, heat conduction, fluid mechanics, and so on. They play a central role in the treatment of inverse problems (see, *e.g.*, Ammari and Kang [1]) and in the computation of the so-called 'topological derivative', a fundamental tool in shape and topological optimization (see, *e.g.*, Novotny and Sokołowsky [31]). Due to the difference in size between the small part removed and the whole domain, the application of standard numerical methods requires the introduction of highly inhomogeneous meshes and often leads to inaccuracy and instability. To overcome this

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difficulty and ensure the validity of the numerical strategies, one has to perform preliminary theoretical studies.

In this paper we will consider the case of a Dirichlet problem for the Laplace equation in a domain with a small hole 'moderately close' to the boundary, *i.e.* with a hole which approaches the outer boundary of the domain at a certain speed and, at the same time, shrinks its size at a faster rate. We will confine ourselves to the case where the dimension n of the Euclidean ambient space is greater than or equal to 3. The analysis of the two-dimensional case requires some specific techniques and it is presented in the forthcoming paper [2].

We begin by describing the geometric setting of our problem. Without loss of generality we set ourself in the upper-half space, which we denote by $\mathbb{R}^n_+$. Namely, we define

$$\mathbb{R}^n_+ \equiv \{\mathsf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n \ : \ x_n > 0\} .$$

(We observe that the boundary $\partial\mathbb{R}^n_+$ coincides with the hyperplane $x_n = 0$.) Then we fix a domain Ω such that

$$\Omega \text{ is an open bounded connected subset of } \mathbb{R}^n_+ \text{ of class } \mathscr{C}^{1,\alpha}, \tag{H_1}$$

where $\alpha \in]0, 1[$ is a regularity parameter. For the definition of functions and sets of the usual Schauder classes $\mathscr{C}^{k,\alpha}$ $(k = 0, 1)$, we refer to Gilbarg and Trudinger [18, §6.2]. We denote by $\partial\Omega$ the boundary of Ω and we set

$$\partial_0\Omega \equiv \partial\Omega \cap \partial\mathbb{R}^n_+, \qquad \partial_+\Omega \equiv \partial\Omega \cap \mathbb{R}^n_+$$

(see Figure 1). Then we consider the following assumption:

$$\partial_0\Omega \text{ is an open neighbourhood of } 0 \text{ in } \partial\mathbb{R}^n_+. \tag{H_2}$$

The set Ω will play the role of the 'unperturbed' domain. To define the hole, we consider another set ω satisfying the following assumption:

$$\omega \text{ is a bounded open connected subset of } \mathbb{R}^n \text{ of class } \mathscr{C}^{1,\alpha} \text{ such that } 0 \in \omega.$$

The set ω represents the shape of the perforation. Then we fix a point

$$\mathsf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n_+, \tag{1.1}$$

and we define the inclusion $\omega_{\boldsymbol{\varepsilon}}$ by

$$\omega_{\boldsymbol{\varepsilon}} \equiv \varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2\omega\,, \qquad \forall \boldsymbol{\varepsilon} \equiv (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2\,.$$

We will exploit the following notation. If $\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}'' \in \mathbb{R}^2$, then we write $\boldsymbol{\varepsilon}' \le \boldsymbol{\varepsilon}''$ (resp. $\boldsymbol{\varepsilon}' < \boldsymbol{\varepsilon}''$) if and only if $\varepsilon'_j \le \varepsilon''_j$ (resp. $\varepsilon'_j < \varepsilon''_j$), for $j = 1, 2$. Accordingly, we denote by $]\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}''[$ the open rectangular domain of the $\boldsymbol{\varepsilon} \in \mathbb{R}^2$ such that $\boldsymbol{\varepsilon}' < \boldsymbol{\varepsilon} < \boldsymbol{\varepsilon}''$. We also set $\mathbf{0} \equiv (0, 0)$. Then one verifies that there is $\boldsymbol{\varepsilon}^{\mathrm{ad}} \in]0, +\infty[^2$ such that

$$\overline{\omega_{\boldsymbol{\varepsilon}}} \subseteq \Omega, \qquad \forall \boldsymbol{\varepsilon} \in]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[.$$

In a sense, $]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$ is a set of admissible parameters for which we can define the perforated domain $\Omega_{\boldsymbol{\varepsilon}}$ obtained by removing from the unperturbed domain Ω the closure $\overline{\omega_{\boldsymbol{\varepsilon}}}$ of $\omega_{\boldsymbol{\varepsilon}}$, *i.e.*

$$\Omega_{\boldsymbol{\varepsilon}} \equiv \Omega \setminus \overline{\omega_{\boldsymbol{\varepsilon}}}, \qquad \forall \boldsymbol{\varepsilon} \in]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[.$$

We observe that, for all $\boldsymbol{\varepsilon} \in \,]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$, $\Omega_{\boldsymbol{\varepsilon}}$ is a bounded connected open domain of class $\mathscr{C}^{1,\alpha}$ with boundary $\partial\Omega_{\boldsymbol{\varepsilon}}$ consisting of two connected components: $\partial\Omega$ and $\partial\omega_{\boldsymbol{\varepsilon}} = \varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2\partial\omega$. The distance of the hole $\omega_{\boldsymbol{\varepsilon}}$ from the boundary $\partial\Omega$ is controlled by ε_1 while its size is controlled by the product $\varepsilon_1\varepsilon_2$. As the pair $\boldsymbol{\varepsilon} \in \,]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$ approaches the degenerate value $(0,0)$, both the size of $\omega_{\boldsymbol{\varepsilon}}$ and its distance from $\partial\Omega$ tend to 0, but the size decreases at a faster speed. Figure 1 illustrates the geometric setting.

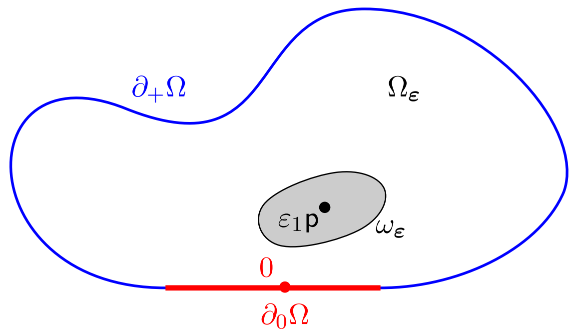

Figure 1: Geometrical settings.

Now that we have the $\boldsymbol{\varepsilon}$-dependent domain $\Omega_{\boldsymbol{\varepsilon}}$, we can introduce a Dirichlet problem on it. To do so, we take a function $g^{\mathrm{o}} \in \mathscr{C}^{1,\alpha}(\partial\Omega)$ and a function $g^{\mathrm{i}} \in \mathscr{C}^{1,\alpha}(\partial\omega)$. Then, for $\boldsymbol{\varepsilon} \in \,]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$ fixed, we consider the following boundary value problem for a function $u \in \mathscr{C}^{1,\alpha}(\overline{\Omega_{\boldsymbol{\varepsilon}}})$:

$$\begin{cases} \Delta u(\mathsf{x}) = 0, & \forall \mathsf{x} \in \Omega_{\boldsymbol{\varepsilon}}\,, \\ u(\mathsf{x}) = g^{\mathrm{o}}(\mathsf{x}), & \forall \mathsf{x} \in \partial\Omega\,, \\ u(\mathsf{x}) = g^{\mathrm{i}}\left(\frac{\mathsf{x}-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right), & \forall \mathsf{x} \in \partial\omega_{\boldsymbol{\varepsilon}}\,. \end{cases} \tag{1.2}$$

As is well known, the solution of (1.2) exists unique in $\mathscr{C}^{1,\alpha}(\overline{\Omega_{\boldsymbol{\varepsilon}}})$. To emphasize its dependence on $\boldsymbol{\varepsilon}$, we will denote it by $u_{\boldsymbol{\varepsilon}}$. The aim of this paper is to investigate the behaviour of $u_{\boldsymbol{\varepsilon}}$ when the parameter $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ approaches the degenerate value $\mathbf{0} \equiv (0,0)$. To formulate a precise question we may for example observe that every point $\mathsf{x} \in \Omega$ stays in $\Omega_{\boldsymbol{\varepsilon}}$ for $\boldsymbol{\varepsilon}$ sufficiently close to $\mathbf{0}$. Accordingly, if we fix a point $\mathsf{x} \in \Omega$, then the evaluation $u_{\boldsymbol{\varepsilon}}(\mathsf{x})$ is well defined for $\boldsymbol{\varepsilon}$ small enough and we can ask:

$$\text{What can be said on the map } \boldsymbol{\varepsilon} \mapsto u_{\boldsymbol{\varepsilon}}(\mathsf{x}) \text{ for } \boldsymbol{\varepsilon} \text{ near } \mathbf{0}\text{?} \tag{1.3}$$

In literature, the analysis of boundary value problems in domains with small holes has been mostly carried out by means of asymptotic expansion methods. We mention, for example, the method of matching outer and inner expansions proposed by Il'in (see [19, 20, 21]) and the compound asymptotic expansion method of Maz'ya, Nazarov, and Plamenevskij [29] and of Kozlov, Maz'ya, and Movchan [22]. Problems with 'clouds' of holes whose size is smaller than the relative distance have been analysed by the mesoscale asymptotic method by Maz'ya, Movchan, and Nieves (see, *e.g.*, [27, 28]). The method of multiscale asymptotic expansions has been exploited to study problems with moderately close holes in the papers by Bonnaillie-Noël, Dambrine, Tordeux, and Vial [5, 6], Bonnaillie-Noël and Dambrine [3], and Bonnaillie-

Noël, Dambrine, and Lacave [4]. We also mention the works of Bonnaillie-Noël, Lacave, and Masmoudi [7], Chesnel and Claeys [8], and Dauge, Tordeux, and Vial [15].

By the asymptotic expansion methods, one typically describes the behavior of solutions of singularly perturbed problems by means of asymptotic approximations. For the same purpose, a different approach proposed by Lanza de Cristoforis, and which we call 'functional analytic approach', aims at employing real analytic functions. For example, the functional analytic approach would answer to question (1.3) by representing the map $\boldsymbol{\varepsilon} \mapsto u_{\boldsymbol{\varepsilon}}(\mathsf{x})$ in terms of real analytic functions of $\boldsymbol{\varepsilon}$ defined in an open neighbourhood of $\mathbf{0}$ and, in case, of singular but known elementary functions of ε_1 and ε_2. This approach has been so far used for various elliptic problems, also with non-linear conditions. For problems concerning the Laplace operator we refer to the papers of Lanza de Cristoforis (see, e.g., [23, 24]), of Dalla Riva and Musolino (see, *e.g.*, [11, 12, 13]), and to the paper of Dalla Riva, Musolino, and Rogosin [14], where the computation of the coefficients of the power series expansion of the resulting analytic maps is reduced to the solution of certain recursive systems of boundary integral equations.

The aim of the present paper is twofold. On the one hand, we use the functional analytic approach to answer to question (1.3), on the other hand, we compute explicit asymptotic expansions by means of the multiscale asymptotic approach.

We wish now to state our main Theorem 1.1. To do so, in addition to (H_1) and (H_2), we will assume that Ω satisfies the following technical condition:

$$\overline{\partial_+\Omega} \text{ is a compact submanifold with boundary of } \mathbb{R}^n \text{ of class } \mathscr{C}^{1,\alpha}. \tag{H_3}$$

We will use (H_3) to identify certain spaces of functions on $\partial\Omega$ via a controlled extension result (cf. Lemma 2.17 below). We also need a regularity assumption on the Dirichlet datum around the origin. We will assume that

$$\text{there exists } r_0 > 0 \text{ such that the restriction } g^{\mathrm{o}}_{|\mathcal{B}(0,r_0)\cap\partial_0\Omega} \text{ is real analytic.} \tag{H_4}$$

Here and in the sequel $\mathcal{B}(\mathsf{x}, r)$ denotes the ball of $\mathbb{R}^n$ centered at x and of radius r. To understand condition (H_4) one may observe that, as it happens for the solution of a Dirichlet problem in a domain with a small hole 'far' from the boundary, the solution $u_{\boldsymbol{\varepsilon}}$ converges as $\boldsymbol{\varepsilon} \to \mathbf{0}$ to the unique solution $u_0 \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$ of the unperturbed problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega\,, \\ u = g^{\mathrm{o}} & \text{on } \partial\Omega\,. \end{cases} \tag{1.4}$$

The function u_0 is harmonic and therefore analytic in the open set Ω, a fact which can be exploited when the hole collapses to an interior point of Ω. However, in our problem (1.2) the hole shrinks to the origin of $\mathbb{R}^n$, which is situated on the boundary of Ω. Then, to prove our analyticity result on $u_{\boldsymbol{\varepsilon}}$ we have to ensure that u_0 has an analytic continuation in a neighborhood of the origin. A fact which is granted by condition (H_4) (see Proposition 2.18 below).

We are now ready to state Theorem 1.1. We observe that, instead of the evaluation of $u_{\boldsymbol{\varepsilon}}$ at a point x, we will consider its restriction to a suitable subset Ω' of Ω and we show that the map $\boldsymbol{\varepsilon} \mapsto u_{\boldsymbol{\varepsilon}|\overline{\Omega'}}$ is the restriction of a real analytic function of $\boldsymbol{\varepsilon}$ defined in a neighbourhood of $(0, 0)$ and which takes values in a suitable Banach space (for the definition of real analytic maps in Banach spaces we refer to Deimling [16, p. 150]).

Theorem 1.1. *Let Ω' be an open subset of Ω such that $0 \notin \overline{\Omega'}$. There exists $\boldsymbol{\varepsilon}' \in]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$ with $\overline{\omega_{\boldsymbol{\varepsilon}}} \cap \overline{\Omega'} = \emptyset$ for all $\boldsymbol{\varepsilon} \in]-\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}'[$ and a real analytic map $\mathfrak{U}_{\Omega'}$ from $]-\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}'[$ to $\mathscr{C}^{1,\alpha}(\overline{\Omega'})$ such that*

$$u_{\boldsymbol{\varepsilon}|\overline{\Omega'}} = \mathfrak{U}_{\Omega'}[\boldsymbol{\varepsilon}] \qquad \forall \boldsymbol{\varepsilon} \in]\mathbf{0}, \boldsymbol{\varepsilon}'[\,. \tag{1.5}$$

Moreover we have

$$\mathfrak{U}_{\Omega'}[\mathbf{0}] = u_{0|\overline{\Omega'}}\,. \tag{1.6}$$

A similar result as the one in Theorem 1.1 is presented in Theorem 3.6 for the behaviour of $u_{\boldsymbol{\varepsilon}}$ close to the boundary of the hole, namely, for the rescaled function $u_{\boldsymbol{\varepsilon}}(\varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2\,\cdot)$. Then, in Theorems 3.7 and 3.9 we show real analytic continuation results also for the energy integral $\int_{\Omega_{\boldsymbol{\varepsilon}}} |\nabla u_{\boldsymbol{\varepsilon}}|^2 d\mathsf{x}$. Since real analytic maps can be expanded into convergent power series, it follows that $u_{\boldsymbol{\varepsilon}|\overline{\Omega'}}$ and the energy $\int_{\Omega_{\boldsymbol{\varepsilon}}} |\nabla u_{\boldsymbol{\varepsilon}}|^2 d\mathsf{x}$ can be written in terms of a convergent power series of ε_1 and ε_2, for ε_1 and ε_2 sufficiently small. As a consequence, we can compute asymptotic approximations for $u_{\boldsymbol{\varepsilon}}$ with the advantage that the convergence is granted by our preliminary analysis. Some explicit asymptotic expansions are presented in Subsection 4.

To conclude this section, we would like to comment on some novel techniques that we bring into the functional analytic approach for the analysis of our problem. To do so, we first describe how the functional analytic approach 'normally' operates on a boundary value problem defined on a domain which depends on a parameter $\boldsymbol{\varepsilon}$ and which degenerates in some sense when $\boldsymbol{\varepsilon}$ tends to a limit value $\mathbf{0}$. The first step consists in applying potential theoretic techniques to transform the boundary value problem into a system of boundary integral equations. Then, possibly after some suitable manipulation, the system of boundary integral equations is written as a functional equation of the form $\mathfrak{L}[\boldsymbol{\varepsilon}, \boldsymbol{\mu}] = 0$, where $\mathfrak{L}$ is a (nonlinear) operator acting from an open subset of a Banach space $\mathscr{R} \times \mathscr{B}_1$ to another Banach space $\mathscr{B}_2$. Here $\mathscr{R}$ is a neighbourhood of $\mathbf{0}$ and the Banach spaces $\mathscr{B}_1$ and $\mathscr{B}_2$ are usually the product of Schauder spaces on the boundaries of certain fixed domains. The next step is to apply the implicit function theorem to the equation $\mathfrak{L}[\boldsymbol{\varepsilon}, \boldsymbol{\mu}] = 0$ in order to understand the dependence of $\boldsymbol{\mu}$ on $\boldsymbol{\varepsilon}$. Then one can recover the dependence of the solution of the original boundary value problem upon $\boldsymbol{\varepsilon}$.

The strategy adopted in this paper differs from the standard application of the functional analytic approach for the features describes in the following two points:

- The first point concerns the potential theory used to transform the problem into a system of integral equations. To take care of the special geometry of the problem, instead of the classical layer potential of the Laplace operator, we exploit layer potentials where the role of the fundamental solution is replaced by the Dirichlet Green function of the upper-half space. Since the hole collapses on $\partial\mathbb{R}^n_+ \cap \partial\Omega$ as $\boldsymbol{\varepsilon}$ tends to $\mathbf{0}$, such a method allows to get rid of the integral equation defined on the part of the boundary of $\Omega_{\boldsymbol{\varepsilon}}$ where the boundary of the hole and the exterior boundary interact for $\boldsymbol{\varepsilon} = \mathbf{0}$. In Section 2 we collect a number of general results on such special layer potentials. In spite of the fact that this paper concerns problem (1.2) only for $n \geq 3$, we include here also the two dimensional case. This further generalization does not cost any particular extra work and provides a potential theoretic basis also for the two-dimensional case (see [2]). We also observe that, if the union of Ω and of its reflection with respect to $\partial\mathbb{R}^n_+$ forms a regular domain, then one does not need to introduce special layer potential and may analyse the problem by a technique based on the functional analytic approach and on a reflection argument (cf. Costabel, Dalla Riva, Dauge, and Musolino [10]).

However, under the assumption of the present paper the union of Ω and of its reflection with respect to $\partial\mathbb{R}^n_+$ produces an edge on $\partial\mathbb{R}^n_+$ and it is not a regular domain.

• The second point is a consequence of the first. Indeed, by exploiting the special layer potentials described above we can transform problem (1.2) into an equation $\mathfrak{L}[\boldsymbol{\varepsilon}, \boldsymbol{\mu}] = 0$ where the operator $\mathfrak{L}$ acts from an open set $]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathscr{B}_1$ into a Banach space $\mathscr{B}_2$ whose construction is in a sense artificial. It is the product of a Schauder space and of the image of a certain integral operator (see Propositions 2.11 and 3.1). Then we have to be particularly careful to check that the image of $\mathfrak{L}$ is actually contained in such a Banach space $\mathscr{B}_2$ and that $\mathfrak{L}$ is a real analytic operator (see Proposition 3.1). We observe that this step is instead quite straightforward in the other applications of the functional analytic approach so far considered (cf., *e.g.*, [13, Prop. 5.4]). Only at this point we will be ready to use the implicit function theorem and deduce the dependence of the solution upon $\boldsymbol{\varepsilon}$.

The paper is organised as follows. In Section 2, we present some preliminary results in potential theory and we introduce and study the layer potentials with integral kernels derived by the Dirichlet Green function. Then Section 3 is devoted to the proof of our analyticity result. In particular, we prove here Theorem 1.1. In the last Section 4 we provide some asymptotic expansion. We have postponed in Appendix some common technical tools which are needed: in Appendix A we prove some decay properties of the Green function and of the associated single layer potential, whereas in Appendix B we show an extension result based on the Cauchy-Kovalevskaya Theorem.

2 Preliminaries of potential theory

In this section, we introduce some technical results and notation that we shall need in the sequel. Most of them are related to potential theory and representation formulas built with the Dirichlet Green function for the upper-half plane.

For the analysis of the present section, we take

$$n \in \mathbb{N} \setminus \{0, 1\}.$$

2.1 Single and double layer potentials

As a first step, we introduce the classical layer potentials for the Laplace equation and thus we denote by S_n the fundamental solution of Δ defined by

$$S_n(\mathsf{x}) \equiv \begin{cases} \frac{1}{s_n} \log |\mathsf{x}| & \text{if } n = 2\,, \\ \frac{1}{(2-n)s_n} |\mathsf{x}|^{2-n} & \text{if } n \geq 3\,, \end{cases} \qquad \forall \mathsf{x} \in \mathbb{R}^n \setminus \{0\}\,,$$

where s_n is the $(n-1)$-dimensional measure of the boundary of the unit ball in $\mathbb{R}^n$.

Let us now introduce the single and double layer potentials for a generic domain $\mathcal{D}$ assumed to be an open bounded connected subset of $\mathbb{R}^n$ of class $\mathscr{C}^{1,\alpha}$.

Definition 2.1 (Definition of the layer potentials)**.** *For any $\phi \in \mathscr{C}^{0,\alpha}(\partial\mathcal{D})$, we define*

$$v_{S_n}[\partial\mathcal{D}, \phi](\mathsf{x}) \equiv \int_{\partial\mathcal{D}} \phi(y) S_n(\mathsf{x} - y)\, d\sigma_y, \qquad \forall \mathsf{x} \in \mathbb{R}^n\,,$$

where $d\sigma$ denotes the area element on $\partial\mathcal{D}$.
The restrictions of $v_{S_n}[\partial\mathcal{D},\phi]$ to $\overline{\mathcal{D}}$ and to $\mathbb{R}^n\setminus\mathcal{D}$ are denoted $v^i_{S_n}[\partial\mathcal{D},\phi]$ and $v^e_{S_n}[\partial\mathcal{D},\phi]$ respectively (the letter 'i' stands for 'interior' while the letter 'e' stands for 'exterior').
For any $\psi\in\mathscr{C}^{1,\alpha}(\partial\mathcal{D})$, we define

$$w_{S_n}[\partial\mathcal{D},\psi](\mathsf{x})\equiv-\int_{\partial\mathcal{D}}\psi(\mathsf{y})\,\nu_{\mathcal{D}}(\mathsf{y})\cdot\nabla S_n(\mathsf{x}-\mathsf{y})\,d\sigma_{\mathsf{y}},\qquad\forall\mathsf{x}\in\mathbb{R}^n\,,$$

where $\nu_{\mathcal{D}}$ denotes the outer unit normal to $\partial\mathcal{D}$ and the symbol $\cdot$ denotes the scalar product in $\mathbb{R}^n$.

Before dealing with the regularity of these layer potentials, let us introduce some notation.

Definition 2.2. *We denote by $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\mathcal{D})$ the space of functions on $\mathbb{R}^n\setminus\mathcal{D}$ whose restrictions to $\overline{\mathcal{O}}$ belong to $\mathscr{C}^{1,\alpha}(\overline{\mathcal{O}})$ for all open bounded subsets $\mathcal{O}$ of $\mathbb{R}^n\setminus\mathcal{D}$.*

Let us now mention some well known regularity properties of the single and double layer potentials.

Proposition 2.3 (Regularity of layer potentials)**.** *The function $v_{S_n}[\partial\mathcal{D},\phi]$ is continuous from $\mathbb{R}^n$ to $\mathbb{R}$. The restrictions $v^i_{S_n}[\partial\mathcal{D},\phi]$ and $v^e_{S_n}[\partial\mathcal{D},\phi]$ belong to $\mathscr{C}^{1,\alpha}(\overline{\mathcal{D}})$ and to $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\mathcal{D})$, respectively.*
The restriction $w_{S_n}[\partial\mathcal{D},\psi]_{|\mathcal{D}}$ extends to a function $w^i_{S_n}[\partial\mathcal{D},\psi]$ of $\mathscr{C}^{1,\alpha}(\overline{\mathcal{D}})$ and the restriction $w_{S_n}[\partial\mathcal{D},\psi]_{|\mathbb{R}^n\setminus\overline{\mathcal{D}}}$ extends to a function $w^e_{S_n}[\partial\mathcal{D},\psi]$ of $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\mathcal{D})$.

Let us recall the classical jump formulas (see, *e.g.*, Folland [17, Chap. 3]).

Proposition 2.4 (Jump relations of layer potentials)**.** *For any $\mathsf{x}\in\partial\mathcal{D}$, $\psi\in\mathscr{C}^{1,\alpha}(\partial\mathcal{D})$, and $\phi\in\mathscr{C}^{0,\alpha}(\partial\mathcal{D})$, we have*

$$\begin{aligned}w^\sharp_{S_n}[\partial\mathcal{D},\psi](\mathsf{x})&=\frac{\mathsf{s}_\sharp}{2}\psi(\mathsf{x})+w_{S_n}[\partial\mathcal{D},\psi](\mathsf{x})\,,\\ \nu_\Omega(\mathsf{x})\cdot\nabla v^\sharp_{S_n}[\partial\mathcal{D},\phi](\mathsf{x})&=-\frac{\mathsf{s}_\sharp}{2}\phi(\mathsf{x})+\int_{\partial\mathcal{D}}\phi(\mathsf{y})\nu_\Omega(\mathsf{x})\cdot\nabla S_n(\mathsf{x}-\mathsf{y})\,d\sigma_{\mathsf{y}}\,,\end{aligned}$$

where $\sharp=i,e$ and $\mathsf{s}_i=1$, $\mathsf{s}_e=-1$.

We will exploit the following classical result of potential theory.

Lemma 2.5. *If $n\geq3$, then the map* $\begin{array}{rcl}\mathscr{C}^{0,\alpha}(\partial\mathcal{D})&\to&\mathscr{C}^{1,\alpha}(\partial\mathcal{D})\\ \phi&\mapsto&v_{S_n}[\partial\mathcal{D},\phi]_{|\partial\mathcal{D}}\end{array}$ *is an isomorphism.*

2.2 Green function for the upper-half space and associated layer potentials

The key tool for the analysis of problem (1.2) are layer potentials built with the Dirichlet Green function for the upper-half space instead of the classical fundamental solution S_n. This will allow to get rid of the integral equation on $\partial_0\Omega$, which is the part of the boundary of $\partial\Omega$ where the inclusion $\omega_{\boldsymbol{\varepsilon}}$ collapses for $\boldsymbol{\varepsilon}=\mathbf{0}$. We now introduce some notation.

We denote by ς the reflexion with respect to the hyperplane $\partial\mathbb{R}^n_+$, so that

$$\varsigma(\mathsf{x})\equiv(x_1,\ldots,x_{n-1},-x_n)\,,\qquad\forall\mathsf{x}=(x_1,\ldots,x_n)\in\mathbb{R}^n\,,$$

and we define $\varsigma(\mathcal{D}) \equiv \{\mathsf{x} \in \mathbb{R}^2 \mid \varsigma(\mathsf{x}) \in \mathcal{D}\}$, for all subsets $\mathcal{D}$ of $\mathbb{R}^n$. Then we denote by G the Green function defined by

$$G(\mathsf{x},\mathsf{y}) \equiv S_n(\mathsf{x}-\mathsf{y}) - S_n(\varsigma(\mathsf{x})-\mathsf{y}), \quad \forall(\mathsf{x},\mathsf{y}) \in \mathbb{R}^n \times \mathbb{R}^n \text{ with } \mathsf{y} \neq \mathsf{x} \text{ and } \mathsf{y} \neq \varsigma(\mathsf{x}).$$

We observe that

$$\begin{cases} G(\mathsf{x},\mathsf{y}) = G(\mathsf{y},\mathsf{x}), & \forall(\mathsf{x},\mathsf{y}) \in \mathbb{R}^n \times \mathbb{R}^n \text{ with } \mathsf{y} \neq \mathsf{x} \text{ and } \mathsf{y} \neq \varsigma(\mathsf{x}), \\ G(\mathsf{x},\mathsf{y}) = 0, & \forall(\mathsf{x},\mathsf{y}) \in \partial\mathbb{R}^n_+ \times \mathbb{R}^n \text{ with } \mathsf{y} \neq \mathsf{x} \text{ and } \mathsf{y} \neq \varsigma(\mathsf{x}). \end{cases}$$

If $\mathcal{D}$ is a subset of $\mathbb{R}^n$, we find convenient to set $\varsigma(\mathcal{D}) \equiv \{\mathsf{x} \in \mathbb{R}^n \mid \varsigma(\mathsf{x}) \in \mathcal{D}\}$. Let us now introduce analogs of the classical layer potentials of Definition 2.2 obtained by replacing S_n by the Green function G. In order to do so, we will need to consider an open bounded connected subset $\mathcal{D}_+$ of $\mathbb{R}^n_+$ of class $\mathscr{C}^{1,\alpha}$.

Definition 2.6 (Definition of layer potentials derived by G)**.** *For any $\phi \in \mathscr{C}^{0,\alpha}(\partial\mathcal{D}_+)$, we define*

$$v_G[\partial\mathcal{D}_+,\phi](\mathsf{x}) \equiv \int_{\partial\mathcal{D}_+} \phi(\mathsf{y}) G(\mathsf{x},\mathsf{y})\, d\sigma_{\mathsf{y}}, \qquad \forall \mathsf{x} \in \mathbb{R}^n\,.$$

The restrictions of $v_G[\partial\mathcal{D}_+,\phi]$ to $\overline{\mathcal{D}}_+$ and $\overline{\mathbb{R}^n_+ \setminus \mathcal{D}_+}$ are denoted $v^i_G[\partial\mathcal{D}_+,\phi]$ and $v^e_G[\partial\mathcal{D}_+,\phi]$ respectively.
For any subset of the boundary $\Gamma \subseteq \partial\mathcal{D}_+$ and for any $\psi \in \mathscr{C}^{1,\alpha}(\partial\mathcal{D}_+)$, we define

$$w_G[\Gamma,\psi](\mathsf{x}) \equiv \int_\Gamma \psi(\mathsf{y})\, \nu_{\mathcal{D}_+}(\mathsf{y}) \cdot \nabla_{\mathsf{y}} G(\mathsf{x},\mathsf{y})\, d\sigma_{\mathsf{y}}, \qquad \forall \mathsf{x} \in \mathbb{R}^n\,.$$

By definition of G, we easily obtain the equalities

$$v_G[\partial\mathcal{D}_+,\phi](\mathsf{x}) = v_{S_n}[\partial\mathcal{D}_+,\phi](\mathsf{x}) - v_{S_n}[\partial\mathcal{D}_+,\phi](\varsigma(\mathsf{x}))\,, \qquad \forall \mathsf{x} \in \mathbb{R}^n,\ \forall \phi \in \mathscr{C}^{0,\alpha}(\partial\mathcal{D}_+)\,,$$

and

$$w_G[\partial\mathcal{D}_+,\psi](\mathsf{x}) = w_{S_n}[\partial\mathcal{D}_+,\psi](\mathsf{x}) - w_{S_n}[\partial\mathcal{D}_+,\psi](\varsigma(\mathsf{x}))\,, \qquad \forall \mathsf{x} \in \mathbb{R}^n\,,\ \forall \psi \in \mathscr{C}^{1,\alpha}(\partial\mathcal{D}_+)\,.$$

Thus one deduces regularity properties and jump formulas for $v_G[\partial\mathcal{D}_+,\phi]$ and $w_G[\partial\mathcal{D}_+,\psi]$.

Proposition 2.7 (Regularity and jump relations for the layer potentials derived by G)**.** *Let $\phi \in \mathscr{C}^{0,\alpha}(\partial\mathcal{D}_+)$ and $\psi \in \mathscr{C}^{1,\alpha}(\partial\mathcal{D}_+)$. Then*

- *the functions $v_G[\partial\mathcal{D}_+,\phi]$ and $w_G[\partial\mathcal{D}_+,\psi]$ are harmonic in $\mathcal{D}_+$, $\varsigma(\mathcal{D}_+)$, and $\mathbb{R}^n \setminus \overline{\mathcal{D}_+ \cup \varsigma(\mathcal{D}_+)}$;*
- *the function $v_G[\partial\mathcal{D}_+,\phi]$ is continuous from $\mathbb{R}^n$ to $\mathbb{R}$ and the restrictions $v^i_G[\partial\mathcal{D}_+,\phi]$ and $v^e_G[\partial\mathcal{D}_+,\phi]$ belong to $\mathscr{C}^{1,\alpha}(\overline{\mathcal{D}_+})$ and to $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+ \setminus \mathcal{D}_+})$, respectively;*
- *the restriction $w_G[\partial\mathcal{D}_+,\psi]_{|\Omega}$ extends to a function $w^i_G[\partial\mathcal{D}_+,\psi]$ of $\mathscr{C}^{1,\alpha}(\overline{\mathcal{D}_+})$ and the restriction $w_G[\partial\mathcal{D}_+,\psi]_{|\mathbb{R}^n_+ \setminus \overline{\mathcal{D}_+}}$ extends to a function $w^e_G[\partial\mathcal{D}_+,\psi]$ of $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+ \setminus \mathcal{D}_+})$.*

The jump formulas for the double layer potential are (with $\sharp = i, e$, $\mathsf{s}_i = 1$, $\mathsf{s}_e = -1$)

$$
\begin{aligned}
w^\sharp_G[\partial\mathcal{D}_+, \psi](\mathsf{x}) &= \frac{\mathsf{s}_\sharp}{2}\psi(\mathsf{x}) + w_G[\partial\mathcal{D}_+, \psi](\mathsf{x}), && \forall \mathsf{x} \in \partial_+\mathcal{D}_+\,,\\
w^i_G[\partial\mathcal{D}_+, \psi](\mathsf{x}) &= \psi(\mathsf{x}), && \forall \mathsf{x} \in \partial_0\mathcal{D}_+\,.
\end{aligned}
$$

In addition we have

$$
\begin{aligned}
v_G[\partial\mathcal{D}_+, \phi](\mathsf{x}) &= 0, \qquad \forall \mathsf{x} \in \partial\mathbb{R}^n_+\,,\\
w^e_G[\partial\mathcal{D}_+, \psi](\mathsf{x}) &= 0, \qquad \forall \mathsf{x} \in \partial\mathbb{R}^n_+ \setminus \partial_0\mathcal{D}_+\,.
\end{aligned}
\tag{2.1}
$$

Here above, $\partial_0\mathcal{D}_+ \equiv \partial\mathcal{D}_+ \cap \partial\mathbb{R}^n_+$ and $\partial_+\mathcal{D}_+ \equiv \partial\mathcal{D}_+ \cap \mathbb{R}^n_+$.

In the following lemma we show how the layer potentials with kernel G introduced in Definition 2.6 allow to prove a corresponding Green-like representation formula.

Lemma 2.8 (Green-like representation formula in $\mathcal{D}_+$)**.** *Let $u^i \in \mathscr{C}^{1,\alpha}(\overline{\mathcal{D}_+})$ be such that $\Delta u^i = 0$ in $\mathcal{D}_+$. Then we have*

$$
w_G[\partial\mathcal{D}_+, u^i_{|\partial\mathcal{D}_+}] - v_G[\partial\mathcal{D}_+, \nu_{\mathcal{D}_+}\cdot\nabla u^i_{|\partial\mathcal{D}_+}] = \begin{cases} u^i & \text{in } \mathcal{D}_+,\\ 0 & \text{in } \mathbb{R}^n \setminus \overline{\mathcal{D}_+ \cup \varsigma(\mathcal{D}_+)}\,. \end{cases}
\tag{2.2}
$$

Proof. Let us first consider $\mathsf{x} \in \mathcal{D}_+$. By the Green representation formula (see, *e.g.*, Folland [17, Chap. 2]), we have

$$
u^i(\mathsf{x}) = -\int_{\partial\mathcal{D}_+} \nu_{\mathcal{D}_+}(\mathsf{y})\cdot\nabla S_n(\mathsf{x}-\mathsf{y})\, u^i(\mathsf{y})\, d\sigma_{\mathsf{y}} - \int_{\partial\mathcal{D}_+} S_n(\mathsf{x}-\mathsf{y})\, \nu_{\mathcal{D}_+}(\mathsf{y})\cdot\nabla u^i(\mathsf{y})\, d\sigma_{\mathsf{y}}, \quad \forall \mathsf{x} \in \mathcal{D}_+\,.
\tag{2.3}
$$

On the other hand, we note that if $\mathsf{x} \in \mathcal{D}_+$ is fixed, then the function $\mathsf{y} \mapsto S_n(\varsigma(\mathsf{x}) - \mathsf{y})$ is of class $\mathscr{C}^1(\overline{\mathcal{D}_+})$ and harmonic in $\mathcal{D}_+$. Therefore, by the Green identity, we have

$$
0 = \int_{\partial\mathcal{D}_+} \nu_{\mathcal{D}_+}(\mathsf{y})\cdot\nabla S_n(\varsigma(\mathsf{x})-\mathsf{y})\, u^i(\mathsf{y})\, d\sigma_{\mathsf{y}} + \int_{\partial\mathcal{D}_+} S_n(\varsigma(\mathsf{x})-\mathsf{y})\, \nu_{\mathcal{D}_+}(\mathsf{y})\cdot\nabla u^i(\mathsf{y})\, d\sigma_{\mathsf{y}} \qquad \forall \mathsf{x} \in \mathcal{D}_+\,.
\tag{2.4}
$$

Then, by summing equalities (2.3) and (2.4) we deduce the validity of (2.2) in $\mathcal{D}_+$.
Let us now consider any fixed $\mathsf{x} \in \mathbb{R}^n \setminus \overline{\mathcal{D}_+ \cup \varsigma(\mathcal{D}_+)}$. We observe that the functions $\mathsf{y} \mapsto S_n(\mathsf{x}-\mathsf{y})$ and $\mathsf{y} \mapsto S_n(\varsigma(\mathsf{x})-\mathsf{y})$ are harmonic on $\mathcal{D}_+$. Accordingly $G(\mathsf{x},\cdot)$ is an harmonic function in $\mathcal{D}_+$. Then a standard argument based on the divergence theorem shows that

$$
\int_{\partial\mathcal{D}_+} u^i(\mathsf{y})\, \nu_{\mathcal{D}_+}(\mathsf{y})\cdot\nabla_{\mathsf{y}} G(\mathsf{x},\mathsf{y}) - G(\mathsf{x},\mathsf{y})\, \nu_{\mathcal{D}_+}(\mathsf{y})\cdot\nabla u^i(\mathsf{y})\, d\sigma_{\mathsf{y}} = 0\,.
$$

□

2.3 Mapping properties of the single layer potential $v_G[\partial\Omega, \cdot]$

In order to analyse the ε-dependent boundary value problem (1.2), we are going to exploit the layer potentials with kernel derived by G in the case when $\mathcal{D} = \Omega_\varepsilon$. Since $\partial\Omega_\varepsilon = \partial\Omega \cup \partial\omega_\varepsilon$, we need to consider layer potentials integrated on $\partial\Omega$ and on $\partial\omega_\varepsilon$. In this section, we will investigate some properties of the single layer potential supported on the boundary of the set Ω which satisfies the assumptions (H_1), (H_2), and (H_3).

First of all, as one can easily see, the single layer potential $v_G[\partial\Omega, \phi]$ does not depend on the values of the density ϕ on $\partial_0\Omega$. In other words, it takes into account only $\phi_{|\partial_+\Omega}$. For this reason, it is convenient to introduce a quotient Banach space.

Definition 2.9. *We denote by $\mathscr{C}^{0,\alpha}_+(\partial\Omega)$ the quotient Banach space*

$$\mathscr{C}^{0,\alpha}(\partial\Omega)/\{\phi \in \mathscr{C}^{0,\alpha}(\partial\Omega) \mid \phi_{|\partial_+\Omega} = 0\}\,.$$

Then we can prove that the single layer potential map

$$\begin{array}{rcl} \mathscr{C}^{0,\alpha}_+(\partial\Omega) & \to & \mathscr{C}^{1,\alpha}(\partial_+\Omega) \\ \phi & \mapsto & v_G[\partial\Omega, \phi]_{|\partial_+\Omega} \end{array}$$

is well defined and one-to-one. Namely we have the following.

Proposition 2.10 (Null space of the single layer potential derived by G)**.** *Let $\phi \in \mathscr{C}^{0,\alpha}(\partial\Omega)$. Then $v_G[\partial\Omega, \phi]_{|\partial_+\Omega} = 0$ if and only if $\phi_{|\partial_+\Omega} = 0$.*

Proof. Let $\phi \in \mathscr{C}^{0,\alpha}(\partial\Omega)$ be such that $\phi_{|\partial_+\Omega} = 0$. As a consequence,

$$v_G[\partial\Omega, \phi](\mathsf{x}) = \int_{\partial_+\Omega} G(\mathsf{x}, \mathsf{y})\phi_{|\partial_+\Omega}(\mathsf{y})\, d\sigma_{\mathsf{y}} = 0 \qquad \forall \mathsf{x} \in \partial\Omega\,.$$

Let now assume that $v_G[\partial\Omega, \phi]_{|\partial_+\Omega} = 0$. With (2.1), we have in particular $v_G[\partial\Omega, \phi]_{|\partial_0\Omega} = 0$ and then $v_G[\partial\Omega, \phi]_{|\partial\Omega} = 0$. By the uniqueness of the solution of the Dirichlet problem we deduce that $v_G[\partial\Omega, \phi] = 0$ in $\overline{\Omega}$. By the harmonicity at infinity of $v_G[\partial\Omega, \phi]$ (cf. Lemma A.2), by equality $v_G[\partial\Omega, \phi]_{|\partial\mathbb{R}^n_+ \cup \partial\Omega} = 0$, and by a standard energy argument based on the divergence theorem, we deduce that $\nabla v_G[\partial\Omega, \phi] = 0$ in $\mathbb{R}^n_+ \setminus \overline{\Omega}$, and that accordingly $v_G[\partial\Omega, \phi]$ is constant in $\mathbb{R}^n_+ \setminus \Omega$. Since $v_G[\partial\Omega, \phi] = 0$ on $\partial\Omega$, we have $v_G[\partial\Omega, \phi] = 0$ in $\mathbb{R}^n_+$. Then, by the jump formulas for the normal derivative of $v_G[\partial\Omega, \phi]$ on $\partial_+\Omega$, it follows that

$$\phi = \nu_\Omega \cdot \nabla v^e_G[\partial\Omega, \phi] - \nu_\Omega \cdot \nabla v^i_G[\partial\Omega, \phi] = 0 \qquad \text{on } \partial_+\Omega\,,$$

and thus the proof is complete. □

By the previous Proposition 2.10 one readily verifies the validity of the following Proposition 2.11 where we introduce image space $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$ of $v_G[\partial\Omega, \cdot]_{|\partial_+\Omega}$.

Proposition 2.11 (Image of the single layer potential derived by G)**.** *Let $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$ denote the vector space*

$$\mathscr{V}^{1,\alpha}(\partial_+\Omega) = \left\{ v_G[\partial\Omega, \phi]_{|\partial_+\Omega},\ \forall \phi \in \mathscr{C}^{0,\alpha}_+(\partial\Omega) \right\}.$$

Let $\|\cdot\|_{\mathscr{V}^{1,\alpha}(\partial_+\Omega)}$ be the norm on $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$ defined by

$$\|f\|_{\mathscr{V}^{1,\alpha}(\partial_+\Omega)} \equiv \|\phi\|_{\mathscr{C}^{0,\alpha}_+(\partial\Omega)}$$

for all $(f, \phi) \in \mathscr{V}^{1,\alpha}(\partial_+\Omega) \times \mathscr{C}^{0,\alpha}_+(\partial\Omega)$ such that $f = v_G[\partial\Omega, \phi]_{|\partial_+\Omega}$. Then the following statements hold.

(i) $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$ endowed with the norm $\|\cdot\|_{\mathscr{V}^{1,\alpha}(\partial_+\Omega)}$ is a Banach space.

(ii) The operator $v_G[\partial\Omega, \cdot]_{|\partial\Omega}$ is an homeomorphism from $\mathscr{C}^{0,\alpha}_+(\partial\Omega)$ to $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$.

2.3.1 Characterisation of the image of the single layer potential

We wish now to characterise the functions of $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$, that is the set of the elements of $\mathscr{C}^{1,\alpha}(\partial\Omega)$ that can be represented as $v_G[\partial\Omega,\phi]_{|\partial_+\Omega}$ for some $\phi\in\mathscr{C}^{0,\alpha}_+(\partial\Omega)$. We do so in the following Proposition 2.12.

Proposition 2.12. *Let $f\in\mathscr{C}^{1,\alpha}(\partial_+\Omega)$. Then f belongs to $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$ if and only if $f=u^e_{|\partial_+\Omega}$, where u^e is a function of $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+\setminus\Omega})$ such that*

$$\begin{cases}\Delta u^e=0 & \text{in } \mathbb{R}^n_+\setminus\overline{\Omega},\\ u^e=0 & \text{on } \partial\mathbb{R}^n_+\setminus\partial_0\Omega,\\ \lim_{\mathsf{x}\to\infty}\frac{1}{|\mathsf{x}|}u^e(\mathsf{x})=0,\\ \lim_{\mathsf{x}\to\infty}\frac{\mathsf{x}}{|\mathsf{x}|}\cdot\nabla u^e(\mathsf{x})=0.\end{cases}\tag{2.5}$$

Proof of Proposition 2.12. We divide the proof in three steps.

• *First step: Green-like representation formulas in $\mathbb{R}^n_+\setminus\overline{\Omega}$.* As a first step, we prove representation formulas for harmonic functions in the set $\mathbb{R}^n_+\setminus\overline{\Omega}$. To do so, we first introduce a suitable Green-like representation formula in the following Lemma 2.13.

Lemma 2.13. *Let $u^e\in\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+\setminus\Omega})$ be such that*

$$\begin{cases}\Delta u^e=0 & \text{in } \mathbb{R}^n_+\setminus\overline{\Omega},\\ \lim_{|\mathsf{x}|\to\infty}\frac{1}{|\mathsf{x}|}u^e(\mathsf{x})=0,\\ \lim_{|\mathsf{x}|\to\infty}\frac{\mathsf{x}}{|\mathsf{x}|}\cdot\nabla u^e(\mathsf{x})=0.\end{cases}$$

Then we have

$$\begin{aligned}-w_G[\partial_+\Omega,u^e_{|\partial_+\Omega}](\mathsf{x})+v_G[\partial\Omega,\nu_\Omega\cdot\nabla u^e_{|\partial\Omega}](\mathsf{x})+\frac{2x_n}{s_n}\int_{\partial\mathbb{R}^n_+\setminus\partial_0\Omega}\frac{u^e(\mathsf{y})}{|\mathsf{x}-\mathsf{y}|^n}\,d\sigma_{\mathsf{y}}&\\ =\begin{cases}u^e(\mathsf{x}) & \forall\mathsf{x}\in\mathbb{R}^n_+\setminus\overline{\Omega},\\ 0 & \forall\mathsf{x}\in\Omega.\end{cases}&\end{aligned}$$

Proof. Let $R>\max_{\mathsf{x}\in\overline{\Omega}}|\mathsf{x}|$. Let $\Omega^{e,+}_R\equiv\mathbb{R}^n_+\cap\mathcal{B}(0,R)\setminus\overline{\Omega}$. Let $\mathsf{x}\in\Omega^{e,+}_R$. Let $r>0$ and $\mathcal{B}(\mathsf{x},r)\subseteq\Omega^{e,+}_R$. Then by Lemma 2.8 we have

$$\begin{aligned}u^e(\mathsf{x})&=w_G[\partial\mathcal{B}(\mathsf{x},r),u^e_{|\partial\mathcal{B}(\mathsf{x},r)}](\mathsf{x})-v_G[\partial\mathcal{B}(\mathsf{x},r),\nu_{\mathcal{B}(\mathsf{x},r)}\cdot\nabla u^e_{|\partial\mathcal{B}(\mathsf{x},r)}](\mathsf{x})\\ &=\int_{\partial\mathcal{B}(\mathsf{x},r)}u^e(\mathsf{y})\,\nu_{\mathcal{B}(\mathsf{x},r)}(\mathsf{y})\cdot\nabla_{\mathsf{y}}G(\mathsf{x},\mathsf{y})-G(\mathsf{x},\mathsf{y})\,\nu_{\mathcal{B}(\mathsf{x},r)}(\mathsf{y})\cdot\nabla u^e(\mathsf{y})\,d\sigma_{\mathsf{y}}\,.\end{aligned}\tag{2.6}$$

Then we observe that $G(\mathsf{x},\cdot)$ is a harmonic function in $\Omega^{e,+}_R\setminus\overline{\mathcal{B}(\mathsf{x},r)}$ and thus by the divergence theorem we have

$$
\begin{aligned}
0 = &\int_{\Omega_R^{e,+}\setminus\overline{\mathcal{B}(\mathsf{x},r)}} u^e(\mathsf{y})\Delta_{\mathsf{y}}G(\mathsf{x},\mathsf{y}) - G(\mathsf{x},\mathsf{y})\Delta u^e(\mathsf{y})\,d\mathsf{x}\\
= &-\int_{\partial\mathcal{B}(\mathsf{x},r)} u^e(\mathsf{y})\;\nu_{\mathcal{B}(\mathsf{x},r)}(\mathsf{y})\cdot\nabla_{\mathsf{y}}G(\mathsf{x},\mathsf{y}) - G(\mathsf{x},\mathsf{y})\;\nu_{\mathcal{B}(\mathsf{x},r)}(\mathsf{y})\cdot\nabla u^e(\mathsf{y})\;d\sigma_{\mathsf{y}}\\
&+\int_{\partial\Omega_R^{e,+}} u^e(\mathsf{y})\;\nu_{\Omega_R^{e,+}}(\mathsf{y})\cdot\nabla_{\mathsf{y}}G(\mathsf{x},\mathsf{y}) - G(\mathsf{x},\mathsf{y})\;\nu_{\Omega_R^{e,+}}(\mathsf{y})\cdot\nabla u^e(\mathsf{y})\;d\sigma_{\mathsf{y}}\\
= &-\int_{\partial\mathcal{B}(\mathsf{x},r)} u^e(\mathsf{y})\;\nu_{\mathcal{B}(\mathsf{x},r)}(\mathsf{y})\cdot\nabla_{\mathsf{y}}G(\mathsf{x},\mathsf{y}) - G(\mathsf{x},\mathsf{y})\;\nu_{\mathcal{B}(\mathsf{x},r)}(\mathsf{y})\cdot\nabla u^e(\mathsf{y})\;d\sigma_{\mathsf{y}}\\
&-\int_{\partial_+\Omega} u^e(\mathsf{y})\;\nu_{\Omega}(\mathsf{y})\cdot\nabla_{\mathsf{y}}G(\mathsf{x},\mathsf{y}) - G(\mathsf{x},\mathsf{y})\;\nu_{\Omega}(\mathsf{y})\cdot\nabla u^e(\mathsf{y})\;d\sigma_{\mathsf{y}}\\
&+\int_{\partial_+\mathcal{B}(0,R)} u^e(\mathsf{y})\;\nu_{\mathcal{B}(0,R)}(\mathsf{y})\cdot\nabla_{\mathsf{y}}G(\mathsf{x},\mathsf{y}) - G(\mathsf{x},\mathsf{y})\;\nu_{\mathcal{B}(0,R)}(\mathsf{y})\cdot\nabla u^e(\mathsf{y})\;d\sigma_{\mathsf{y}}\\
&-\int_{\partial\mathbb{R}^n_+\cap\mathcal{B}(0,R)\setminus\partial_0\Omega} u^e(\mathsf{y})\;\partial_{y_n}G(\mathsf{x},\mathsf{y}) - G(\mathsf{x},\mathsf{y})\;\partial_{y_n}u^e(\mathsf{y})\;d\sigma_{\mathsf{y}}.
\end{aligned}
$$

Using Definition 2.6 and the fact that $G(\mathsf{x},\mathsf{y}) = 0$ and $\partial_{y_n}G(\mathsf{x},\mathsf{y}) = -2x_n s_n^{-1}|\mathsf{x}-\mathsf{y}|^{-n}$ for all $\mathsf{y}\in\partial\mathbb{R}^n_+$, we deduce

$$
\begin{aligned}
0 = &-\int_{\partial\mathcal{B}(\mathsf{x},r)} u^e(\mathsf{y})\;\nu_{\mathcal{B}(\mathsf{x},r)}(\mathsf{y})\cdot\nabla_{\mathsf{y}}G(\mathsf{x},\mathsf{y}) - G(\mathsf{x},\mathsf{y})\;\nu_{\mathcal{B}(\mathsf{x},r)}(\mathsf{y})\cdot\nabla u^e(\mathsf{y})\;d\sigma_{\mathsf{y}}\\
&- w_G[\partial_+\Omega, u^e_{|\partial_+\Omega}](\mathsf{x}) + v_G[\partial_+\Omega, \nu_\Omega\cdot\nabla u^e_{|\partial_+\Omega}](\mathsf{x})\\
&+\int_{\partial_+\mathcal{B}(0,R)} u^e(\mathsf{y})\;\nu_{\mathcal{B}(0,R)}(\mathsf{y})\cdot\nabla_{\mathsf{y}}G(\mathsf{x},\mathsf{y}) - G(\mathsf{x},\mathsf{y})\;\nu_{\mathcal{B}(0,R)}(\mathsf{y})\cdot\nabla u^e(\mathsf{y})\;d\sigma_{\mathsf{y}}\\
&+\frac{2x_n}{s_n}\int_{\partial\mathbb{R}^n_+\cap\mathcal{B}(0,R)\setminus\partial_0\Omega}\frac{u^e(\mathsf{y})}{|\mathsf{x}-\mathsf{y}|^n}\,d\sigma_{\mathsf{y}}.
\end{aligned}
$$

Then we observe that the maps $\mathsf{y}\mapsto|\mathsf{y}|^{n-1}G(\mathsf{x},\mathsf{y})$ and $\mathsf{y}\mapsto|\mathsf{y}|^n\nabla_{\mathsf{y}}G(\mathsf{x},\mathsf{y})$ are bounded in $\partial\mathbb{R}^n_+$. Thus, by taking the limit as $R\to\infty$ we obtain

$$
\begin{aligned}
0 = &-\int_{\partial\mathcal{B}(x,r)} u^e(\mathsf{y})\;\nu_{\mathcal{B}(\mathsf{x},r)}(\mathsf{y})\cdot\nabla_{\mathsf{y}}G(\mathsf{x},\mathsf{y}) - G(\mathsf{x},\mathsf{y})\;\nu_{\mathcal{B}(\mathsf{x},r)}(\mathsf{y})\cdot\nabla u^e(\mathsf{y})\;d\sigma_{\mathsf{y}}\\
&- w_G[\partial_+\Omega, u^e_{|\partial_+\Omega}](\mathsf{x}) + v_G[\partial\Omega, \nu_\Omega\cdot\nabla u^e_{|\partial\Omega}](\mathsf{x}) + \frac{2x_n}{s_n}\int_{\partial\mathbb{R}^n_+\setminus\partial_0\Omega}\frac{u^e(\mathsf{y})}{|\mathsf{x}-\mathsf{y}|^n}\,d\sigma_{\mathsf{y}}\,.
\end{aligned}
\tag{2.7}
$$

Then by summing (2.6) and (2.7) we show the validity of the first equality in the statement. The proof of the second equality is similar and accordingly omitted. □

Incidentally, we observe that under the assumptions of Lemma 2.13 the integral

$$
\int_{\partial\mathbb{R}^n_+\setminus\partial_0\Omega}\frac{u^e(\mathsf{y})}{|\mathsf{x}-\mathsf{y}|^n}\,d\sigma_{\mathsf{y}}
$$

exists finite for all $\mathsf{x}\in\mathbb{R}^n_+\setminus\partial\Omega$.

• *Second step: representation in terms of single layer potentials plus an extra term.* In the following Proposition 2.14, we introduce a representation formula for a suitable family of functions of $\mathscr{C}^{1,\alpha}(\partial\Omega)$. More precisely, we show that the restriction to $\partial_+\Omega$ of a function $f \in \mathscr{C}^{1,\alpha}(\partial\Omega)$ which satisfies certain assumptions can be written as the sum of a single layer potential with kernel G plus an extra term.

Proposition 2.14. *Let $f \in \mathscr{C}^{1,\alpha}(\partial\Omega)$ with $f_{|\partial_0\Omega} = 0$. Assume that there exists a function $u^e \in \mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+ \setminus \Omega})$ such that*

$$\begin{cases} \Delta u^e = 0 & \text{in } \mathbb{R}^n_+ \setminus \overline{\Omega}, \\ u^e = f & \text{on } \partial_+\Omega, \\ \lim_{\mathsf{x}\to\infty} \frac{1}{|\mathsf{x}|} u^e(\mathsf{x}) = 0, \\ \lim_{\mathsf{x}\to\infty} \frac{\mathsf{x}}{|\mathsf{x}|} \cdot \nabla u^e(\mathsf{x}) = 0. \end{cases}$$

Then there exists $\phi \in \mathscr{C}^{0,\alpha}(\partial\Omega)$ such that

$$v_G[\partial\Omega, \phi](\mathsf{x}) + \frac{2x_n}{s_n} \int_{\partial\mathbb{R}^n_+ \setminus \partial_0\Omega} \frac{u^e(\mathsf{y})}{|\mathsf{x}-\mathsf{y}|^n}\, d\sigma_{\mathsf{y}} = f(\mathsf{x}), \qquad \forall \mathsf{x} \in \partial_+\Omega. \tag{2.8}$$

Proof. Let $u^i \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$ be the solution of the Dirichlet problem with boundary datum f. By Lemma 2.8 we have

$$0 = w_G[\partial\Omega, u^i_{|\partial\Omega}](\mathsf{x}) - v_G[\partial\Omega, \nu_\Omega \cdot \nabla u^i_{|\partial\Omega}](\mathsf{x}), \qquad \forall \mathsf{x} \in \mathbb{R}^n_+ \setminus \overline{\Omega}.$$

Since $u^i_{|\partial_0\Omega} = f_{|\partial_0\Omega} = 0$ we deduce that

$$0 = w_G[\partial_+\Omega, f_{|\partial_+\Omega}](\mathsf{x}) - v_G[\partial\Omega, \nu_\Omega \cdot \nabla u^i_{|\partial\Omega}](\mathsf{x}), \qquad \forall \mathsf{x} \in \mathbb{R}^n_+ \setminus \overline{\Omega}. \tag{2.9}$$

By Lemma 2.13 we have

$$u^e(\mathsf{x}) = -w_G[\partial_+\Omega, f_{|\partial_+\Omega}](\mathsf{x}) + v_G[\partial\Omega, \nu_\Omega \cdot \nabla u^e_{|\partial\Omega}](\mathsf{x}) + \frac{2x_n}{s_n} \int_{\partial\mathbb{R}^n_+ \setminus \partial_0\Omega} \frac{u^e(\mathsf{y})}{|\mathsf{x}-\mathsf{y}|^n}\, d\sigma_{\mathsf{y}} \tag{2.10}$$

for all $\mathsf{x} \in \mathbb{R}^n_+ \setminus \overline{\Omega}$. Then by taking the sum of (2.9) and (2.10) and by the continuity properties of the (Green) single layer potential one verifies that the proposition holds with

$$\phi = \nu_\Omega \cdot \nabla u^e_{|\partial\Omega} - \nu_\Omega \cdot \nabla u^i_{|\partial\Omega}.$$

□

• *Last step: vanishing of the extra term in* (2.8). In order to understand what can be represented just by means of the single layer potential, the final step is to understand when such an extra term vanishes. So let $f \in \mathscr{C}^{1,\alpha}(\partial_+\Omega)$ be such that $f = u^e_{|\partial_+\Omega}$, where u^e is a function of $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+ \setminus \Omega})$ such that (2.5) holds. Then

$$\frac{2x_n}{s_n} \int_{\partial\mathbb{R}^n_+ \setminus \partial_0\Omega} \frac{u^e(\mathsf{y})}{|\mathsf{x}-\mathsf{y}|^n}\, d\sigma_{\mathsf{y}} = 0, \qquad \forall \mathsf{x} \in \partial_+\Omega,$$

and thus (2.8) implies that $f \in \mathscr{V}^{1,\alpha}(\partial_+\Omega)$. Conversely, if $f \in \mathscr{V}^{1,\alpha}(\partial_+\Omega)$ then there exists $\phi \in \mathscr{C}^{0,\alpha}_+(\partial\Omega)$ such that $f = v_G[\partial\Omega,\phi]_{|\partial_+\Omega}$ and the function $u^e \equiv v_G[\partial\Omega,\phi]_{|\overline{\mathbb{R}^n_+\setminus\Omega}}$ satisifies (2.5). This concludes the proof of Proposition 2.12. □

Now that Proposition 2.12 is proved, we observe that if u^e is as in Proposition 2.14, then

$$\lim_{t\to 0^+} \frac{2t}{s_n}\int_{\partial\mathbb{R}^n_+\setminus\partial_0\Omega} \frac{u^e(\mathsf{y})}{|\mathsf{x}+t\mathsf{e}_n-\mathsf{y}|^n}\,d\sigma_{\mathsf{y}} = u^e(\mathsf{x}), \qquad \forall \mathsf{x}\in\partial\mathbb{R}^n_+\setminus\partial_0\Omega\,, \tag{2.11}$$

where e_n denotes the vector $(0,\dots,0,1)\in\mathbb{R}^n$. The limit in (2.11) can be computed by exploiting known results in potential theory (cf. Cialdea [9, Thm. 1]). A consequence of (2.11) is that the second term in the left hand side of (2.8) vanishes on $\partial_+\Omega$ only if $u^e_{|\partial\mathbb{R}^n_+\setminus\partial_0\Omega} = 0$. Namely, we have the following

Proposition 2.15. *Let u^e be as in Proposition 2.14. Then we have*

$$\frac{2x_n}{s_n}\int_{\partial\mathbb{R}^n_+\setminus\partial_0\Omega} \frac{u^e(\mathsf{y})}{|\mathsf{x}-\mathsf{y}|^n}\,d\sigma_{\mathsf{y}} = 0, \qquad \forall \mathsf{x}\in\partial_+\Omega \tag{2.12}$$

if and only if

$$u^e_{|\partial\mathbb{R}^n_+\setminus\partial_0\Omega} = 0\,. \tag{2.13}$$

Proof. One immediately verifies that (2.13) implies (2.12). To prove that (2.12) implies (2.13), we denote by U^+ the function of $\mathsf{x}\in\mathbb{R}^n\setminus(\partial\mathbb{R}^n_+\setminus\partial_0\Omega)$ defined by the left hand side of (2.12). Then, we observe that, by the properties of integral operators with real analytic kernel and no singularity, U^+ is harmonic in $\mathbb{R}^n\setminus(\partial\mathbb{R}^n_+\setminus\partial_0\Omega)$ and vanishes on $\partial_0\Omega$. Thus, (2.12) implies that $U^+=0$ on the whole of $\partial\Omega$ and by the uniqueness of the solution of the Dirichlet problem we have that $U^+=0$ on Ω. By the identity principle for analytic functions it follows that $U^+=0$ on $\mathbb{R}^n\setminus(\partial\mathbb{R}^n_+\setminus\partial_0\Omega)$ and thus, by (2.11), we have

$$u^e(\mathsf{x}) = \lim_{t\to 0^+} U^+(\mathsf{x}+t\mathsf{e}_n) = 0 \qquad \forall \mathsf{x}\in\partial\mathbb{R}^n_+\setminus\partial_0\Omega.$$

□

In Remark 2.16 here below we observe that a function u^e which satisfies the conditions of Proposition 2.14 actually exists and that the second term in the left hand side of (2.8) cannot be in general omitted.

Remark 2.16. *Let $f\in\mathscr{C}^{1,\alpha}(\partial\Omega)$ with $f_{|\partial_0\Omega}=0$ and let $u_\#\in\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega)$ be the unique solution of the Dirichlet problem in $\mathbb{R}^n\setminus\overline{\Omega}$ with boundary datum f which satisfies the decay condition $\lim_{\mathsf{x}\to\infty}u_\#(\mathsf{x})=0$ if $n\geq 3$ and such that $u_\#$ is bounded if $n=2$ (i.e., $u_\#$ is harmonic at ∞). Then the function $u^e_\#\equiv u_{\#|\overline{\mathbb{R}^n_+\setminus\Omega}}$ satisfies the conditions of Proposition 2.14. In addition, $u^e_{\#|\partial\mathbb{R}^n_+\setminus\partial_0\Omega}=0$ only if $f=0$, and thus the corresponding second term in the left hand side of (2.8) is 0 only if $f=0$ (cf. Proposition 2.15). The latter fact can be proved by observing that if $u^e_{\#|\partial\mathbb{R}^n_+\setminus\partial_0\Omega}=0$, then $u_{\#|\partial\mathbb{R}^n_+\setminus\partial_0\Omega}=0$ and thus $u_{\#|\partial\mathbb{R}^n_+}=0$ (because $u_{\#|\partial_0\Omega}=f_{|\partial_0\Omega}=0$ by our assumptions on f). Then, by the decay properties of $u_\#$ and by the divergence theorem we have*

$$\int_{\mathbb{R}^n_-}|\nabla u_\#|^2\,d\mathsf{x} = \lim_{R\to\infty}\left(\int_{\partial\mathcal{B}(0,R)\cap\mathbb{R}^n_-} u\,\nu_{\partial\mathcal{B}(0,R)}\cdot\nabla u\,d\sigma + \int_{\partial\mathbb{R}^n_+\cap\mathcal{B}(0,R)} u\,\partial_{x_n}u\,d\sigma\right) = 0\,.$$

It follows that $u_{\#|\mathbb{R}^n_-} = 0$, which in turn implies that $u_\# = 0$ by the identity principle of real analytic functions. Hence $f = u_{\#|\partial\Omega} = 0$.

2.4 Additional useful results

2.4.1 Extending functions from $\mathscr{C}^{k,\alpha}(\overline{\partial_+\Omega})$ to $\mathscr{C}^{k,\alpha}(\partial\Omega)$

We will need to pass from functions defined on $\partial_+\Omega$ to functions defined on $\partial\Omega$, and viceversa. On the one hand, the restriction operator from $\mathscr{C}^{k,\alpha}(\partial\Omega)$ to $\mathscr{C}^{k,\alpha}(\overline{\partial_+\Omega})$ is linear and continuous for $k = 0, 1$. On the other hand, we have the following extension result.

Lemma 2.17. *There exist linear and continuous extension operators $E^{k,\alpha}$ from $\mathscr{C}^{k,\alpha}(\overline{\partial_+\Omega})$ to $\mathscr{C}^{k,\alpha}(\partial\Omega)$, for $k = 0, 1$.*

A proof can be effected by arguing as in Troianiello [32, proof of Lem. 1.5, p. 16] and by exploiting condition (H_3). We observe that as a consequence of Lemma 2.17 we can identify $\mathscr{C}^{0,\alpha}_+(\partial\Omega)$ and $\mathscr{C}^{0,\alpha}(\overline{\partial_+\Omega})$.

2.4.2 On the boundary analyticity of u_0

As explained in the Introduction, the regularity condition (H_4) on the Dirichlet datum g^{o} ensures that the solution u_0 of the limit problem (1.4) has a real analytic continuation in a neighbourhood of the origin in $\mathbb{R}^n$. Indeed, by a classical argument based on the Cauchy-Kovalevskaya Theorem one can prove the following proposition (cf. Appendix B).

Proposition 2.18. *There exists $r_1 \in]0, r_0]$ and a function U_0 from $\overline{\mathcal{B}(0, r_1)}$ to $\mathbb{R}$ such that $(\mathcal{B}(0, r_1) \cap \mathbb{R}^n_+) \subseteq \Omega$ and*

$$\begin{cases} \Delta U_0 = 0 & \text{in } \mathcal{B}(0, r_1), \\ U_0 = u_0 & \text{in } \overline{\mathcal{B}(0, r_1) \cap \mathbb{R}^n_+}. \end{cases}$$

Possibly shrinking $\boldsymbol{\varepsilon}^{\mathrm{ad}}$ we can assume:

$$\varepsilon_1 \mathsf{p} + \varepsilon_1 \varepsilon_2 \overline{\omega} \subseteq \mathcal{B}(0, r_1), \qquad \forall \boldsymbol{\varepsilon} \in] - \boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[. \tag{2.14}$$

3 Asymptotic behaviour of the solution of (1.2)

In this section, we investigate the asymptotic behaviour of the solution of problem (1.2) as $\boldsymbol{\varepsilon} \to \mathbf{0}$. For the whole Section 3, the dimension n is assumed to be greater than or equal to 3. Namely,

$$n \in \mathbb{N} \setminus \{0, 1, 2\}.$$

Our strategy is here to reformulate the problem as an equation $\mathfrak{L}[\boldsymbol{\varepsilon}, \boldsymbol{\mu}] = 0$ where $\mathfrak{L}$ is a real analytic function and to use the implicit function theorem.

3.1 Defining the operator $\mathfrak{L}$

Let $\boldsymbol{\varepsilon} \in]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$. We start from the Green-like representation formula of Lemma 2.8. For the solution $u_{\boldsymbol{\varepsilon}}$ of (1.2) we can write:

$$\begin{aligned} u_{\boldsymbol{\varepsilon}} =& w^i_G[\partial\Omega_{\boldsymbol{\varepsilon}}, u_{\boldsymbol{\varepsilon}|\partial\Omega_{\boldsymbol{\varepsilon}}}] - v^i_G[\partial\Omega_{\boldsymbol{\varepsilon}}, \nu_{\Omega_{\boldsymbol{\varepsilon}}} \cdot \nabla u_{\boldsymbol{\varepsilon}|\partial\Omega_{\boldsymbol{\varepsilon}}}] \\ =& w^i_G[\partial\Omega, g^{\mathrm{o}}] - w^e_G\Big[\partial\omega_{\boldsymbol{\varepsilon}}, g^{\mathrm{i}}\big(\frac{\cdot - \varepsilon_1 \mathsf{p}}{\varepsilon_1\varepsilon_2}\big)\Big] - v^i_G[\partial\Omega, \nu_\Omega \cdot \nabla u_{\boldsymbol{\varepsilon}|\partial\Omega}] + v^e_G[\partial\omega_{\boldsymbol{\varepsilon}}, \nu_{\omega_{\boldsymbol{\varepsilon}}} \cdot \nabla u_{\boldsymbol{\varepsilon}|\partial\omega_{\boldsymbol{\varepsilon}}}] \,. \end{aligned}$$

By adding and subtracting $v^i_G[\partial\Omega, \nu_\Omega \cdot \nabla u_{0|\partial\Omega}]$ we get

$$\begin{aligned} u_{\boldsymbol{\varepsilon}} =& w^i_G[\partial\Omega, g^{\mathrm{o}}] - v^i_G[\partial\Omega, \nu_\Omega \cdot \nabla u_{0|\partial\Omega}] - w^e_G\Big[\partial\omega_{\boldsymbol{\varepsilon}}, g^{\mathrm{i}}\big(\frac{\cdot - \varepsilon_1 \mathsf{p}}{\varepsilon_1\varepsilon_2}\big)\Big] \\ & - v^i_G[\partial\Omega, \nu_\Omega \cdot \nabla u_{\boldsymbol{\varepsilon}|\partial\Omega} - \nu_\Omega \cdot \nabla u_{0|\partial\Omega}] + v^e_G[\partial\omega_{\boldsymbol{\varepsilon}}, \nu_{\omega_{\boldsymbol{\varepsilon}}} \cdot \nabla u_{\boldsymbol{\varepsilon}|\partial\omega_{\boldsymbol{\varepsilon}}}] \,. \end{aligned} \tag{3.1}$$

Then we note that

$$u_0 = w^i_G[\partial\Omega, g^{\mathrm{o}}] - v^i_G[\partial\Omega, \nu_\Omega \cdot \nabla u_{0|\partial\Omega}]$$

and we think to the functions

$$\nu_\Omega \cdot \nabla u_{\boldsymbol{\varepsilon}|\partial\Omega} - \nu_\Omega \cdot \nabla u_{0|\partial\Omega} \,, \qquad \nu_{\omega_{\boldsymbol{\varepsilon}}} \cdot \nabla u_{\boldsymbol{\varepsilon}|\partial\omega_{\boldsymbol{\varepsilon}}}$$

as to unknown densities which have to be determined in order to solve problem (1.2). Accordingly, inspired by (3.1) and by the rule of change of variables in integrals, we look for a solution of problem (1.2) in the form

$$\begin{aligned} u_0(\mathsf{x}) - \varepsilon_1^{n-1}\varepsilon_2^{n-1} \int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot (\nabla_{\mathsf{y}} G)(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y}) g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}} - \int_{\partial_+\Omega} G(\mathsf{x},\mathsf{y})\mu_1(\mathsf{y})\, d\sigma_{\mathsf{y}} \\ + \varepsilon_1^{n-2}\varepsilon_2^{n-2} \int_{\partial\omega} G(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y})\mu_2(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \,, \qquad \mathsf{x} \in \Omega_{\boldsymbol{\varepsilon}}, \end{aligned} \tag{3.2}$$

where the pair $(\mu_1, \mu_2) \in \mathscr{C}^{0,\alpha}_+(\partial\Omega) \times \mathscr{C}^{0,\alpha}(\partial\omega)$ has to be determined. We set $\boldsymbol{\mu} \equiv (\mu_1, \mu_2)$ and $\mathscr{B}_1 \equiv \mathscr{C}^{0,\alpha}_+(\partial\Omega) \times \mathscr{C}^{0,\alpha}(\partial\omega)$. Since the function in (3.2) is harmonic in $\Omega_{\boldsymbol{\varepsilon}}$ for all $\boldsymbol{\mu} \in \mathscr{B}_1$, we just need to choose $\boldsymbol{\mu} \in \mathscr{B}_1$ such that the boundary conditions are satisfied. By the jump properties of the layer potentials derived by G, this is equivalent to ask that $\boldsymbol{\mu} \in \mathscr{B}_1$ solves

$$\mathfrak{L}[\boldsymbol{\varepsilon}, \boldsymbol{\mu}] = 0, \tag{3.3}$$

where $\mathfrak{L}[\boldsymbol{\varepsilon}, \boldsymbol{\mu}] \equiv (\mathfrak{L}_1[\boldsymbol{\varepsilon}, \boldsymbol{\mu}], \mathfrak{L}_2[\boldsymbol{\varepsilon}, \boldsymbol{\mu}])$ is defined by

$$\begin{aligned} \mathfrak{L}_1[\boldsymbol{\varepsilon}, \boldsymbol{\mu}](\mathsf{x}) \equiv& v_G[\partial\Omega, \mu_1](\mathsf{x}) \\ & - \varepsilon_1^{n-2}\varepsilon_2^{n-2} \int_{\partial\omega} G(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y})\, \mu_2(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \\ & + \varepsilon_1^{n-1}\varepsilon_2^{n-1} \int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot (\nabla_{\mathsf{y}} G)(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y})\, g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \qquad \forall \mathsf{x} \in \partial_+\Omega \,, \\ \mathfrak{L}_2[\boldsymbol{\varepsilon}, \boldsymbol{\mu}](t) \equiv& v_{S_n}[\partial\omega, \mu_2](\mathsf{X}) - \varepsilon_2^{n-2} \int_{\partial\omega} S_n(-2p_n\mathsf{e}_n + \varepsilon_2(\varsigma(\mathsf{X}) - \mathsf{Y}))\, \mu_2(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \\ & - \int_{\partial_+\Omega} G(\varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{X}, \mathsf{y})\, \mu_1(\mathsf{y})\, d\sigma_{\mathsf{y}} \\ & - \varepsilon_2^{n-1} \int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot \nabla S_n(-2p_n\mathsf{e}_n + \varepsilon_2(\varsigma(\mathsf{X}) - \mathsf{Y}))\, g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \\ & + U_0(\varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{X}) - w_{S_n}[\partial\omega, g^{\mathrm{i}}](\mathsf{X}) - \frac{g^{\mathrm{i}}(\mathsf{X})}{2} \qquad \forall \mathsf{X} \in \partial\omega \,, \end{aligned}$$

with p_n as in (1.1) (note that $p_n > 0$ by the membership of p in $\mathbb{R}^n_+$) and U_0 as in Proposition 2.18.

3.2 Real analyticity of the operator $\mathfrak{L}$

By the equivalence of the boundary value problem (1.2) and the functional equation (3.3), we can deduce results for the map $\boldsymbol{\varepsilon} \mapsto u_{\boldsymbol{\varepsilon}}$ by studying the dependence of $\boldsymbol{\mu}$ upon $\boldsymbol{\varepsilon}$ in (3.3). To do so, we plan to apply the implicit function theorem for real analytic maps and, as a first step, we wish to prove that the operator $\mathfrak{L}$ is real analytic.

Proposition 3.1 (Real analyticity of $\mathfrak{L}$)**.** *The map*

$$\begin{array}{rcl}]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathscr{B}_1 & \to & \mathscr{V}^{1,\alpha}(\partial_+\Omega) \times \mathscr{C}^{1,\alpha}(\partial\omega) \\ (\boldsymbol{\varepsilon}, \boldsymbol{\mu}) & \mapsto & \mathfrak{L}[\boldsymbol{\varepsilon}, \boldsymbol{\mu}] \end{array}$$

is real analytic.

Proof. We split the proof component by component.

Study of $\mathfrak{L}_1$. Here we prove that $\mathfrak{L}_1$ is real analytic from $]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathscr{B}_1$ to $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$. *First step: the range of $\mathfrak{L}_1$ is a subset of $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$.* Let $U^e[\boldsymbol{\varepsilon}, \boldsymbol{\mu}]$ denote the function from $\overline{\mathbb{R}^n_+ \setminus \Omega}$ to $\mathbb{R}$ defined by

$$\begin{aligned} U^e[\boldsymbol{\varepsilon}, \boldsymbol{\mu}](\mathsf{x}) \equiv\, & v^e_G[\partial\Omega, \mu_1](\mathsf{x}) \\ & - \varepsilon_1^{n-2}\varepsilon_2^{n-2} \int_{\partial\omega} G(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y})\, \mu_2(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \\ & + \varepsilon_1^{n-1}\varepsilon_2^{n-1} \int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot (\nabla_{\mathsf{y}} G)(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y})\, g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}}\,. \end{aligned}$$

Then, by the properties of the (Green) single layer potential and by the properties of integral operators with real analytic kernel and no singularity one verifies that $U^e[\boldsymbol{\varepsilon}, \boldsymbol{\mu}] \in \mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+ \setminus \Omega})$. In addition, one has

$$\begin{cases} \Delta U^e[\boldsymbol{\varepsilon}, \boldsymbol{\mu}] = 0 & \text{in } \mathbb{R}^n_+ \setminus \overline{\Omega}, \\ U^e[\boldsymbol{\varepsilon}, \boldsymbol{\mu}] = 0 & \text{on } \partial\mathbb{R}^n_+ \setminus \partial_0\Omega, \\ \lim_{\mathsf{x}\to\infty} U^e[\boldsymbol{\varepsilon}, \boldsymbol{\mu}](\mathsf{x}) = 0, & \\ \lim_{\mathsf{x}\to\infty} \frac{\mathsf{x}}{|\mathsf{x}|} \cdot \nabla U^e[\boldsymbol{\varepsilon}, \boldsymbol{\mu}](\mathsf{x}) = 0 & \end{cases}$$

Thus U^e satisfies the conditions of Proposition 2.12. Accordingly, we conclude that $\mathfrak{L}_1[\boldsymbol{\varepsilon}, \boldsymbol{\mu}] = U^e[\boldsymbol{\varepsilon}, \boldsymbol{\mu}]_{|\partial_+\Omega}$ belongs to $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$.

Second step: $\mathfrak{L}_1$ is real analytic. We decompose $\mathfrak{L}_1$ and study each part separately.

- By the definition of $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$ in Proposition 2.11, one readily verifies that the map $\mu_1 \mapsto v_G[\partial\Omega, \mu_1]_{|\partial_+\Omega}$ is linear and continuous from $\mathscr{C}^{0,\alpha}_+(\partial\Omega)$ to $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$ and therefore real analytic.

- We now consider the map which takes $(\boldsymbol{\varepsilon}, \mu_2)$ to the function $f[\boldsymbol{\varepsilon}, \mu_2](\mathsf{x})$ of $\mathsf{x} \in \overline{\partial_+\Omega}$ defined by

$$f[\boldsymbol{\varepsilon}, \mu_2](\mathsf{x}) \equiv \int_{\partial\omega} G(\mathsf{x}, \varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2 \mathsf{Y})\, \mu_2(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \qquad \forall \mathsf{x} \in \overline{\partial_+\Omega}.$$

We wish to prove that f is real analytic from $]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathscr{C}^{0,\alpha}(\partial\Omega)$ to $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$ by showing that there is a real analytic function

$$\phi :]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathscr{C}^{0,\alpha}(\partial\Omega) \to \mathscr{C}^{0,\alpha}_+(\partial\Omega) \tag{3.4}$$

such that

$$f[\boldsymbol{\varepsilon}, \mu_2] = v_G[\partial\Omega, \phi[\boldsymbol{\varepsilon}, \mu_2]]_{|\partial_+\Omega} \tag{3.5}$$

for all $(\boldsymbol{\varepsilon}, \mu_2) \in]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathscr{C}^{0,\alpha}(\partial\Omega)$. Then the real analyticity of f follows by the definition of $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$ in Proposition 2.11.

We will obtain ϕ as the sum of two real analytic terms. To find the first we observe that, by the properties of integral operators with real analytic kernel and no singularity, the map $(\boldsymbol{\varepsilon}, \mu_2) \mapsto f[\boldsymbol{\varepsilon}, \mu_2]$ is real analytic from $]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathscr{C}^{0,\alpha}(\partial\omega)$ to $\mathscr{C}^{1,\alpha}(\overline{\partial_+\Omega})$ (see Lanza de Cristoforis and Musolino [25, Prop. 4.1 (ii)]). Then, by the extension Lemma 2.17, we deduce that the map

$$\begin{array}{rcl}]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathscr{C}^{0,\alpha}(\partial\Omega) & \to & \mathscr{C}^{1,\alpha}(\partial\Omega) \\ (\boldsymbol{\varepsilon}, \mu_2) & \mapsto & E^{1,\alpha} f[\boldsymbol{\varepsilon}, \mu_2] \end{array}$$

is real analytic. Let now $u^i[\boldsymbol{\varepsilon}, \mu_2]$ denote the unique solution of the Dirichlet problem for the Laplace equation in Ω with boundary datum $E^{1,\alpha} f[\boldsymbol{\varepsilon}, \mu_2]$. As is well-known, the map from $\mathscr{C}^{1,\alpha}(\partial\Omega)$ to $\mathscr{C}^{1,\alpha}(\overline{\Omega})$ which takes a function ψ to the unique solution of the Dirichlet problem for the Laplace equation in Ω with boundary datum ψ is linear and continuous. It follows that the map from $]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathscr{C}^{0,\alpha}(\partial\omega)$ to $\mathscr{C}^{1,\alpha}(\overline{\Omega})$ which takes $(\boldsymbol{\varepsilon}, \mu_2)$ to $u^i[\boldsymbol{\varepsilon}, \mu_2]$ is real analytic. Thus the map

$$\begin{array}{rcl}]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathscr{C}^{0,\alpha}(\partial\omega) & \to & \mathscr{C}^{0,\alpha}_+(\partial\Omega) \\ (\boldsymbol{\varepsilon}, \mu_2) & \mapsto & \nu_\Omega \cdot \nabla u^i[\boldsymbol{\varepsilon}, \mu_2]_{|\partial\Omega} \end{array} \tag{3.6}$$

is real analytic.

The function in (3.6) is the first term in the sum that gives ϕ. To obtain the second term we define

$$u^e[\boldsymbol{\varepsilon}, \mu_2](\mathsf{x}) \equiv \int_{\partial\omega} G(\mathsf{x}, \varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2 \mathsf{Y})\, \mu_2(\mathsf{Y})\, d\sigma_{\mathsf{Y}}\,, \qquad \forall \mathsf{x} \in \overline{\mathbb{R}^n_+ \setminus \Omega}\,.$$

Then, by standard properties of integral operators with real analytic kernels and no singularity one verifies that the map from $]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathscr{C}^{0,\alpha}(\partial\omega)$ to $\mathscr{C}^{0,\alpha}(\overline{\partial_+\Omega})$ which takes $(\boldsymbol{\varepsilon}, \mu_2)$ to

$$\nu_\Omega \cdot \nabla u^e[\boldsymbol{\varepsilon}, \mu_2](\mathsf{x}) = \nu_\Omega(\mathsf{x}) \cdot \int_{\partial\omega} \nabla_{\mathsf{x}} G(\mathsf{x}, \varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2 \mathsf{Y})\, \mu_2(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \qquad \forall \mathsf{x} \in \overline{\partial_+\Omega}$$

is real analytic (see Lanza de Cristoforis and Musolino [25, Prop. 4.1 (ii)]). Thus, by the extension Lemma 2.17, we can show that the map

$$\begin{aligned}]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathscr{C}^{0,\alpha}(\partial\omega) &\to \mathscr{C}^{0,\alpha}_{+}(\partial\Omega) \\ (\boldsymbol{\varepsilon}, \mu_2) &\mapsto \nu_\Omega \cdot \nabla u^e[\boldsymbol{\varepsilon}, \mu_2]_{|\partial\Omega} \end{aligned} \tag{3.7}$$

is real analytic.

We now have our two terms and we can define ϕ by taking

$$\phi[\boldsymbol{\varepsilon}, \mu_2] = \nu_\Omega \cdot \nabla u^e[\boldsymbol{\varepsilon}, \mu_2]_{|\partial_+\Omega} - \nu_\Omega \cdot \nabla u^i[\boldsymbol{\varepsilon}, \mu_2]_{|\partial_+\Omega}$$

for all $(\boldsymbol{\varepsilon}, \mu_2) \in]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathscr{C}^{0,\alpha}(\partial\Omega)$. Since the functions in (3.6) and (3.7) are real analytic, it follows that ϕ is real analytic from $]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathscr{C}^{0,\alpha}(\partial\Omega)$ to $\mathscr{C}^{0,\alpha}_{+}(\partial\Omega)$. In addition, since $u^e[\boldsymbol{\varepsilon}, \mu_2](\mathsf{x}) = 0$ for all $\mathsf{x} \in \partial\mathbb{R}^n_+ \setminus \partial_0\Omega$, we can argue as in the proof of Proposition 2.14 and show, by the representation formulas in interior of Ω (cf. Lemma 2.8) and in the exterior of Ω (cf. Lemma 2.13), that (3.5) holds true.

- Finally, we have to consider the function which takes $\boldsymbol{\varepsilon}$ to the function $g[\boldsymbol{\varepsilon}]$ defined on $\overline{\partial_+\Omega}$ by

$$g[\boldsymbol{\varepsilon}](\mathsf{x}) \equiv \int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot (\nabla_{\mathsf{y}} G)(\mathsf{x}, \varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2 \mathsf{Y})\, g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \qquad \forall \mathsf{x} \in \overline{\partial_+\Omega}\,.$$

  By arguing as we have done above for $f[\boldsymbol{\varepsilon}, \mu_2]$, we can verify that the map $\boldsymbol{\varepsilon} \mapsto g[\boldsymbol{\varepsilon}]$ is real analytic from $]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[$ to $\mathscr{V}^{1,\alpha}(\partial_+\Omega)$.

This proves the analyticity of $\mathfrak{L}_1$.

Study of $\mathfrak{L}_2$. The analyticity of $\mathfrak{L}_2$ from $]-\boldsymbol{\varepsilon}^{\mathrm{ad}}, \boldsymbol{\varepsilon}^{\mathrm{ad}}[\times \mathscr{B}_1$ to $\mathscr{C}^{1,\alpha}(\partial\omega)$ is a consequence of Proposition 2.18 (see also assumption (2.14)) and of the mapping properties of the single layer potential (see Lanza de Cristoforis and Rossi [26, Thm. 3.1] and Miranda [30]) and of the integral operators with real analytic kernels and no singularity (see Lanza de Cristoforis and Musolino [25, Prop. 4.1 (ii)]). □

3.3 Functional analytic representation theorems

To investigate problem (1.2) for $\boldsymbol{\varepsilon}$ close to $\mathbf{0}$, we consider in the following Proposition 3.2 the equation in (3.3) for $\boldsymbol{\varepsilon} = \mathbf{0}$.

Proposition 3.2. *There exists a unique pair of functions $\boldsymbol{\mu}^* \equiv (\mu_1^*, \mu_2^*) \in \mathscr{B}_1$ such that*

$$\mathfrak{L}[\mathbf{0}, \boldsymbol{\mu}^*] = 0,$$

and we have

$$\mu_1^* = 0 \qquad \textit{and} \qquad v_{S_n}[\partial\omega, \mu_2^*]_{|\partial\omega} = -g^{\mathrm{o}}(0) + w_{S_n}[\partial\omega, g^{\mathrm{i}}]_{|\partial\omega} + \frac{g^{\mathrm{i}}}{2}\,.$$

Proof. First of all, we observe that for all $\boldsymbol{\mu} \in \mathscr{B}_1$, we have

$$\begin{cases} \mathfrak{L}_1[\mathbf{0}, \boldsymbol{\mu}](\mathsf{x}) = v_G[\partial\Omega, \mu_1](\mathsf{x}), & \forall \mathsf{x} \in \partial_+\Omega\,, \\ \mathfrak{L}_2[\mathbf{0}, \boldsymbol{\mu}](\mathsf{X}) = v_{S_n}[\partial\omega, \mu_2](\mathsf{X}) + g^{\mathrm{o}}(0) - w_{S_n}[\partial\omega, g^{\mathrm{i}}](\mathsf{X}) - \frac{g^{\mathrm{i}}(\mathsf{X})}{2}, & \forall \mathsf{X} \in \partial\omega\,. \end{cases}$$

By Proposition 2.10, the unique function in $\mathscr{C}^{0,\alpha}_+(\partial\Omega)$ such that $v_G[\partial\Omega, \mu_1] = 0$ on $\partial_+\Omega$ is $\mu_1 = 0$. On the other hand, by classical potential theory and Lemma 2.5, there exists a unique function $\mu_2 \in \mathscr{C}^{0,\alpha}(\partial\omega)$ such that

$$v_{S_n}[\partial\omega, \mu_2](\mathsf{X}) = -g^{\mathrm{o}}(0) + w_{S_n}[\partial\omega, g^{\mathrm{i}}](\mathsf{X}) + \frac{g^{\mathrm{i}}(\mathsf{X})}{2} \qquad \forall \mathsf{X} \in \partial\omega.$$

Now the validity of the proposition is proved. □

Now are ready to study the dependence of the solution of (3.3) upon $\boldsymbol{\varepsilon}$. Indeed, by exploiting the implicit function theorem for real analytic maps (see Deimling [16, Thm. 15.3]) one proves the following.

Theorem 3.3. *There exist $0 < \boldsymbol{\varepsilon}^* < \boldsymbol{\varepsilon}^{\mathrm{ad}}$, an open neighbourhood $\mathcal{U}_*$ of $\boldsymbol{\mu}^* \in \mathscr{B}_1$ and a real analytic map $\Phi \equiv (\Phi_1, \Phi_2)$ from $]-\boldsymbol{\varepsilon}^*, \boldsymbol{\varepsilon}^*[$ to $\mathcal{U}_*$ such that the set of zeros of $\mathfrak{L}$ in $]-\boldsymbol{\varepsilon}^*, \boldsymbol{\varepsilon}^*[\times\mathcal{U}_*$ coincides with the graph of Φ.*

Proof. The partial differential of $\mathfrak{L}$ with respect to $\boldsymbol{\mu}$ evaluated at $(\mathbf{0}, \boldsymbol{\mu}^*)$ is delivered by

$$\begin{aligned} \partial_{\boldsymbol{\mu}}\mathfrak{L}_1[\mathbf{0}, \boldsymbol{\mu}^*](\bar{\boldsymbol{\mu}}) &= v_G[\partial\Omega, \bar{\mu}_1]_{|\partial_+\Omega}\,, \\ \partial_{\boldsymbol{\mu}}\mathfrak{L}_2[\mathbf{0}, \boldsymbol{\mu}^*](\bar{\boldsymbol{\mu}}) &= v_{S_n}[\partial\omega, \bar{\mu}_2]\,, \end{aligned}$$

for all $\bar{\boldsymbol{\mu}} \in \mathscr{B}_1$. Then by Proposition 2.11 and by the properties of the (classical) single layer potential we deduce that $\partial_{\boldsymbol{\mu}}\mathfrak{L}[\mathbf{0}, \boldsymbol{\mu}^*]$ is an isomorphism from $\mathscr{B}_1$ to $\mathscr{V}^{1,\alpha}(\partial_+\Omega) \times \mathscr{C}^{1,\alpha}(\partial\omega)$. Then the theorem follows by the implicit function theorem (see Deimling [16, Thm. 15.3]) and by Proposition 3.1. □

3.3.1 Macroscopic behavior

In the following remark, we exploit the maps Φ_1 and Φ_2 of Theorem 3.3 for the representation of the solution $u_{\boldsymbol{\varepsilon}}$.

Remark 3.4. *Let the assumptions of Theorem 3.3 hold. Then*

$$\begin{aligned} u_{\boldsymbol{\varepsilon}}(\mathsf{x}) = & u_0(\mathsf{x}) - \varepsilon_1^{n-1}\varepsilon_2^{n-1} \int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot (\nabla_{\mathsf{y}} G)(\mathsf{x}, \varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y}) g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \\ & - \int_{\partial_+\Omega} G(\mathsf{x}, \mathsf{y})\Phi_1[\boldsymbol{\varepsilon}](\mathsf{y})\, d\sigma_{\mathsf{y}} \\ & + \varepsilon_1^{n-2}\varepsilon_2^{n-2} \int_{\partial\omega} G(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y})\Phi_2[\boldsymbol{\varepsilon}](\mathsf{Y})\, d\sigma_{\mathsf{Y}} \qquad \forall \mathsf{x} \in \Omega_{\boldsymbol{\varepsilon}}, \end{aligned}$$

for all $\boldsymbol{\varepsilon} \in]\mathbf{0}, \boldsymbol{\varepsilon}^*[$.

As a consequence of Remark 3.4 one can prove that for all fixed $\mathsf{x} \in \Omega$ the function $u_{\boldsymbol{\varepsilon}}(\mathsf{x})$ can be written in terms of a convergent power series of $\boldsymbol{\varepsilon}$ for ε_1 and ε_2 positive and small. If Ω' is an open subset of Ω such that $0 \notin \overline{\Omega'}$, then a similar result holds for the restriction $u_{\boldsymbol{\varepsilon}|\overline{\Omega'}}$, which describes the 'macroscopic' behaviour of $u_{\boldsymbol{\varepsilon}}$ far from the hole. Namely, we are now ready to prove Theorem 1.1 with $\boldsymbol{\varepsilon}^*$ as in Theorem 3.3.

Proof of Theorem 1.1. Let $\boldsymbol{\varepsilon}^*$ be as in Theorem 3.3. We take $\boldsymbol{\varepsilon}' \in]\mathbf{0}, \boldsymbol{\varepsilon}^*[$ small enough so that $\overline{\omega_{\boldsymbol{\varepsilon}}} \cap \overline{\Omega'} = \emptyset$ for all $\boldsymbol{\varepsilon} \in]-\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}'[$. Then we define

$$\begin{aligned}\mathfrak{U}_{\Omega'}[\boldsymbol{\varepsilon}](\mathsf{x}) \equiv u_0(\mathsf{x}) &- \varepsilon_1^{n-1}\varepsilon_2^{n-1}\int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot (\nabla_{\mathsf{y}} G)(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y}) g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \\ &- \int_{\partial_+\Omega} G(\mathsf{x},\mathsf{y})\Phi_1[\boldsymbol{\varepsilon}](\mathsf{y})\, d\sigma_{\mathsf{y}} \\ &+ \varepsilon_1^{n-2}\varepsilon_2^{n-2}\int_{\partial\omega} G(\mathsf{x}, \varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{Y})\Phi_2[\boldsymbol{\varepsilon}](\mathsf{Y})\, d\sigma_{\mathsf{Y}},\end{aligned}$$

for all $\mathsf{x} \in \overline{\Omega'}$ and for all $\boldsymbol{\varepsilon} \in]-\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}'[$. Then, by Theorem 3.3, by the properties of the integral operators with real analytic kernel and no singularity (see Lanza de Cristoforis and Musolino [25, Prop. 4.1]), by the mapping properties of the single layer potential (see Lanza de Cristoforis and Rossi [26, Thm. 3.1] and Miranda [30]), and by standard calculus in Banach space, one deduces that $\mathfrak{U}_{\Omega'}$ is real analytic from $]-\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}'[$ to $\mathscr{C}^{1,\alpha}(\overline{\Omega'})$. The validity of (1.5) follows by Remark 3.4 and the validity of (1.6) can be deduced by Proposition 3.2, by Theorem 3.3, and by a straightforward computation. $\square$

3.3.2 Microscopic behaviour

As a consequence of Remark 3.4 and of the rule of change of variable in integrals we provide below a representation of the solution $u_{\boldsymbol{\varepsilon}}$ in proximity of the perforation.

Remark 3.5. *Let the assumptions of Theorem 3.3 hold. Then*

$$\begin{aligned}u_{\boldsymbol{\varepsilon}}(\varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{X}) = u_0(\varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{X}) &- w^e_{S_n}[\partial\omega, g^{\mathrm{i}}](\mathsf{X}) \\ &- \varepsilon_2^{n-1}\int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot \nabla S_n(-2p_n\mathsf{e}_n + \varepsilon_2(\varsigma(\mathsf{X}) - \mathsf{Y}))\, g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \\ &- \int_{\partial_+\Omega} G(\varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{X}, \mathsf{y})\, \Phi_1[\boldsymbol{\varepsilon}](\mathsf{y})\, d\sigma_{\mathsf{y}} \\ &+ v_{S_n}[\partial\omega, \Phi_2[\boldsymbol{\varepsilon}]](\mathsf{X}) - \varepsilon_2^{n-2}\int_{\partial\omega} S_n(-2p_n\mathsf{e}_n + \varepsilon_2(\varsigma(\mathsf{X}) - \mathsf{Y}))\, \Phi_2[\boldsymbol{\varepsilon}](\mathsf{Y})\, d\sigma_{\mathsf{Y}}\end{aligned}$$

for all $\mathsf{X} \in \mathbb{R}^n \setminus \omega$ and all $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in]\mathbf{0}, \boldsymbol{\varepsilon}^*[$ such that $\varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{X} \in \overline{\Omega_{\boldsymbol{\varepsilon}}}$.

Then we can prove the following theorem, where we characterise the 'microscopic' behaviour of $u_{\boldsymbol{\varepsilon}}$ close to hole, *i.e.* $u_{\boldsymbol{\varepsilon}}(\varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\,\cdot\,)$ as $\boldsymbol{\varepsilon} \to 0$.

Theorem 3.6. *Let the assumptions of Theorem 3.3 hold. Let ω' be an open bounded subset of $\mathbb{R}^n \setminus \overline{\omega}$. Let $\boldsymbol{\varepsilon}''$ be such that $0 < \boldsymbol{\varepsilon}'' < \boldsymbol{\varepsilon}^*$ and $(\varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\overline{\omega'}) \subseteq \mathcal{B}(0, r_1)$ for all $\boldsymbol{\varepsilon} \in]-\boldsymbol{\varepsilon}'', \boldsymbol{\varepsilon}''[$. Then there exists a real analytic map $\mathfrak{V}_{\omega'}$ from $]-\boldsymbol{\varepsilon}'', \boldsymbol{\varepsilon}''[$ to $\mathscr{C}^{1,\alpha}(\overline{\omega'})$ such that*

$$u_{\boldsymbol{\varepsilon}}(\varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\,\cdot\,)_{|\overline{\omega'}} = \mathfrak{V}_{\omega'}[\boldsymbol{\varepsilon}] \qquad \forall \boldsymbol{\varepsilon} \in]\mathbf{0}, \boldsymbol{\varepsilon}''[\,. \tag{3.8}$$

Moreover we have

$$\mathfrak{V}_{\omega'}[\mathbf{0}] = v_{0|\overline{\omega'}} \tag{3.9}$$

where $w_0 \in \mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \omega)$ *is the unique solution of*

$$\begin{cases} \Delta w_0 = 0 & \text{in } \mathbb{R}^n \setminus \omega\,, \\ w_0 = g^{\mathrm{i}} & \text{on } \partial\omega \\ \lim_{\mathsf{X}\to\infty} w_0(\mathsf{X}) = g^{\mathrm{o}}(0)\,. \end{cases}$$

Proof. We define

$$\begin{aligned} \mathfrak{V}_{\omega'}[\boldsymbol{\varepsilon}](\mathsf{X}) \equiv\, & U_0(\varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{X}) - w^e_{S_n}[\partial\omega, g^{\mathrm{i}}](\mathsf{X}) \\ & - \varepsilon_2^{n-1}\int_{\partial\omega} \nu_\omega(\mathsf{Y})\cdot\nabla S_n(-2p_n\mathsf{e}_n + \varepsilon_2(\varsigma(\mathsf{X}) - \mathsf{Y}))\, g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \\ & - \int_{\partial_+\Omega} G(\varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{X}, \mathsf{y})\, \Phi_1[\boldsymbol{\varepsilon}](\mathsf{y})\, d\sigma_{\mathsf{y}} \\ & + v_{S_n}[\partial\omega, \Phi_2[\boldsymbol{\varepsilon}]](\mathsf{X}) - \varepsilon_2^{n-2}\int_{\partial\omega} S_n(-2p_n\mathsf{e}_n + \varepsilon_2(\varsigma(\mathsf{X}) - \mathsf{Y}))\, \Phi_2[\boldsymbol{\varepsilon}](\mathsf{Y})\, d\sigma_{\mathsf{Y}} \end{aligned}$$

for all $\mathsf{X} \in \overline{\omega'}$ and for all $\boldsymbol{\varepsilon} \in\,]-\boldsymbol{\varepsilon}'', \boldsymbol{\varepsilon}''[$. Then, by Proposition 2.18, by Theorem 3.3, by the properties of the integral operators with real analytic kernel and no singularity (see Lanza de Cristoforis and Musolino [25, Prop. 4.1]), by the mapping properties of the single layer potential (see Lanza de Cristoforis and Rossi [26, Thm. 3.1] and Miranda [30]), and by standard calculus in Banach space, one deduces that $\mathfrak{V}_{\omega'}$ is real analytic from $]-\boldsymbol{\varepsilon}'', \boldsymbol{\varepsilon}''[$ to $\mathscr{C}^{1,\alpha}(\overline{\omega'})$. The validity of (3.8) follows by Remark 3.5. By a straightforward computation and by Proposition 3.2 one verifies that

$$\mathfrak{V}_{\omega'}[\mathbf{0}](\mathsf{X}) = g^{\mathrm{o}}(0) - w^e_{S_n}[\partial\omega, g^{\mathrm{i}}](\mathsf{X}) + v_{S_n}[\partial\omega, \Phi_2[\mathbf{0}]](\mathsf{X}), \tag{3.10}$$

for all $\mathsf{X} \in \overline{\omega'}$. Then, by Proposition 3.2 and by the jump properties of the double layer potential we deduce that the right hand side of (3.10) equals g^{i} on $\partial\omega$. Hence, by the decaying properties at ∞ of the single and double layer potentials and by the uniqueness of the solution of the exterior Dirichlet problem, we deduce the validity of (3.9). □

3.3.3 Energy integral

We now turn to study the behaviour of the energy integral $\int_{\Omega_{\boldsymbol{\varepsilon}}} |\nabla u_{\boldsymbol{\varepsilon}}|^2\, d\mathsf{x}$ by representing it in terms of a real analytic function. In Theorem 3.7 here below we consider the case when $g^{\mathrm{o}} = 0$.

Theorem 3.7. *Let* $g^{\mathrm{o}} = 0$. *Let the assumptions of Theorem 3.3 hold. Then there exist* $\boldsymbol{\varepsilon}^{\mathfrak{F}} \in\,]\mathbf{0}, \boldsymbol{\varepsilon}^*[$ *and a real analytic map* $\mathfrak{F}$ *from* $]-\boldsymbol{\varepsilon}^{\mathfrak{F}}, \boldsymbol{\varepsilon}^{\mathfrak{F}}[$ *to* $\mathbb{R}$ *such that*

$$\int_{\Omega_{\boldsymbol{\varepsilon}}} |\nabla u_{\boldsymbol{\varepsilon}}|^2\, d\mathsf{x} = \varepsilon_1^{n-2}\varepsilon_2^{n-2}\mathfrak{F}(\boldsymbol{\varepsilon}) \qquad \forall \boldsymbol{\varepsilon} \in\,]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathfrak{F}}[\tag{3.11}$$

and

$$\mathfrak{F}(\mathbf{0}) = \int_{\mathbb{R}^n\setminus\omega} |\nabla w_0|^2\, d\mathsf{x}\,. \tag{3.12}$$

Proof. We take ω' as in Theorem 3.6 which in addition satisfies the condition $\partial\omega \subseteq \overline{\omega'}$. Then we set $\boldsymbol{\varepsilon}^{\mathfrak{F}} \equiv \boldsymbol{\varepsilon}''$ with $\boldsymbol{\varepsilon}''$ as in Theorem 3.6 and we define

$$\mathfrak{F}(\boldsymbol{\varepsilon}) \equiv -\int_{\partial\omega} g^{\mathrm{i}}\, \nu_\omega \cdot \nabla \mathfrak{V}_{\omega'}[\boldsymbol{\varepsilon}]\, d\sigma \qquad \forall \boldsymbol{\varepsilon} \in\,]-\boldsymbol{\varepsilon}^{\mathfrak{F}}, \boldsymbol{\varepsilon}^{\mathfrak{F}}[\,.$$

By Theorem 3.6 and by standard calculus in Banach spaces it follows that $\mathfrak{F}$ is real analytic from $]-\boldsymbol{\varepsilon}^{\mathfrak{F}}, \boldsymbol{\varepsilon}^{\mathfrak{F}}[$ to $\mathbb{R}$. By a computation based on the divergence theorem and on the rule of change of variable in integrals one shows the validity of (3.11). The validity of (3.12) follows by (1.6) and by the divergence theorem. □

We now consider the case when $g^{\mathrm{o}} \neq 0$. To do so, we need the following technical Lemma 3.8 which can be proved by the properties of integral operators with harmonic kernel (and no singularity).

Lemma 3.8. *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ such that $\overline{\mathcal{O}} \cap \mathbb{R}^n_+$ is contained in Ω. Then $w_G[\partial_+\Omega, \psi]$ is harmonic on $\mathcal{O}$ for all $\psi \in \mathscr{C}^{1,\alpha}(\partial\Omega)$.*

Theorem 3.9. *Let the assumptions of Theorem 3.3 hold. Then there exist $\boldsymbol{\varepsilon}^{\mathfrak{E}} \in\,]\mathbf{0}, \boldsymbol{\varepsilon}^*[$ and a real analytic map $\mathfrak{E}$ from $]-\boldsymbol{\varepsilon}^{\mathfrak{E}}, \boldsymbol{\varepsilon}^{\mathfrak{E}}[$ to $\mathbb{R}$ such that*

$$\int_{\Omega_{\boldsymbol{\varepsilon}}} |\nabla u_{\boldsymbol{\varepsilon}}|^2\, d\mathsf{x} = \mathfrak{E}(\boldsymbol{\varepsilon}) \qquad \forall \boldsymbol{\varepsilon} \in\,]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathfrak{E}}[\tag{3.13}$$

and

$$\mathfrak{E}(\mathbf{0}) = \int_{\Omega} |\nabla u_0|^2\, d\mathsf{x}\,. \tag{3.14}$$

Proof. We take ω' as in Theorem 3.6 which in addition satisfies the condition $\partial\omega \subseteq \overline{\omega'}$. Then we set $\boldsymbol{\varepsilon}^{\mathfrak{E}} \equiv \boldsymbol{\varepsilon}''$ with $\boldsymbol{\varepsilon}''$ as in Theorem 3.6 and we define

$$\begin{aligned}
\mathfrak{E}_1(\boldsymbol{\varepsilon}) &\equiv \int_{\partial\Omega} g^{\mathrm{o}}\, \nu_\Omega \cdot \nabla(u_0 - v_G[\partial_+\Omega, \Phi_1[\varepsilon]])\, d\sigma\,,\\
\mathfrak{E}_2(\boldsymbol{\varepsilon}) &\equiv \int_{\partial\omega} g^{\mathrm{i}}(\mathsf{Y})\, \nu_\omega(\mathsf{Y}) \cdot \nabla w_G[\partial_+\Omega, g^{\mathrm{o}}](\varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2 \mathsf{Y})\, d\sigma_{\mathsf{Y}}\,,\\
\mathfrak{E}_3(\boldsymbol{\varepsilon}) &\equiv -\int_{\partial\omega} \Phi_2[\boldsymbol{\varepsilon}](\mathsf{Y})\, w_G[\partial_+\Omega, g^{\mathrm{o}}](\varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2 \mathsf{Y})\, d\sigma_{\mathsf{Y}}\,,\\
\mathfrak{E}_4(\boldsymbol{\varepsilon}) &\equiv -\int_{\partial\omega} g^{\mathrm{i}}\, \nu_\omega \cdot \nabla \mathfrak{V}_{\omega'}[\boldsymbol{\varepsilon}]\, d\sigma
\end{aligned}$$

and

$$\mathfrak{E}(\boldsymbol{\varepsilon}) \equiv \mathfrak{E}_1(\boldsymbol{\varepsilon}) + \varepsilon_1^{n-2}\varepsilon_2^{n-2}\left(\mathfrak{E}_2(\boldsymbol{\varepsilon}) + \mathfrak{E}_3(\boldsymbol{\varepsilon}) + \mathfrak{E}_4(\boldsymbol{\varepsilon})\right)$$

for all $\boldsymbol{\varepsilon} \in\,]-\boldsymbol{\varepsilon}^{\mathfrak{E}}, \boldsymbol{\varepsilon}^{\mathfrak{E}}[$. By Theorems 3.3 and 3.6, by Lemma 3.8, by the properties of integral operators with real analytic kernel (and no singularity), and by standard calculus in Banach spaces it follows that the $\mathfrak{E}_i$'s and $\mathfrak{E}$ are real analytic from $]-\boldsymbol{\varepsilon}^{\mathfrak{E}}, \boldsymbol{\varepsilon}^{\mathfrak{E}}[$ to $\mathbb{R}$. By the definition of $w_G[\partial_+\Omega, g^{\mathrm{o}}]$ and by the Fubini theorem one verifies that

$$\mathfrak{E}_2(\boldsymbol{\varepsilon}) = -\int_{\partial\Omega} g^{\mathrm{o}}(\mathsf{x})\, \nu_\Omega(\mathsf{x}) \cdot \nabla_{\mathsf{x}} \left(\int_{\partial\omega} \nu_\omega(\mathsf{Y}) \cdot (\nabla_{\mathsf{y}} G)(\mathsf{x}, \varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2 \mathsf{Y}) g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \right) d\sigma_{\mathsf{x}}$$

and

$$\mathfrak{E}_3(\boldsymbol{\varepsilon}) = \int_{\partial\Omega} g^{\mathrm{o}}(\mathsf{x})\, \nu_\Omega(\mathsf{x}) \cdot \nabla_{\mathsf{x}} \left(\int_{\partial\omega} G(\mathsf{x}, \varepsilon_1 \mathsf{p} + \varepsilon_1\varepsilon_2 \mathsf{Y}) \Phi_2[\boldsymbol{\varepsilon}](\mathsf{Y})\, d\sigma_{\mathsf{Y}} \right) d\sigma_{\mathsf{x}}$$

for all $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in]\mathbf{0}, \boldsymbol{\varepsilon}^{\mathfrak{E}}[$. Then, by Remark 3.4, by Theorem 3.6, and by a computation based on the divergence theorem and on the rule of change of variable in integrals one shows the validity of (3.13). To prove (3.14) we observe that by Proposition 3.2 and Theorem 3.3, we have $\Phi_1[\mathbf{0}] = 0$. Thus (3.14) implies that $\mathfrak{E}(\mathbf{0}) = \int_{\partial\Omega} g^{\mathrm{o}} \nu_\Omega \cdot \nabla u_0\, d\sigma$ and (3.14) follows by the divergence theorem. □

3.4 Real analytic continuation in global spaces

In view of the interaction between the functional analytic approach and the multiscale asymptotic expansion method, we find convenient to follow an idea of Costabel, Dalla Riva, Dauge, and Musolino [10] and express the solution u_ε in terms of real analytic maps of $\boldsymbol{\varepsilon}$ with values in spaces of functions on $\overline{\Omega}$ and $\mathbb{R}^n \setminus \omega$. To do so, we introduce the space $\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\mathbb{R}^n \setminus \omega)$ defined by

$$\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\mathbb{R}^n \setminus \omega) \equiv \left\{ v \in \mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \omega) \,:\, \Delta v = 0 \text{ in } \mathbb{R}^n \setminus \overline{\omega}\,, \text{ and } \lim_{\mathsf{x}\to\infty} v(\mathsf{x}) = 0 \right\}.$$

Since the solution of a Dirichlet problem in $\mathbb{R}^n \setminus \omega$ which decays at infinity exists and is unique (cf., *e.g.*, Folland [17, Chap. 2]), the map which takes a function v to its trace $v_{|\partial\omega}$ is a linear isomorphism between $\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\mathbb{R}^n \setminus \omega)$ and $\mathscr{C}^{1,\alpha}(\partial\omega)$. Accordingly, we can endow $\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\mathbb{R}^n \setminus \omega)$ with the norm

$$\|v\|_{\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\mathbb{R}^n\setminus\omega)} \equiv \|v_{|\partial\omega}\|_{\mathscr{C}^{1,\alpha}(\partial\omega)} \qquad \forall v \in \mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\mathbb{R}^n \setminus \omega)\,.$$

With such norm, $\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\mathbb{R}^n \setminus \omega)$ is a Banach space. In addition, by exploiting elliptic a priori estimates, one verifies that the topology generated by $\|\cdot\|_{\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\mathbb{R}^n\setminus\omega)}$ is equivalent to the topology induced by (the Fréchet space) $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \omega)$. We retain a similar notation for

$$\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\overline{\Omega}) \equiv \left\{ v \in \mathscr{C}^{1,\alpha}(\overline{\Omega}) \,:\, \Delta v = 0 \text{ in } \Omega \right\},$$

which is a closed subspace of $\mathscr{C}^{1,\alpha}(\overline{\Omega})$ and therefore a Banach space.

As in Remark 3.4, we exploit the maps Φ_1 and Φ_2 of Theorem 3.3 to write the solution $u_{\boldsymbol{\varepsilon}}$ in a suitable way.

Remark 3.10. *Let the assumptions of Theorem 3.3 hold. Then*

$$\begin{aligned} u_{\boldsymbol{\varepsilon}}(\mathsf{x}) =& u_0(\mathsf{x}) - \int_{\partial_+\Omega} G(\mathsf{x}, \mathsf{y}) \Phi_1[\boldsymbol{\varepsilon}](\mathsf{y})\, d\sigma_{\mathsf{y}} \\ & - w^e_{S_n}[\partial\omega, g^{\mathrm{i}}] \left(\frac{\mathsf{x} - \varepsilon_1 \mathsf{p}}{\varepsilon_1 \varepsilon_2} \right) + v_{S_n}[\partial\omega, \Phi_2[\boldsymbol{\varepsilon}]] \left(\frac{\mathsf{x} - \varepsilon_1 \mathsf{p}}{\varepsilon_1 \varepsilon_2} \right) \\ & - \int_{\partial\omega} \nu_\omega(s) \cdot \nabla S_n \left(\varsigma \left(\frac{\mathsf{x} - \varepsilon_1 \mathsf{p}}{\varepsilon_1 \varepsilon_2} \right) - \frac{2 p_n \mathsf{e}_n}{\varepsilon_2} - \mathsf{Y} \right) g^{\mathrm{i}}(\mathsf{Y})\, d\sigma_{\mathsf{Y}} \\ & - \int_{\partial\omega} S_n \left(\varsigma \left(\frac{\mathsf{x} - \varepsilon_1 \mathsf{p}}{\varepsilon_1 \varepsilon_2} \right) - \frac{2 p_n \mathsf{e}_n}{\varepsilon_2} - \mathsf{Y} \right) \Phi_2[\boldsymbol{\varepsilon}](\mathsf{Y})\, d\sigma_{\mathsf{Y}} \end{aligned}$$

for all $\mathsf{x} \in \overline{\Omega_{\boldsymbol{\varepsilon}}}$ *and for all* $\boldsymbol{\varepsilon} \in]\mathbf{0}, \boldsymbol{\varepsilon}^*[$.

We are now ready to state the following

Theorem 3.11. *Let the assumptions of Theorem 3.3 hold. There exist and are unique a real analytic map $\mathfrak{U}$ from $]-\boldsymbol{\varepsilon}^*,\boldsymbol{\varepsilon}^*[$ to $\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\overline{\Omega})$ and a real analytic map $\mathfrak{V}$ from $]-\boldsymbol{\varepsilon}^*,\boldsymbol{\varepsilon}^*[$ to $\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\mathbb{R}^n\setminus\omega)$ such that*

$$u_{\boldsymbol{\varepsilon}}(\mathsf{x}) = \mathfrak{U}[\boldsymbol{\varepsilon}](\mathsf{x}) + \mathfrak{V}[\boldsymbol{\varepsilon}]\left(\frac{\mathsf{x}-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) - \mathfrak{V}[\boldsymbol{\varepsilon}]\left(\frac{\varsigma(\mathsf{x})-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) \qquad \forall \mathsf{x}\in\overline{\Omega_{\boldsymbol{\varepsilon}}}\,,\ \boldsymbol{\varepsilon}\in\,]\mathbf{0},\boldsymbol{\varepsilon}^*[\,. \tag{3.15}$$

In addition,

$$\mathfrak{U}[\mathbf{0}] = u_0 \qquad \textit{and} \qquad \mathfrak{V}[\mathbf{0}] = v_0\,, \tag{3.16}$$

where v_0 is the unique element of $\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\mathbb{R}^n\setminus\omega)$ such that $v_{0|\partial\omega} = g^{\mathrm{i}} - g^{\mathrm{o}}(0)$.

Proof. We define

$$\mathfrak{U}[\boldsymbol{\varepsilon}](\mathsf{x}) \equiv u_0(\mathsf{x}) - \int_{\partial_+\Omega} G(\mathsf{x},\mathsf{y})\Phi_1[\boldsymbol{\varepsilon}](\mathsf{y})\,d\sigma_{\mathsf{y}}$$

for all $\mathsf{x}\in\overline{\Omega}$ and for all $\boldsymbol{\varepsilon}\in\,]-\boldsymbol{\varepsilon}^*,\boldsymbol{\varepsilon}^*[$, and

$$\mathfrak{V}[\boldsymbol{\varepsilon}](\mathsf{X}) \equiv -w^e_{S_n}[\partial\omega, g^{\mathrm{i}}](\mathsf{X}) + v_{S_n}[\partial\omega,\Phi_2[\boldsymbol{\varepsilon}]](\mathsf{X}),$$

for all $\mathsf{X}\in\mathbb{R}^n\setminus\omega$ and for all $\boldsymbol{\varepsilon}\in\,]-\boldsymbol{\varepsilon}^*,\boldsymbol{\varepsilon}^*[$. Then, by Theorem 3.3, by the mapping properties of the single layer potential (see Lanza de Cristoforis and Rossi [26, Thm. 3.1] and Miranda [30]), and by standard calculus in Banach space, one deduces that $\mathfrak{U}$ is real analytic from $]-\boldsymbol{\varepsilon}^*,\boldsymbol{\varepsilon}^*[$ to $\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\overline{\Omega})$ and that $\mathfrak{V}$ is real analytic from $]-\boldsymbol{\varepsilon}^*,\boldsymbol{\varepsilon}^*[$ to $\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\mathbb{R}^n\setminus\omega)$. The validity of (3.15) follows by Remark 3.10 and by noting that

$$\varsigma\left(\frac{\mathsf{x}-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) - \frac{2p_n\mathsf{e}_n}{\varepsilon_2} = \frac{\varsigma(\mathsf{x})-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\,.$$

The validity of the first equality in (3.16) can be deduced by Proposition 3.2, by Theorem 3.3, and by a straightforward computation. To prove the second equality in (3.16) we observe that, by Proposition 3.2 we have

$$\mathfrak{V}[\mathbf{0}] = -w^e_{S_n}[\partial\omega, g^{\mathrm{i}}] + v^e_{S_n}[\partial\omega,\Phi_2[\mathbf{0}]]\,. \tag{3.17}$$

Then, by Proposition 3.2 and by the jump properties of the double layer potential we deduce that the right hand side of (3.17) equals $g^{\mathrm{i}} - g^{\mathrm{o}}(0)$ on $\partial\omega$. Hence, by the decaying properties at ∞ of the single and double layer potentials and by the uniqueness of the solution of the exterior Dirichlet problem, we deduce the validity of (3.9).

To verify that $\mathfrak{U}$ and $\mathfrak{V}$ are unique we observe that if we have two functions $\phi\in\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\overline{\Omega})$ and $\psi\in\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\mathbb{R}^n\setminus\omega)$ such that

$$\phi(\mathsf{x}) + \psi\left(\frac{\mathsf{x}-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) - \psi\left(\frac{\varsigma(\mathsf{x})-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) = 0 \qquad \forall\mathsf{x}\in\overline{\Omega_{\boldsymbol{\varepsilon}}} \tag{3.18}$$

for some $\boldsymbol{\varepsilon}\in\,]\mathbf{0},\boldsymbol{\varepsilon}^*[$, then

$$\psi\left(\frac{\mathsf{x}-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) = -\phi(\mathsf{x}) + \psi\left(\frac{\varsigma(\mathsf{x})-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) \qquad \forall\mathsf{x}\in\overline{\Omega_{\boldsymbol{\varepsilon}}},$$

and thus $\psi\left(\frac{\cdot-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right)$ has an harmonic extension on the whole of $\mathbb{R}^n$. Since $\lim_{\mathsf{x}\to\infty}\psi\left(\frac{\mathsf{x}-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right)=0$, it follows that $\psi=0$. Then, $\phi=0$ by (3.18) and by the identity principle. $\square$

As a consequence of Theorem 3.11, there exist $\boldsymbol{\varepsilon}^{**}\in\,]\mathbf{0},\boldsymbol{\varepsilon}^*[$ and two countable families of functions $\{U_{i,j}\}_{i,j\in\mathbb{N}^2}\subseteq\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\overline{\Omega})$ and $\{V_{i,j}\}_{i,j\in\mathbb{N}^2}\subseteq\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\mathbb{R}^n\setminus\omega)$ such that the series

$$\sum_{i,j\in\mathbb{N}^2}\varepsilon_1^i\varepsilon_2^j\,U_{i,j}\qquad\text{and}\qquad\sum_{i,j\in\mathbb{N}^2}\varepsilon_1^i\varepsilon_2^j\,V_{i,j}$$

converge for $\boldsymbol{\varepsilon}\in\,]-\boldsymbol{\varepsilon}^{**},\boldsymbol{\varepsilon}^{**}[$ in $\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\overline{\Omega})$ and in $\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\mathbb{R}^n\setminus\omega)$, respectively, and such that

$$u_{\boldsymbol{\varepsilon}}(\mathsf{x})=\sum_{i,j\in\mathbb{N}^2}\varepsilon_1^i\varepsilon_2^j\,U_{i,j}(\mathsf{x})+\sum_{i,j\in\mathbb{N}^2}\varepsilon_1^i\varepsilon_2^j\,V_{i,j}\left(\frac{\mathsf{x}-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right)-\sum_{i,j\in\mathbb{N}^2}\varepsilon_1^i\varepsilon_2^j\,V_{i,j}\left(\frac{\varsigma(\mathsf{x})-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right)\tag{3.19}$$

for all $\mathsf{x}\in\overline{\Omega_{\boldsymbol{\varepsilon}}}$ and all $\boldsymbol{\varepsilon}\in\,]\mathbf{0},\boldsymbol{\varepsilon}^{**}[$. Moreover, by the uniqueness of $\mathfrak{U}$ and $\mathfrak{V}$ and by the identity principle for real analytic functions, one verifies that such families $\{U_{i,j}\}_{i,j\in\mathbb{N}^2}$ and $\{V_{i,j}\}_{i,j\in\mathbb{N}^2}$ are univocally identified by (3.19).

In addition, we denote by $\mathscr{C}(\overline{\Omega})$ the space of continuous functions on $\overline{\Omega}$ and by $\mathscr{C}_0(\mathbb{R}^3\setminus\omega)$ the space of continuous functions on $\mathbb{R}^3\setminus\omega$ which vanish at infinite.

Proposition 3.12. *Let $N\in\mathbb{N}$ and assume that we have functions $u_{i,j}\in\mathscr{C}(\overline{\Omega})$ and $v_{i,j}\in\mathscr{C}_0(\mathbb{R}^n\setminus\omega)$, with $(i,j)\in\mathbb{N}^2$ and $i+j\leq N$, such that*

$$\|u_{\boldsymbol{\varepsilon}}-W_{\boldsymbol{\varepsilon},N}\|_{\mathscr{L}^\infty(\Omega_{\boldsymbol{\varepsilon}})}=o\biggl(\sum_{i+j=N}\varepsilon_1^i\varepsilon_2^j\biggr)\qquad\text{as }\boldsymbol{\varepsilon}\to\mathbf{0}\text{ in }]\mathbf{0},\boldsymbol{\varepsilon}^*[\,,\tag{3.20}$$

where

$$W_{\boldsymbol{\varepsilon},N}(\mathsf{x})\equiv\sum_{i+j\leq N}\varepsilon_1^i\varepsilon_2^j\,u_{i,j}(\mathsf{x})+\sum_{i+j\leq N}\varepsilon_1^i\varepsilon_2^j\,v_{i,j}\left(\frac{\mathsf{x}-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right)-\sum_{i+j\leq N}\varepsilon_1^i\varepsilon_2^j\,v_{i,j}\left(\frac{\varsigma(\mathsf{x})-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right)\quad\forall\mathsf{x}\in\overline{\Omega_{\boldsymbol{\varepsilon}}}\,.$$

Then $u_{i,j}=U_{i,j}$ and $v_{i,j}=V_{i,j}$ for all $(i,j)\in\mathbb{N}^2$ with $i+j\leq N$.

Proof. For $0<\eta_*<1$ sufficiently small we consider the curve in $]\mathbf{0},\boldsymbol{\varepsilon}^*[$ which takes $\eta\in]0,\eta_*[$ to $\boldsymbol{\varepsilon}(\eta)=(\varepsilon_1(\eta),\varepsilon_2(\eta))\equiv(\eta,-1/\log\eta)$. Then we observe that $\varepsilon_1^i(\eta)\varepsilon_2^j(\eta)=o(\varepsilon_1^h(\eta)\varepsilon_2^k(\eta))$ as $\eta\to0$ if and only if either one of the following conditions holds true:

$$i>h\qquad\text{or}\qquad i=h\text{ and }j>k\,.\tag{3.21}$$

Accordingly, we endow $\mathbb{N}^2$ with a total order relation by saying that $(i,j)>(h,k)$ if and only if (3.21) is verified, and, as usual, $(i,j)\geq(h,k)$ if $(i,j)=(h,k)$ or $(i,j)>(h,k)$. An elementary argument shows that every subset of $\mathbb{N}^2$ has a minimum with respect to $\geq$. Therefore, we are in the position to prove the lemma by an induction argument on (i,j).

We first prove that $u_{0,0}=U_{0,0}$. We note that by the membership of $v_{i,j}$ in $\mathscr{C}_0(\mathbb{R}^n\setminus\omega)$ we have

$$\lim_{\eta\to0}v_{i,j}\left(\frac{\mathsf{x}-\varepsilon_1(\eta)\mathsf{p}}{\varepsilon_1(\eta)\varepsilon_2(\eta)}\right)=0\quad\text{and}\quad\lim_{\eta\to0}v_{i,j}\left(\frac{\varsigma(\mathsf{x})-\varepsilon_1(\eta)\mathsf{p}}{\varepsilon_1(\eta)\varepsilon_2(\eta)}\right)=0\,,\qquad\forall\mathsf{x}\in\overline{\Omega}\,,\tag{3.22}$$

for all $(i,j) \in \mathbb{N}^2$ with $i+j \leq N$. Similarly, by the membership of $V_{i,j}$ in $\mathscr{C}^{1,\alpha}_{\mathrm{harm}}(\mathbb{R}^n \setminus \omega)$ we have

$$\lim_{\eta\to 0} V_{i,j}\left(\frac{\mathsf{x}-\varepsilon_1(\eta)\mathsf{p}}{\varepsilon_1(\eta)\varepsilon_2(\eta)}\right) = 0 \quad \text{and} \quad \lim_{\eta\to 0} V_{i,j}\left(\frac{\varsigma(\mathsf{x})-\varepsilon_1(\eta)\mathsf{p}}{\varepsilon_1(\eta)\varepsilon_2(\eta)}\right) = 0\,, \qquad \forall \mathsf{x}\in\overline{\Omega}\,, \tag{3.23}$$

for all $(i,j) \in \mathbb{N}^2$. Then, by (3.22) we compute that

$$\lim_{\eta\to 0} W_{\boldsymbol{\varepsilon}(\eta),N}(\mathsf{x}) = u_{0,0}(\mathsf{x})\,, \qquad \forall \mathsf{x}\in\overline{\Omega}$$

and, by (3.19) and (3.23) we also have

$$\lim_{\eta\to 0} u_{\boldsymbol{\varepsilon}(\eta)}(\mathsf{x}) = U_{0,0}(\mathsf{x})\,, \qquad \forall \mathsf{x}\in\overline{\Omega}\,.$$

By (3.20) it follows that $u_{0,0} = U_{0,0}$.

To prove that $v_{0,0} = V_{0,0}$, we observe that

$$\lim_{\eta\to 0} v_{i,j}\left(\frac{\varsigma(\varepsilon_1(\eta)\mathsf{p}+\varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{X})-\varepsilon_1(\eta)\mathsf{p}}{\varepsilon_1(\eta)\varepsilon_2(\eta)}\right) = \lim_{\eta\to 0} v_{i,j}\left(\varsigma(\mathsf{X})-\frac{2p_n\mathsf{e}_n}{\varepsilon_2(\eta)}\right) = 0\,, \qquad \forall \mathsf{X}\in\mathbb{R}^n\setminus\omega\,, \tag{3.24}$$

for all $(i,j) \in \mathbb{N}^2$ with $i+j \leq N$, and similarly

$$\lim_{\eta\to 0} V_{i,j}\left(\frac{\varsigma(\varepsilon_1(\eta)\mathsf{p}+\varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{X})-\varepsilon_1(\eta)\mathsf{p}}{\varepsilon_1(\eta)\varepsilon_2(\eta)}\right) = \lim_{\eta\to 0} V_{i,j}\left(\varsigma(\mathsf{X})-\frac{2p_n\mathsf{e}_n}{\varepsilon_2(\eta)}\right) = 0\,, \qquad \forall \mathsf{X}\in\mathbb{R}^n\setminus\omega\,, \tag{3.25}$$

for all $(i,j) \in \mathbb{N}^2$. It follows that

$$\lim_{\eta\to 0} W_{\boldsymbol{\varepsilon}(\eta),N}(\varepsilon_1(\eta)\mathsf{p}+\varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{X}) = u_{0,0}(0)+v_{0,0}(\mathsf{X})\,, \qquad \forall \mathsf{X}\in\mathbb{R}^n\setminus\omega\,, \tag{3.26}$$

and that

$$\lim_{\eta\to 0} u_{\boldsymbol{\varepsilon}}(\varepsilon_1(\eta)\mathsf{p}+\varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{X}) = U_{0,0}(0)+V_{0,0}(\mathsf{X})\,, \qquad \forall \mathsf{X}\in\mathbb{R}^n\setminus\omega \tag{3.27}$$

(see also (3.19)). Then we observe that (3.20) is equivalent to

$$\|u_{\boldsymbol{\varepsilon}}(\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\,\cdot)-W_{\boldsymbol{\varepsilon},N}(\varepsilon_1\mathsf{p}+\varepsilon_1\varepsilon_2\,\cdot)\|_{\mathscr{L}^\infty(\Omega^{\boldsymbol{\varepsilon}}\setminus\overline{\omega})} = o\left(\sum_{i+j=N}\varepsilon_1^i\varepsilon_2^j\right) \qquad \text{as } \boldsymbol{\varepsilon}\to\mathbf{0}\,, \tag{3.28}$$

where $\Omega^{\boldsymbol{\varepsilon}} \equiv (\Omega-\varepsilon_1\mathsf{p})/\varepsilon_1\varepsilon_2$. By (3.26) and (3.27) and by equality $u_{0,0}(0) = U_{0,0}(0)$, it follows that $v_{0,0} = V_{0,0}$.

If $N = 0$ the proof is complete. We now assume that $N > 0$ and that $u_{i,j} = U_{i,j}$ and $v_{i,j} = V_{i,j}$ for all $(i,j) \leq (i_*,j_*)$, where (i_*,j_*) is a multi-index with $i_*+j_* \leq N$ and $i_* < N$ (note that $(N,0)$ is the 'biggest' multi-index which appears in the sums of (3.20)). For all $\boldsymbol{\varepsilon} \in \,]\mathbf{0},\boldsymbol{\varepsilon}^*[$ we denote by $W_{\boldsymbol{\varepsilon},(i_*,j_*)}$ the function of $\mathsf{x}\in\Omega_{\boldsymbol{\varepsilon}}$ defined by

$$\begin{aligned} &W_{\boldsymbol{\varepsilon},(i_*,j_*)}(\mathsf{x}) \\ &\quad\equiv \sum_{(i,j)\leq(i_*,j_*)}\varepsilon_1^i\varepsilon_2^j\,u_{i,j}(\mathsf{x}) + \sum_{(i,j)\leq(i_*,j_*)}\varepsilon_1^i\varepsilon_2^j\,v_{i,j}\left(\frac{\mathsf{x}-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) - \sum_{(i,j)\leq(i_*,j_*)}\varepsilon_1^i\varepsilon_2^j\,v_{i,j}\left(\frac{\varsigma(\mathsf{x})-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) \end{aligned} \tag{3.29}$$

for all $\mathsf{x} \in \overline{\Omega_{\boldsymbol{\varepsilon}}}$. Then we denote by (i^*, j^*) the minimum of the multi-indexes (i,j) with $i+j \le N$ which are strictly bigger than (i_*, j_*). By (3.22) we compute that

$$\lim_{\eta\to 0} \frac{W_{\boldsymbol{\varepsilon}(\eta),N}(\mathsf{x}) - W_{\boldsymbol{\varepsilon}(\eta),(i_*,j_*)}(\mathsf{x})}{\varepsilon_1^{i^*}(\eta)\varepsilon_2^{j^*}(\eta)} = u_{i^*,j^*}(\mathsf{x})\,, \qquad \forall \mathsf{x} \in \overline{\Omega}\,,$$

and by (3.19) and (3.23) we have

$$\lim_{\eta\to 0} \frac{u_{\boldsymbol{\varepsilon}(\eta)}(\mathsf{x}) - W_{\boldsymbol{\varepsilon}(\eta),(i_*,j_*)}(\mathsf{x})}{\varepsilon_1^{i^*}(\eta)\varepsilon_2^{j^*}(\eta)} = U_{i^*,j^*}(\mathsf{x})\,, \qquad \forall \mathsf{x} \in \overline{\Omega}\,.$$

Accordingly, (3.20) implies that $u_{i^*,j^*} = U_{i^*,j^*}$. To show that $v_{i^*,j^*} = V_{i^*,j^*}$ we use (3.24) to verify that

$$\lim_{\eta\to 0} \frac{W_{\boldsymbol{\varepsilon}(\eta),N}(\varepsilon_1(\eta)\mathsf{p} + \varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{X}) - W_{\boldsymbol{\varepsilon}(\eta),(i_*,j_*)}(\varepsilon_1(\eta)\mathsf{p} + \varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{X})}{\varepsilon_1^{i^*}(\eta)\varepsilon_2^{j^*}(\eta)} = u_{i^*,j^*}(0) + v_{i^*,j^*}(\mathsf{X})$$

for all $\mathsf{X} \in \mathbb{R}^n \setminus \omega$. Similarly, by (3.19) and (3.25) we deduce that

$$\lim_{\eta\to 0} \frac{u_{\boldsymbol{\varepsilon}(\eta)}(\varepsilon_1(\eta)\mathsf{p} + \varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{X}) - W_{\boldsymbol{\varepsilon}(\eta),(i_*,j_*)}(\varepsilon_1(\eta)\mathsf{p} + \varepsilon_1(\eta)\varepsilon_2(\eta)\mathsf{X})}{\varepsilon_1^{i^*}(\eta)\varepsilon_2^{j^*}(\eta)} = U_{i^*,j^*}(0) + V_{i^*,j^*}(\mathsf{X})$$

for all $\mathsf{X} \in \mathbb{R}^n \setminus \omega$. Then (3.28) and equality $u_{i^*,j^*}(0) = U_{i^*,j^*}(0)$ imply that $v_{i^*,j^*} = V_{i^*,j^*}$. Our proof is now complete. □

Proposition 3.12 shows that the solution $u_{\boldsymbol{\varepsilon}}$ can be approximated by a finite sum of slow variable terms, fast variable terms, and reflected fast variable terms. In the following Remark 3.13 we investigate the possibility of approximating the solution $u_{\boldsymbol{\varepsilon}}$ by a finite sum of slow variable and fast variable terms, and no reflected fast variable terms. By showing an example in $\mathbb{R}^3$, we prove that in general this is not possible, at least not if we want the terms in the sum to be continuous functions.

Remark 3.13. *Let $n = 3$ and assume that $v_0(\mathsf{x}) = 1/|\mathsf{x}|$ for all $\mathsf{x} \in \mathbb{R}^3 \setminus \omega$. Then, there are no functions $u^\#_{0,0}, u^\#_{0,1}, u^\#_{1,0} \in \mathscr{C}(\overline{\Omega})$ and $v^\#_{0,0}, v^\#_{0,1}, v^\#_{1,0} \in \mathscr{C}_0(\mathbb{R}^3 \setminus \omega)$ such that*

$$\left\| u_{\boldsymbol{\varepsilon}} - W^\#_{\boldsymbol{\varepsilon},1} \right\|_{\mathscr{L}^\infty(\Omega_{\boldsymbol{\varepsilon}})} = o(\varepsilon_1 + \varepsilon_2) \qquad \text{as } \boldsymbol{\varepsilon} \to \mathbf{0} \text{ in }]\mathbf{0}, \boldsymbol{\varepsilon}^*[\,, \tag{3.30}$$

where

$$\begin{aligned} &W^\#_{\boldsymbol{\varepsilon},1}(\mathsf{x}) \\ &\equiv u^\#_{0,0}(\mathsf{x}) + v^\#_{0,0}\left(\frac{\mathsf{x} - \varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) + \varepsilon_2\, u^\#_{0,1}(\mathsf{x}) + \varepsilon_2\, v^\#_{0,1}\left(\frac{\mathsf{x} - \varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) + \varepsilon_1\, u^\#_{1,0}(\mathsf{x}) + \varepsilon_1\, v^\#_{1,0}\left(\frac{\mathsf{x} - \varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) \end{aligned}$$

for all $\mathsf{x} \in \overline{\Omega_{\boldsymbol{\varepsilon}}}$.

Proof. We assume by contradiction that there are such $u^\#_{0,0}, u^\#_{0,1}, u^\#_{1,0} \in \mathscr{C}(\overline{\Omega})$ and $v^\#_{0,0}, v^\#_{0,1}, v^\#_{1,0} \in \mathscr{C}_0(\mathbb{R}^3 \setminus \omega)$ for which (3.30) holds true. As in the proof of Proposition 3.12, we take $0 <$

$\eta_* < 1$ small enough and we consider the curve in $]\mathbf{0}, \boldsymbol{\varepsilon}^*[$ which takes $\eta \in]0, \eta_*[$ to $\boldsymbol{\varepsilon}(\eta) = (\varepsilon_1(\eta), \varepsilon_2(\eta)) \equiv (\eta, -1/\log \eta)$. By the membership of $v^\#_{0,0}, v^\#_{0,1}, v^\#_{1,0}$ in $\mathscr{C}_0(\mathbb{R}^3 \setminus \omega)$ we have

$$\lim_{\eta\to 0} v^\#_{i,j}\left(\frac{\mathsf{x} - \varepsilon_1(\eta)\mathsf{p}}{\varepsilon_1(\eta)\varepsilon_2(\eta)}\right) = 0 \qquad \forall \mathsf{x} \in \overline{\Omega}\,, (i,j) \in \{(0,0), (0,1), (1,0)\}\,. \tag{3.31}$$

Then, by arguing as we have done in the proof of Proposition 3.12 to show that $u_{0,0} = U_{0,0}$ and $v_{0,0} = V_{0,0}$, we can verify that $u^\#_{0,0} = U_{0,0}$ and $v^\#_{0,0} = V_{0,0}$. In particular, $u^\#_{0,0} = u_0$ and $v^\#_{0,0}(\mathsf{x}) = v_0(\mathsf{x}) = 1/|\mathsf{x}|$ for all $\mathsf{x} \in \mathbb{R}^3 \setminus \omega$ (cf. Theorem 3.11). Since $\varepsilon_1(\eta) = o(\varepsilon_2(\eta))$ as $\eta \to 0$ the asymptotic relation (3.30) and the limit (3.31) imply that

$$\lim_{\eta\to 0} \frac{1}{\varepsilon_2(\eta)}\left(u_{\boldsymbol{\varepsilon}}(\mathsf{x}) - W^\#_{\boldsymbol{\varepsilon}(\eta),(0,1)}(\mathsf{x})\right) = 0 \qquad \forall \mathsf{x} \in \overline{\Omega}\,, \tag{3.32}$$

where

$$W^\#_{\boldsymbol{\varepsilon}(\eta),(0,1)}(\mathsf{x}) \equiv u_0(\mathsf{x}) + v_0\left(\frac{\mathsf{x} - \varepsilon_1(\eta)\mathsf{p}}{\varepsilon_1(\eta)\varepsilon_2(\eta)}\right) + \varepsilon_2(\eta)\, u^\#_{0,1}(\mathsf{x}) + \varepsilon_2(\eta)\, v^\#_{0,1}\left(\frac{\mathsf{x} - \varepsilon_1(\eta)\mathsf{p}}{\varepsilon_1(\eta)\varepsilon_2(\eta)}\right)$$

for all $\mathsf{x} \in \overline{\Omega_{\boldsymbol{\varepsilon}(\eta)}}$. By (3.31) we deduce that

$$\lim_{\eta\to 0} \frac{1}{\varepsilon_2(\eta)}\left(W^\#_{\boldsymbol{\varepsilon}(\eta),(0,1)}(\mathsf{x}) - u_0(\mathsf{x}) - v_0\left(\frac{\mathsf{x} - \varepsilon_1(\eta)\mathsf{p}}{\varepsilon_1(\eta)\varepsilon_2(\eta)}\right)\right) = u^\#_{0,1}(\mathsf{x}) \qquad \forall \mathsf{x} \in \overline{\Omega}\,. \tag{3.33}$$

Then we observe that

$$\lim_{\eta\to 0} \frac{1}{\varepsilon_2(\eta)} v_0\left(\frac{\varsigma(\mathsf{x}) - \varepsilon_1(\eta)\mathsf{p}}{\varepsilon_1(\eta)\varepsilon_2(\eta)}\right) = \lim_{\eta\to 0} \frac{\varepsilon_1(\eta)}{|\varsigma(\mathsf{x}) - \varepsilon_1(\eta)\mathsf{p}|} = 0 \qquad \forall \mathsf{x} \in \overline{\Omega} \setminus \{0\}$$

(note that we cannot take $\mathsf{x} = 0$ in this limit relation) and we deduce by (3.19) and (3.23) that

$$\lim_{\eta\to 0} \frac{1}{\varepsilon_2(\eta)}\left(u_{\boldsymbol{\varepsilon}(\eta)}(\mathsf{x}) - u_0(\mathsf{x}) - v_0\left(\frac{\mathsf{x} - \varepsilon_1(\eta)\mathsf{p}}{\varepsilon_1(\eta)\varepsilon_2(\eta)}\right)\right) = U_{0,1}(\mathsf{x}) \qquad \forall \mathsf{x} \in \overline{\Omega} \setminus \{0\}\,. \tag{3.34}$$

Now (3.32), (3.33), and (3.34) imply that $u^\#_{0,1}(\mathsf{x}) = U_{0,1}(\mathsf{x})$ for all $\mathsf{x} \in \overline{\Omega} \setminus \{0\}$. Since both $u^\#_{0,1}$ and $U_{0,1}$ are continuous functions on $\overline{\Omega}$, it follows that

$$u^\#_{0,1} = U_{0,1} \text{ on the whole of } \overline{\Omega}.$$

Then by (3.19) we compute that

$$\begin{aligned}
u_{\boldsymbol{\varepsilon}(\eta)}(\mathsf{x}) - W^\#_{\boldsymbol{\varepsilon}(\eta),(0,1)}(\mathsf{x}) = &- v_0\left(\frac{\varsigma(\mathsf{x}) - \varepsilon_1(\eta)\mathsf{p}}{\varepsilon_1(\eta)\varepsilon_2(\eta)}\right) - \varepsilon_2(\eta) v^\#_{0,1}\left(\frac{\mathsf{x} - \varepsilon_1(\eta)\mathsf{p}}{\varepsilon_1(\eta)\varepsilon_2(\eta)}\right) \\
&+ \sum_{(i,j)>(0,1)} \varepsilon_1^i(\eta)\varepsilon_2^j(\eta)\, U_{i,j}(\mathsf{x}) \\
&+ \sum_{(i,j)\geq(0,1)} \varepsilon_1^i(\eta)\varepsilon_2^j(\eta)\, V_{i,j}\left(\frac{\mathsf{x} - \varepsilon_1(\eta)\mathsf{p}}{\varepsilon_1(\eta)\varepsilon_2(\eta)}\right) \\
&- \sum_{(i,j)\geq(0,1)} \varepsilon_1^i(\eta)\varepsilon_2^j(\eta)\, V_{i,j}\left(\frac{\varsigma(\mathsf{x}) - \varepsilon_1(\eta)\mathsf{p}}{\varepsilon_1(\eta)\varepsilon_2(\eta)}\right) \qquad \forall \mathsf{x} \in \overline{\Omega_{\boldsymbol{\varepsilon}(\eta)}}\,.
\end{aligned}$$

(As in the proof of Proposition 3.12 we order $\mathbb{N}^2$ by saying that $(i,j) > (h,k)$ if (3.21) is verified.) It follows that for $\mathsf{x} = 0$ we have

$$u_{\boldsymbol{\varepsilon}(\eta)}(0) - W^{\#}_{\boldsymbol{\varepsilon}(\eta),(0,1)}(0) = -v_0\left(-\frac{\mathsf{p}}{\varepsilon_2(\eta)}\right) + \varepsilon_2(\eta) v^{\#}_{0,1}\left(-\frac{\mathsf{p}}{\varepsilon_2(\eta)}\right) + o(\varepsilon_2(\eta)) \qquad \text{as } \eta \to 0\,.$$

Then, by the membership of $v^{\#}_{0,1}$ in $\mathscr{C}_0(\mathbb{R}^3 \setminus \omega)$ and by equality $v_0(\mathsf{x}) = 1/|\mathsf{x}|$, we have

$$\lim_{\eta\to 0} \frac{1}{\varepsilon_2(\eta)} \left(u_{\boldsymbol{\varepsilon}(\eta)}(0) - W^{\#}_{\boldsymbol{\varepsilon}(\eta),(0,1)}(0)\right) = -\frac{1}{|\mathsf{p}|} \neq 0\,.$$

The latter limit relation contradicts (3.32) and the remark is proved. □

We observe that in Remark (3.13) the condition $v_0(\mathsf{x}) = 1/|\mathsf{x}|$ can be relaxed by taking $\lim_{\mathsf{x}\to\infty} |\mathsf{x}| v_0(\mathsf{x}) \neq 0$. We also observe that one has $\lim_{\mathsf{x}\to\infty} |\mathsf{x}| v_0(\mathsf{x}) = 0$ if and only if v_0 is given by a double layer potential, or equivalently, if and only if $\int_{\partial\omega} \nu_\omega \cdot \nabla v_0 \, d\sigma = 0$. To have $v_0(\mathsf{x}) = 1/|\mathsf{x}|$ one may for example take ω to be the unite ball in $\mathbb{R}^3$ and g^{i} to be identically equal to 1.

The Remark 3.13 here above is stronger than the previous Remarks 3.12 and 3.13, which have been accordingly omitted in the present version of the paper. Those remarks were saying that we cannot get rid of the reflected fast variable term if we want to have converging series. This is true, but something more is also true. Indeed, the present Remark 3.13 is saying that we cannot approximate the solution just with slow and fast variables terms, we need to use also reflected fast variable terms if we want that the coefficients to be continuous functions. One may think that an approximation with slow and fast variables can be obtained if we allow the coefficients to be discontinuous, but this is not so easy to achieve, because with discontinuous coefficients we cannot use the maximum principle to bring the error from the boundary to the interior....

Therefore, the best strategy for the multiscale approach could be to modify the algorithm as follows: assume that we start from the corrector at scale 1, then we

(S1) achieve a Taylor series expansion on that corrector at 0;

(S2) extend the traces on $\partial\omega_\varepsilon$ of these expansions to the complement of ω thanks to E_ω and subtract the reflected counterpart. This defines correctors at the scale $\varepsilon_1\varepsilon_2$. The boundary condition is now satisfied on $\partial\omega_{\boldsymbol{\varepsilon}}$ and also on $\partial_0\Omega$ but it is not the case anymore on $\partial_+\Omega$;

(S3) decompose each of the previous correctors as a sum of homogeneous harmonic functions;

(S4) extend the traces on $\partial\Omega$ of the homogeneous harmonic functions to the whole Ω thanks to E_Ω. This defines correctors at the scale 1. The boundary condition is now satisfied on $\partial\Omega$ but it is not the case anymore on $\partial\omega_{\boldsymbol{\varepsilon}}$. Go back to (S1).

(Note that the difference is only in (S2).)

Let me try to sketch the first steps without entering into details. To begin with, we observe that we already know that the first slow variable coefficient will be

$$U_{0,0}(\mathsf{x}) = u_0(\mathsf{x}) \qquad \forall \mathsf{x} \in \overline{\Omega}\,,$$

the first fast variable coefficient will be

$$V_{0,0}(\mathsf{x}) = v_0(\mathsf{x}) \qquad \forall \mathsf{x} \in \mathbb{R}^n \setminus \omega\,,$$

and the first fast variable term, which is equal to $V_{0,0}$. Then one observes that $V_{0,0}\left(\frac{\mathsf{x}-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) - V_{0,0}\left(\frac{\varsigma(\mathsf{x})-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right)$ gives an error of order ε_2^{n-2} on $\partial\omega$, of order $\varepsilon_1(\varepsilon_1\varepsilon_2)^{n-2}$ on $\partial_+\Omega$, and it gives no contribution on $\partial_0\Omega$. Then we correct the error on $\partial_+\Omega$ with a corrector R_1 which is of order $\varepsilon_1(\varepsilon_1\varepsilon_2)^{n-2}$ on $\partial_+\Omega$ and it is 0 on $\partial_0\Omega$. Then R_1 gives an error of order $\varepsilon_1^2(\varepsilon_1\varepsilon_2)^{n-2}$ on $\partial\omega$. We introduce the corrector r_1 of order $\varepsilon_1^2(\varepsilon_1\varepsilon_2)^{n-2}$ on $\partial\omega$. Instead of 'lifting' just r_1 we consider $r_1\left(\frac{\mathsf{x}-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) - r_1\left(\frac{\varsigma(\mathsf{x})-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right)$. This will give an error of order $\varepsilon_2^{n-2}\varepsilon_1^2(\varepsilon_1\varepsilon_2)^{n-2}$ on $\partial\omega$, of order $\varepsilon_1^3(\varepsilon_1\varepsilon_2)^{2(n-2)}$ on $\partial_+\Omega$, and again, it gives no contribution on $\partial_0\Omega$. Then the corrector R_2 will be of order $\varepsilon_1^3(\varepsilon_1\varepsilon_2)^{2(n-2)}$ on $\partial_+\Omega$ and it will be 0 on $\partial_0\Omega$, accordingly it gives an error of order $\varepsilon_1^4(\varepsilon_1\varepsilon_2)^{2(n-2)}$ on $\partial\omega$ ....

4 The multiscale asymptotic approach

As in Section 3 we assume that

$$n \in \mathbb{N} \setminus \{0,1,2\}\,.$$

If Ω' is an open subset of Ω such that $0 \notin \overline{\Omega'}$, then by Theorem 1.1 there are functions $u_{i,j}$ defined on Ω' such that

$$u_{\boldsymbol{\varepsilon}|\overline{\Omega'}} = \sum_{i,j\geq 0} \varepsilon_1^i {\varepsilon_2}^j \mathsf{u}_{i,j}, \tag{4.1}$$

where the series converges in $\mathscr{C}^{1,\alpha}(\overline{\Omega'})$ norm for ε_1 and ε_2 small enough. In this section, we aim at computing the first functions $\mathsf{u}_{i,j}$. Let us emphasize that the analyticity result (Theorem 1.1) allows us to be formal and to just identify the coefficients.

We choose here to follow the ideas of multiscale asymptotic expansions of [5, 6]. The aim of this method is to derive an asymptotic expansion of the solution both for the macroscopic scale (that is, away from the origin of $\mathbb{R}^n$) and for the microscopic scale (close to the origin of $\mathbb{R}^n$). We remark that this is a completely different point of view with respect to the functional analytic approach developed in the previous sections.

4.1 Preliminary results and notation

V : I introduce the macro `\pt` for X but it is probably the same than `\ps`

If v is an analytic function in Ω, there are functions $\mathsf{v}_{i,j}$ with $i,j \in \mathbb{N}$ defined on $\partial\omega$ such that

$$v(\varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{X}) = \sum_{i,j\geq 0} \varepsilon_1^i(\varepsilon_1\varepsilon_2)^j \mathsf{v}_{i,j}(\mathsf{X}), \qquad \forall \mathsf{X} \in \partial\omega. \tag{4.2}$$

In our expansion method we need to build correctors for traces on the boundary of Ω and on the inclusion's boundary. To do so, we will exploit the extension operators E_Ω and E_ω defined as follows: E_Ω takes a function $\phi \in \mathscr{C}^{1,\alpha}(\partial\Omega)$ to the unique solution $v \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$ of the Dirichlet boundary value problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega\,, \\ v = \phi & \text{on } \partial\Omega\,, \end{cases}$$

and E_ω takes a function $\psi \in \mathscr{C}^{1,\alpha}(\partial\omega)$ to the unique solution $u \in \mathscr{C}^{1,\alpha}_{\mathrm{loc},\infty}(\mathbb{R}^n \setminus \omega)$ of the exterior Dirichlet boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\omega}\,, \\ u = \psi & \text{on } \partial\omega\,, \end{cases} \tag{4.3}$$

PROBLEM IN THE FOLLOWING SENTENCE!! where $\mathscr{C}^{1,\alpha}_{\mathrm{loc},\infty}(\mathbb{R}^n \setminus \omega)$ consists of the function of We now observe that the solution u of (4.3) can be written as the sum of homogeneous harmonic functions $\{u\}_k$ for $k \geq n-2$. This can be verified by introducing a sphere $\partial\mathcal{B}(0,R)$ in $\mathbb{R}^n \setminus \omega$ and by considering the decomposition of $u_{|\partial\mathcal{B}(0,R)}$ into spherical harmonics. For each element of such decomposition, we can use the separation of variables to find the solution of the corresponding exterior Dirichlet problem in $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \omega)$ which vanishes at ∞. Doing so, we obtain a family of functions $a_k[u]$ with $k \geq n-2$ defined on the unit sphere and such that

$$u(\mathsf{x}) = \sum_{k \geq n-2} \{u\}_k(\mathsf{x}) \qquad \text{with} \qquad \{u\}_k(\mathsf{x}) = a_k[u]\left(\frac{\mathsf{x}}{|\mathsf{x}|}\right)|\mathsf{x}|^{-k}. \tag{4.4}$$

Choose between $|\mathsf{x}|$ (in Section 2) and $\|\mathsf{x}\|(here--previously)$ We notice that $a_k[u]$ belongs to the space of spherical harmonics of order $k-n+2$ and it is therefore analytic (as the trace of a polynomial on the unit sphere). If ω is a ball, then the computation of the $a_k[u]$'s is easily deduced by the projection of $\mathscr{L}^2(\partial\omega)$ on the space generated by the spherical harmonics of order $k-n+2$. However, for a general geometry, there is not such an elementary derivation.

4.2 An iterative construction

The strategy to derive the multiscale expansion is then to follow an iterative construction consisting in four substeps. We start from a function v that satisfies $\Delta v = 0$ in $\Omega_{\boldsymbol{\varepsilon}}$ and the boundary condition $v = g^{\circ}$ on the fixed boundary $\partial\Omega$ but not the boundary condition on the defect:

(S1) achieve a Taylor series expansion on that function at 0,

(S2) extend the traces on $\partial\omega_\varepsilon$ of these expansions to the complement of ω thanks to E_ω. This defines correctors at the scale $\varepsilon_1\varepsilon_2$. The boundary condition is now satisfied on $\partial\omega_{\boldsymbol{\varepsilon}}$ but it is not the case anymore on $\partial\Omega$.

(S3) decompose each of the previous correctors as a sum of homogeneous harmonic functions.

(S4) extend the traces on $\partial\Omega$ of the homogeneous harmonic functions to the whole Ω thanks to E_Ω. This defines correctors at the scale 1. The boundary condition is now satisfied on $\partial\Omega$ but it is not the case anymore on $\partial\omega_{\boldsymbol{\varepsilon}}$.

4.3 The first iteration

Let us build the first terms in this asymptotic expansion. We start by $u_{0,0} = u_0$ according to Theorem 1.1 and set $r_0 = u_\varepsilon - u_0$. Then, we compute:

$$\begin{cases} \Delta r_0 = 0 & \text{in } \Omega_{\boldsymbol{\varepsilon}}, \\ r_0 = g^{\mathrm{o}} - g^{\mathrm{o}} = 0 & \text{on } \partial\Omega, \\ r_0 = g^{\mathrm{i}}\left(\dfrac{\cdot - \varepsilon_1 \mathsf{p}}{\varepsilon_1\varepsilon_2}\right) - u_0 & \text{on } \partial\omega_{\boldsymbol{\varepsilon}}. \end{cases}$$

Notice that the error on $\partial\omega_{\boldsymbol{\varepsilon}}$ is of order 1.

Step (S1). We decompose the trace of r_0 on $\partial\omega_{\boldsymbol{\varepsilon}}$ as

$$g^{\mathrm{i}}(\mathsf{X}) - u_0\left(\varepsilon_1\mathsf{p} + \varepsilon_1\varepsilon_2\mathsf{X}\right) = \sum_{i,j\geq 0} \varepsilon_1^i(\varepsilon_1\varepsilon_2)^j[r_0]_{i,j}(\mathsf{X}).$$

With this notation, the first terms are:

$$\begin{aligned} &[r_0]_{0,0}(\mathsf{X}) = g^{\mathrm{i}}(\mathsf{X}) - g^{\mathrm{o}}(0), \\ &[r_0]_{1,0}(\mathsf{X}) = -\nabla u_0(0)\cdot\mathsf{p}, \qquad [r_0]_{0,1}(\mathsf{X}) = -\nabla u_0(0)\cdot\mathsf{X}, \\ &[r_0]_{2,0}(\mathsf{X}) = -\frac{1}{2}D^2u_0(0)\cdot(\mathsf{p},\mathsf{p}), \qquad [r_0]_{1,1}(\mathsf{X}) = -D^2u_0(0)\cdot(\mathsf{p},\mathsf{X}), \\ &\qquad [r_0]_{0,2}(\mathsf{X}) = -\frac{1}{2}D^2u_0(0)\cdot(\mathsf{X},\mathsf{X}). \end{aligned}$$

Step (S2). It is then natural to lift the misfits and to introduce correctors to compensate the leading errors. This is achieved thanks to E_ω. We then define a function $\bar{r}$ by

$$\bar{r} = r_0 - \sum_{i,j\geq 0}\varepsilon_1^i(\varepsilon_1\varepsilon_2)^j E_\omega\left([r_0]_{i,j}\right)\left(\frac{\cdot - \varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right),$$

and compute:

$$\Delta\bar{r} = 0 \text{ in } \Omega_{\boldsymbol{\varepsilon}}, \qquad \bar{r} = 0 \text{ on } \partial\omega_{\boldsymbol{\varepsilon}}.$$

On the fixed boundary $\partial\Omega$, it holds:

$$\bar{r} = g^{\mathrm{o}} - g^{\mathrm{o}} - \sum_{i,j\geq 0}\varepsilon_1^i(\varepsilon_1\varepsilon_2)^j E_\omega\left([r_0]_{i,j}\right)\left(\frac{\cdot - \varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) = -\sum_{i,j\geq 0}\varepsilon_1^i(\varepsilon_1\varepsilon_2)^j E_\omega\left([r_0]_{i,j}\right)\left(\frac{\cdot - \varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right).$$

Step (S3). According to (4.4), we decompose each $E_\omega\left([r_0]_{i,j}\right)$ into homogeneous terms as

$$E_\omega\left([r_0]_{i,j}\right) = \sum_{k\geq n-2}\left\{E_\omega\left([r_0]_{i,j}\right)\right\}_k.$$

An important question arises then, what is the order of the trace of $\left\{E_\omega\left([r_0]_{i,j}\right)\right\}_k\left(\frac{\cdot - \varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right)$ on $\partial\Omega$? Since we are dealing with homogeneous functions, this is directly connected to the distance between the source point $\varepsilon_1 \mathsf{p}$ and the boundary $\partial\Omega$ since for $\mathsf{y} \in \partial\Omega$, $i, j \geq 0$ and $k \geq n-2$, there is a spherical harmonic a_{ijk} defined on the unit sphere such that

$$\{E_\omega([r_0]_{i,j})\}_k \left(\frac{\mathsf{y}-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) = (\varepsilon_1\varepsilon_2)^k\, a_{ijk}\left(\frac{\mathsf{y}-\varepsilon_1\mathsf{p}}{|\mathsf{y}-\varepsilon_1\mathsf{p}|}\right)|\mathsf{y}-\varepsilon_1\mathsf{p}|^{-k}$$

Let us compute this distance: let y be a point on $\partial\Omega$, it holds

$$d(\mathsf{y},\varepsilon_1\mathsf{p})^2 = |\mathsf{y}|^2 + 2\varepsilon_1\mathsf{p}\cdot\mathsf{y} + \varepsilon_1^2|\mathsf{p}|^2$$

then for $k \geq n-2$,

$$d(\mathsf{y},\varepsilon_1\mathsf{p})^{-k} = |\mathsf{y}|^{-k}\left(1 - \frac{k}{2}2\varepsilon_1\frac{\mathsf{p}\cdot\mathsf{y}}{|\mathsf{y}|^2} + o(\varepsilon_1)\right).$$

The previous expansion fails at the vicinity of 0. Hence, we consider $\mathsf{y} = \varepsilon_1 Y$ for Y in $\partial\Omega$, then $d(\mathsf{y},\varepsilon_1\mathsf{p}) = \varepsilon_1 d(Y,\mathsf{p})$. Therefore, the order of the trace is $(\varepsilon_1\varepsilon_2)^k\varepsilon_1^{-k} = \varepsilon_2^k$. We therefore normalize the homogeneous functions so that their $\mathscr{L}^\infty$ norm on $\partial\Omega$ is of size 1:

$$\{E_\omega\widetilde{([r_0]_{i,j})}\}_k\left(\frac{\cdot-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) = \varepsilon_1^k\,\{E_\omega([r_0]_{i,j})\}_k\left(\frac{\cdot-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right).$$

We therefore have the normalized decomposition

$$E_\omega([r_0]_{i,j})\left(\frac{\cdot-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) = \sum_{k\geq n-2}\varepsilon_2^k\{E_\omega\widetilde{([r_0]_{i,j})}\}_k\left(\frac{\cdot-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right).$$

On the fixed boundary $\partial\Omega$, it holds:

$$\bar{r} = -\sum_{i,j\geq 0}\sum_{k\geq n-2}\varepsilon_1^{i+j}\varepsilon_2^{j+k}\,\{E_\omega\widetilde{([r_0]_{i,j})}\}_k\left(\frac{\cdot-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right).$$

Step (S4). Now, we lift each previous normalized term thanks to E_Ω to correct the boundary condition on the fixed boundary $\partial\Omega$ and define

$$\begin{aligned} r_1 = r_0 &- \sum_{i,j\geq 0}\varepsilon_1^{i+j}\varepsilon_2^j\,E_\omega([r_0]_{i,j})\left(\frac{\cdot-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right) \\ &+ \sum_{i,j\geq 0}\sum_{k\geq n-2}\varepsilon_1^{i+j}\varepsilon_2^{j+k}\,E_\Omega\left(\{E_\omega\widetilde{([r_0]_{i,j})}\}_k\left(\frac{\cdot-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right)\right). \end{aligned} \tag{4.5}$$

Clearly, r_1 is harmonic in Ω_ε with 0 trace on $\partial\Omega$. Concerning the trace on $\partial\omega_\varepsilon$, it is easy to check that the function $E_\Omega\left(\{E_\omega\widetilde{([r_0]_{i,j})}\}_k\left(\frac{\cdot-\varepsilon_1\mathsf{p}}{\varepsilon_1\varepsilon_2}\right)\right)$ is of size 1. Hence, r_1 is of size ε_2^{n-2} on $\partial\omega_\varepsilon$. After one iteration, we have a set $I_1 \subset \mathbb{N}\times\mathbb{N}$ of active indices (that is associated with *a priori* nonzero coefficient).

4.3.1 The general iterations

Once one has build a reminder r_l for $l \geq 1$, applying the previous construction leads to r_{k+1} defined as

$$r_{l+1} = r_l - \sum_{i,j\geq 0} \varepsilon_1^{i+j} \varepsilon_2^j \, E_\omega\left([r_l]_{i,j}\right)\left(\frac{\cdot - \varepsilon_1 \mathsf{p}}{\varepsilon_1\varepsilon_2}\right)$$
$$+ \sum_{i,j\geq 0}\sum_{k\geq n-2} \varepsilon_1^{i+j}\varepsilon_2^{j+k}\, E_\Omega\left(\widetilde{\{E_\omega\left([r_l]_{i,j}\right)\}_k}\left(\frac{\cdot - \varepsilon_1 \mathsf{p}}{\varepsilon_1\varepsilon_2}\right)\right).$$

It is easy to check that r_l is of size $\varepsilon_2^{l(n-2)}$ on $\partial\omega_{\boldsymbol{\varepsilon}}$. After l iterations, we have a set $I_l \subset \mathbb{N}\times\{(k-1)(n-2),\cdots\}$ of active indices.

Therefore, to deduce the coefficient $\mathsf{u}_{i,j}(\mathsf{x})$, one has to proceed to $\lfloor \frac{j}{n-2}\rfloor$ iterations of the construction. Here, $\lfloor t\rfloor$ denotes the integer part of the real t. An important remark is that the expansion (4.1) does not see *fast variable* term. One should rewrite these terms. At a fixed point $\mathsf{x}\in\Omega$, one has

$$E_\omega\left([r_l]_{i,j}\right)\left(\frac{\mathsf{x} - \varepsilon_1 \mathsf{p}}{\varepsilon_1\varepsilon_2}\right) = \sum_{k\geq n-2}\{E_\omega\left([r_l]_{i,j}\right)\}_k\left(\frac{\mathsf{x} - \varepsilon_1 \mathsf{p}}{\varepsilon_1\varepsilon_2}\right)$$
$$= \sum_{k\geq n-2} (\varepsilon_1\varepsilon_2)^k\, a_k[[r_l]_{i,j}]\left(\frac{\mathsf{x} - \varepsilon_1 \mathsf{p}}{|\mathsf{x} - \varepsilon_1 \mathsf{p}|}\right)|\mathsf{x} - \varepsilon_1 \mathsf{p}|^{-k}.$$

This means that to compute the contribution of *fast variable* term in the expansion (4.1), one needs to decompose each term in homogeneous harmonic functions according to the previous formula and use the product of the Taylor series of the functions $\mathsf{x}\mapsto a_k[[r_l]_{i,j}]\left(\frac{\mathsf{x}-\varepsilon_1\mathsf{p}}{\|\mathsf{x}-\varepsilon_1\mathsf{p}\|}\right)$ and $\mathsf{x}\mapsto\|\mathsf{x}-\varepsilon_1\mathsf{p}\|^{-k}$ that are analytic in Ω'.

4.4 Derivation of an asymptotic expansion

If one is interested in a asymptotic expansion up to a given order, it is mandatory to precise the respective order of ε_1 and ε_2. To reduce to the situation to a single parameter, we define a curve $\boldsymbol{\eta} : \eta \mapsto \boldsymbol{\varepsilon}(\eta)$ in the quarter $\{\boldsymbol{\varepsilon} = (\varepsilon_1,\varepsilon_2), \varepsilon_i \geq 0,\ i = 1,2\}$ starting from 0. Up to a reparametrization of the curve $\boldsymbol{\eta}$, we can assume it to be of the type $\eta\mapsto(\varepsilon_1(\eta),\varepsilon_2(\eta)=\eta)$. We moreover assume that there is a real number γ such that $\varepsilon_1(\eta)=\eta^\gamma$. Therefore, in order to get an expansion of order K in $\eta=\varepsilon_2$, the terms of order (i,j) such that $\gamma i + j \leq K$ have to be taken into account and hence requires $\lfloor K/(n-2)\rfloor$ iterations of the previous construction.

Let us take an example: $n = 3$, $K = 2$ and $\gamma = 2$ that is $\eta\mapsto\boldsymbol{\varepsilon}(\eta) = (\eta^2,\eta)$. Two iterations are needed. Let us consider the first iteration of the construction. According to (4.5), we have to consider the terms:

- in fast variables such that $2(i+j)+j \leq 2$ that is $(i,j) = (0,0),\ (1,0),\ (0,1),\ (0,2)$.
- in slow variable such that $2(i+j)+j+k \leq 2$ that is $(i,j,k) = (0,0,1),\ (0,1,1)$.

This generates the terms:

$$
\begin{aligned}
& u_0(\mathsf{x}) - E_\omega\left(g^{\mathrm{i}} - g^{\mathrm{o}}(0)\right)\left(\frac{\cdot - \eta^2 p}{\eta^3}\right) \\
& + \eta\left(E_\omega\left(\nabla u_0 \cdot \mathsf{X}\right)\left(\frac{\cdot - \eta^2 p}{\eta^3}\right) + E_\Omega\left(a_1[g^{\mathrm{i}} - g^{\mathrm{o}}(0)]\left(\frac{\mathsf{x} - \eta^2 \mathsf{p}}{|\mathsf{x} - \eta^2 \mathsf{p}|}\right)|\mathsf{x} - \eta^2\mathsf{p}|^{-1}\right)\right) \\
& + \eta^2\left(E_\omega\left(\nabla u_0 \cdot \mathsf{p} + \frac{1}{2}D^2 u_0(0).(\mathsf{X},\mathsf{X})\right)\left(\frac{\cdot - \eta^2 p}{\eta^3}\right) - E_\Omega\left(a_1[\nabla u_0 \cdot \mathsf{X}]\left(\frac{\mathsf{x} - \eta^2 \mathsf{p}}{|\mathsf{x} - \eta^2 \mathsf{p}|}\right)|\mathsf{x} - \eta^2\mathsf{p}|^{-1}\right)\right)
\end{aligned}
$$

A second iteration of the construction is needed. The same procedure has to be initiated from r_1. By construction, the trace on $\partial\omega_\varepsilon$ contains only the contribution of the slow variable correctors $E_\Omega\left(\{E_\omega\widetilde{([r_l]_{i,j})}\}_k\left(\frac{\cdot - \varepsilon_1 \mathsf{p}}{\varepsilon_1\varepsilon_2}\right)\right)$. Hence, we has to add the terms

$$
\begin{aligned}
\eta \quad & E_\omega\left(E_\Omega\left(a_1[g^{\mathrm{i}} - g^{\mathrm{o}}(0)]\left(\frac{\mathsf{x} - \eta^2 \mathsf{p}}{|\mathsf{x} - \eta^2 \mathsf{p}|}\right)|\mathsf{x} - \eta^2\mathsf{p}|^{-1}\right)\right)\left(\frac{\cdot - \eta^2 p}{\eta^3}\right) \\
+\eta^2 \quad & E_\Omega\left(E_\omega\left(E_\Omega\left(a_1[g^{\mathrm{i}} - g^{\mathrm{o}}(0)]\left(\frac{\mathsf{x} - \eta^2 \mathsf{p}}{|\mathsf{x} - \eta^2 \mathsf{p}|}\right)|\mathsf{x} - \eta^2\mathsf{p}|^{-1}\right)\right)\left(\frac{\cdot - \eta^2 p}{\eta^3}\right)\right) \\
-\eta^2 \quad & E_\omega\left(E_\Omega\left(a_1[\nabla u_0 \cdot \mathsf{X}]\left(\frac{\mathsf{x} - \eta^2 \mathsf{p}}{|\mathsf{x} - \eta^2 \mathsf{p}|}\right)|\mathsf{x} - \eta^2\mathsf{p}|^{-1}\right)\right)\left(\frac{\cdot - \eta^2 p}{\eta^3}\right).
\end{aligned}
$$

A Decay properties of the Green function and of the associated single layer potential

In the following Lemma A.1 we present a result for the Green function G which allows us to study the behaviour of $v_G[\partial\Omega, \phi]$ at infinity.

Lemma A.1. *Let $n \in \mathbb{N} \setminus \{0,1\}$. Let $d \equiv 2\sup_{\mathsf{y}\in\Omega}|\mathsf{y}|$. Then the function*

$$
\begin{array}{rcl}
(\mathbb{R}^n \setminus \mathcal{B}(0,d)) \times \overline{\Omega} & \to & \mathbb{R} \\
(\mathsf{x},\mathsf{y}) & \mapsto & |\mathsf{x}|^{n-1}\, G(\mathsf{x},\mathsf{y})
\end{array}
$$

is bounded.

Proof. We observe that, for all $(\mathsf{x},\mathsf{y}) \in (\mathbb{R}^n \setminus \mathcal{B}(0,d)) \times \overline{\Omega}$, we have

$$
|\mathsf{x}-\mathsf{y}|^2 - |\varsigma(\mathsf{x})-\mathsf{y}|^2 = \sum_{j=1}^{n-1}(x_j - y_j)^2 + (x_n - y_n)^2 - \sum_{j=1}^{n-1}(x_j - y_j)^2 - (x_n + y_n)^2 = -4x_n y_n.
$$

Let us first consider $n = 2$. By exploiting the inequality $|\mathsf{x}| > 2|\mathsf{y}|$, we calculate that for any $(\mathsf{x},\mathsf{y}) \in (\mathbb{R}^n \setminus \mathcal{B}(0,d)) \times \overline{\Omega}$,

$$
\begin{aligned}
|G(\mathsf{x},\mathsf{y})| &= \frac{1}{2\pi}\left|\log|\mathsf{x}-\mathsf{y}| - \log|\varsigma(\mathsf{x})-\mathsf{y}|\right| = \frac{1}{4\pi}\left|\log|\mathsf{x}-\mathsf{y}|^2 - \log|\varsigma(\mathsf{x})-\mathsf{y}|^2\right| \\
&\leq \frac{1}{4\pi}\frac{1}{\min\{|\mathsf{x}-\mathsf{y}|^2\,,\,|\varsigma(\mathsf{x})-\mathsf{y}|^2\}}\left||\mathsf{x}-\mathsf{y}|^2 - |\varsigma(\mathsf{x})-\mathsf{y}|^2\right| \leq \frac{1}{\pi}\frac{|x_2 y_2|}{\min\{|\mathsf{x}-\mathsf{y}|^2\,,\,|\varsigma(\mathsf{x})-\mathsf{y}|^2\}} \\
&\leq \frac{1}{\pi}\frac{|\mathsf{x}|\,|\mathsf{y}|}{(|\mathsf{x}|-|\mathsf{y}|)^2} = \frac{1}{\pi}\frac{|\mathsf{y}|}{(1-|\mathsf{y}|/|\mathsf{x}|)^2}\frac{1}{|\mathsf{x}|} \leq \frac{4|\mathsf{y}|}{\pi}\frac{1}{|\mathsf{x}|} \leq \frac{2d}{\pi}\frac{1}{|\mathsf{x}|}.
\end{aligned}
$$

To prove the statement for $n \geq 3$ we observe that

$$
\begin{aligned}
|G(\mathsf{x},\mathsf{y})| &= \frac{1}{(n-2)s_n}\left||\mathsf{x}-\mathsf{y}|^{2-n} - |\varsigma(\mathsf{x})-\mathsf{y}|^{2-n}\right| \\
&= \frac{1}{(n-2)s_n}\frac{\left||\mathsf{x}-\mathsf{y}|^2 - |\varsigma(\mathsf{x})-\mathsf{y}|^2\right|}{|\mathsf{x}-\mathsf{y}|\,|\varsigma(\mathsf{x})-\mathsf{y}|(|\mathsf{x}-\mathsf{y}|+|\varsigma(\mathsf{x})-\mathsf{y}|)}\sum_{j=0}^{n-3}|\mathsf{x}-\mathsf{y}|^{j+3-n}|\varsigma(\mathsf{x})-\mathsf{y}|^{-j} \leq \frac{2^n d}{s_n}\frac{1}{|\mathsf{x}|^{n-1}}
\end{aligned}
$$

for all $(\mathsf{x},\mathsf{y}) \in (\mathbb{R}^n \setminus \mathcal{B}(0,d)) \times \overline{\Omega}$. Hence $|\mathsf{x}|^{n-1}\,|G(\mathsf{x},\mathsf{y})| \leq 2^n d/s_n$ for all $(\mathsf{x},\mathsf{y}) \in (\mathbb{R}^n \setminus \mathcal{B}(0,d)) \times \overline{\Omega}$ and for all $n \in \mathbb{N} \setminus \{0,1\}$. $\square$

Then, by Lemma A.1 one readily deduces the validity of the following.

Lemma A.2. *Let $n \in \mathbb{N} \setminus \{0,1\}$. Let $\phi \in \mathscr{C}^{0,\alpha}(\partial\Omega)$. Then the function which takes $\mathsf{x} \in \mathbb{R}^n \setminus (\Omega \cup \varsigma(\Omega))$ to $|\mathsf{x}|^{n-1}\, v_G[\partial\Omega,\phi](\mathsf{x})$ is bounded. In particular, $v_G[\partial\Omega,\phi]$ is harmonic at infinity.*

B An extension result

In this Appendix we prove Proposition 2.18. We find convenient to set $\mathcal{B}^+(0,r) \equiv \mathcal{B}(0,r) \cap \mathbb{R}^n_+$ and $\mathcal{B}^-(0,r) \equiv \mathcal{B}(0,r) \setminus \overline{\mathcal{B}^+(0,r)}$ for all $r > 0$. Then, possibly shrinking r_0 we can assume that $\mathcal{B}^+(0,r_0) \subseteq \Omega$. By assumption (H_4) and by a standard argument based on the Cauchy-Kovalevskaya Theorem we shows the validity of the following

Lemma B.1. *Let $n \in \mathbb{N} \setminus \{0,1\}$. There exist $r_1 \in]0,r_0]$ and a function H from $\overline{\mathcal{B}(0,r_1)}$ to $\mathbb{R}$ such that $\Delta H = 0$ in $\mathcal{B}(0,r_1)$ and $H_{|\overline{\mathcal{B}(0,r_1)}\cap\partial_0\Omega} = g^{\mathrm{o}}_{|\overline{\mathcal{B}(0,r_1)}\cap\partial_0\Omega}$.*

Proof. By the Cauchy-Kovalevskaya Theorem there exists $r_1 \in]0,r_0]$, a function H^+ from $\overline{\mathcal{B}^+(0,r_1)}$ to $\mathbb{R}$, and a function H^- from $\overline{\mathcal{B}^-(0,r_1)}$ to $\mathbb{R}$, such that

$$
\begin{aligned}
&\Delta H^+ = 0 \text{ in } \mathcal{B}^+(0,r_1),\ H^+_{|\overline{\mathcal{B}(0,r_1)}\cap\partial_0\Omega} = g^{\mathrm{o}}_{|\overline{\mathcal{B}(0,r_1)}\cap\partial_0\Omega}, \text{ and } \partial_{x_n} H^+_{|\overline{\mathcal{B}(0,r_1)}\cap\partial_0\Omega} = 0, \\
&\Delta H^- = 0 \text{ in } \mathcal{B}^-(0,r_1),\ H^-_{|\overline{\mathcal{B}(0,r_1)}\cap\partial_0\Omega} = g^{\mathrm{o}}_{|\overline{\mathcal{B}(0,r_1)}\cap\partial_0\Omega}, \text{ and } \partial_{x_n} H^-_{|\overline{\mathcal{B}(0,r_1)}\cap\partial_0\Omega} = 0.
\end{aligned}
$$

We now define

$$
H(\mathsf{x}) \equiv \begin{cases} H^+(\mathsf{x}) & \text{if } \mathsf{x} \in \overline{\mathcal{B}^+(0,r_1)}\,, \\ H^-(\mathsf{x}) & \text{if } \mathsf{x} \in \overline{\mathcal{B}^-(0,r_1)}\,, \end{cases}
$$

for all $\mathsf{x} \in \overline{\mathcal{B}(0,r_1)}$. Note that H is well defined and $H(\mathsf{x}) = g^{\mathrm{o}}(\mathsf{x})$ for $\mathsf{x} \in \mathcal{B}(0,r_1) \cap \partial_0\Omega$. Then one observes that

$$
\begin{aligned}
\int_{\mathcal{B}(0,r_1)} H\,\Delta\varphi\,d\mathsf{x} &= \int_{\mathcal{B}^+(0,r_1)} H^+\,\Delta\varphi\,d\mathsf{x} + \int_{\mathcal{B}^-(0,r_1)} H^-\,\Delta\varphi\,d\mathsf{x} \\
&= -\int_{\mathcal{B}(0,r_1)\cap\partial_0\Omega} H^+\,\partial_{x_n}\varphi\,d\sigma + \int_{\mathcal{B}(0,r_1)\cap\partial_0\Omega} H^-\,\partial_{x_n}\varphi\,d\sigma \\
&\quad + \int_{\mathcal{B}(0,r_1)\cap\partial_0\Omega} (\partial_{x_n} H^+)\,\varphi\,d\sigma - \int_{\mathcal{B}(0,r_1)\cap\partial_0\Omega} (\partial_{x_n} H^-)\,\varphi\,d\sigma = 0
\end{aligned}
$$

for all test functions $\varphi \in C^\infty_c(\mathcal{B}(0,r_1))$. Hence the lemma is proved. $\square$

We are now ready to prove Proposition 2.18.

Proof of Proposition 2.18. Let H be as in Lemma B.1. Let $V^+ \equiv u_{0|\overline{\mathcal{B}^+(0,r_1)}} - H_{|\overline{\mathcal{B}^+(0,r_1)}}$. Then we have $\Delta V^+ = 0$ in $\mathcal{B}^+(0,r_1)$ and $V^+_{|\overline{\mathcal{B}(0,r_1)}\cap\partial_0\Omega} = 0$. Then we define $V^-(\mathsf{x}) \equiv -V^+(\varsigma(\mathsf{x}))$ for all $\mathsf{x} \in \overline{\mathcal{B}^-(0,r_1)}$. Then one verifies that $\Delta V^- = 0$ in $\mathcal{B}^-(0,r_1)$ and $V^-_{|\overline{\mathcal{B}(0,r_1)}\cap\partial_0\Omega} = 0$. In addition we have $\partial_{x_n} V^+_{|\overline{\mathcal{B}(0,r_1)}\cap\partial_0\Omega} = \partial_{x_n} V^-_{|\overline{\mathcal{B}(0,r_1)}\cap\partial_0\Omega}$. Then we set

$$V(\mathsf{x}) \equiv \begin{cases} V^+(\mathsf{x}) & \text{if } \mathsf{x} \in \overline{\mathcal{B}^+(0,r_1)}\,, \\ V^-(\mathsf{x}) & \text{if } \mathsf{x} \in \overline{\mathcal{B}^-(0,r_1)}\,, \end{cases}$$

for all $\mathsf{x} \in \overline{\mathcal{B}(0,r_1)}$. Hence we compute

$$\begin{aligned} \int_{\mathcal{B}(0,r_1)} V\,\Delta\varphi\,d\mathsf{x} &= \int_{\mathcal{B}^+(0,r_1)} V^+\,\Delta\varphi\,d\mathsf{x} + \int_{\mathcal{B}^-(0,r_1)} V^-\,\Delta\varphi\,d\mathsf{x} \\ &= -\int_{\mathcal{B}(0,r_1)\cap\partial_0\Omega} V^+\,\partial_{x_n}\varphi\,d\sigma + \int_{\mathcal{B}(0,r_1)\cap\partial_0\Omega} V^-\,\partial_{x_n}\varphi\,d\sigma \\ &\quad + \int_{\mathcal{B}(0,r_1)\cap\partial_0\Omega} (\partial_{x_n}V^+)\,\varphi\,d\sigma - \int_{\mathcal{B}(0,r_1)\cap\partial_0\Omega} (\partial_{x_n}V^-)\,\varphi\,d\sigma = 0 \end{aligned}$$

for all test functions $\varphi \in C^\infty_c(\mathcal{B}(0,r_1))$. So that $\Delta V = 0$ in $\mathcal{B}(0,r_1)$. Finally we take $U_0 \equiv V + H$ and we readily verify that the statement of Proposition 2.18 is verified (see also Lemma B.1). □

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References

[1] H. Ammari and H. Kang, *Polarization and moment tensors*, volume **162** of *Applied Mathematical Sciences*, Springer, New York, 2007.

[2] V. Bonnaillie-Noël, M. Dalla Riva, M. Dambrine, and P. Musolino, *A small hole close to the boundary for the two-dimensional Laplace Dirichlet problem*, in preparation.

[3] V. Bonnaillie-Noël and M. Dambrine, *Interactions between moderately close circular inclusions: the Dirichlet-Laplace equation in the plane*, Asymptot. Anal., **84** (2013), 197–227.

[4] V. Bonnaillie-Noël, M. Dambrine, and C. Lacave, *Interactions Between Moderately Close Inclusions for the Two-Dimensional Dirichlet–Laplacian*, Appl. Math. Res. Express. AMRX, **2016** (2016), 1–23.

[5] V. Bonnaillie-Noël, M. Dambrine, S. Tordeux, and G. Vial, *On moderately close inclusions for the Laplace equation.*, C. R. Math. Acad. Sci. Paris, **19** (2007), 609–614.

[6] V. Bonnaillie-Noël, M. Dambrine, S. Tordeux, and G. Vial, *Interactions between moderately close inclusions for the Laplace equation*, Math. Models Methods Appl. Sci., **19** (2009), 1853–1882.

[7] V. Bonnaillie-Noël, C. Lacave, and N. Masmoudi, *Permeability through a perforated domain for the incompressible 2D Euler equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **32** (2015), 159–182.

[8] L. Chesnel and X. Claeys, *A numerical approach for the Poisson equation in a planar domain with a small inclusion*, Bit Numer. Math., **56**, 4, (2016), 1237–1256.

[9] A. Cialdea, *A general theory of hypersurface potentials.*, Ann. Mat. Pura Appl. **168** (1995), 37–61.

[10] M. Costabel, M. Dalla Riva, M. Dauge, and P. Musolino, *Converging expansions for self-similar perforations of a plane sector*, typewritten manuscript.

[11] M. Dalla Riva and P. Musolino, *Real analytic families of harmonic functions in a domain with a small hole*, J. Differential Equations, **252** (2012), 6337–6355.

[12] M. Dalla Riva and P. Musolino, *Real analytic families of harmonic functions in a planar domain with a small hole*, J. Math. Anal Appl., **422** (2015), 37–55.

[13] M. Dalla Riva and P. Musolino, *A mixed problem for the Laplace operator in a domain with moderately close holes*, Comm. Partial Differential Equations, **41** (2016), 812–837.

[14] M. Dalla Riva, P. Musolino, and S.V. Rogosin, *Series expansions for the solution of the Dirichlet problem in a planar domain with a small hole*, Asymptot. Anal., **92** (2015), 339–361.

[15] M. Dauge, S. Tordeux, and G. Vial, *Selfsimilar perturbation near a corner: matching versus multiscale expansions for a model problem.* In *Around the research of Vladimir Maz'ya. II*, 95–134, Int. Math. Ser. (N. Y.), **12**, Springer, New York, 2010.

[16] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, 1985.

[17] G.B. Folland, *Introduction to partial differential equations*, Second edition, Princeton University Press, Princeton N.J., 1995.

[18] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer Verlag, Berlin, *etc.*, 1983.

[19] A.M. Il'in, *A boundary value problem for a second-order elliptic equation in a domain with a narrow slit. I. The two-dimensional case*, Math. USSR Sb., **28** (1978), pp. 459–480.

[20] A.M. Il'in, *Matching of asymptotic expansions of solutions of boundary value problems*, Translations of Mathematical Monographs **102**, American Mathematical Society, Providence, 1992.

[21] A.M. Il'in, *The boundary layer*, in: Fedoryuk MV (ed.) *Partial Differential Equations. V. Asymptotic Methods for Partial Differential Equations*, Encylopaedia of Mathematical Sciences **34**, Springer-Verlag, Berlin, 1999, pp. 173–210.

[22] V. Kozlov, V. Maz'ya, and A. Movchan, *Asymptotic analysis of fields in multi-structures*, Oxford Mathematical Monographs, the Clarendon Press Oxford University Press, New York, 1999.

[23] M. Lanza de Cristoforis, *Asymptotic behavior of the solutions of a nonlinear Robin problem for the Laplace operator in a domain with a small hole: a functional analytic approach*, Complex Var. Elliptic Equ., **52** (2007), 945–977.

[24] M. Lanza de Cristoforis, *Asymptotic behavior of the solutions of a non-linear transmission problem for the Laplace operator in a domain with a small hole. A functional analytic approach*, Complex Var. Elliptic Equ., **55** (2010), 269–303.

[25] M. Lanza de Cristoforis and P. Musolino, *A real analyticity result for a nonlinear integral operator*, J. Integral Equations Appl., **25** (2013), 21–46.

[26] M. Lanza de Cristoforis and L. Rossi, *Real analytic dependence of simple and double layer potentials upon perturbation of the support and of the density*, J. Integral Equations Appl., **16** (2004), 137–174.

[27] V. Maz'ya, A. Movchan, and M. Nieves, *Green's kernels and meso-scale approximations in perforated domains*, Lecture Notes in Mathematics **2077**, Springer, Berlin, 2013.

[28] V. Maz'ya, A. Movchan, and M. Nieves, *Mesoscale Models and Approximate Solutions for Solids Containing Clouds of Voids.* Multiscale Modeling & Simulation **14**, (2016), 138–172.

[29] V. Maz'ya, S. Nazarov, and B. Plamenevskij, *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains. Vols. I, II*, volumes **111, 112** of *Operator Theory: Advances and Applications*, Birkhäuser Verlag, Basel, 2000.

[30] C. Miranda, *Sulle proprietà di regolarità di certe trasformazioni integrali*, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I, **7** (1965), 303–336.

[31] A.A. Novotny and J. Sokołowski, *Topological derivatives in shape optimization*, Interaction of Mechanics and Mathematics, Springer, Heidelberg, 2013.

[32] G.M. Troianiello, *Elliptic differential equations and obstacle problems*, Plenum Press, New York and London, 1987.