From Euler to Monge and vice versa.

Y.B. Lecture 2:
From Euler to Vlasov through Monge-Ampère and Kantorovich.

Yann Brenier, Mikaela Iacobelli, Filippo Santambrogio,
Paris, Durham, Lyon.

MFO SEMINAR 1842, 14-20/10/2018.
EULER’S MODEL OF INCOMPRESSIBLE FLUIDS

One can describe the motion of an incompressible fluid inside a bounded domain $D$ in $\mathbb{R}^d$ by a time-dependent family $t \rightarrow \mathcal{X}_t$ of maps belonging to the Hilbert space $H = L^2(D, \mathbb{R}^d)$, valued in the subset $\text{VPM}(D) \subset H$ of all Lebesgue measure-preserving maps.
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$$\text{VPM}(D) = \{ \mathcal{X} \in H, \quad \int_D q(\mathcal{X}(a)) da = \int_D q(a) da, \quad \forall q \in \mathcal{C}(\mathbb{R}^d) \}$$
EULER’S MODEL OF INCOMPRESSIBLE FLUIDS

One can describe the motion of an incompressible fluid inside a bounded domain \( D \) in \( \mathbb{R}^d \) by a time-dependent family \( t \to \mathcal{X}_t \) of maps belonging to the Hilbert space \( H = L^2(D, \mathbb{R}^d) \), valued in the subset \( \text{VPM}(D) \subset H \) of all Lebesgue measure-preserving maps

\[
\text{VPM}(D) = \{ \mathcal{X} \in H, \quad \int_D q(\mathcal{X}(a)) \, da = \int_D q(a) \, da, \quad \forall q \in C(\mathbb{R}^d) \}
\]

The Euler model, introduced in 1755, correspond to those curves \( t \to \mathcal{X}_t \in \text{VPM}(D) \) for which there is a "pressure field" \( p_t(x) \) s.t.

\[
\frac{d^2 \mathcal{X}_t}{dt^2} + (\nabla p_t) \circ \mathcal{X}_t = 0
\]
three maps of the periodic square: one is area preserving
THE PRINCIPLE OF LEAST ACTION

(easy) THEOREM Let $D$ be convex and $(x_t, p_t)$ be a solution of the Euler equations, with $D_{x}^{2}p_t \leq C\mathbb{I}$.

Then, as long as $C|t_{1}−t_{0}|^2 < \pi$, $X|_{[t_{0}, t_{1}]}$ is the unique minimizer, among all curves along $VPM(D)$ that coincide with $X_t$ at $t = t_{0}, t = t_{1}$, of the following ACTION

$$\int_{t_{0}}^{t_{1}}||dX_{t}||^2H_{t}dt,$$

$H = L^2(D, Rd)$. In other words, such a curve is nothing but a (constant speed) minimizing geodesic along $VPM(D)$, with respect to the metric induced by $H = L^2(D, Rd)$ on $VPM(D)$.

(easy) THEOREM Let $D$ be convex and $(\mathcal{X}_t, p_t)$ be a solution of the Euler equations, with $D^2 \mathcal{X} p_t \leq C \Pi$. Then, as long as $C |t_1 - t_0|^2 < \pi$, $\mathcal{X}_{[t_0, t_1]}$ is the unique minimizer, among all curves along $\text{VPM}(D)$ that coincide with $\mathcal{X}_t$ at $t = t_0, t = t_1$, of the following ACTION

$$\int_{t_0}^{t_1} \left\| \frac{d\mathcal{X}_t}{dt} \right\|_H^2 \, dt, \quad H = \mathcal{L}^2(D, \mathbb{R}^d)$$
THE PRINCIPLE OF LEAST ACTION

(easy) THEOREM Let $D$ be convex and $(x_t, p_t)$ be a solution of the Euler equations, with $D_x p_t \leq C$. Then, as long as $C|t_1 - t_0|^2 < \pi$, $x|[t_0, t_1]$ is the unique minimizer, among all curves along $VPM(D)$ that coincide with $x_t$ at $t = t_0, t = t_1$, of the following ACTION

$$\int_{t_0}^{t_1} \| \frac{d x_t}{dt} \|_H^2 \ dt,$$

where $H = L^2(D, \mathbb{R}^d)$.

In other words, such a curve is nothing but a (constant speed) minimizing geodesic along $VPM(D)$, with respect to the metric induced by $H = L^2(D, \mathbb{R}^d)$ on $VPM(D)$.
THE PRINCIPLE OF LEAST ACTION

(easy) THEOREM Let $D$ be convex and $(\chi_t, p_t)$ be a solution of the Euler equations, with $D^2 p_t \leq C$. Then, as long as $C|t_1 - t_0|^2 < \pi$, $\chi|_{[t_0, t_1]}$ is the unique minimizer, among all curves along $VPM(D)$ that coincide with $\chi_t$ at $t = t_0, t = t_1$, of the following ACTION

$$\int_{t_0}^{t_1} \left\| \frac{d\chi_t}{dt} \right\|_H^2 dt,$$

$H = L^2(D, \mathbb{R}^d)$

In other words, such a curve is nothing but a (constant speed) minimizing geodesic along $VPM(D)$, with respect to the metric induced by $H = L^2(D, \mathbb{R}^d)$ on $VPM(D)$.

Fix $D = [0, 1]^d$ and consider its dyadic decomposition by $N = 2^n$ sub-cubes $D(\alpha)$, of barycenters $A(\alpha)$, $\alpha = 1, \ldots, N$. 

For numerical purposes, we approximate the set $VPM(D)$ of all volume-preserving maps by the discrete subset $P_N(D)$ of all rigid rearrangements of the $N$ sub-cubes, namely maps of form:

$$s(a) = a - A(\alpha) + A(\sigma(\alpha)),$$

$a \in D(\alpha)$, $\alpha = 1, \ldots, N$, where $S_N$ is the set of all permutations of \{1, 2, \ldots, N\}. 

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VOLUME-PRESERVING MAPS: APPROXIMATION PAR PERMUTATIONS

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For numerical purposes, we approximate the set $VPM(D)$ of all volume-preserving maps by the discrete subset $P_N(D)$ of all rigid rearrangements of the $N$ sub-cubes, namely maps of form:

$$s(a) = a - A(\alpha) + A(\sigma(\alpha)), \quad a \in D(\alpha), \quad \alpha = 1, \ldots, N, \quad \sigma \in S_N$$

where $S_N$ is the set of all permutations of $\{1, \ldots, N\}$. 
REARRANGEMENTS OF N=16 SUB-CUBES AS EXAMPLES OF VOLUME PRESERVING MAPS

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PENALIZATION OF THE EULER ACTION

Since minimizing geodesics along a discrete set such as the set of rigid permutations $P_N(D)$ do not make much sense, we rather consider a penalized version of the Euler action (*)

\[ \int_{t_0}^{t_1} \left( \| \frac{d\mathbf{x}_t}{dt} \|_H^2 + \epsilon^{-1} \inf_{s \in P_N(D)} \| \mathbf{x}_t - s \|_H^2 \right) dt, \quad H = L^2(D, \mathbb{R}^d) \]
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Since minimizing geodesics along a discrete set such as the set of rigid permutations $P_N(D)$ do not make much sense, we rather consider a penalized version of the Euler action (*)

$$\int_{t_0}^{t_1} \left( \| \frac{d\mathcal{X}_t}{dt} \|_H^2 + \epsilon^{-1} \inf_{s \in P_N(D)} \| \mathcal{X}_t - s \|_H^2 \right) dt, \quad H = L^2(D, \mathbb{R}^d)$$

(*) For smooth sets, this is a consistent approximation to minimizing geodesics (cf. Rubin-Ungar, CPAM 1957).
FINITE-DIMENSIONAL REDUCTION

It is consistent to limit ourself to piecewise affine maps of form

\[ \mathcal{X}_t(a) = a - A(\alpha) + X_t(\alpha), \quad a \in D(\alpha), \]

Here \( X_t \in (\mathbb{R}^d)^N \) becomes the new, finite-dimensional, unknown. Accordingly, the penalized action can be easily computed

\[ \int_{t_1}^{t_0} \left( \|dX_t\|_2^2 + \epsilon^{-1} \inf_{\sigma \in S_N} \|X_t - A_{\sigma}(\alpha)\|_2^2 \right) dt \]

Here \( \|\cdot\| \) denotes the euclidean norm in \( H = (\mathbb{R}^d)^N \), \( S_N \) is the set of all permutations of \( \{1, \ldots, N\} \) and \( A_{\sigma}(\alpha) = A(\sigma(\alpha)) \), \( \alpha = 1, \ldots, N \).
It is consistent to limit ourself to piecewise affine maps of form

\[ x_t(a) = a - A(\alpha) + x_t(\alpha), \quad a \in D(\alpha), \quad x_t(\alpha) = x_t(A(\alpha)), \quad \alpha = 1, \ldots, N \]

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It is consistent to limit ourself to piecewise affine maps of form

\[ \mathcal{X}_t(a) = a - A(\alpha) + X_t(\alpha), \quad a \in D(\alpha), \quad X_t(\alpha) = \mathcal{X}_t(A(\alpha)), \quad \alpha = 1, \ldots, N \]

Here \( X_t \in (\mathbb{R}^d)^N \) becomes the new, finite-dimensional, unknown. Accordingly, the penalized action can be easily computed

\[ \int_{t_0}^{t_1} \left( \left\| \frac{dX_t}{dt} \right\|^2 + \epsilon^{-1} \inf_{\sigma \in S_N} \left\| X_t - A_\sigma \right\|^2 \right) dt \]

Here \( \| \cdot \| \) denotes the euclidean norm in \( H = (\mathbb{R}^d)^N \), \( S_N \) is the set of all permutations of \( \{1, \cdots, N\} \) and \( A_\sigma(\alpha) = A(\sigma(\alpha)), \quad \alpha = 1, \ldots, N \).
Using the least-action principle, we end up with the following finite-dimensional dynamical system

\[ \epsilon \frac{d^2 X_t(\alpha)}{dt^2} = X_t(\alpha) - A(\sigma_{opt}(\alpha)), \quad \alpha = 1, \ldots, N \]

This can be used for numerical purposes! See related work by Mérigot and Mirebeau arXiv:1505.03306, based on Mérigot’s fast Monge-Ampère solver.

The explicit time discrete version was introduced in Y.B. CMP 2000 for \( \epsilon < 0 \), with convergence to the Euler model as \( |\epsilon| \to 0, \quad N \geq C|\epsilon|^{-8d}, \quad \delta t \leq C|\epsilon|^{-4} \).
The continuous version, involving the Monge-Ampère equation, was introduced in B. and Loeper (GAFA 2004), studied by Cullen, Gangbo, Pisante (Arma 2007), Ambrosio-Gangbo (CPAM 2008)...

\[
\partial_t f(t, x, \xi) + \nabla_x \cdot (\xi f(t, x, \xi)) - \nabla_\xi \cdot (\nabla_x \varphi(t, x)f(t, x, \xi)) = 0
\]

\[
\det(I + \epsilon D^2_x \varphi(t, x)) = \int_{\mathbb{R}^d} f(t, x, \xi) d\xi, \quad (t, x, \xi) \in \mathbb{R}^{1+d+d}
\]
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It is a fully nonlinear correction of the well-known Vlasov-Poisson system describing Newtonian gravitation as \(d = 3\).
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It is a fully nonlinear correction of the well-known Vlasov-Poisson system describing Newtonian gravitation as \( d = 3 \). As \( \epsilon = 0 \) we recover the "kinetic" formulation of the Euler equations.
PART II: A PURELY STOCHASTIC ORIGIN OF THE (discrete) VLASOV-MONGE-AMPERE MODEL

Using large deviation principles and the concept of "onde pilote" (coming from quantum mechanics), we will recover this discrete dynamical system from the trivial stochastic model of a Brownian point cloud.
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Using large deviation principles and the concept of "onde pilote" (coming from quantum mechanics), we will recover this discrete dynamical system from the trivial stochastic model of a Brownian point cloud.

As a consequence and in some sense, the Euler model of incompressible fluids can be obtained out of pure noise!
LET US RECALL SOME VERY OLD IDEAS ABOUT RANDOMNESS BY LUCRETIUS

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When atoms move straight down through the void by their own weight, they deflect a bit in space at a quite uncertain time and in uncertain places, just enough that you could say that their motion has changed. But if they were not in the habit of swerving, they would all fall straight down through the depths of the void, like drops of rain, and no collision would occur, nor would any blow be produced among the atoms. In that case, nature would never have produced anything. (Lucretius, ∼ 99 – 55 BC.)
We consider a cubic lattice \( \{ A(\alpha) \in \mathbb{R}^d, \alpha = 1, \cdots, N \} \) subject to brownian vibrations

\[
A(\alpha) + \sqrt{\epsilon} B_t(\alpha), \quad \alpha = 1, \cdots, N
\]

We define a point cloud as a finite set of indistinguishable points, i.e. as a point in the quotient space \((\mathbb{R}^d)^N / S_N\).
WHERE IS THE LOCATION OF THE LATTICE AT TIME $T$?

At a fixed time $T > 0$, the probability for the point cloud

$$Y_t(\alpha) = A(\alpha) + \sqrt{\epsilon} B_t(\alpha), \quad \alpha = 1, \cdots, N$$

to be observed at $X_T = (X_T(\alpha), \alpha = 1, \cdots, N) \in \mathbb{R}^{dN}$ has density

$$Z \sum_{\sigma \in S_N} \prod_{\alpha=1}^{N} \exp(-\frac{|X_T(\alpha) - A(\sigma(\alpha))|^2}{2\epsilon T})$$
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$$\frac{1}{Z} \sum_{\sigma \in S_N} \prod_{\alpha=1}^N \exp\left( - \frac{|X_T(\alpha) - A(\sigma(\alpha))|^2}{2\epsilon T} \right) = \frac{1}{Z} \sum_{\sigma \in S_N} \exp\left( - \frac{||X_T - A_\sigma||^2}{2\epsilon T} \right)$$
WHERE IS THE LOCATION OF THE LATTICE AT TIME $T$?

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$S_N = \{\text{permutations}\}$, $|\cdot|$ and $||\cdot||$ = euclidean norms in $\mathbb{R}^d$ and $\mathbb{R}^{Nd}$. $Z = (2\pi\epsilon T)^{-Nd/2}N!$. We crucially used the indistinguishability of the particles.
LET US "SURF" THE "HEAT WAVE"

This density is just the solution of the heat equation in $\mathbb{R}^{Nd}$

$$\frac{\partial \rho}{\partial t}(t, X) = \frac{\epsilon}{2} \triangle \rho(t, X), \quad \rho(t = 0, X) = \frac{1}{N!} \sum_{\sigma \in S_N} \delta(X - A_\sigma).$$
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Given $X_{t_0} \in \mathbb{R}^{Nd}$ at $t_0 > 0$ we follow the "heat wave" by solving

$$\frac{dX_t}{dt} = v(t, X_t), \quad v(t, X) = -\frac{\epsilon}{2} \nabla_X \log \rho(t, X), \quad t \geq t_0$$
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Given $X_{t_0} \in \mathbb{R}^{Nd}$ at $t_0 > 0$ we follow the "heat wave" by solving

$$\frac{dX_t}{dt} = v(t, X_t), \quad v(t, X) = -\frac{\epsilon}{2} \nabla X \log \rho(t, X), \quad t \geq t_0$$

This is an adaptation of de Broglie’s "onde pilote" concept. As a matter of fact, a similar calculation also works for the free Schrödinger equation:

$$(i \partial_t + \triangle)\psi = 0, \quad \psi(0, X) = \sum_\sigma \exp(-||X - A_\sigma||^2/a^2), \quad v = \nabla \text{Im} \log \psi$$
We get the "onde pilote" system, setting $t = \exp(2\theta)$,

$$\frac{dX_\theta}{d\theta} = X_\theta - \langle A \rangle \quad \langle A \rangle = \frac{\sum_{\sigma \in S_N} A_\sigma \exp\left(\frac{-\|X_\theta - A_\sigma\|^2}{2\epsilon \exp(2\theta)}\right)}{\sum_{\sigma \in S_N} \exp\left(\frac{-\|X_\theta - A_\sigma\|^2}{2\epsilon \exp(2\theta)}\right)}$$
As $\epsilon$ goes to zero, we get the first order dynamical system

\[
\frac{dX_\theta}{d\theta} = X_\theta - A_{\sigma_{opt}}, \quad \sigma_{opt} = \text{Arginf}_{\sigma \in \mathcal{S}_N} \|X_\theta - A_\sigma\|^2
\]
As \( \epsilon \) goes to zero, we get the first order dynamical system

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\]

i.e.

\[
\frac{d_+X_\theta}{d\theta} = -\nabla \Phi(X_\theta)
\]

which is the "gradient flow" of the semi-convex function

\[
\Phi(X) = -\inf_{\sigma \in \mathcal{S}_N} \|X - A_\sigma\|^2 / 2
\]

N.B. this formulation automatically include 1D sticky collisions.
Sticky collisions

horizontal : 51 grid points in x / vertical : 60 grid points in t
From free (Bosonic) Schrödinger to sticky particles
LARGE DEVIATIONS OF THE "HEAT WAVE ODE"

Let us surf the heat wave with some additional noise $\eta > 0$

\[
\frac{dX_\theta^\epsilon}{d\theta} = X_\theta^\epsilon - < A > + \eta \frac{dB_\theta}{d\theta}, \quad < A > = \frac{\sum_{\sigma \in S_N} A_\sigma \exp\left(\frac{-||X_\theta^\epsilon - A_\sigma||^2}{2\epsilon \exp(2\theta)}\right)}{\sum_{\sigma \in S_N} \exp\left(\frac{-||X_\theta^\epsilon - A_\sigma||^2}{2\epsilon \exp(2\theta)}\right)}
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LARGE DEVIATIONS OF THE "HEAT WAVE ODE"

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$$\frac{dX^\epsilon_{\theta}}{d\theta} = X^\epsilon_{\theta} - <A> + \eta \frac{dB_{\theta}}{d\theta}, \quad <A> = \frac{\sum_{\sigma \in S_N} A_{\sigma} \exp\left(\frac{-||X^\epsilon_{\theta} - A_{\sigma}||^2}{2\epsilon \exp(2\theta)}\right)}{\sum_{\sigma \in S_N} \exp\left(\frac{-||X^\epsilon_{\theta} - A_{\sigma}||^2}{2\epsilon \exp(2\theta)}\right)}$$

For $\epsilon$ fixed, we first use the Freidlin-Vencel theory to get the "good rate function" for the large deviations of the system as $\eta \to 0$. 

(*) thanks to L. Ambrosio, private communication.
LARGE DEVIATIONS OF THE "HEAT WAVE ODE"

Let us surf the heat wave with some additional noise \( \eta > 0 \)

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\frac{dX^\epsilon_\theta}{d\theta} = X^\epsilon_\theta - < A > + \eta \frac{dB_\theta}{d\theta}, \quad < A > = \frac{\sum_{\sigma \in S_N} A_\sigma \exp\left( -\frac{||X^\epsilon_\theta - A_\sigma||^2}{2\epsilon \exp(2\theta)} \right)}{\sum_{\sigma \in S_N} \exp\left( -\frac{||X^\epsilon_\theta - A_\sigma||^2}{2\epsilon \exp(2\theta)} \right)}
\]

For \( \epsilon \) fixed, we first use the Freidlin-Vencel theory to get the "good rate function" for the large deviations of the system as \( \eta \to 0 \).

Then, we may pass to the limit \( \epsilon \to 0 \) (*) and obtain as "\( \Gamma \)–limit"

\[
\int || \frac{dX_\theta}{d\theta} ||^2 + || \nabla \Phi(X_\theta) ||^2 d\theta, \quad \Phi(X) = - \inf_{\sigma \in S_N} || X - A_\sigma ||^2 / 2
\]

(*) thanks to L. Ambrosio, private communication.
LEAST ACTION PRINCIPLE

The least action principle applied to

\[ \int \left\| \frac{dX_{\theta}}{d\theta} \right\|^2 + \left\| \nabla \Phi(X_{\theta}) \right\|^2 d\theta, \quad \Phi(X) = -\inf_{\sigma \in S_N} \left\| X - A_{\sigma} \right\|^2 / 2 \]

(formally) leads to the following dynamical system

\[ \frac{d^2 X_{\theta}}{d\theta^2} = \nabla \left( \frac{\left\| \nabla \Phi \right\|^2}{2} \right)(X_{\theta}) \]
The least action principle applied to

\[ \int \left( \left\| \frac{dX_\theta}{d\theta} \right\|^2 + \left\| \nabla \Phi(X_\theta) \right\|^2 \right) d\theta, \quad \Phi(X) = -\inf_{\sigma \in S_N} \left\| X - A_{\sigma} \right\|^2 / 2 \]

(formally) leads to the following dynamical system

\[ \frac{d^2 X_\theta}{d\theta^2} = \nabla \left( \frac{\left\| \nabla \Phi \right\|^2}{2} \right)(X_\theta) = -\left( \nabla \Phi \right)(X_\theta) \]

Indeed \( \left\| \nabla \Phi \right\|^2 = -2\Phi \) because \(-2\Phi\) is a squared distance function.
THE RESULTING DYNAMICAL SYSTEM

So, we have finally obtained

\[
\frac{d^2 X_\theta(\alpha)}{d\theta^2} = X_\theta(\alpha) - A(\sigma_{opt}(\alpha)) , \quad X_\theta(\alpha) \in \mathbb{R}^d, \quad \alpha = 1, \ldots, N
\]

\[
\sigma_{opt} = \text{Arginf}_{\sigma \in S_N} \sum_{\alpha=1}^{N} |X_\theta(\alpha) - A(\sigma(\alpha))|^2
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which was precisely the dynamical system we introduced to discretize the Euler equations with permutations.
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Berman’s papers (related to Kählerian Geometry)