From Euler to Monge and vice versa: Link between MFG and initial value problems (Y.B.)

Yann Brenier, Mikaela Iacobelli, Filippo Santambrogio, Paris, Durham, Lyon.

MFO SEMINAR 1842, 14-20/10/2018.
Yesterday, FS discussed the variational MFG

\[ \partial_t \mu + \nabla \cdot (\mu \nabla \phi) = \nu \Delta \mu, \quad \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + \nu \Delta \phi = f'(\mu), \]

\( t \in [0, T], \ x \in D = \mathbb{T}^d, \ \mu(t, x) \geq 0, \ \phi(t, x), \) respectively prescribed at \( t = 0 \) and \( t = T. \)
Yesterday, FS discussed the variational MFG

\[
\partial_t \mu + \nabla \cdot (\mu \nabla \phi) = \nu \Delta \mu, \quad \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + \nu \Delta \phi = f'(\mu),
\]

\[t \in [0, T], \ x \in D = \mathbb{T}^d, \ \mu(t, x) \geq 0, \ \phi(t, x), \text{ respectively prescribed at } t = 0 \text{ and } t = T.\]

**CONVEXITY of** \( f \) **was CRUCIAL** for theory (and for numerics).
Yesterday, FS discussed the variational MFG

\[ \partial_t \mu + \nabla \cdot (\mu \nabla \phi) = \nu \Delta \mu, \quad \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + \nu \Delta \phi = f'(\mu), \]

\( t \in [0, T], \ x \in D = \mathbb{T}^d, \ \mu(t, x) \geq 0, \ \phi(t, x), \) respectively prescribed at \( t = 0 \) and \( t = T. \)

**CONVEXITY** of \( f \) was CRUCIAL for theory (and for numerics).

With \( \nu = 0 \) and written in terms of \( \nu = \nabla \phi \), these equations read

\[ \partial_t \mu + \nabla \cdot (\mu \nabla \phi) = 0, \quad \partial_t \nu + (\nu \cdot \nabla)\nu = \nabla(f'(\mu)), \]

and looks like the equations written by Euler in 1755-57 for compressible fluids.
The Euler equations written in conservation form

\[ \partial_t \mu + \nabla \cdot q = 0, \quad q = \mu v, \]

\[ \partial_t q + \nabla \cdot \left( \frac{q \otimes q}{\mu} \right) = -\nabla(p(\mu)), \quad p'(w) = -wf''(w) \]
The Euler equations written in conservation form

\[ \partial_t \mu + \nabla \cdot q = 0, \quad q = \mu v, \]

\[ \partial_t q + \nabla \cdot \left( \frac{q \otimes q}{\mu} \right) = -\nabla (p(\mu)), \quad p'(w) = -wf''(w) \]

were introduced by Euler in 1755 for Fluid Mechanics.
The Euler equations written in conservation form

\[ \partial_t \mu + \nabla \cdot q = 0, \quad q = \mu v, \]

\[ \partial_t q + \nabla \cdot \left( \frac{q \otimes q}{\mu} \right) = -\nabla (p(\mu)), \quad p'(w) = -wf''(w) \]

were introduced by Euler in 1755 for Fluid Mechanics.

(This way, Euler introduced at once the first set of PDEs and the first field theory ever!)
The Euler equations written in conservation form

\[ \partial_t \mu + \nabla \cdot q = 0, \quad q = \mu v, \]

\[ \partial_t q + \nabla \cdot \left( \frac{q \otimes q}{\mu} \right) = -\nabla (p(\mu)), \quad p'(w) = -wf''(w) \]

were introduced by Euler in 1755 for Fluid Mechanics.
(This way, Euler introduced at once the first set of PDEs and the first field theory ever!)

CONCAVITY of \( f \) is needed to get a WELL-POSED INITIAL VALUE PROBLEM, with boundary conditions only at \( t = 0 \) and none at \( t = T \), in contrast with MFG.
OUR GOAL

We want to solve the initial value problem for a large class of equations including Euler’s ones by a variational approach based on convexity.
OUR GOAL

We want to solve the initial value problem for a large class of equations including Euler’s ones by a variational approach based on convexity.

This will be possible through a GENERALIZED MFG, involving vector-potentials (and measures taking values in the cone of semi-definite positive matrices).
The class of "entropic conservation laws"
The class of "entropic conservation laws"

\[ \partial_t U + \nabla \cdot (F(U)) = 0, \quad U = U(t, x) \in \mathcal{W} \subset \mathbb{R}^m, \quad x \in D \]

(where F is given so that \( \sum_{\beta=1}^{m} \partial_\beta \mathcal{E}(W) \partial_\alpha F^\beta(W) = \partial_\alpha Q^i(W), \quad \forall W \in \mathcal{W}, \)

for some \((\mathcal{E}, Q) : \mathcal{W} \to \mathbb{R}^{1+d}, \) with \(\mathcal{W}\) open convex and "entropy" \(\mathcal{E}\) strictly convex, which implies: \( \partial_t (\mathcal{E}(U)) + \nabla \cdot (Q(U)) = 0, \) for all smooth solutions \( U \)

contains the Euler equations, for which: \( \mathcal{E}(\mu, q) = \frac{|q|^2}{2\mu} - f(\mu), \quad \mu > 0, \) with \( f \) concave.
Inviscid Burgers equation: \( \partial_t u + \partial_x (u^2/2) = 0 \), \( u = u(t, x) \), \( x \in \mathbb{R}/\mathbb{Z} \), \( t \geq 0 \).

Formation of two shock waves. (Vertical axis: \( t \in [0, 1/4] \), horizontal axis: \( x \in \mathbb{T} \).)
A MFG approach to the Cauchy problem
A MFG approach to the Cauchy problem

Given $U_0$ on $D = \mathbb{R}^d / \mathbb{Z}^d$ and $T > 0$, we minimize the entropy among all weak solutions $U$ of the Cauchy pb:
A MFG approach to the Cauchy problem

Given $U_0$ on $D = \mathbb{R}^d/\mathbb{Z}^d$ and $T > 0$, we minimize the entropy among all weak solutions $U$ of the Cauchy pb:

$$\inf_{U} \int_0^T \int_D \mathcal{E}(U), \quad U = U(t, x) \in \mathcal{W} \subset \mathbb{R}^m$$

subject to
A MFG approach to the Cauchy problem

Given $U_0$ on $D = \mathbb{R}^d / \mathbb{Z}^d$ and $T > 0$, we minimize the entropy among all weak solutions $U$ of the Cauchy pb:

$$\inf \int_0^T \int_D \mathcal{E}(U), \quad U = U(t, x) \in \mathcal{W} \subset \mathbb{R}^m$$

subject to

$$\int_0^T \int_D \partial_t A \cdot U + \nabla A \cdot F(U) + \int_D A(0, \cdot) \cdot U_0 = 0$$

for all smooth $A = A(t, x) \in \mathbb{R}^m$ with $A(T, \cdot) = 0$. 
A MFG approach to the Cauchy problem

Given $U_0$ on $D = \mathbb{R}^d/\mathbb{Z}^d$ and $T > 0$, we minimize the entropy among all weak solutions $U$ of the Cauchy pb:

$$\inf_U \int_0^T \int_D \mathcal{E}(U), \quad U = U(t, x) \in \mathcal{W} \subset \mathbb{R}^m \text{ subject to}$$

$$\int_0^T \int_D \partial_t A \cdot U + \nabla A \cdot F(U) + \int_D A(0, \cdot) \cdot U_0 = 0$$

for all smooth $A = A(t, x) \in \mathbb{R}^m$ with $A(T, \cdot) = 0$.

The problem is not trivial since there may be many weak solutions starting from $U_0$ which are not entropy-preserving (by "convex integration" à la De Lellis-Székelyhidi).
The resulting saddle-point problem

\[
\inf_U \sup_A \int_0^T \int_D E(U) - \partial_t A \cdot U - \nabla A \cdot F(U) - \int_D A(0, \cdot) \cdot U_0
\]

where \( A = A(t, x) \in \mathbb{R}^m \) is smooth with \( A(T, \cdot) = 0 \).

Here \( U_0 \) is the initial condition and \( T \) the final time.

N.B. The supremum in \( A \) exactly encodes that \( U \) is a weak solution with initial condition \( U_0 \), all test functions \( A \) acting like Lagrange multipliers.
The resulting saddle-point problem

\[
\inf_U \sup_A \int_0^T \int_D \mathcal{E}(U) - \partial_t A \cdot U - \nabla A \cdot F(U) \\
- \int_D A(0, \cdot) \cdot U_0
\]

where \( A = A(t, x) \in \mathbb{R}^m \) is smooth with \( A(T, \cdot) = 0 \).
Here \( U_0 \) is the initial condition and \( T \) the final time.
The resulting saddle-point problem

\[ \inf_U \sup_A \int_0^T \int_D \mathcal{E}(U) - \partial_t A \cdot U - \nabla A \cdot F(U) \]

\[ - \int_D A(0, \cdot) \cdot U_0 \]

where \( A = A(t, x) \in \mathbb{R}^m \) is smooth with \( A(T, \cdot) = 0 \).

Here \( U_0 \) is the initial condition and \( T \) the final time.

N.B. The supremum in \( A \) exactly encodes that \( U \) is a weak solution with initial condition \( U_0 \), all test functions \( A \) acting like Lagrange multipliers.
Reversing infimum and supremum...
Reversing infimum and supremum...

leads to a *concave* maximization problem in $\mathcal{A}$, namely

$$\sup_{A(T, \cdot) = 0} \inf_U \int_0^T \int_D \mathcal{E}(U) - \partial_t A \cdot U - \nabla A \cdot F(U) - \int_D A(0, \cdot) \cdot U_0$$

Notice that $G$ is automatically convex.

YB-MI-FS (Paris, Durham, Lyon.) From Euler to Monge and vice versa MFO Seminar 14-20/10/2018 9 / 24
Reversing infimum and supremum...

leads to a *concave* maximization problem in $A$, namely

$$
\sup_{A(T, \cdot) = 0} \inf_U \int_0^T \int_D \mathcal{E}(U) - \partial_t A \cdot U - \nabla A \cdot F(U) - \int_D A(0, \cdot) \cdot U_0 
$$

$$
= \sup_{A(T, \cdot) = 0} \int_0^T \int_D -G(\partial_t A, \nabla A) - \int_D A(0, \cdot) \cdot U_0 
$$

$$
G(E, B) = \sup_{V \in \mathcal{W} \subset \mathbb{R}^m} E \cdot V + B \cdot F(V) - \mathcal{E}(V), \ (E, B) \in \mathbb{R}^m \times \mathbb{R}^{m \times d}.
$$
Reversing infimum and supremum...

leads to a \textit{concave} maximization problem in $A$, namely

$$\sup_{A(T, \cdot) = 0} \inf_U \int_0^T \int_D \mathcal{E}(U) - \partial_t A \cdot U - \nabla A \cdot F(U) - \int_D A(0, \cdot) \cdot U_0$$

$$= \sup_{A(T, \cdot) = 0} \int_0^T \int_D -G(\partial_t A, \nabla A) - \int_D A(0, \cdot) \cdot U_0$$

$$G(E, B) = \sup_{V \in \mathcal{W} \subset \mathbb{R}^m} E \cdot V + B \cdot F(V) - \mathcal{E}(V), \ (E, B) \in \mathbb{R}^m \times \mathbb{R}^{m \times d}.$$  

Notice that $G$ is automatically convex.
Comparison with variational MFG

\[
\sup_{\phi} \int_0^T \int_D -G(\partial_t \phi + \nu \Delta \phi, \nabla \phi) - <\mu_0, \phi_0>
\]

is the variational MFG (in primal form) with \(\mu_0\) and \(\phi_T\) prescribed.
Comparison with variational MFG

\[
\sup_{\phi} \int_0^T \int_D -G(\partial_t \phi + \nu \Delta \phi, \nabla \phi) - <\mu_0, \phi_0>
\]

is the variational MFG (in primal form) with \(\mu_0\) and \(\phi_T\) prescribed. Now, we rather have

\[
\sup_{A(T, \cdot)=0} \int_0^T \int_D -G(\partial_t A, \nabla A) - \int_D A(0, \cdot) \cdot U_0
\]

where \(\nu = 0\) and the vector-potential \(A\) substitutes for the scalar potential \(\phi\).
Comparison with variational MFG

\[
\sup_\phi \int_0^T \int_D -G(\partial_t \phi + \nu \Delta \phi, \nabla \phi) - \langle \mu_0, \phi_0 \rangle
\]

is the variational MFG (in primal form) with \( \mu_0 \) and \( \phi_T \) prescribed. Now, we rather have

\[
\sup_A \int_0^T \int_D -G(\partial_t A, \nabla A) - \int_D A(0, \cdot) \cdot U_0
\]

where \( \nu = 0 \) and the vector-potential \( A \) substitutes for the scalar potential \( \phi \).

Thus our dual maximization problem to solve the initial value problem can be interpreted as a generalized variational 1st order-MFG with vector-valued potential.
Main results

Theorem 1: If $U$ is a smooth solution to the Cauchy problem and $T$ is not too large (*), then $U$ can be recovered from the concave maximization problem which admits $A(t, x) = (t - T)E'(U(t, x))$ as solution.

Theorem 2: For the Burgers equation, all entropy solutions can be recovered, for arbitrarily large $T$.

(*) more precisely if, $\forall t, x, V \in W$, $E'(V) - (T - t)F(V) \cdot \nabla (E'(U(t, x))) > 0$. 
Main results

Theorem 1: If $U$ is a smooth solution to the Cauchy problem and $T$ is not too large, then $U$ can be recovered from the concave maximization problem which admits $A(t,x) = (t - T)E'(U(t,x))$ as solution.

Theorem 2: For the Burgers equation, all entropy solutions can be recovered, for arbitrarily large $T$.

(*) more precisely if, $\forall t, x, V \in W, E'(V) - (T - t) F'(V) \cdot \nabla (E'(U(t,x))) > 0$. 

YB-MI-FS (Paris, Durham, Lyon.) From Euler to Monge and vice versa MFO Seminar 14-20/10/2018 11 / 24
Main results

Theorem 1: If $U$ is a smooth solution to the Cauchy problem and $T$ is not too large (*), then $U$ can be recovered from the concave maximization problem which admits $A(t, x) = (t - T)\mathcal{E}'(U(t, x))$ as solution.

(*) more precisely if, $\forall t, x, V \in W,$ $E'(V) - (T - t)F(V) \cdot \nabla (E'(U(t, x))) > 0.$
Main results

Theorem 1: If \( U \) is a smooth solution to the Cauchy problem and \( T \) is not too large (*), then \( U \) can be recovered from the concave maximization problem which admits \( A(t, x) = (t - T)E'(U(t, x)) \) as solution.

Theorem 2: For the Burgers equation, all entropy solutions can be recovered, for arbitrarily large \( T \).

(*) more precisely if, \( \forall \ t, x, V \in \mathcal{W}, \ E''(V) - (T - t)F''(V) \cdot \nabla(E'(U(t, x))) > 0 \).
The elementary example of the Burgers equation
The elementary example of the Burgers equation

Then, the maximization problem in $A$ simply reads

$$\sup_A \int_{[0,T] \times \mathbb{T}} - \frac{(\partial_t A)^2}{2(1 - \partial_x A)} - \int_{\mathbb{T}} A(0, \cdot) u_0.$$ 

with $A = A(t, x) \in \mathbb{R}$ subject to $A(T, \cdot) = 0$, $\partial_x A \leq 1$. 
The elementary example of the Burgers equation

Then, the maximization problem in $A$ simply reads

$$\sup_{A} \int_{[0,T] \times \mathbb{T}} \left( \frac{(\partial_t A)^2}{2(1 - \partial_x A)} \right) - \int_{\mathbb{T}} A(0, \cdot) u_0.$$ 

with $A = A(t, x) \in \mathbb{R}$ subject to $A(T, \cdot) = 0, \partial_x A \leq 1$.

Introducing $\mu = 1 - \partial_x A \geq 0, q = \partial_t A$, we get the MFG

$$\sup_{(\mu, q)} \left\{ \int_{[0,T] \times \mathbb{T}} \left( -\frac{q^2}{2\mu} - qu_0 \right) \mid \partial_t \mu + \partial_x q = 0, \mu(T, \cdot) = 1 \right\}.$$
Generalized MFG for the Euler equations

\[
\begin{align*}
\frac{\partial}{\partial t} \mu + \nabla \cdot q &= 0, \\
\frac{\partial}{\partial t} q + \nabla \cdot (q \otimes q \mu) &= -\nabla (p(\mu))
\end{align*}
\]

where

\[
f(w) = w - \log w,
\]

\[
p'(w) = -wf(f(w)) \rightarrow p(w) = w.
\]
Generalized MFG for the Euler equations

Let us compute the generalized MFG in the particular case of the Euler equations of isothermal fluids

\[ \partial_t \mu + \nabla \cdot q = 0, \quad \partial_t q + \nabla \cdot \left( \frac{q \otimes q}{\mu} \right) = -\nabla (p(\mu)) \]
Generalized MFG for the Euler equations

Let us compute the generalized MFG in the particular case of the Euler equations of isothermal fluids

\[ \partial_t \mu + \nabla \cdot q = 0, \quad \partial_t q + \nabla \cdot \left( \frac{q \otimes q}{\mu} \right) = -\nabla (p(\mu)) \]

where

\[ f(w) = w - \log w, \quad p'(w) = -wf''(w) \rightarrow p(w) = w. \]
Generalized MFG for isothermal Euler equations
Generalized MFG for isothermal Euler equations

Then, the generalized MFG amounts to minimizing

\[
\int_{[0,T] \times D} \exp(u) \exp\left(\frac{1}{2} Q \cdot M^{-1} \cdot Q\right) + \int_D \sigma_0 \mu_0 + w_0 \cdot q_0,
\]
Generalized MFG for isothermal Euler equations

Then, the generalized MFG amounts to minimizing

$$\int_{[0,T] \times D} \exp(u) \exp \left( \frac{1}{2} Q \cdot M^{-1} \cdot Q \right) + \int_D \sigma_0 \mu_0 + w_0 \cdot q_0,$$

among all fields $u = u(t, x) \in \mathbb{R}$, $Q = Q(t, x) \in \mathbb{R}^d$, $M = M(t, x) = M^t(t, x) \in \mathbb{R}^{d \times d}$, $M \geq 0$, $M_0 \geq 0$. YB-MI-FS (Paris, Durham, Lyon.) From Euler to Monge and vice versa MFO Seminar 14-20/10/2018 14 / 24
Generalized MFG for isothermal Euler equations

Then, the generalized MFG amounts to minimizing

$$\int_{[0,T] \times D} \exp(u) \exp\left(\frac{1}{2} Q \cdot M^{-1} \cdot Q\right) + \int_D \sigma_0 \mu_0 + w_0 \cdot q_0,$$

among all fields $u = u(t, x) \in \mathbb{R}$, $Q = Q(t, x) \in \mathbb{R}^d$, $M = M(t, x) = M^t(t, x) \in \mathbb{R}^{d \times d}$, $M \geq 0$, of form:

$$u = \partial_t \sigma + \partial^i w_i, \quad Q_i = \partial_t w_i + \partial_i \sigma, \quad M_{ij} = \delta_{ij} - \partial_i w_j - \partial_j w_i,$$

where $\sigma$ and $w$ must vanish at $t = T$. 
The Euler equations of incompressible fluids

\[ \frac{\partial}{\partial t} q + \nabla \cdot (q \otimes q) = -\nabla p, \]
\[ \nabla \cdot q = 0, \]

where \( q \) is prescribed at \( t = 0 \) and \( p \) is now a Lagrange multiplier ("price") for constraint \( \nabla \cdot q = 0 \).

We get again a generalized MFG for measures valued in the cone of semi-definite symmetric matrices: YB-MI-FS (Paris, Durham, Lyon.) From Euler to Monge and vice versa MFO Seminar 14-20/10/2018 15 / 24
The Euler equations of incompressible fluids

The same method also applies to the Euler equations of incompressible fluids ("saturated congestion")

\[ \partial_t q + \nabla \cdot (q \otimes q) = -\nabla p, \quad \nabla \cdot q = 0, \]

where \( q \) is prescribed at \( t = 0 \) and \( p \) is now a Lagrange multiplier ("price") for constraint \( \nabla \cdot q = 0 \).
The Euler equations of incompressible fluids

The same method also applies to the Euler equations of incompressible fluids ("saturated congestion")

$$\partial_t q + \nabla \cdot (q \otimes q) = -\nabla p, \quad \nabla \cdot q = 0,$$

where $q$ is prescribed at $t = 0$ and $p$ is now a Lagrange multiplier ("price") for constraint $\nabla \cdot q = 0$.

We get again a generalized MFG for measures valued in the cone of semi-definite symmetric matrices:
Generalized MFG for incompressible fluids
Generalized MFG for incompressible fluids

This generalized (variational) MFG reads

$$
\sup_{(M,Q)} - \int_{[0,T] \times D} q_0 \cdot Q + \frac{1}{2} Q \cdot M^{-1} \cdot Q,
$$

where now $Q$ is a vector field (not necessarily divergence-free) and $M = M_t \geq 0$ is a field of semi-definite symmetric matrices subject to

$$
M_{ij}(T, \cdot) = \delta_{ij}, \quad \partial_t M_{ij} = \partial_j Q_i + \partial_i Q_j + 2 \partial_i \partial_j (-\Delta) - \frac{1}{2} \partial_k Q_k.
$$
Generalized MFG for incompressible fluids

This generalized (variational) MFG reads

$$\sup_{(M,Q)} - \int_{[0,T] \times D} q_0 \cdot Q + \frac{1}{2} Q \cdot M^{-1} \cdot Q,$$

where now $Q$ is a vector field (not necessarily divergence-free) and $M = M^t \geq 0$ is a field of semi-definite symmetric matrices subject to
Generalized MFG for incompressible fluids

This generalized (variational) MFG reads

$$
\sup_{(M,Q)} - \int_{[0,T] \times D} q_0 \cdot Q + \frac{1}{2} Q \cdot M^{-1} \cdot Q,
$$

where now $Q$ is a vector field (not necessarily divergence-free) and $M = M^t \geq 0$ is a field of semi-definite symmetric matrices subject to

$$
M_{ij}(T, \cdot) = \delta_{ij}, \quad \partial_t M_{ij} = \partial_j Q_i + \partial_i Q_j + 2 \partial_i \partial_j (-\Delta)^{-1} \partial_k Q^k.
$$
Extension to some parabolic equations

Using the quadratic change of time $t \rightarrow \theta = t^2/2$, as in Y. B., X. Duan (Arma 2018), we may derive from the Euler equations, with pressure $p = \rho^2$, the "porous medium" equation

$$\frac{\partial \theta}{\partial t} \rho = \Delta \rho^2$$

and, therefore, we get for it a corresponding convex minimization problem:

$$\inf \left\{ \int_0^T \int \sigma - \rho_0 q \cdot \partial \theta \sigma + \Delta q = 0, \sigma(T, \cdot) = 1 \right\}.$$ 

This is again similar to an optimal transport problem.
Extension to some parabolic equations

Using the quadratic change of time $t \rightarrow \theta = t^2/2$, as in Y.B., X. Duan (Arma 2018), we may derive from the Euler equations, with pressure $p = \rho^2$, the "porous medium" equation $\partial_\theta \rho = \Delta \rho^2$ and, therefore, we get for it a corresponding convex minimization problem:

$$\inf\left\{ \int_{[0, T] \times \mathbb{T}^d} \frac{q^2}{4\sigma} - \rho_0 q, \text{ s.t. } \partial_\theta \sigma + \Delta q = 0, \quad \sigma(T, \cdot) = 1 \right\}.$$
Extension to some parabolic equations

Using the quadratic change of time $t \rightarrow \theta = t^2/2$, as in Y.B., X. Duan (Arma 2018), we may derive from the Euler equations, with pressure $p = \rho^2$, the "porous medium" equation $\partial_\theta \rho = \Delta \rho^2$ and, therefore, we get for it a corresponding convex minimization problem:

$$\inf \left\{ \int_{[0,T] \times \mathbb{T}^d} \frac{q^2}{4\sigma} - \rho_0 q, \text{ s.t. } \partial_\theta \sigma + \Delta q = 0, \sigma(T, \cdot) = 1 \right\}.$$  

This is again similar to an optimal transport problem.
Let us move back to the Burgers equation

\[ \frac{\partial \mu}{\partial t} + \frac{\partial \mu}{\partial x} q = 0, \; \mu(T, \cdot) = 1 \]

It turns out that, for arbitrarily large $T$, we may recover, through this problem, the correct "entropy solution" à la Kruzhkov-Panov, but only at time $T$ and (surprisingly enough) not for $t < T$, once shocks have formed.)
Let us move back to the Burgers equation

We have already obtained the (elementary) MFG

$$\sup_{(\mu,q)} \left\{ \int_{[0,T] \times \mathbb{T}} \left( -\frac{q^2}{2\mu} - qu_0 \right) \mid \partial_t \mu + \partial_x q = 0, \mu(T, \cdot) = 1 \right\}.$$
Let us move back to the Burgers equation

We have already obtained the (elementary) MFG

$$\sup_{(\mu,q)} \left\{ \int_{[0,T] \times \mathbb{T}} -\frac{q^2}{2\mu} - qu_0 \mid \partial_t \mu + \partial_x q = 0, \ \mu(T, \cdot) = 1 \right\}.$$ 

It turns out that, for arbitrarily large $T$, we may recover, through this problem, the correct "entropy solution" à la Kruzhkov-Panov, but only at time $T$ and (surprisingly enough) not for $t < T$, once shocks have formed.
Inviscid Burgers equation: $\partial_t u + \partial_x (u^2/2) = 0$, $u = u(t, x)$, $x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.

Formation of two shock waves. (Vertical axis: $t \in [0, 1/4]$, horizontal axis: $x \in \mathbb{T}$.)
Inviscid Burgers equation: \( \partial_t u + \partial_x (\frac{u^2}{2}) = 0 \), \( u = u(t, x), x \in \mathbb{R}/\mathbb{Z}, t \geq 0 \).

Recovery of the solution at time \( T=0.1 \) by convex optimization.

Observe the formation of a first vacuum zone as the first shock has formed.
Inviscid Burgers equation: \( \partial_t u + \partial_x (u^2/2) = 0, \ u = u(t, x), \ x \in \mathbb{R}/\mathbb{Z}, \ t \geq 0. \)

Recovery of the solution at time \( T=0.16 \) by convex optimisation.

Observe the formation of a second vacuum zone as the second shock has formed.
Inviscid Burgers equation: $\partial_t u + \partial_x (u^2/2) = 0$, $u = u(t, x)$, $x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.

Recovery of the solution at time $T=0.225$ by convex optimisation.

Observe the extension of the two vacuum zones.
Numerics: 2 lines of code differ from a standard (Benamou-B.) OT solver!
Analogy with mountain climbing: going from Everest to Lhotse without following the crest! (Partial credit to Thomas Gallouët for this analogy.)
Analogy with mountain climbing: going from Everest to Lhotse without following the crest! (Partial credit to Thomas Gallouët for this analogy.)

Thanks for your attention!
Analogy with mountain climbing: going from Everest to Lhotse without following the crest! (Partial credit to Thomas Gallouët for this analogy.)

**Thanks for your attention!** For more details, voir Y.B. ArXiv Oct. 2017, to appear in CMP.