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ARITHMETIC AMPLENESS AND AN ARITHMETIC BERTINI THEOREM

BY François CHARLES

ABSTRACT. – Let $\mathcal{X}$ be a projective arithmetic variety of dimension at least 2. If $\overline{\mathcal{L}}$ is an ample hermitian line bundle on $\mathcal{X}$, we prove that the proportion of those effective sections σ of $\overline{\mathcal{L}}^{\otimes n}$ such that the divisor of σ on $\mathcal{X}$ is irreducible tends to 1 as n tends to ∞. We prove variants of this statement for schemes mapping to such an $\mathcal{X}$.

On the way to these results, we discuss some general properties of arithmetic ampleness, including restriction theorems, and upper bounds for the number of effective sections of hermitian line bundles on arithmetic varieties.

RÉSUMÉ. – Soit $\mathcal{X}$ une variété arithmétique projective de dimension au moins 2, et soit $\overline{\mathcal{L}}$ un fibré hermitien sur $\mathcal{X}$. Si $\overline{\mathcal{L}}$ est ample, on démontre que la proportion des sections effectives de $\overline{\mathcal{L}}^n$ qui définissent un diviseur irréductible sur $\mathcal{X}$ tend vers 1 quand n tend vers ∞. On démontre également des variantes de ce résultat pour des schémas admettant un morphisme vers $\mathcal{X}$.

On prouve par ailleurs un certain nombre de propriétés générales de l'amplitude arithmétique, autour notamment de théorèmes de restriction et d'estimées pour le nombre de sections effectives de fibrés en droites hermitiens.

1. Introduction

1.1. Bertini theorems over fields

Let k be an infinite field, and let X be an irreducible variety over k with dimension at least 2. Given an embedding of X in some projective space over k, the classical Bertini theorem [23, Theorem 3.3.1] shows, in its simplest form, that infinitely many hyperplane sections of X are irreducible.

In the case where k is finite, the Bertini theorem can fail, since the finitely many hyperplane sections of X can all be reducible. As was first explained in [26] in the setting of smoothness theorems, this phenomenon can be dealt with by replacing hyperplane sections with ample hypersurfaces of higher degree. We can state the main result of [11]—see Theorem 1.6 in [11] and the discussion in the proof of Theorem 6.1 below—as follows: let k be a finite field, let

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X be a projective variety over k and let L be an ample line bundle on X. Let Y be an integral scheme of finite type over k together with a morphism $f : Y \to X$. Assume that the image of f has dimension at least 2. If Z is a subscheme of Y, write Z_{horiz} for the union of those irreducible components of Z that do not map to a closed point of X. Then the set

$$\mathcal{P} = \Big\{\sigma \in \bigcup_{n>0} H^0(X, L^{\otimes n}),\ \mathrm{div}(f^*\sigma)_{\mathrm{horiz}} \text{ is irreducible}\Big\}$$

has density 1, in the sense that

$$\lim_{n\to\infty} \frac{|\mathcal{P} \cap H^0(X, L^{\otimes n})|}{|H^0(X, L^{\otimes n})|} = 1.$$

Here if S is a set, we denote by $|S|$ its cardinality. When Y is a subscheme of X, we can disregard the horizontality subscript.

1.2. The arithmetic case

In this paper, we deal with an arithmetic version of Bertini theorems as above. Let $\mathcal{X}$ be an arithmetic variety, that is, an integral scheme which is separated, flat of finite type over $\operatorname{Spec}\mathbb{Z}$. Assume that $\mathcal{X}$ is projective, and let $\mathcal{L}$ be a relatively ample line bundle on $\mathcal{X}$. As is well known, sections of $\mathcal{L}$ over $\mathcal{X}$ are not the analogue of global sections of a line bundle over a projective variety over a field. Indeed, it is more natural to consider a hermitian line bundle $\overline{\mathcal{L}}$ with underlying line bundle $\mathcal{L}$ and consider the sets

$$H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$$

of sections with norm at most 1 everywhere. We discuss ampleness for hermitian line bundles in Section 2, which we refer to for definitions.

Given finite sets $(X_n)_{n>0}$, and a subset $\mathcal{P}$ of $\bigcup_{n>0} X_n$, say that $\mathcal{P}$ has density ρ if the following equality holds:

$$\lim_{n\to\infty} \frac{|\mathcal{P} \cap X_n|}{|X_n|} = \rho.$$

The main result of this paper is the following arithmetic Bertini theorem. Again, given a morphism of schemes $f : Y \to X$, and if Z is a subscheme of Y, we denote by Z_{horiz} the union of those irreducible components of Z that do not map to a closed point of X. If $\overline{\mathcal{M}} = (\mathcal{M}, ||.||)$ is a hermitian vector bundle and δ is a real number, write $\overline{\mathcal{M}}(\delta)$ for the hermitian vector bundle $(\mathcal{M}, e^{-\delta}||.||)$. Write $||\sigma||_\infty$ for the sup norm of a section of a hermitian vector bundle.

THEOREM 1.1. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Let $\mathcal{Y}$ be an integral scheme of finite type over* $\operatorname{Spec}\mathbb{Z}$ *together with a morphism $f : \mathcal{Y} \to \mathcal{X}$ which is generically smooth over its image. Assume that the image of $\mathcal{Y}$ has dimension at least* 2. *Let ε be a positive real number. Then the set*

$$\Big\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}(\varepsilon)^{\otimes n}),\ ||\sigma_{|f(\mathcal{Y}(\mathbb{C}))}||_\infty \leq 1 \text{ and } \mathrm{div}(f^*\sigma)_{\mathrm{horiz}} \text{ is irreducible}\Big\}$$

has density 1 *in* $\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}(\varepsilon)^{\otimes n}),\ ||\sigma_{|f(\mathcal{Y}(\mathbb{C}))}||_\infty \leq 1\}$.

Recall that by definition, the condition $\sigma \in H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}(\varepsilon)^{\otimes n})$ means

$$||\sigma||_\infty \leq \varepsilon^n.$$

Remark 1.2. – A Weil divisor is said to be irreducible if it comes from an integral codimension 1 subscheme.

Remark 1.3. – The hypothesis that f is generically smooth over its image is necessary: when f is the Frobenius morphism of a fiber $\mathcal{X}_p$, all $\mathrm{div}(f^*\sigma)$ have components with multiplicities divisible by p. Of course, it holds when $\mathcal{Y}$ is flat over $\mathrm{Spec}\,\mathbb{Z}$. Without this hypothesis on f, the conclusion is only that the support of $\mathrm{div}(f^*\sigma)$ is irreducible for a density 1 set of σ.

An important special case of the theorem deals with the special case where f is dominant. In this case, generic smoothness is automatic.

Theorem 1.4. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Let $\mathcal{Y}$ be an integral scheme of finite type over* $\mathrm{Spec}\,\mathbb{Z}$ *together with a morphism $f : \mathcal{Y} \to \mathcal{X}$. Assume that the image of $\mathcal{Y}$ has dimension at least 2 and f is dominant. Then the set*

$$\Big\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}),\ \mathrm{div}(f^*\sigma)_{\mathrm{horiz}} \text{ is irreducible}\Big\}$$

has density 1 in $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$.

Remark 1.5. – We will prove Theorem 1.1 as a consequence of Theorem 1.4. However, the latter is a special case of the former. Indeed, with the notation of Theorem 1.1, when f is dominant, if $\sigma \in H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}(\varepsilon)^{\otimes n}$, then the condition

$$||\sigma|_{f(\mathcal{Y}(\mathbb{C}))}||_\infty \leq 1$$

is equivalent to

$$||\sigma||_\infty \leq 1,$$

i.e., $\sigma \in H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$.

The case where $\mathcal{Y} = \mathcal{X}$ is particularly significant. We state it independently below. Most of this paper will be devoted to its proof, and we will prove 1.1 and 1.4 as consequences.

Theorem 1.6. – *Let $\mathcal{X}$ be a projective arithmetic variety of dimension at least 2, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Then the set*

$$\Big\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}),\ \mathrm{div}(\sigma) \text{ is irreducible}\Big\}$$

has density 1 in $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$.

Theorem 1.6 is stronger than the Bertini irreducibility theorem of [11, Theorem 1.1], as we explain in Section 3. Note however that we use the results of [11] in our proofs.

In Theorem 1.6, the case where $\mathcal{X}$ has dimension at least 3—that is, relative dimension at least 2 over $\operatorname{Spec}\mathbb{Z}$—is easier. Indeed, if p is a large enough prime number, we can apply the Bertini irreducibility theorems over finite fields to the reduction of $\mathcal{X}$ modulo p, which with moderate work is enough to prove the theorem. However, when $\mathcal{X}$ is an arithmetic surface, Theorem 1.6 is genuinely different from its finite field counterpart. Note that the hardest case of the main result of [11] is the surface case as well.

Theorem 1.1 should be compared to the Hilbert irreducibility theorem, which implies, if $\mathcal{L}$ is very ample on the generic fiber of $\mathcal{X}$ and $\mathcal{Y}$ is flat over $\mathbb{Z}$, the existence of many sections σ of $\mathcal{L}$ such that the generic fiber of $\operatorname{div}(f^*\sigma)$ is irreducible. However, the Hilbert irreducibility theorem does not guarantee that these sections have small norm. To our knowledge, Theorem 1.1 does not imply the Hilbert irreducibility theorem, nor does it follow from it.

1.3. Previous results and applications

Arithmetic Bertini theorems, in the setting of both general arithmetic geometry and Arakelov geometry, have appeared in the literature. In [26], Poonen is able to prove a Bertini regularity theorem for ample line bundles on regular quasi-projective schemes over $\operatorname{Spec}\mathbb{Z}$ under the abc conjecture and technical assumptions. The statement does not involve hermitian metrics but still involves a form of density.

In [24], Moriwaki proves a Bertini theorem showing the existence of at least one effective section of large powers of an arithmetically ample line bundle that defines a generically smooth divisor—this was reproved and generalized in [19]. As a byproduct of our discussion of arithmetic ampleness in Section 2 and Poonen's result over finite fields, we will give a short proof of a more precise version of this result.

Theorem 1.7. – *Let $\mathcal{X}$ be a projective arithmetic variety with smooth generic fiber, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Then the set*

$$\Big\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}),\ \operatorname{div}(\sigma)_{\mathbb{Q}} \text{ is smooth}\Big\}$$

has density 1 *in* $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$.

Of course, this result can be combined with Theorem 1.6 if $\mathcal{X}$ has dimension at least 2.

Weaker Bertini theorems over rings of integers in number fields have been used in higher class-field theory, under the form of the Bloch-Raskind-Kerz approximation lemma proved in [5, 28, 35, 21]—see [33, Lemme 5.2] for a discussion. These results can be obtained easily as a special case of our Corollary 3.6 (or its variant corresponding to Theorem 1.1 for Wiesend's version)—this corollary allows us furthermore to control the cohomology class of the irreducible subvarieties involved.

An arithmetic Bertini theorem has been proved by Autissier in [2, 3]. Counts of irreducible divisors on arithmetic varieties have been provided by many authors, starting with Faltings in [14].

Our Bertini theorem is expected to be used in its precise form in upcoming work of Hrushovski on the model theory of number fields. We hope to discuss its applications to both general Arakelov geometry and number-theoretic problems in future work.

1.4. Strategy of the proofs

The starting point of our proof is that, as in [11], the Bertini irreducibility theorem is susceptible to a counting approach: to show that most divisors are irreducible, simply bound the number of the reducible ones.

To carry on this approach, we need to translate in Arakelov geometry results from classical geometry. The two key results in that respect are the study of restriction maps for powers of ample hermitian line bundles we prove in 2.3 and bounds for sections of hermitian line bundles on surfaces in 4.2. We hope that these results have independent interest.

Even with these tools at our disposal, we are not able to adapt the methods of [11], for two reasons. First, the error terms in the various estimates we deal with (including arithmetic Hilbert-Samuel) are big enough that we need a more involved sieving technique than in [11] involving the analysis of simultaneous restriction of sections modulo infinite families of subschemes. Second, given a section of a hermitian line bundle with reducible divisor on a regular arithmetic surface, we need to construct a corresponding decomposition of the hermitian line bundle, which involves constructing suitable metrics. The relevant analysis is dealt with in 4.1 and can only be applied when suitable geometric bounds hold. To get a hold of the geometry, we need a careful analysis dealing with infinite families of curves over finite fields—coming from the reduction of our given arithmetic surface modulo many primes. This is the content of 5.2.

1.5. Notation and definitions

If S is a set, we denote by $|S|$ the cardinal of S.

If $\mathcal{X}$ is a scheme of finite type over $\operatorname{Spec}\mathbb{Z}$, we denote by $\mathcal{X}^{\mathrm{an}}$ the associated complex analytic space.

By an arithmetic variety, we mean an integral scheme which is flat of finite type over $\operatorname{Spec}\mathbb{Z}$. A projective arithmetic variety of dimension 2 is an arithmetic surface. If $\mathcal{X}$ is a scheme over $\operatorname{Spec}\mathbb{Z}$ and if p is a prime number, we will denote by $\mathcal{X}_p$ the reduction of $\mathcal{X}$ modulo p. If $f : X \to Y$ is a morphism of noetherian schemes, we say that an irreducible component of X is vertical if its image is a closed point of Y, and horizontal if not. We denote by X_{horiz} the union of the horizontal components of X.

Let X be a complex analytic space. A hermitian vector bundle $\overline{M} = (M, ||.||)$ is a pair consisting of a vector bundle M on X and a hermitian metric on the restriction of M to the reduced subspace X_{red}. We require furthermore for the metric to be smooth, i.e., if X is of pure dimension d, given any holomorphic map from the unit disk D^d in $\mathbb{C}^d$ to X, the metric pulled-back from X to D^d is smooth.

Let $\mathcal{X}$ be a scheme of finite type over $\operatorname{Spec}\mathbb{Z}$. A hermitian vector bundle $\overline{\mathcal{M}}$ on $\mathcal{X}$ is a pair $\overline{\mathcal{M}} = (\mathcal{M}, ||.||)$ where $\mathcal{M}$ is a vector bundle on $\mathcal{X}$ and $||.||$ is a smooth metric on the restriction of $\mathcal{M}$ to the complex space $\mathcal{X}(\mathbb{C})$. If $\overline{\mathcal{M}}$ is a hermitian vector bundle over a scheme $\mathcal{X}$ of finite type over $\mathbb{Z}$, we will denote by $\mathcal{M}$ the underlying vector bundle. Note that if the generic fiber $\mathcal{X}_{\mathbb{Q}}$ is empty, i.e., if $\mathcal{X}$ is vertical, a hermitian vector bundle on $\mathcal{X}$ is nothing but a vector bundle. If $\overline{\mathcal{M}} = (\mathcal{M}, ||.||)$ is a hermitian vector bundle and δ is a real number, write $\overline{\mathcal{M}}(\delta)$ for the hermitian vector bundle $(\mathcal{M}, e^{-\delta}||.||)$.

Let $\overline{\mathcal{M}}$ be a hermitian vector bundle on a proper scheme $\mathcal{X}$ over $\mathbb{Z}$. If σ is a section of $\mathcal{M}$ on $\mathcal{X}$, we will often denote by $||\sigma||_\infty$ the sup norm of σ, that is, σ is the supremum of the $||\sigma(x)||$ as x runs through all complex points of x. We will call $||\sigma||_\infty$ the norm of σ.

If $||\sigma||_\infty \leq 1$ (resp. $||\sigma||_\infty < 1$), we say that σ is effective (resp. strictly effective). We denote by $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}})$ the set of effective sections of $\overline{\mathcal{L}}$, and write

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}}) = \log |H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}})|.$$

If $\mathcal{X}_{\mathbb{Q}}$ is generically reduced, then $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}})$ is finite. Note again that if $\mathcal{X}_{\mathbb{Q}}$ is empty, then

$$H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}}) = H^0(\mathcal{X}, \mathcal{M}).$$

We say that a hermitian line bundle on $\mathcal{X}$ is effective if it has a regular effective section.

1.6. Outline of the paper

Section 2 is devoted to a general discussion of arithmetic ampleness. After setting definitions, we recall aspects of the arithmetic Hilbert-Samuel theorem, taking care of error terms. We then prove a number of results concerning the image of restriction maps for sections of large powers of ample hermitian line bundles.

In the short Section 3, we make use of the previous section to discuss consequences and variants of the main theorems. We prove Theorem 1.7.

In Section 4, we gather general results—both analytic and arithmetic—dealing with hermitian line bundles on Riemann surfaces and arithmetic surfaces. We prove norm estimates in the spirit of [10], and we prove a basic upper bound on the number of effective sections for positive line bundles—in some sense—on arithmetic surfaces, making use of the θ-invariants of Bost, as well as a result on the effective cone of arithmetic surfaces.

Section 5 is devoted to the proof of Theorem 1.6, and Section 6 to the remaining theorems of the introduction.

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2. Some results on arithmetic ampleness

In this section, we gather some results on ample hermitian line bundles on arithmetic varieties. Most results are well-known and can be found in a similar form in the literature. Aside from a precise statement regarding error terms in the arithmetic Hilbert-Samuel theorem, our main original contribution consists in the results of 2.3 that deals with restriction maps for sections of ample line bundles.

2.1. Definitions and basic properties

We recall basic properties of arithmetic ampleness as used in the work of Zhang [39].

Definition 2.1. – *Let X be a complex analytic space, and let $\overline{L} = (L, ||.||)$ be a hermitian line bundle on X. We say that $\overline{L}$ is* semipositive *if for any open subset U of X, and any section s of $\overline{\mathcal{L}}$ on U, the function* $-\log ||s||^2$ *is plurisubharmonic on U.*

Remark 2.2. – Since for any holomorphic function f, the function $-\log |f|^2$ is harmonic, it is readily checked that $\overline{L}$ is *semipositive* if X admits a covering by open subsets U such that there exists a nowhere vanishing section s of L on U such that the function $-\log ||s||^2$ is plurisubharmonic on U. In particular, semipositivity is a local property on X.

Definition 2.3. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be a hermitian line bundle on $\mathcal{X}$. We say that $\overline{\mathcal{L}}$ is* ample *if $\mathcal{L}$ is ample, $\overline{\mathcal{L}}$ is semipositive on $\mathcal{X}^{\mathrm{an}}$ and for any large enough integer n, there exists a basis of $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$ consisting of strictly effective sections.*

Proposition 2.4. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Let $\overline{\mathcal{M}} = (\mathcal{M}, ||.||)$ be a hermitian vector bundle on $\mathcal{X}$, and let $\mathcal{F}$ be a coherent subsheaf of $\mathcal{M}$. Then there exists a positive real number ε such that for any large enough integer n, there exists a basis of $H^0(\mathcal{X}, \mathcal{L}^{\otimes n} \otimes \mathcal{F}) \subset H^0(\mathcal{X}, \mathcal{L}^{\otimes n} \otimes \mathcal{M})$ consisting of sections with norm at most $e^{-n\varepsilon}$.*

Proof. – Since $\mathcal{L}$ is relatively ample, for any large enough integers a and b, the morphism

$$H^0(\mathcal{X}, \mathcal{L}^{\otimes a}) \otimes H^0(\mathcal{X}, \mathcal{L}^{\otimes b} \otimes \mathcal{F}) \to H^0(\mathcal{X}, \mathcal{L}^{\otimes a+b} \otimes \mathcal{F})$$

is surjective. As a consequence, for any two large enough integers a and b, and any positive integer n, the morphism

$$H^0(\mathcal{X}, \mathcal{L}^{\otimes a})^{\otimes n} \otimes H^0(\mathcal{X}, \mathcal{L}^{\otimes b} \otimes \mathcal{F}) \to H^0(\mathcal{X}, \mathcal{L}^{\otimes an+b} \otimes \mathcal{F})$$

is surjective.

Choose a large enough so that the space $H^0(\mathcal{X}, \mathcal{L}^{\otimes a})$ has a basis consisting of sections with norm at most α for some $\alpha < 1$. Choose $b_1, \ldots, b_a$ large enough integers that form a complete residue system modulo l. We can assume that the maps

$$H^0(\mathcal{X}, \mathcal{L}^{\otimes a})^{\otimes n} \otimes H^0(\mathcal{X}, \mathcal{L}^{\otimes b_i} \otimes \mathcal{F}) \to H^0(\mathcal{X}, \mathcal{L}^{\otimes an+b_i} \otimes \mathcal{F})$$

are surjective for all positive integer n and all i between 1 and a. Now choose bases for the spaces $H^0(\mathcal{X}, \mathcal{L}^{\otimes b_i} \otimes \mathcal{F})$ as i varies between 1 and a, and let β be an upper bound for the norm of any element of these bases. Taking products of elements of these bases, we find a subspace of full rank in $H^0(\mathcal{X}, \mathcal{L}^{\otimes an+b_i} \otimes \mathcal{F})$ which has a basis whose elements have norm at most $\alpha^n\beta$. By [38, Lemma 1.7], this implies that $H^0(\mathcal{X}, \mathcal{L}^{\otimes an+b_i} \otimes \mathcal{F})$ has a basis whose elements have norm at most $r\alpha^n\beta$, where r is the rank of $H^0(\mathcal{X}, \mathcal{L}^{\otimes an+b_i} \otimes \mathcal{F})$.

The theory of Hilbert polynomials shows that the rank of $H^0(\mathcal{X}, \mathcal{L}^{an+b_i})$ is bounded above by a polynomial in $an + b_i$. Since $\alpha < 1$, the number $r\alpha^n\beta$ decreases exponentially as $an+b_i$ grows, which shows the result since any integer can be written as $an+b_i$ for some i and n. □

Corollary 2.5. – *Let $\mathcal{X}$ be a projective arithmetic variety. Let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$ and let $\overline{\mathcal{M}}$ be a hermitian line bundle on $\mathcal{X}$. Then for any large enough integer n, the hermitian line bundle $\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}$ is ample if and only if it is semipositive.*

Proof. – Since $\mathcal{L}$ is relatively ample, for any large enough integer n, the line bundle $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is relatively ample and the morphisms

$$H^0(\mathcal{X}, \mathcal{L}^{\otimes n} \otimes \mathcal{M})^{\otimes m} \to H^0(\mathcal{X}, (\mathcal{L}^{\otimes n} \otimes \mathcal{M})^{\otimes m})$$

are surjective for any positive integer m.

For large enough n, Proposition 2.4 guarantees that there is a basis for $H^0(\mathcal{X}, \mathcal{L}^{\otimes n} \otimes \mathcal{M})$ consisting of sections with norm at most $e^{-n\varepsilon}$ for some positive number ε. As a consequence, we can find a subspace of full rank in $H^0(\mathcal{X}, (\mathcal{L}^{\otimes n} \otimes \mathcal{M})^{\otimes m})$ with a basis consisting of sections with norm at most $e^{-mn\varepsilon}$. By [38, Lemma 7.1], this implies that $H^0(\mathcal{X}, (\mathcal{L}^{\otimes n} \otimes \mathcal{M})^{\otimes m})$ itself has a basis whose elements have norm at most $re^{-mn\varepsilon}$, where r is the rank of $H^0(\mathcal{X}, (\mathcal{L}^{\otimes n} \otimes \mathcal{M})^{\otimes m})$. Since again r is bounded above by a polynomial in mn, this shows the result. □

Corollary 2.6. – *Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of projective arithmetic varieties, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Then there exists a positive real number ε such that for any large enough integer n, there exists a basis of $H^0(\mathcal{Y}, f^*\mathcal{L}^{\otimes n})$ consisting of sections with norm bounded above by $e^{-n\varepsilon}$.*

Proof. – By the projection formula, for any integer k, we have a canonical isomorphism

$$H^0(\mathcal{Y}, f^*\mathcal{L}^{\otimes k}) \simeq H^0(\mathcal{X}, \mathcal{L}^{\otimes k} \otimes f_*\mathcal{O}_{\mathcal{Y}}).$$

Since $\mathcal{L}$ is relatively ample, for any two large enough integers n and k, the map

$$H^0(\mathcal{X}, \mathcal{L}^{\otimes n}) \otimes H^0(\mathcal{X}, \mathcal{L}^{\otimes k} \otimes f_*\mathcal{O}_{\mathcal{Y}}) \to H^0(\mathcal{X}, \mathcal{L}^{\otimes (n+k)} \otimes f_*\mathcal{O}_{\mathcal{Y}})$$

is surjective, which means that the natural map

$$H^0(\mathcal{X}, \mathcal{L}^{\otimes n}) \otimes H^0(\mathcal{Y}, f^*\mathcal{L}^{\otimes k}) \to H^0(\mathcal{Y}, f^*\mathcal{L}^{\otimes (n+k)})$$

is surjective.

Fix a large enough integer k for the previous assumption to hold. By Proposition 2.4, the space $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$ admits a basis consisting of elements with norm decreasing exponentially with n, which shows that the same property holds for $H^0(\mathcal{Y}, f^*\mathcal{L}^{\otimes (n+k)})$. □

Corollary 2.7. – *Let f be a finite morphism between projective arithmetic varieties. The pullback of an ample hermitian line bundle by f is ample.*

Proof. – By the previous results, we only have to show that if $f : X \to Y$ is a finite map between complex projective varieties, and if $\overline{L}$ is a semipositive hermitian line bundle on Y, then $f^*\overline{L}$ is semipositive.

Let U be an open subset of Y on which L is trivial, and let s be a nowhere vanishing section of L on U. Then f^*s is a nowhere vanishing section of f^*L on $f^{-1}(U)$, and the function

$$-\log ||f^*s||^2 = (-\log ||s||^2) \circ f$$

is plurisubharmonic on $f^{-1}(U)$, being the composition of a holomorphic function and a plurisubharmonic function. Remark 2.2 shows that $f^*\overline{L}$ is semipositive. □

Let n be a positive integer, and consider the complex projective space $\mathbb{CP}^n$. The line bundle $\mathcal{O}(1)$ on $\mathbb{CP}^n$ is endowed with the *Fubini-Study metric* $||.||$ defined as follows. Let x be a point of $\mathbb{CP}^n$ with homogeneous coordinates $[x_0 : \cdots : x_n]$. The fiber of $\mathcal{O}(1)$ at x may be identified with linear forms $\mathbb{C}(x_0, \ldots, x_1) \to \mathbb{C}$. Endow the line $\mathbb{C}(x_0, \ldots, x_1)$ with the norm induced by the standard hermitian norm on $\mathbb{C}^{n+1}$. Then the Fubini-Study metric on $\mathcal{O}(1)$ is the one that corresponds to the operator norm on linear forms.

The following is the basic example of an ample hermitian line bundle.

Proposition 2.8. – *Let n be a positive integer, and let $\overline{\mathcal{O}(1)}$ be the hermitian line bundle on $\mathbb{P}^n_{\mathbb{Z}}$ corresponding to the line bundle $\mathcal{O}(1)$ endowed with the Fubini-Study metric. Then for any $\varepsilon > 0$, the hermitian line bundle $\overline{\mathcal{O}(1)}(\varepsilon)$ is ample on $\mathbb{P}^n_{\mathbb{Z}}$.*

Proof. – The line bundle $\mathcal{O}(1)$ is ample on $\mathbb{P}^n_{\mathbb{Z}}$, and the Fubini-Study metric is known to have positive curvature.

Let $X_0^{d_0} \cdots X_n^{d_n}$ be a monomial of total degree $d > 0$, seen as a section of $\mathcal{O}(d)$. With respect to the Fubini-Study metric, if x is a point of $\mathbb{CP}^n$ with homogeneous coordinates $[x_0 : \cdots : x_n]$, we have

$$||X_0^{d_0} \cdots X_n^{d_n}(x)|| = \frac{|x_0^{d_0} \cdots x_n^{d_n}|}{(|x_0|^2 + \cdots + |x_n|^2)^{d/2}} \leq 1.$$

This shows that $H^0(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{O}(d))$ has a basis consisting of sections of norm bounded above by 1, and proves the result. □

The following follows immediately from Proposition 2.8 and Corollary 2.7.

Corollary 2.9. – *Let $\mathcal{X}$ be an arithmetic variety, and let $\mathcal{L}$ be a relatively ample line bundle on $\mathcal{X}$. Then there exists a metric $||.||$ on $\mathcal{L}_{\mathbb{C}}$, invariant under complex conjugation, such that the hermitian line bundle $(\mathcal{L}, ||.||)$ is ample.*

Proof. – Some positive power $\mathcal{L}^{\otimes n}$ of $\mathcal{L}$ is the pullback of the line bundle $\mathcal{O}(1)$ on some projective space. By Proposition 2.8 and Corollary 2.7, the pullback of the Fubini-Study metric, scaled by $e^{-\varepsilon}$ for some $\varepsilon > 0$, to $\mathcal{L}^{\otimes n}$ gives $\mathcal{L}^{\otimes n}$ the structure of an ample hermitian line bundle.

Endow $\mathcal{L}$ with the hermitian metric $||.||$ whose n-th power is the metric above. We get a hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, ||.||)$ such that $\overline{\mathcal{L}}^{\otimes n}$ is ample. This implies that $\overline{\mathcal{L}}$ is ample. □

2.2. Arithmetic Hilbert-Samuel

We turn to the arithmetic Hilbert-Samuel theorem, giving an estimate for $h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$, where $\overline{\mathcal{L}}$ is ample and n is large. This has been proved by Gillet-Soulé in [17, Theorem 8 and Theorem 9] and extended by [39, Theorem (1.4)], see also [1] and [6]. In later arguments, we will need an estimate for the error term in the arithmetic Hilbert-Samuel theorem. In the case where the generic fiber of $\mathcal{X}$ is smooth, such an estimate follows from the work of Gillet-Soulé and Bismut-Vasserot. The general case does not seem to be worked out. However, for arithmetic surfaces, an argument of Vojta gives us enough information for our later needs.

We start with a proposition relating the Hilbert-Samuel function of a hermitian line bundle and its pullback under a birational morphism.

Proposition 2.10. – *Let $f : \mathcal{Y} \to \mathcal{X}$ be a birational morphism of projective arithmetic varieties, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Then there exists a positive integer k and a positive real number C such that for any integer n and any hermitian vector bundle $\overline{\mathcal{M}}$ on $\mathcal{X}$, the following equality holds:*

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) \leq h^0_{\mathrm{Ar}}(\mathcal{Y}, f^*(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}})) \leq h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes (n+k)} \otimes \overline{\mathcal{M}}(C)).$$

Proof. – Pullback of sections induces an injective map

$$f^* : H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) \to H^0_{\mathrm{Ar}}(\mathcal{Y}, f^*(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}})),$$

which proves the first inequality.

The coherent sheaf $\mathcal{H}\wr\Updownarrow(f_*\mathcal{O}_{\mathcal{Y}}, \mathcal{O}_{\mathcal{X}})$ is non-zero. As a consequence, there exists a positive integer k such that the sheaf

$$\mathcal{H}\wr\Updownarrow(f_*\mathcal{O}_{\mathcal{Y}}, \mathcal{O}_{\mathcal{X}}) \otimes \mathcal{L}^{\otimes k} = \mathcal{H}\wr\Updownarrow(f_*\mathcal{O}_{\mathcal{Y}}, \mathcal{L}^{\otimes k})$$

has a nonzero global section ϕ. Since f is birational, the morphism

$$\phi : f_*\mathcal{O}_{\mathcal{Y}} \to \mathcal{L}^{\otimes k}$$

is injective. If U is an open subset of the compact complex manifold $\mathcal{X}(\mathbb{C})$ and n is an integer, we endow the spaces

$$H^0(U, \mathcal{L}^{\otimes n} \otimes \mathcal{M} \otimes f_*\mathcal{O}_{\mathcal{Y}}) = H^0(f^{-1}(U), f^*\mathcal{L}^{\otimes n} \otimes \mathcal{M})$$

and

$$H^0(U, \mathcal{L}^{\otimes (n+k)} \otimes \mathcal{M})$$

with the sup norm—which might take the value ∞. Since $\mathcal{X}(\mathbb{C})$ is compact, we can find a constant C such that the maps

$$\phi_U : H^0(U, f_*\mathcal{O}_{\mathcal{Y}}) \to \mathcal{H}^0(U, \mathcal{L}^{\otimes k})$$

all have norm bounded above by e^C. As a consequence, all the maps

$$\mathrm{Id} \otimes \phi_U : H^0(U, \mathcal{L}^{\otimes n} \otimes \mathcal{M} \otimes f_*\mathcal{O}_{\mathcal{Y}}) \to H^0(U, \mathcal{L}^{\otimes (n+k)} \otimes \mathcal{M})$$

have norm bounded above by e^C as well, and the induced map

$$H^0(\mathcal{Y}, f^*(\mathcal{L}^{\otimes n} \otimes \mathcal{M})) \to H^0(\mathcal{X}, \mathcal{L}^{\otimes (n+k)} \otimes \mathcal{M})$$

has norm bounded above by e^C. Since this map is injective, we have an injection

$$H^0_{\mathrm{Ar}}(\mathcal{Y}, f^*(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}})) \to H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes (n+k)} \otimes \overline{\mathcal{M}}(C)),$$

which shows the second inequality. □

We may now state some forms of the arithmetic-Hilbert-Samuel theorem. For the purposes of this paper, the key statement is (iii). We will need the more precise estimate on the error term it provides compared to (i).

Theorem 2.11. – *Let $\mathcal{X}$ be a projective arithmetic variety of dimension d, let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$, and let $\overline{\mathcal{M}}$ be a hermitian vector bundle of rank r.*

(i) *As n tends to ∞, we have*

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) = \frac{r}{d!}\overline{\mathcal{L}}^d n^d + o(n^d);$$

(ii) *if $\mathcal{X}_{\mathbb{Q}}$ is smooth and the curvature form of $\overline{\mathcal{L}}$ is positive, then*

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) = \frac{r}{d!}\overline{\mathcal{L}}^d n^d + O(n^{d-1}\log n)$$

as n tends to ∞;

(iii) *if $d = 2$, then*

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) \geq \frac{r}{2}\overline{\mathcal{L}}^2 n^2 + O(n\log n)$$

as n tends to ∞.

Proof. – The first statement can be found in [36, Corollary 2.7(1)]. It is a consequence of the extension by Zhang in [39, Theorem (1.4)] of the arithmetic Hilbert-Samuel theorem of Gillet-Soulé of [17, Theorem 8], together with [16, Theorem 1].

Let us prove the second statement. Choose a Kähler metric on $\mathcal{X}(\mathbb{C})$, assumed to be invariant under complex conjugation, and write $\chi_{L^2}(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}})$ (resp. $\chi_{sup}(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}})$) for the logarithm of the covolume of $H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}})$ for the associated L^2 norm (resp. for the sup norm). Then by [17, Theorem 8], we have

$$\chi_{L^2}(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) = \frac{r}{d!}\overline{\mathcal{L}}^d n^d + O(n^{d-1}\log n).$$

By the Gromov inequality as in for instance [36, Corollary 2.7(2)], this implies

$$\chi_{sup}(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) = \frac{r}{d!}\overline{\mathcal{L}}^d n^d + O(n^{d-1}\log n).$$

Consider the lattice $\Lambda = H^0(\mathcal{X}, \mathcal{L}^{\otimes n} \otimes \mathcal{M})$, endowed with the sup norm. Then since Λ is generated by elements of norm strictly smaller than 1, the dual of Λ does not contain any nonzero element of norm smaller than 1. Furthermore, the geometric version of the Hilbert-Samuel theorem shows that the rank of Λ has the form $O(n^{d-1})$. By [16, Theorem 1], we get

$$\left|h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) - \chi_{sup}(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}})\right| = O(n^{d-1}\log n),$$

which proves the desired result.

We now prove the last statement. Let $f : \mathcal{Y} \to \mathcal{X}$ be the normalization of $\mathcal{X}$, so that f is birational, finite, and $\mathcal{Y}$ has smooth generic fiber.

Since f is finite, the line bundle $f^*\mathcal{L}$ is ample. Let $\overline{\mathcal{L}}'$ be $f^*\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}'$ be $f^*\overline{\mathcal{M}}$. Choose a Kähler metric on $\mathcal{Y}(\mathbb{C})$, assumed to be invariant under complex conjugation, and again write

$\chi_{L^2}(\overline{\mathcal{L}}'^{\otimes n} \otimes \overline{\mathcal{M}}')$ for the logarithm of the covolume of $H^0(\mathcal{Y}, \mathcal{L}'^{\otimes n} \otimes \mathcal{M}')$ for the associated L^2 norm.

By Proposition 2.10, we can find a constant C and an integer k such that for any integer n greater or equal to k, we have

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) \geq h^0_{\mathrm{Ar}}\big(\mathcal{Y}, \overline{\mathcal{L}}'^{\otimes n-k} \otimes \overline{\mathcal{M}}'(-C)\big). \tag{2.1}$$

Applying the arithmetic Riemann-Roch theorem for n large enough so that the higher cohomology groups of $\mathcal{L}'^{\otimes n} \otimes \mathcal{M}'$ vanish, we get the following equality:

$$\chi_{L^2}\big(\overline{\mathcal{L}}'^{\otimes n} \otimes \overline{\mathcal{M}}'(-C)\big) - \frac{1}{2}T_n = \frac{r}{2}\overline{\mathcal{L}}^2 n^2 + O(n),$$

where by T_n we denote the analytic torsion of the hermitian vector bundle $\overline{\mathcal{L}}'^{\otimes n} \otimes \overline{\mathcal{M}}'$. The equality above is proved via the computations of [17, Theorem 8], or [15, Théorème 1]. In contrast with the usual setting of Hilbert-Samuel, note that the curvature form of $\overline{\mathcal{L}}'$ might not be positive everywhere, so that we cannot apply the estimates of [4] for T_n. However, since the dimension of $\mathcal{X}$ is 2, we have

$$T_n = \zeta'_{1,n}(0),$$

where ζ_1 is the zeta function of the Laplace operator acting on forms of type $(0,1)$ with values in $\overline{\mathcal{L}}'^{\otimes n} \otimes \overline{\mathcal{M}}'(-C)$. We can use an estimate of Vojta to control the analytic torsion T_n. Indeed, by [34, Proposition 2.7.6], we have

$$\zeta'_{1,n}(0) \geq -Kn\log n$$

for some constant K, so that

$$\chi_{L^2}\big(\overline{\mathcal{L}}'^{\otimes n} \otimes \overline{\mathcal{M}}'(-C)\big) = \frac{r}{2}\overline{\mathcal{L}}^2 n^2 + \frac{1}{2}T_n + O(n) \geq \frac{r}{2}\overline{\mathcal{L}}^2 n^2 + O(n\log n). \tag{2.2}$$

Combining as above Gromov's inequality, [16, Theorem 1], Corollary 2.6 and the geometric version of Hilbert-Samuel, we can write

$$\big|h^0_{\mathrm{Ar}}\big(\mathcal{Y}, \overline{\mathcal{L}}'^{\otimes n} \otimes \overline{\mathcal{M}}'(-C)\big) - \chi_{L^2}\big(\overline{\mathcal{L}}'^{\otimes n} \otimes \overline{\mathcal{M}}'(-C)\big)\big| = O(n\log n),$$

which together with (2.2) gives the estimate

$$h^0_{\mathrm{Ar}}\big(\mathcal{Y}, \overline{\mathcal{L}}'^{\otimes n} \otimes \overline{\mathcal{M}}'(-C)\big) \geq \frac{r}{2}\overline{\mathcal{L}}^2 n^2 + O(n\log n).$$

From (2.1), we finally obtain

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) \geq \frac{r}{2}\overline{\mathcal{L}}^2 (n-k)^2 + O(n\log n) \geq \frac{r}{2}\overline{\mathcal{L}}^2 n^2 + O(n\log n). \qquad \square$$

2.3. Restriction of sections

Let k be a field, and let X be a projective variety over k. Let $\mathcal{L}$ be an ample line bundle on X. If Y is any closed subscheme of X, consider the restriction maps

$$\phi_n : H^0(X, \mathcal{L}^{\otimes n}) \to H^0(Y, \mathcal{L}_{|Y}^{\otimes n}).$$

The map ϕ_n is surjective if n is large enough and, obviously, there are bijections between the different fibers of ϕ_n when it is surjective.

In this section, we give arithmetic analogues of these results, looking at H^0_{Ar} instead of H^0—this is Theorem 2.17. Furthermore, we show in Theorem 2.21 that the lower bound on the dimension of the image of the restriction map can be given to be independent of Y.

In the following, let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. If n is an integer, we denote by Λ_n the space $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$ endowed with the sup norm, and we write r_n for its rank. If R is a nonnegative real number, let $B_n(R)$ be the closed ball of radius R in Λ_n. In particular, we have

$$B_n(1) = H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}).$$

Let $B_n(R)_{\mathbb{R}}$ be the closed ball of radius R in $\Lambda_n \otimes \mathbb{R}$. Let Vol denote the volume with respect to the unique translation-invariant measure on $\Lambda_n \otimes \mathbb{R}$ for which $\mathrm{Vol}(B_n(1)_{\mathbb{R}}) = 1$.

If $\mathcal{I}$ is a quasi-coherent sheaf of ideals on $\mathcal{X}$, we write $\Lambda_n^{\mathcal{I}}$ for the subgroup $H^0(\mathcal{X}, \mathcal{L}^{\otimes n} \otimes \mathcal{I})$ of $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$, endowed with the induced norm. We write $r_n^{\mathcal{I}}$, $B_n(R)^{\mathcal{I}}$, $B_n(R)_{\mathbb{R}}^{\mathcal{I}}$, $\mathrm{Vol}^{\mathcal{I}}$ for the corresponding objects.

We gather a few results regarding point counting in the lattices $\Lambda_n^{\mathcal{I}}$. The following is a basic estimate.

Lemma 2.12. – *Let η be a positive real number. Let C be any real number. Then, as n goes to infinity, we have, for any positive R,*

$$\mathrm{Vol}^{\mathcal{I}}\big(B_n(R + Ce^{-n\eta})_{\mathbb{R}}^{\mathcal{I}}\big) = R^{r_n^{\mathcal{I}}}\big(1 + CR^{-1}r_n^{\mathcal{I}}e^{-n\eta} + o(R^{-1}r_n^{\mathcal{I}}e^{-n\eta})\big),$$

where the implied constant in o depends only on C.

Proof. – We write

$$\begin{aligned}\mathrm{Vol}^{\mathcal{I}}\big(B_n(R + Ce^{-n\eta})_{\mathbb{R}}^{\mathcal{I}}\big) &= (R + Ce^{-n\eta})^{r_n^{\mathcal{I}}}\\ &= R^{r_n^{\mathcal{I}}}\exp\big(r_n^{\mathcal{I}}\log(1 + CR^{-1}e^{-n\eta})\big)\\ &= R^{r_n^{\mathcal{I}}}\exp\big(CR^{-1}r_n^{\mathcal{I}}e^{-n\eta} + o(R^{-1}r_n^{\mathcal{I}}e^{-n\eta})\big)\\ &= R^{r_n^{\mathcal{I}}}\big(1 + CR^{-1}r_n^{\mathcal{I}}e^{-n\eta} + o(R^{-1}r_n^{\mathcal{I}}e^{-n\eta})\big).\end{aligned}$$ □

Fix $\mathcal{I}$ as above. Let n be a large enough integer. By Proposition 2.4, we can find a positive number $\varepsilon^{\mathcal{I}}$, independent of n, and a basis $\sigma_1, \ldots, \sigma_{r_n^{\mathcal{I}}}$ of $\Lambda_n^{\mathcal{I}}$ such that $\|\sigma_i\|_\infty \leq e^{-n\varepsilon^{\mathcal{I}}}$ for all $i \in \{1, \ldots, r_n^{\mathcal{I}}\}$. Consider the fundamental domain

$$D_n^{\mathcal{I}} = \Big\{\sum_{i=1}^{r_n^{\mathcal{I}}}\lambda_i\sigma_i | \forall i \in \{1, \ldots, n\},\ 0 \leq \lambda_i < 1\Big\}. \tag{2.3}$$

Proposition 2.13. – *Let α be a positive number with $0 < \alpha < 1$. As n tends to ∞, we have, for any $R > e^{-n^{\alpha}}$,*

$$|B_n(R)^{\mathcal{I}}|\mathrm{Vol}^{\mathcal{I}}(D_n^{\mathcal{I}}) \sim R^{r_n^{\mathcal{I}}}.$$

Proof. – Let n be a large enough integer. As σ runs through the elements of $\Lambda_n^{\mathcal{I}}$, the sets $\sigma + D_n^{\mathcal{I}}$ are pairwise disjoint and cover $\Lambda_n^{\mathcal{I}} \otimes \mathbb{R}$. Furthermore, the diameter of $D_n^{\mathcal{I}}$ is bounded above by $r_n^{\mathcal{I}}e^{-n\varepsilon^{\mathcal{I}}}$. As a consequence, if σ is any element of $\Lambda_n^{\mathcal{I}}$, then

$$\sigma + D_n^{\mathcal{I}} \subset B_n(\|\sigma\|_\infty + r_n^{\mathcal{I}}e^{-n\varepsilon^{\mathcal{I}}})_{\mathbb{R}}^{\mathcal{I}}$$

and

$$(\sigma + D_n^{\mathcal{I}}) \cap B_n(\|\sigma\|_\infty - r_n^{\mathcal{I}}e^{-n\varepsilon^{\mathcal{I}}})_{\mathbb{R}}^{\mathcal{I}} = \emptyset.$$

As a consequence, we have

$$\mathrm{Vol}^{\mathcal{I}}\big(B_n(R - r_n^{\mathcal{I}} e^{-n\varepsilon^{\mathcal{I}}})^{\mathcal{I}}_{\mathbb{R}}\big) \leq |B_n(r)^{\mathcal{I}}|\mathrm{Vol}^{\mathcal{I}}(D_n^{\mathcal{I}}) \leq \mathrm{Vol}^{\mathcal{I}}\big(B_n(R + r_n^{\mathcal{I}} e^{-n\varepsilon^{\mathcal{I}}})^{\mathcal{I}}_{\mathbb{R}}\big).$$

By Riemann-Roch, the rank $r_n^{\mathcal{I}}$ grows at most polynomially in n. As a consequence, $R^{-1} r_n^{\mathcal{I}} e^{-n\epsilon^{\mathcal{I}}}$ goes to 0 as n goes to infinity, and Lemma 2.12 shows that both the left and right terms are equivalent to $R^{r_n^{\mathcal{I}}}$ as n goes to infinity. □

Proposition 2.14. – *Let α and η be positive real numbers with $0 < \alpha < 1$. Let C be any real number. Then, as n tends to ∞, there exists a positive real number η' such that we have, for any positive $R > e^{-n^{\alpha}}$,*

$$\frac{\big||B_n(R + Ce^{-n\eta})^{\mathcal{I}}| - |B_n(R)^{\mathcal{I}}|\big|}{|B_n(R)^{\mathcal{I}}|} = O(e^{-n\eta'}),$$

where the implied constants depend on α, C and η.

Proof. – We assume that C is positive. The case where C is negative (or zero) can be treated by the same computations.

Let η' be a positive number strictly smaller than both $\varepsilon^{\mathcal{I}}$ and η. Since the $\sigma + D_n^{\mathcal{I}}$ are pairwise disjoint as σ runs through the elements of $\Lambda_n^{\mathcal{I}}$, we get, for large enough n

$$\begin{aligned}
\big(|B_n(R + Ce^{-n\eta})^{\mathcal{I}}| &- |B_n(R)^{\mathcal{I}}|\big)\mathrm{Vol}^{\mathcal{I}}(D_n^{\mathcal{I}}) \\
&\leq \mathrm{Vol}^{\mathcal{I}}(B_n(R + Ce^{-n\eta} + r_n^{\mathcal{I}} e^{-n\varepsilon^{\mathcal{I}}})^{\mathcal{I}}_{\mathbb{R}}) - \mathrm{Vol}^{\mathcal{I}}(B_n(R - r_n^{\mathcal{I}} e^{-n\varepsilon^{\mathcal{I}}})^{\mathcal{I}}_{\mathbb{R}}) \\
&\leq \mathrm{Vol}^{\mathcal{I}}(B_n(R + e^{-n\eta'})^{\mathcal{I}}_{\mathbb{R}}) - \mathrm{Vol}^{\mathcal{I}}(B_n(R - e^{-n\eta'})^{\mathcal{I}}_{\mathbb{R}}) \\
&\sim 2R^{r_n^{\mathcal{I}}-1} r_n^{\mathcal{I}} e^{-n\eta'},
\end{aligned}$$

where in the last line we applied Lemma 2.12, using that $R^{-1} r_n^{\mathcal{I}} e^{-n\eta'}$ tends to 0 as n tends to ∞.

Putting the previous estimate together with Proposition 2.13 and replacing η' with a smaller positive number, we get the desired result. □

The following is a first step in controlling restriction maps.

Proposition 2.15. – *Let α be a positive number with $0 < \alpha < 1$. There exists a positive constant η such that for any large enough integer n, if N is any positive integer with $N < e^{n^{\alpha}}$, then the following statements hold:*

(i) *the map $\phi_{n,N} : H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}) \to \Lambda_n / N\Lambda_n$ is surjective;*

(ii) *for any two s, s' in $\Lambda_n / N\Lambda_n$, we have*

$$\frac{\big||\phi_{n,N}^{-1}(s)| - |\phi_{n,N}^{-1}(s')|\big|}{|\phi_{n,N}^{-1}(s)|} \leq e^{-n\eta}.$$

Proof. – Let r_n be the rank of Λ_n. Let n be a positive integer, which will be chosen large enough, and let N be an integer bounded above by $e^{n^{\alpha}}$.

By Proposition 2.4, we can find a positive number ε, independent of n, and a basis $\sigma_1, \dots, \sigma_{r_n}$ of Λ_n such that $||\sigma_i||_{\infty} \leq e^{-n\varepsilon}$ for all $i \in \{1, \dots, r_n\}$. Any element of $\Lambda_n / N\Lambda_n$ is the image of an element of Λ_n of the form

$$\sigma = \lambda_1 \sigma_1 + \dots + \lambda_{r_n} \sigma_{r_n},$$

where the λ_i are integers between 0 and $N-1$. We have

$$||\sigma||_\infty < Nr_n e^{-n\varepsilon} \leq r_n e^{n^\alpha - n\varepsilon}.$$

We know that r_n is a polynomial in n for large enough n and $\alpha < 1$ by assumption, so that any σ as above is strictly effective for large enough n. This shows that the map $\phi_{n,N}$ is surjective and proves (i).

We now proceed to the proof of (ii). Let n be a large enough integer. By the discussion above, we can find a positive real number ε' such that for any large enough integer n, and any s in $\Lambda_n/N\Lambda_n$, there exists an element σ_0 in Λ_n with $||\sigma_0||_\infty \leq e^{-n\varepsilon'}$ that restricts to s. We have

$$\phi_{n,N}^{-1}(s) = \{\sigma_0 + N\sigma | \sigma \in \Lambda_n, ||\sigma_0 + N\sigma||_\infty \leq 1\},$$

so that, up to replacing ε' by a smaller positive number

$$|B_n(1/N - e^{-n\varepsilon'})| \leq |\phi_{n,N}^{-1}(s)| \leq |B_n(1/N + e^{-n\varepsilon'})|$$

and

$$\big| |\phi_{n,N}^{-1}(s)| - |\phi_{n,N}^{-1}(s')| \big| \leq |B_n(1/N + e^{-n\varepsilon'})| - |B_n(1/N - e^{-n\varepsilon'})| \tag{2.4}$$

for any two s, s' in $\Lambda_n/N\Lambda_n$. We conclude by applying Proposition 2.14. □

We need a variant of a theorem of Bost.

PROPOSITION 2.16. – *Let X be a reduced complex analytic space, Y a closed reduced subspace of X, L an ample line bundle on X and $||.||$ a semipositive smooth metric on L. Then for any $\varepsilon > 0$, any large enough integer n, and any $s \in H^0(Y, L_{|Y}^{\otimes n})$, we can find $\sigma \in H^0(X, L^{\otimes n})$ such that $\sigma_{|Y} = s$ and*

$$||\sigma||_\infty \leq e^{\varepsilon n} ||s||_\infty.$$

Proof. – If the metric $||.||$ is positive, then this is the content of [8, Theorem A.1] (1). Since L is ample, it admits a positive hermitian metric, so that we can find a smooth function $\phi : X \to \mathbb{R}$ such that $||.||e^{-\phi}$ is positive. Since $||.||$ is semipositive, the metric $||.||e^{-\delta\phi}$ is positive for any $\delta > 0$.

Let ε be a positive real number, and choose $\delta > 0$ such that

$$\forall x \in X, |\delta\phi(x)| \leq \varepsilon.$$

Apply [8, Theorem A.1] to the line bundle L with the positive metric $||.||e^{-\delta\phi}$: if n is large enough, and $s \in H^0(Y, L^{\otimes n})$, we can find $\sigma \in H^0(X, L^{\otimes n})$ such that $\sigma_{|Y} = s$ and

$$||\sigma||_\infty \leq e^{3\varepsilon n} ||s||_\infty.$$

This shows the result. □

The following is a key property of ample line bundles.

---

(1) The only assumption necessary in [8] is that $\mathcal{Y}_{\mathbb{C}}$ and $\mathcal{X}_{\mathbb{C}}$ are reduced, see [27] for a statement that makes this explicit.

THEOREM 2.17. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample line bundle on $\mathcal{X}$. Let $\mathcal{Y}$ be a closed subscheme of $\mathcal{X}$, such that $\mathcal{Y}_{\mathbb{Q}}$ is reduced. If n is a positive integer, let*

$$\phi_n : H^0(\mathcal{X}, \mathcal{L}^{\otimes n}) \to H^0(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})$$

be the restriction map. For any positive ε, define

$$\Lambda_n^{\varepsilon} = H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}) \cap \phi_n^{-1}\big(H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})\big),$$

that is, Λ_n^{ε} is the space of effective sections σ of $\overline{\mathcal{L}}^{\otimes n}$ such that the restriction of σ to $\mathcal{Y}$ has norm at most $e^{-n\varepsilon}$. Write $\psi_n := (\phi_n)|_{\Lambda_n^{\varepsilon}}$. Then the following statements hold:

(i) *for any large enough integer n, the restriction map*

$$\psi_n : \Lambda_n^{\varepsilon} \to H^0_{\mathrm{Ar}}\big(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n}\big)$$

is surjective;

(ii) *there exists a positive constant η such that for any large enough integer n, and any two s, s' in $H^0_{\mathrm{Ar}}\big(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n}\big)$, we have*

$$\frac{\big||\phi_n^{-1}(s)| - |\phi_n^{-1}(s')|\big|}{|\phi_n^{-1}(s)|} \leq e^{-n\eta};$$

(iii) *η being chosen as above, for any $s \in H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})$, we have*

$$\left|\phi_n^{-1}(s) - \frac{|\Lambda_n^{\varepsilon}|}{|H^0_{\mathrm{Ar}}\big(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)^{\otimes n}\big)|}\right| \leq e^{-n\eta}|\phi_n^{-1}(s)|.$$

Proof. – Fix $\varepsilon > 0$. The group

$$\{\sigma \in H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n}), ||\sigma_{\mathbb{C}}|| = 0\}$$

is the torsion subgroup of $H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})$, which we denote by $H^0_{\mathrm{Ar}}(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})_{\mathrm{tor}}$—note that this group does not depend on ε nor the hermitian metric. This is a finite group. Let N be a positive integer with

$$NH^0_{\mathrm{Ar}}(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})_{\mathrm{tor}} = 0.$$

Assume n is large enough. The restriction map

$$\phi_n : \Lambda_n = H^0(\mathcal{X}, \mathcal{L}^{\otimes n}) \to H^0(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})$$

is surjective since $\mathcal{L}$ is relatively ample. The map

$$\Lambda_n / N\Lambda_n \to H^0_{\mathrm{Ar}}(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})_{\mathrm{tor}}$$

is well-defined and surjective as well. Applying Proposition 2.15, this shows that the image of ψ_n contains $H^0_{\mathrm{Ar}}(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})_{\mathrm{tor}}$.

Let s be an element of $H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})$. Let ε' be a real number with $0 < \varepsilon' < \varepsilon$. Apply Proposition 2.16 to the closed subspace $\mathcal{Y}_{\mathbb{C}}$ of $\mathcal{X}_{\mathbb{C}}$. If n is large, we can find a section σ of $\mathcal{L}^{\otimes n}$ on $\mathcal{X}_{\mathbb{C}}$ with $||\sigma||_{\infty} \leq e^{-n\varepsilon'}$ and $\sigma|_{\mathcal{Y}_{\mathbb{C}}} = s_{\mathbb{C}}$. Up to replacing σ with $\sigma + \overline{\sigma}$, and making ε' smaller, we may assume that σ is a section of $\mathcal{L}^{\otimes n}$ over $\mathcal{X}_{\mathbb{R}}$, that is,

$$\sigma \in B_n(e^{-n\varepsilon'})_{\mathbb{R}}.$$

Let $\mathcal{I}$ be the ideal of $\mathcal{Y}$ in $\mathcal{X}$. The kernel of the—surjective when n is large enough—restriction map

$$\phi_n : \Lambda_n \to H^0(\mathcal{Y}, \mathcal{L}_{|\mathcal{Y}}^{\otimes n})$$

is $\Lambda_n^{\mathcal{I}}$. Let σ' be an element of Λ_n mapping to s. Then $\sigma \in (\Lambda_n^{\mathcal{I}})_{\mathbb{R}} + \sigma'$.

The fundamental domain $D_n^{\mathcal{I}}$ defined in (2.3) has diameter bounded above by $r_n e^{-n\varepsilon^{\mathcal{I}}}$—note that $r^n \geq r_n^{\mathcal{I}}$. In particular, we can find $\sigma'' \in \Lambda_n^{\mathcal{I}} + \sigma'$ with

$$||\sigma'' - \sigma||_\infty \leq r_n e^{-n\varepsilon^{\mathcal{I}}},$$

so that

$$||\sigma''||_\infty \leq e^{-n\varepsilon'} + r_n e^{-n\varepsilon^{\mathcal{I}}} < 1$$

for large enough n. We have $\psi_n(\sigma)_{\mathbb{C}} = s_{\mathbb{C}}$, i.e., $\psi_n(\sigma) - \sigma$ is torsion. This shows that the image of ψ_n maps surjectively onto the quotient of $H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)_{|\mathcal{Y}}^{\otimes n})$ by $H^0_{\mathrm{Ar}}(\mathcal{Y}, \mathcal{L}_{|\mathcal{Y}}^{\otimes n})_{\mathrm{tor}}$. Since we showed above that it contains $H^0_{\mathrm{Ar}}(\mathcal{Y}, \mathcal{L}_{|\mathcal{Y}}^{\otimes n})_{\mathrm{tor}}$, this proves that ψ_n is surjective.

Apply statement (i) after replacing $\overline{\mathcal{L}}$ with $\overline{\mathcal{L}}(-\delta)$, where $\delta > 0$ is chosen small enough so that $\overline{\mathcal{L}}(-\delta)$ is ample. Then if $\varepsilon > \delta$ and n is large enough, for any $s \in H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)_{|\mathcal{Y}}^{\otimes n})$, we can find $\sigma_0 \in H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}(-\delta)^{\otimes n})$ that restricts to s.

To prove (ii), we argue as in Proposition 2.15. Let s and σ_0 be as above. Then

$$\psi_n^{-1}(s) = \{\sigma_0 + \sigma | \sigma \in \Lambda_n^{\mathcal{I}}, ||\sigma_0 + \sigma||_\infty \leq 1\}$$

and

$$|B_n(1 - e^{-n\delta})^{\mathcal{I}}| \leq |\psi_n^{-1}(s)| \leq B_n(1 + e^{-n\delta})^{\mathcal{I}}.$$

Using Proposition 2.14 again, this proves (ii).

To prove (iii), write

$$|\Lambda_n^\varepsilon| = \sum_{s \in H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)_{|\mathcal{Y}}^{\otimes n})} |\psi_n^{-1}(s)|,$$

so that for any large enough n and any $s \in H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)_{|\mathcal{Y}}^{\otimes n})$, we have

$$\left| |\Lambda_n^\varepsilon| - |\psi_n^{-1}(s)|\, |H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)_{|\mathcal{Y}}^{\otimes n})| \right| \leq e^{-n\eta}\, |\psi_n^{-1}(s)|\ |H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)_{|\mathcal{Y}}^{\otimes n})|. \qquad \square$$

We keep the notation of the theorem.

COROLLARY 2.18. – *Let E be a subset of $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)_{|\mathcal{Y}}^{\otimes n})$. Set*

$$E' := \left\{\sigma \in \bigcup_{n>0} \Lambda_n^\varepsilon, \sigma_{|\mathcal{Y}} \in E\right\}.$$

For any $0 \leq \rho \leq 1$, the set E has density ρ in $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)_{|\mathcal{Y}}^{\otimes n})$ if and only if E' has density ρ in $\bigcup_{n>0} \Lambda_n^\varepsilon$.

Proof. – For any positive integer n, define

$$E_n := E \cap H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n}),\ E'_n := E' \cap H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}) = E' \cap \Lambda^{\varepsilon}_n.$$

Denoting by ψ_n the restriction maps as before, we can write

$$|E'_n| = \sum_{s \in E_n} |\psi_n^{-1}(s)|.$$

Summing the estimate of Theorem 2.17, (iii) over all $s \in E_n$ for large enough n, we can find a positive constant η such that, for large enough n,

$$\left| |E'_n| - \frac{|E_n|}{|H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})|} |\Lambda^{\varepsilon}_n| \right| \leq e^{-n\eta} |E'_n|$$

and, dividing by $|\Lambda^{\varepsilon}_n| \leq |E'_n|$,

$$\left| \frac{|E'_n|}{|\Lambda^{\varepsilon}_n|} - \frac{|E_n|}{|H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})|} \right| \leq e^{-n\eta}.$$

Letting n tend to ∞ gives us the result we were looking for. ☐

As a special case of the theorem, we get the following.

Corollary 2.19. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample line bundle on $\mathcal{X}$. Let Y be a closed subscheme of $\mathcal{X}$ lying over $\mathbb{Z}/N\mathbb{Z}$ for some positive integer N. Then for any large enough integer n, the restriction map*

$$\phi_n : H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}) \to H^0(Y, \mathcal{L}|_Y^{\otimes n})$$

is surjective and there exists a positive constant η such that for any $s \in H^0(Y, \mathcal{L}^{\otimes n})$, we have

$$\left| \phi_n^{-1}(s) - \frac{|H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})|}{|H^0(Y, \mathcal{L}|_Y^{\otimes n})|} \right| \leq e^{-n\eta} |\phi_n^{-1}(s)|.$$

We now turn to uniform lower bounds on the image of restriction maps. We first deal with a geometric result.

Proposition 2.20. – *Let S be a noetherian scheme, and let X be a projective scheme over S. Let $\mathcal{L}$ be a line bundle on X, relatively ample over S. Let d be a positive integer. Then there exists an integer N and a positive constant C such that for any point s of S, any closed subscheme Y of X_s of dimension d, and any $n \geq N$, the image of the restriction map*

$$H^0(X_s, \mathcal{L}^{\otimes n}) \to H^0(Y, \mathcal{L}|_Y^{\otimes n})$$

has dimension at least Cn^d.

Proof. – Since S is noetherian, we can find an integer M such that for any point s of S and any integer $n \geq M$, the restriction of $\mathcal{L}^{\otimes n}$ to X_s is very ample.

Let s be a point of S, and let Y be a closed subscheme of X_s of positive dimension d. Let k be an infinite field containing the residue field of s, and write X_k for the base change of X_s to k.

Since $\mathcal{L}^{\otimes n}$ is very ample on X_k and k is infinite, we can find a $d+1$-dimensional subspace $V \subset H^0(X, \mathcal{L}^{\otimes n})$ such that the restriction to Y of the rational map

$$\phi : X \dashrightarrow \mathbb{P}(V^*)$$

is dominant. Let $\sigma_0, \dots, \sigma_d$ be a basis of V, and let H_∞ be the divisor $\mathrm{div}(\sigma_0)$. Identify the subspace of $\mathbb{P}(V^*)$ defined by $\sigma_0 \neq 0$ to the standard affine space $\mathbb{A}^d_k$ with coordinates $x_1, \dots, x_d$. Then the map ϕ is defined outside H_∞—as certainly the base locus of V is contained in H_∞, and maps onto $\mathbb{A}^d_k$.

For any positive integer r and any integer $n \geq (r+1)M$, the line bundle $\mathcal{L}^{\otimes n}(-rH_\infty) \simeq \mathcal{L}^{\otimes n-rM}$ is very ample. In particular, we can find a section σ of $\mathcal{L}^{\otimes n}$ that vanishes to the order r along H_∞, but does not vanish on Y.

Let $P \in k[x_1, \dots, x_d]$ be a polynomial of degree at most r, considered as a morphism $\mathbb{A}^d_k \to \mathbb{A}^1_k$. Since σ vanishes to the order r along H_∞, the section $(P \circ \phi)\sigma$ of $\mathcal{L}^{\otimes n}$, which is a priori defined only outside H_0, defines a global section of $\mathcal{L}^{\otimes n}$. Because σ does not vanish on Y, the restrictions $(P \circ \phi)\sigma_{|Y}$ are linearly independent as sections of $L^{\otimes n}{}_{|Y}$ as P varies. In particular, the image of the restriction map

$$H^0(X, \mathcal{L}^{\otimes n}) \to H^0(Y, \mathcal{L}_{|Y}^{\otimes n})$$

has dimension at least equal to the dimension of the space of polynomials of degree at most r in $x_1, \dots, x_d$, so that it has dimension at least

$$\binom{r+d}{d} = \frac{1}{d!} r^d + O(r^{d-1})$$

for any r with $r + 1 \leq n/M$. This proves the result. □

THEOREM 2.21. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. If $\mathcal{Y}$ is a subscheme of $\mathcal{X}$, let*

$$\phi_{n,\mathcal{Y}} : H^0(\mathcal{X}, \mathcal{L}^{\otimes n}) \to H^0(\mathcal{Y}, \mathcal{L}_{|\mathcal{Y}}^{\otimes n})$$

be the restriction map.

There exists an integer N and a positive real number η such that for any $n \geq N$ and any closed subscheme $\mathcal{Y}$ of $\mathcal{X}$ of dimension $d > 0$, we have

$$\frac{|\operatorname{Ker}(\phi_{n,\mathcal{Y}}) \cap H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})|}{|H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})|} = O(e^{-n^d \eta}),$$

where the implied constant depends on $\mathcal{X}$ and $\overline{\mathcal{L}}$, but not on $\mathcal{Y}$.

Proof. – We only have to consider those $\mathcal{Y}$ that are irreducible. Let us first assume that $\mathcal{Y}$ is flat over $\operatorname{Spec} \mathbb{Z}$. Let H_n be the kernel of the restriction map

$$\Lambda_n = H^0(\mathcal{X}, \mathcal{L}^{\otimes n}) \to H^0(\mathcal{Y}, \mathcal{L}_{|\mathcal{Y}}^{\otimes n}),$$

and let k_n be the corank of H_n, i.e., the rank of the image of the restriction map. By Proposition 2.20 applied to $\mathcal{X}_\mathbb{Q}$ and $\mathcal{Y}_\mathbb{Q}$, there exists a positive constant C, independent of $\mathcal{Y}$, such that if n is larger than some integer N, independent of $\mathcal{Y}$, then

$$k_n \geq C n^{d-1}. \tag{2.5}$$

Up to enlarging N, Proposition 2.4 allows us to assume that for $n \geq N$, Λ_n has a basis consisting of elements with norm at most $e^{-n\varepsilon}$ for some $\varepsilon > 0$. For $n \geq N$, we can find elements $\sigma_1, \dots, \sigma_{k_n}$ of Λ_n that are linearly independent in Λ_n/H_n and satisfy $||\sigma_i||_\infty \leq e^{-n\varepsilon}$ for $i \in \{1, \dots, k_n\}$.

Let η be a positive number smaller than ε. Then for any $\sigma \in H_n \cap B_n(1)_{\mathbb{R}}$ and any integers $\lambda_1, \dots, \lambda_{k_n}$ with $|\lambda_i| \leq e^{n\eta}$ for $i \in \{1, \dots, k_n\}$, we have

$$||\sigma + \sum_{i=1}^{k_n} \lambda_i \sigma_i||_\infty \leq 1 + k_n e^{-n(\varepsilon - \eta)}.$$

Furthermore, as σ runs through the elements of H_n, and $\lambda_1, \dots, \lambda_{k_n}$ run through the integers, the $\sigma + \sum_{i=1}^{k_n} \lambda_i \sigma_i$ are pairwise distinct. As a consequence, we have

$$e^{nk_n\eta} |H \cap B_n(1)_{\mathbb{R}}| \leq \left|B_n(1 + k_n e^{-n(\varepsilon-\eta)})\right|.$$

Applying Proposition 2.14 and noting that k_n is bounded above by r_n, which is a polynomial in n, we get

$$\frac{|B_n(1) \cap H_n|}{|B_n(1)|} = O(e^{-nk_n\eta}).$$

Together with (2.5), this shows the required estimate.

Now assume that $\mathcal{Y}$ is not flat over $\mathbb{Z}$. Since $\mathcal{Y}$ is irreducible, it lies over a closed point p of $\operatorname{Spec} \mathbb{Z}$. By Proposition 2.20, we can find an integer N and a constant C, independent of $\mathcal{Y}$ and p, such that for any $n \geq N$, the kernel of the restriction map

$$H^0(\mathcal{X}_p, \mathcal{L}^{\otimes n}) \to H^0(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})$$

has codimension at least Cn^d as a vector space over $\mathbb{F}_p$. Let k_n be this codimension. Then

$$k_n \geq Cn^d. \tag{2.6}$$

Again, by Proposition 2.4, up to enlarging N, we can find a positive number ε, depending only on $\mathcal{X}$ and $\overline{\mathcal{L}}$, such that for any $n \geq N$, there exist sections $\sigma_1, \dots, \sigma_{k_n}$ of $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$ with $||\sigma_i||_\infty \leq e^{-n\varepsilon}$ for all $i \in \{1, \dots, k_n\}$, such that the images of $\sigma_1, \dots, \sigma_{k_n}$ in $H^0(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})$ are linearly independent over $\mathbb{F}_p$.

Let H_n be the kernel of the restriction map $\phi_{n,\mathcal{Y}}$. If σ is an element of H_n, and if $\lambda_1, \dots, \lambda_{k_n}$ are integers running through $\{0, \dots, p-1\}$, then $\sigma + \lambda_1\sigma_1 + \dots + \lambda_{k_n}\sigma_{k_n}$ belongs to H_n if and only if all the λ_i vanish. Furthermore, the elements $\sigma + \lambda_1\sigma_1 + \dots + \lambda_{k_n}\sigma_{k_n}$ are pairwise disjoint. As a consequence, considering only those λ_i that are 0 or 1, we have

$$2^{k_n} |H_n \cap H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})| \leq |B_n(1 + k_n e^{-n\varepsilon})|.$$

Again, applying Proposition 2.14 and noting that k_n is bounded above by r_n, which is a polynomial in n, we get

$$\frac{|B_n(1) \cap H_n|}{|B_n(1)|} = O(e^{-k_n\eta}).$$

Together with (2.6), this shows the required estimate. □

3. Variants and consequences

3.1. The irreducibility theorem over finite fields

The arithmetic Bertini theorems we prove are stronger than their finite fields counterparts. Since the latter are already known, we give only an example to illustrate how one can deduce them.

PROPOSITION 3.1. – *Assume Theorem 1.1. Let k be a finite field, and let X be an irreducible projective variety over k of dimension at least* 2. *Let L be a very ample line bundle on X. Then the set*

$$\{\sigma \in \bigcup_{n>0} H^0(X, L^{\otimes n}),\ \operatorname{div}(\sigma) \text{ is irreducible}\}$$

has density 1 *in* $\bigcup_{n>0} H^0(X, L^{\otimes n})$.

Proof. – Since L is very ample, we can find a positive integer N and a closed embedding $i : X \to \mathbb{P}^N_k$ such that $L = i^*\mathcal{O}(1)$. Apply Theorem 1.1 to the composition

$$f : X \to \mathbb{P}^N_k \to \mathbb{P}^N_{\mathcal{O}_K},$$

where K is a number field together with a finite prime $\mathfrak{p}$ such that $\mathcal{O}_K/\mathfrak{p} = k$, and the line bundle $\overline{\mathcal{L}} = \overline{\mathcal{O}_{\mathbb{P}^N_{\mathcal{O}_K}}(1)}(\varepsilon)$, $\varepsilon > 0$, endowed with the Fubini-Study metric scaled by $e^{-\varepsilon}$. The hermitian line bundle $\overline{\mathcal{L}}$ is the pullback of $\overline{\mathcal{O}(1)}$ by the finite map $\mathbb{P}^N_{\mathcal{O}_K} \to \mathbb{P}^N_{\mathbb{Z}}$. By Proposition 2.8 and Corollary 2.7, $\overline{\mathcal{L}}$ is ample.

Since $f(X)$ is supported over a closed point of $\operatorname{Spec}\mathbb{Z}$, Theorem 1.1 guarantees that the set

$$\left\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathbb{P}^N_{\mathcal{O}_K}, \overline{\mathcal{L}}^{\otimes n}),\ \operatorname{div}(\sigma_{|X}) \text{ is irreducible}\right\}$$

has density 1 in $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathbb{P}^N_{\mathcal{O}_K}, \overline{\mathcal{L}}^{\otimes n})$. By Corollary 2.18, the theorem holds. □

Note that since on a scheme X defined over a finite field, every line bundle is a hermitian line bundle, and every section is effective, we can remove the flatness assumptions on the theorems of the introduction and have uniform statements that cover both the results of this paper and those of [11].

3.2. Generic smoothness

We first state the Bertini smoothness theorem of Poonen [26] in the form we need—see [13] for the proof of this version.

THEOREM 3.2. – *Let X be a smooth projective variety over a finite field k, and let L be an ample line bundle on X. Then the density of those $\sigma \in \bigcup_{n>0} H^0(X, L^{\otimes n})$ such that* $\operatorname{div}(\sigma)$ *is smooth is equal to $\zeta_X\big(1 + \dim(X)\big)^{-1}$, where ζ_X is the zeta function of X.*

Applying the above result together with the restriction results of Corollary 2.18, we find the following.

Proposition 3.3. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Let p be a prime number such that $\mathcal{X}_p$ is smooth over $\mathbb{F}_p$. Then the density of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that $\mathrm{div}(\sigma_{|\mathcal{X}_p})$ is smooth is equal to $\zeta_p(\dim(\mathcal{X}))^{-1}$, where ζ_p is the zeta function of $\mathcal{X}_p$.*

Proof. – We apply Corollary 2.18 to the subspace $\mathcal{Y} = \mathcal{X}_p$ of $\mathcal{X}$ and the subset E of $\bigcup_{n>0} H^0(\mathcal{X}_p, \mathcal{L}_{|\mathcal{X}_p}^{\otimes n})$ consisting of sections with smooth divisor. Theorem 3.2 shows that E has density $\zeta_p(1+\dim(\mathcal{X}_p))^{-1} = \zeta_p(\dim(\mathcal{X}))^{-1}$, which implies the result. □

We may prove Theorem 1.7.

Proof of Theorem 1.7. – In the situation of the theorem, we know that $\mathcal{X}_p$ is smooth for all large enough p. Furthermore, denoting again the zeta function of $\mathcal{X}_p$ by ζ_p, we have

$$\lim_{p\to\infty} \zeta_p(x) = 1$$

for any $x > 1$ by [30, 1.3]. This shows that the density of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that there exists p with $\mathcal{X}_p$ smooth and $\mathrm{div}(\sigma_{|\mathcal{X}_p})$ smooth is equal to 1. For any such σ, the divisor $\mathrm{div}(\sigma)_{\mathbb{Q}}$ is smooth, which proves the result. □

3.3. Irreducibility theorems with local conditions

We can give variants of the irreducibility theorems with conditions at prescribed subschemes. For an easier formulation, we give them in the setting of Theorem 1.6.

Proposition 3.4. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Let Z_1 be a finite subscheme of $\mathcal{X}$, and let Z_2 be a positive-dimensional subscheme of $\mathcal{X}$. Choose a trivialization $\phi : \mathcal{L}_{|Z_1} \simeq \mathcal{O}_{Z_1}$, and let T be a subset of $H^0(Z_1, \mathcal{O}_{Z_1})$. Then the density of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that $\sigma_{|Z_1}$ belongs to T (under the trivialization ϕ) and σ does not vanish identically on any component of Z_2 is equal to*

$$\frac{|T|}{|H^0(Z_1, \mathcal{O}_{Z_1})|}.$$

Proof. – By Corollary 2.18, the density of those σ such that $\sigma_{|Z_1}$ belongs to T is indeed $\frac{|T|}{|H^0(Z_1, \mathcal{O}_{Z_1})|}$. On the other hand, Theorem 2.21 ensures that the density of those σ that do not vanish identically on any component of Z_2 is equal to 1. □

Given Theorem 1.6—proven in the last section of this paper—and Theorem 1.7, we find the two following results.

Corollary 3.5. – *Let $\mathcal{X}$ be a projective arithmetic variety of dimension at least 2, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Let Z_1 be a finite subscheme of $\mathcal{X}$, and let Z_2 be a positive-dimensional subscheme of $\mathcal{X}$. Choose a trivialization $\phi : \mathcal{L}_{|Z_1} \simeq \mathcal{O}_{Z_1}$, and let T be a subset of $H^0(Z_1, \mathcal{O}_{Z_1})$. Then the density of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that the following conditions hold:*

(i) $\sigma_{|Z_1}$ *belongs to T (under the trivialization ϕ);*

(ii) σ *does not vanish identically on any component of* Z_2*;*

(iii) $\operatorname{div}(\sigma)$ *is irreducible,*

is equal to

$$\frac{|T|}{|H^0(Z_1, \mathcal{O}_{Z_1})|}.$$

Corollary 3.6. – *Let* $\mathcal{X}$ *be a projective arithmetic variety with smooth generic fiber, and let* $\overline{\mathcal{L}}$ *be an ample hermitian line bundle on* $\mathcal{X}$*. Let* Z_1 *be a finite subscheme of* $\mathcal{X}$*, and let* Z_2 *be a positive-dimensional subscheme of* $\mathcal{X}$*. Choose a trivialization* $\phi : \mathcal{L}|_{Z_1} \simeq \mathcal{O}_{Z_1}$*, and let* T *be a subset of* $H^0(Z_1, \mathcal{O}_{Z_1})$*. Then the density of those* $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ *such that the following conditions hold:*

(i) $\sigma|_{Z_1}$ *belongs to* T *(under the trivialization* ϕ*);*

(ii) σ *does not vanish identically on any component of* Z_2*;*

(iii) $\operatorname{div}(\sigma)_{\mathbb{Q}}$ *is smooth,*

is equal to

$$\frac{|T|}{|H^0(Z_1, \mathcal{O}_{Z_1})|}.$$

4. Preliminary estimates

This section gathers preliminary material on hermitian line bundles on arithmetic surfaces, which will be used in the proof of Theorem 1.6. In 4.1, we give lower bounds for the norm of products of sections of hermitian line bundles. In 4.2, we give an upper bound for the number of effective sections of a hermitian line bundle in terms of its degree with respect to a positive enough hermitian line bundle. Such a result is closely related to the effective bounds of [37]. Our proof is better expressed in terms of the θ-invariants of Bost [9], which we only consider in a finite-dimensional setting. In 4.3, we give an estimate for the number of effective hermitian line bundles satisfying certain boundedness properties.

4.1. Norm estimates for sections of hermitian line bundles

Let X be a compact connected Riemann surface. Let ω be a real semipositive 2-form of type $(1,1)$ on X with

$$\int_X \omega = 1.$$

Define

$$d^c = \frac{1}{2i\pi}(\partial - \overline{\partial}),$$

so that

$$dd^c = \frac{i}{\pi}\partial\overline{\partial}.$$

Let s be a section of a hermitian line bundle on X. In what follows, we will write $||s||$ for the function $P \mapsto ||s(P)||$.

Let $\overline{L} = (L, ||.||)$ be a hermitian line bundle on X. If s is a nonzero section of L, the Lelong-Poincaré formula gives us the equality of currents

$$-dd^c \log ||s|| = c_1(\overline{L}) - \delta_D,$$

where $c_1(\overline{L})$ is the curvature form of $\overline{L}$, D is the divisor of s and δ_D is the current of integration along D.

Define, following [10, (1.4.8)]

$$||s||_0 = \exp\Big(\int_X \log ||s||\omega\Big).$$

Since $\int_X \omega = 1$, the following inequality holds:

$$||s||_0 \leq ||s||_\infty.$$

Say that $\overline{L}$ is *ω-admissible*, or admissible for short, if $c_1(\overline{L})$ is proportional to ω. If $\overline{L}$ is admissible, then the Gauss-Bonnet formula shows

$$c_1(\overline{L}) = (\deg L)\omega.$$

Let M be any line bundle on X. By the $\partial\overline{\partial}$ lemma, we can find a hermitian metric $||.||$ on M such that the hermitian line bundle $(M, ||.||)$ is admissible. Given a nonzero global section s of M, there exists a unique such metric such that $||s||_0 = 1$.

If D is an effective divisor on X, let σ_D be the section of $\mathcal{O}(D)$ that is the image of 1 under the natural morphism $\mathcal{O}_X \to \mathcal{O}(D)$. The discussion above shows that there exists a unique admissible hermitian line bundle $\overline{\mathcal{O}(D)} = (\mathcal{O}(D), ||.||)$ on X such that $||\sigma_D||_0 = 1$. Of course, if D_1 and D_2 are effective divisors, we have

$$\overline{\mathcal{O}(D_1 + D_2)} = \overline{\mathcal{O}(D_1)} \otimes \overline{\mathcal{O}(D_2)}.$$

The functions σ_P satisfy basic uniformities in the point P of X which are readily proved by the following argument using Green functions.

PROPOSITION 4.1. – *Endow X with a Riemannian metric with induced geodesic distance d. Then there exist positive constants C, C' and η such that the following inequalities hold:*

(i) $\forall P \in X$, $||\sigma_P||_\infty \leq C$;

(ii) $\forall (P, Q) \in X \times X$, $\sigma_P(Q) \geq \min(C'd(P, Q), \eta)$.

Proof. – Let $\Delta \subset X \times X$ be the diagonal. Let α be a real closed form of type $(1, 1)$ on $X \times X$ of the form

$$\alpha = p_1^*\omega + p_2^*\omega + \sum_{i \in I} p_1^*\beta_i \wedge p_2^*\gamma_i,$$

where p_1 and p_2 are the two projections from $X \times X$ to X and the β_i (resp. γ_i) are 1-forms on X. Choose the β_i and γ_i so that α is symmetric with respect to the involution of $X \times X$ that exchanges the two factors, and that it is cohomologous to the class of the diagonal Δ in the de Rham cohomology of $X \times X$. By the $\partial\overline{\partial}$ lemma, we can find a hermitian metric on the line bundle $\mathcal{O}(\Delta)$ with curvature form α. For any P in X, this hermitian metric induces a hermitian metric on $\mathcal{O}(P)$ by restriction to $\{P\} \times X$.

Let σ_Δ be the global section of $\mathcal{O}(\Delta)$ corresponding to the constant function 1. For any P in X, write τ_P for the section of $\mathcal{O}(P)$.

$$\tau_P : Q \mapsto \sigma_\Delta(P, Q).$$

Then

$$-dd^c \log ||\tau_P|| = \alpha_{|_{\{P\}\times X}} - \delta_P = \omega - \delta_P,$$

which shows that the metric on $\mathcal{O}(P)$ coming from that on $\mathcal{O}(\Delta)$ differs from the canonical one defined above by a homothety. In particular, we can find a continuous function $X \to \mathbb{R}_+^*$, $P \mapsto \lambda(P)$ such that

$$\forall (P, Q) \in X \times X, ||\sigma_P(Q)|| = \lambda(P)||\sigma_\Delta(P, Q)||.$$

Since $(P, Q) \mapsto \sigma_\Delta(P, Q)$ is a smooth section of $\mathcal{O}(\Delta)$ that vanishes with the order 1 along Δ, this shows the result (2). □

We will make use of the uniformity above to prove inequalities between norms. The following is a variant of [10, Corollary 1.4.3].

PROPOSITION 4.2. – *Let* $\overline{L} = (L, ||.||)$ *be an admissible hermitian line bundle on* X. *Let* P *be a point of* X, *and let* s *be a section of* L. *Then*

$$||s(P)|| \leq ||s||_0 \, ||\sigma_P||_\infty^{\deg L}.$$

In particular, there exists a positive constant C_1 *such that*

$$||s||_\infty \leq C_1^{\deg L} ||s||_0.$$

Proof. – We can assume that s is nonzero. Let D be the divisor of s. Define

$$g = -\log ||s||$$

and

$$g_P = -\log ||\sigma_P||.$$

By Lelong-Poincaré, we have

$$dd^c g = (\deg L)\omega - \delta_D$$

and

$$dd^c g_P = \omega - \delta_P.$$

The Stokes formula

$$\int_X g \, dd^c g_P = \int_X g_P \, dd^c g$$

gives us

$$\text{-}\log ||s||_0 + \log ||s(P)|| = -\deg L \log ||\sigma_P||_0 + \log ||\sigma_P(D)|| = \log ||\sigma_P(D)||,$$

where, if $D = \sum_i n_i P_i$, we wrote

$$||\sigma_P(D)|| = \prod_i ||\sigma(P_i)^{n_i}||.$$

---

(2) Actually, a straightforward computation shows that $||\tau_P||_0$ is a constant function of P, so that λ is constant.

Since the degree of D is equal to the degree of L, we get the first inequality. The second one follows from the first and Proposition 4.1. □

LEMMA 4.3. – *Let* $\overline{L} = (L, ||.||)$ *be an admissible hermitian line bundle on* X *with positive degree. Then for any section* s *of* L*, and any* P *in* X*, the following inequality holds:*

$$||s||_\infty \leq C_2(\deg L)||s\sigma_P||_\infty,$$

where C_2 *is a positive constant depending only on* X *and* ω*.*

Proof. – Let B be the ball $\{z \in \mathbb{C} | \, |z| < 3\}$. Let $(U_i)_{i \in I}$ be a finite cover of X by open subsets such that there exist biholomorphic functions

$$f_i : B \to U_i$$

and assume that X is covered by the $f_i(\{z \in \mathbb{C} | \, |z| < 1\})$. For all $i \in I$, choose a smooth function $\phi_i : B \to \mathbb{R}$ such that $-dd^c\phi_i = f_i^*\omega$.

Since $\overline{L}$ is admissible, the curvature form of $\overline{L}$ is $(\deg L)\omega$ and for any $i \in I$, we can find an isomorphism of hermitian line bundles on B

$$f_i^*\overline{L} \simeq (\mathcal{O}, e^{-\lambda\phi_i}),$$

where $|.|$ is the standard absolute value and $\lambda = \deg L$.

Choose an element $i \in I$, and a complex number z with $|z| < 1$ such that

$$|f_i^*s(z)|e^{-\lambda\phi_i(z)} = ||s||_\infty,$$

where $f_i^*s(z)$ is considered as a complex number via the isomorphism above. By Proposition 4.1, we can find positive constants $\varepsilon < \lambda$ and η depending only on X and ω such that either $|\sigma_P(z)| \geq \frac{\eta}{\lambda}$ or,

$$\forall z' \in \mathbb{C}, |z' - z| = \frac{\varepsilon}{\lambda} \implies |\sigma_P(z')| \geq \frac{\eta}{\lambda}.$$

If $\sigma_P(z) \geq \frac{\eta}{\lambda}$, then

$$||s\sigma_P||_\infty \geq ||s\sigma_P(f_i(z))|| \geq \frac{\eta}{\lambda}||s(f_i(z))|| = \frac{\eta}{\deg L}||s||_\infty.$$

Assume on the contrary $\sigma_P(z) < \frac{\eta}{\lambda}$. By the maximum principle, we can find a complex number z' with $|z' - z| = \frac{\varepsilon}{\lambda}$ and $|f_i^*s(z')| \geq |f_i^*s(z)|$. In particular, $|z'| < 2$ and

$$||s\sigma_P||_\infty \geq ||s\sigma_P(f_i(z'))|| \geq \frac{\eta}{\lambda}||s||_\infty e^{-\lambda(\phi_i(z) - \phi_i(z'))} \geq e^{-C}\frac{\eta}{\deg L}||s||_\infty,$$

where C is an upper bound for the differential of the ϕ_i on the ball $\{z \in \mathbb{C} | \, |z| < 2\}$ as i varies through the finite set I. □

PROPOSITION 4.4. – *Let* $\overline{L} = (L, ||.||)$ *and* $\overline{M} = (M, ||.||)$ *be two admissible hermitian line bundles on* X*. Then for any two sections* s *and* σ *of* L *and* M *respectively, the following inequality holds:*

$$||s||_\infty||\sigma||_0 \leq (C_2(\deg L + \deg M))^{\deg M}||s\sigma||_\infty,$$

where C_2 *is a positive constant depending only on* X *and* ω*.*

Proof. – Let D be the divisor of σ. Then M is isomorphic to $\mathcal{O}(D)$, and the hermitian line bundles $\overline{M}$ and $\overline{\mathcal{O}(D)}$, as well as the sections σ and σ_D differ by a homothethy. Since the inequality we want to prove is invariant under scaling, we can assume that $\overline{M} = \overline{\mathcal{O}(D)}$ and $\sigma = \sigma_D$. If $D = \sum_i n_i P_i$, we have

$$\overline{\mathcal{O}(D)} = \bigotimes_i \overline{\mathcal{O}(P_i)}^{\otimes n_i}$$

and

$$\sigma_D = \Pi_i \sigma_{P_i}^{n_i},$$

so that the result follows from successive applications of Lemma 4.3. □

4.2. An upper bound for the number of sections

Let $\mathcal{X}$ be a projective arithmetic variety with smooth generic fiber. Choose a Kähler form on $\mathcal{X}(\mathbb{C})$ which is invariant under complex conjugation and has volume 1. If $\overline{\mathcal{L}}$ is a hermitian line bundle on $\mathcal{X}$, we write $h^0_\theta(\mathcal{X}, \overline{\mathcal{L}})$ for $h^0_\theta(H^0_{L^2}(\mathcal{X}, \overline{\mathcal{L}}))$, where the hermitian vector bundle $H^0_{L^2}(\mathcal{X}, \overline{\mathcal{L}})$ over $\operatorname{Spec}\mathbb{Z}$ is endowed with the L^2 norm induced by the Kähler metric on $\mathcal{X}$.

We will need a comparison result between the sup norm and the L^2 norm on the space of sections of hermitian line bundles, which we will obtain through a minor generalization of Gromov's lemma [17, Lemma 30]. We follow the proof of Gillet-Soulé and start with a local result.

In the following, if z is an element of $\mathbb{C}^d$, we write $z_1, \ldots, z_d \in \mathbb{C}$ for its coordinate, and, for any $k \in \{1, \ldots, d\}$, we write $z_k = x_k + iy_k$, where x_k and y_k are real.

LEMMA 4.5. – *Let d be a positive integer, and let B be the open ball $\{z \in \mathbb{C}^d \,|\, |z| < 3\}$ in $\mathbb{C}^d$. Let ϕ be a real-valued smooth function on B, and let g be a smooth positive function on B. Then there exists a positive constant C depending only on ϕ and g such that for any real number $\lambda \geq 1$, any holomorphic function f on B and any w in B with $|w| < 1$,*

$$\int \cdots \int_{|z-w|<1} |f(z)|^2 e^{-2\lambda\phi(z)} g(z) dx_1 \cdots dy_d \geq C|f(w)|^2 e^{-2\lambda\phi(w)} \lambda^{-2d}.$$

Proof. – If λ is an integer, the inequality is the "local statement" proved in the beginning of the proof of [17, Lemma 30].

To prove our result, after adding a negative constant to ϕ, we can assume that ϕ is negative on the ball $|z| < 2$. Let C' be a lower bound for the values of ϕ on the ball $|z| < 2$. If $\lambda > 1$ is arbitrary, write $\lambda = n + r$, with $0 \leq r < 1$. Then

$$e^{-2\lambda\phi(z)} = e^{-2n\phi(z)} e^{-2r\phi(z)} \geq e^{-2n\phi(z)}$$

for any z with $|z| < 2$, so that

$$\begin{aligned}
\int \cdots \int_{|z-w|<1} |f(z)|^2 e^{-2\lambda\phi(z)} g(z) dx_1 \cdots dy_d &\geq \int \cdots \int_{|z-w|<1} |f(z)|^2 e^{-2n\phi(z)} g(z) dx_1 \cdots dy_d \\
&\geq C|f(w)|^2 e^{-2n\phi(w)} n^{-2d} \\
&\geq C|f(w)|^2 e^{-2\lambda\phi(w)} e^{2r\phi(w)} \lambda^{-2d} \\
&\geq Ce^{2C'} |f(w)|^2 e^{-2\lambda\phi(w)} \lambda^{-2d}.
\end{aligned}$$

Replacing C with $Ce^{2C'}$, we get the result. □

PROPOSITION 4.6. – *Let X be a compact connected riemannian complex manifold of dimension d, let ω be a real form of type $(1,1)$ on X. Then there exists a positive constant C such that for any hermitian line bundle $\overline{\mathcal{L}}$ on X with positive degree and curvature form $\lambda\omega$ with $|\lambda| > 1$, and any section s of $\mathcal{L}$ over X, we have*

$$||s||_{L^2} \geq C|\lambda|^{-d}||s||_{\infty},$$

where $||s||_{L^2}$ denotes the L^2 norm of s with respect to the given metric on X.

In particular, if $d = 1$, there exists a positive constant C' such that for any hermitian line bundle $\overline{\mathcal{L}}$ with curvature form proportional to ω and positive degree, and any section s of $\mathcal{L}$, we have

$$||s||_{L^2} \geq C'(\deg \mathcal{L})^{-1}||s||_{\infty}.$$

Proof. – As above, let B be the open ball $\{z \in \mathbb{C}^d \,|\, |z| < 3\}$ in $\mathbb{C}^d$. Let $(U_i)_{i\in I}$ be a finite cover of X by open subsets such that there exists biholomorphic functions

$$f_i : B \to U_i$$

and assume that X is covered by the $f_i(\{z \in \mathbb{C}^d \,|\, |z| < 1\})$. For any $i \in I$, we can find a positive smooth function g_i such that the pullback of the standard metric of X to B by f_i is $g_i\, dx_1 \cdots dy_d$.

For all $i \in I$, choose a function ϕ_i on B such that $-dd^c\phi_i = f_i^*\omega$. Let $\overline{\mathcal{L}}$ be a hermitian line bundle with curvature form $\lambda\omega$ for some real number λ with $|\lambda| > 1$. Then, for any $i \in I$, we can fix an isomorphism of hermitian line bundles

$$f_i^*\overline{\mathcal{L}} \simeq (\mathcal{O}_B, e^{-\lambda\phi_i}|.|),$$

where $|.|$ is the standard absolute value. Applying Lemma 4.5 (up to replacing ϕ_i by $-\phi_i$ if λ is negative), we can find a positive constant K, independent of $\overline{\mathcal{L}}$, such that, given any section s of $\overline{\mathcal{L}}$, for any $i \in I$ and any w in B with $|w| < 1$, we have

$$\int |_{z-w|<1} |f_i^*s(z)|^2 e^{-\lambda\phi_i(z)} g(z) dx_1 \cdots dy_d \geq K|f_i^*s(w)|^2 e^{-\lambda\phi_i(w)}|\lambda|^{-2d},$$

where we consider f_i^*s as a holomorphic function via the local trivializations of $\mathcal{L}$. This inequality means

$$\int |_{z-w|<1} ||s(f_i(z))||^2 g(z) dx_1 \cdots dy_d \geq K|\lambda|^{-2d}||s(f_i(w))||^2,$$

so that

$$||s||_{L^2}^2 \geq \int |_{z-w|<1} ||s(f_i(z))||^2 g(z) dx_1 \cdots dy_d \geq K|\lambda|^{-2d}||s(f_i(w))||^2$$

for any w, which proves the first result.

The second result is a consequence of the first one and the Gauss-Bonnet formula. □

Given a real form ω of type $(1,1)$, write $\widehat{\mathrm{Pic}}_{\omega}(\mathcal{X})$ for the group of ω-admissible hermitian line bundles on $\mathcal{X}$, that is, hermitian line bundles whose curvature form is proportional to ω.

PROPOSITION 4.7. – *Let $\mathcal{X}$ be a regular projective arithmetic surface. Choose a Kähler form on $\mathcal{X}(\mathbb{C})$ which is invariant under complex conjugation, and let $\overline{\mathcal{B}}$ be a hermitian line bundle on $\mathcal{X}$. Let ω be a real form of type $(1,1)$ on $\mathcal{X}(\mathbb{C})$ with $\int_{\mathcal{X}(\mathbb{C})}\omega \neq 0$. Assume that the following conditions hold:*

(i) *Some positive power of $\overline{\mathcal{B}}$ is effective;*

(ii) $\overline{\mathcal{B}}.\overline{\mathcal{B}} > 0$*;*

(iii) *If $\overline{\mathcal{M}}$ is an effective hermitian line bundle on $\mathcal{X}$, then $\overline{\mathcal{B}}.\overline{\mathcal{M}} \geq 0$.*

Then for any effective $\overline{\mathcal{M}} \in \widehat{\mathrm{Pic}}_{\omega}(\mathcal{X})$, we have

$$h^0_\theta(\mathcal{X},\overline{\mathcal{M}}) \leq \frac{(\overline{\mathcal{B}}.\overline{\mathcal{M}})^2}{2\overline{\mathcal{B}}.\overline{\mathcal{B}}} + O(\overline{\mathcal{M}}.\overline{\mathcal{B}}\log(1+\overline{\mathcal{M}}.\overline{\mathcal{B}})) + O(\deg\mathcal{M}_{\mathbb{Q}}\log(1+\deg\mathcal{M}_{\mathbb{Q}})) + O(1),$$

where the implied constants depend on $\mathcal{X}$, $\overline{\mathcal{B}}$ and ω, but not on $\overline{\mathcal{M}}$.

REMARK 4.8. – Using the precise computations of [9, Chapter 3], it would be possible to make the implied constants above effective.

REMARK 4.9. – If $\overline{\mathcal{M}}$ is effective, then both $\overline{\mathcal{B}}.\overline{\mathcal{M}}$ and $\deg\mathcal{M}_{\mathbb{Q}}$ are nonnegative.

Proof. – Let $\overline{\mathcal{M}}$ be an effective, ω-admissible, hermitian line bundle. If $\mathcal{M}_{\mathbb{Q}}$ has degree zero, then the curvature form of $\overline{\mathcal{M}}$ vanishes, so that $\overline{\mathcal{M}}$ is isomorphic to $\overline{\mathcal{O}}_{\mathcal{X}}$ and the inequality of the proposition holds. We can assume that the degree of $\mathcal{M}_{\mathbb{Q}}$ is positive. Let us write d for the degree of $\mathcal{M}_{\mathbb{Q}}$.

After replacing $\overline{\mathcal{B}}$ by a positive power, we can assume that $\overline{\mathcal{B}}$ is effective. Let σ be a nonzero effective section of $\overline{\mathcal{B}}$ with divisor D. We have an exact sequence of line bundles

$$0 \to \mathcal{M}\otimes\mathcal{B}^{\otimes -1} \to \mathcal{M} \to \mathcal{M}_{|D} \to 0,$$

in which the first map is multiplication by σ and the second one is restriction of sections. Taking global sections, we get an exact sequence

$$0 \to H^0(\mathcal{X},\mathcal{M}\otimes\mathcal{B}^{\otimes -1}) \to H^0(\mathcal{X},\mathcal{M}) \to H^0(D,\mathcal{M}_{|D}).$$

The map of lattices

$$i : H^0_{L^2}(\mathcal{X},\overline{\mathcal{M}}\otimes\overline{\mathcal{B}}^{\otimes -1}) \to H^0_{L^2}(\mathcal{X},\overline{\mathcal{M}})$$

is the multiplication by the section σ, whose sup norm is bounded above by 1, so the operator norm of i is bounded above by 1.

Endow $H^0(D,\overline{\mathcal{M}}_{|D})$ with the L^2 norm

$$||t||^2_{L^2} = \sum_{z\in D(\mathbb{C})} ||t(z)||^2$$

for $t \in H^0(D,\overline{\mathcal{M}}_{|D})$. Then for any section t of $\mathcal{M}$ over D, we have

$$||t||^2_\infty \geq \frac{1}{\deg D_{\mathbb{Q}}}||t||^2_{L^2}.$$

If s is a global section of $\overline{\mathcal{M}}$ on $\mathcal{X}$, then certainly we have, for the sup norms

$$||s||_\infty \geq ||s_{|D}||_\infty$$

and consequently

$$||s||_\infty \geq \frac{1}{\deg D_{\mathbb{Q}}} ||s_{|D}||_{L^2}^2.$$

By Proposition 4.6, with s as above, we have

$$||s||_{L^2} \geq C\, d^{-1} ||s||_\infty,$$

where we recall that d is the degree of $\mathcal{M}_{\mathbb{Q}}$ and C is a positive constant independent of $\overline{\mathcal{M}}$. We obtain

$$||s||_{L^2} \geq \frac{C}{\deg D_{\mathbb{Q}}} d^{-1} ||s_{|D}||_{L^2}^2.$$

In other words, the operator norm of the map of lattices

$$r : H^0_{L^2}(\mathcal{X}, \overline{\mathcal{M}}) \to H^0_{L^2}(D, \overline{\mathcal{M}}_{|D}),$$

given by restricting sections to D is bounded above by $C'd$, where C' is a positive constant independent of $\overline{\mathcal{M}}$. In other words, the induced map of lattices

$$H^0_{L^2}(\mathcal{X}, \overline{\mathcal{M}}) \to H^0_{L^2}(D, \overline{\mathcal{M}}_{|D})(\log C' + \log d)$$

has norm at most 1—here if Λ is a lattice and δ a real number, we write $\Lambda(\delta)$ for the lattice Λ with the metric scaled by $e^{-\delta}$. Note that from [9, Corollary 3.3.5, (2)], we have

$$h^0_\theta(H^0_{L^2}(D, \overline{\mathcal{M}}_{|D})(\log C' + \log d)) \leq h^0_\theta(D, \overline{\mathcal{M}}_{|D}) + \deg D_{\mathbb{Q}}(\log C' + \log d).$$

From the monotonicity and the subadditivity of θ-invariants proved in [9, Proposition 3.3.2, Proposition 3.8.1], we get

$$h^0_\theta(\mathcal{X}, \overline{\mathcal{M}}) \leq h^0_\theta(\mathcal{X}, \overline{\mathcal{M}} \otimes \overline{\mathcal{B}}^{\otimes -1}) + h^0_\theta(D, \overline{\mathcal{M}}_{|D}) + O(\log d) + O(1), \tag{4.1}$$

where the implied constants are independent of $\overline{\mathcal{M}}$.

By [9, Proposition 3.7.1, Proposition 3.7.2], we have [3]

$$h^0_\theta(D, \overline{\mathcal{M}}_{|D}) \leq \max(\deg \overline{\mathcal{M}}_{|D}, 0) + O(1) \leq \overline{\mathcal{M}}.\overline{\mathcal{B}} + O(1)$$

since we assumed that $\overline{\mathcal{M}}.\overline{\mathcal{B}} \geq 0$ and since D is the zero locus of an effective section of $\overline{\mathcal{B}}$. Together with (4.1), we obtain

$$h^0_\theta(\mathcal{X}, \overline{\mathcal{M}}) \leq h^0_\theta(\mathcal{X}, \overline{\mathcal{M}} \otimes \overline{\mathcal{B}}^{\otimes -1}) + \overline{\mathcal{M}}.\overline{\mathcal{B}} + O(\log d) + O(1). \tag{4.2}$$

Now let m be the smallest integer such that $m\overline{\mathcal{B}}.\overline{\mathcal{B}} > \overline{\mathcal{M}}.\overline{\mathcal{B}}$, so that

$$m \leq \lfloor \overline{\mathcal{M}}.\overline{\mathcal{B}} / \overline{\mathcal{B}}.\overline{\mathcal{B}} \rfloor + 1.$$

Applying the argument above inductively to $\overline{\mathcal{L}}.\overline{\mathcal{B}}^{\otimes -i}$ as i runs from 0 to $m-1$, we get

$$h^0_\theta(\mathcal{X}, \overline{\mathcal{M}}) \leq h^0_\theta(\mathcal{X}, \overline{\mathcal{M}} \otimes \overline{\mathcal{B}}^{\otimes -m}) + \frac{(\overline{\mathcal{M}}.\overline{\mathcal{B}})^2}{2\overline{\mathcal{B}}.\overline{\mathcal{B}}} + O(\overline{\mathcal{M}}.\overline{\mathcal{B}} \log d) + O(\overline{\mathcal{M}}.\overline{\mathcal{B}}) + O(1). \tag{4.3}$$

By construction, $\overline{\mathcal{B}}.(\overline{\mathcal{M}} \otimes \overline{\mathcal{B}}^{\otimes -m}) < 0$, so that condition (iii) ensures that $\overline{\mathcal{M}} \otimes \overline{\mathcal{B}}^{\otimes -m}$ is not effective. By [9, Corollary 4.1.2], we get

$$h^0_\theta(\mathcal{X}, \overline{\mathcal{M}} \otimes \overline{\mathcal{B}}^{\otimes -m}) \leq O(d \log d) + O(1)$$

---

(3) In [9], hermitian vector bundles are only considered over the ring of integers of number fields. However, this assumption is irrelevant, and can be removed by considering the pullback of $\overline{\mathcal{M}}_{|D}$ to the normalization of D.

since the rank of $H^0(\mathcal{X}, \mathcal{M} \otimes \mathcal{B}^{\otimes -m})$ is certainly bounded above by $O(d)$.

Finally, we have

$$h^0_\theta(\mathcal{X}, \overline{\mathcal{M}}) \leq \frac{(\overline{\mathcal{M}}.\overline{\mathcal{B}})^2}{2\overline{\mathcal{B}}.\overline{\mathcal{B}}} + O(\overline{\mathcal{M}}.\overline{\mathcal{B}} \log d) + O(d \log d) + O(\overline{\mathcal{M}}.\overline{\mathcal{B}}) + O(1), \tag{4.4}$$

which shows the result. □

Corollary 4.10. – *Let $\mathcal{X}$ be a regular projective arithmetic surface, and let $\overline{\mathcal{B}}$ be a hermitian line bundle on $\mathcal{X}$. Let ω be a real form of type $(1,1)$ on $\mathcal{X}(\mathbb{C})$ with $\int_{\mathcal{X}(\mathbb{C})} \omega \neq 0$. Assume that the following conditions hold:*

(i) *Some positive power of $\overline{\mathcal{B}}$ is effective;*

(ii) $\overline{\mathcal{B}}.\overline{\mathcal{B}} > 0$*;*

(iii) *If $\overline{\mathcal{M}}$ is an effective hermitian line bundle on $\mathcal{X}$, then $\overline{\mathcal{B}}.\overline{\mathcal{M}} \geq 0$.*

Then for any effective $\overline{\mathcal{M}} \in \widehat{\mathrm{Pic}}_\omega(\mathcal{X})$, we have

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}}) \leq \frac{(\overline{\mathcal{B}}.\overline{\mathcal{M}})^2}{2\overline{\mathcal{B}}.\overline{\mathcal{B}}} + O(\overline{\mathcal{M}}.\overline{\mathcal{B}} \log(\overline{\mathcal{M}}.\overline{\mathcal{B}})) + O(\deg \mathcal{M}_{\mathbb{Q}} \log(\deg \mathcal{M}_{\mathbb{Q}})) + O(1),$$

where the implied constants depend on $\mathcal{X}$ and $\overline{\mathcal{B}}$, but not on $\overline{\mathcal{M}}$.

Proof. – From Proposition 4.7 and [9, Theorem 4.1.1], we find that the inequality holds if one replaces $h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}})$ with $h^0_{\mathrm{Ar},L^2}(\mathcal{X}, \overline{\mathcal{M}})$—this expression being defined as the logarithm of the number of sections of $\overline{\mathcal{M}}$ with L^2 norm bounded above by 1. Choosing the Kähler form on $\mathcal{X}$ to have volume 1, we have

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}}) \leq h^0_{\mathrm{Ar},L^2}(\mathcal{X}, \overline{\mathcal{M}}),$$

which finishes the proof. □

Remark 4.11. – In [37, Theorem A], Yuan and Zhang prove an explicit upper bound for $h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}})$ from which one can deduce—via log-concavity of volumes—special cases of our inequality.

4.3. An upper bound for the number of effective hermitian line bundles

Lemma 4.12. – *Let $\mathcal{X}$ be a projective arithmetic surface, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. If $\overline{\mathcal{M}}$ is an effective hermitian line bundle on $\mathcal{X}$ which is not isomorphic to $\overline{\mathcal{O}}_{\mathcal{X}}$, then*

$$\overline{\mathcal{L}}.\overline{\mathcal{M}} > 0.$$

Proof. – Let s be an effective section of $\overline{\mathcal{M}}$, and let D be the divisor of s. Then by the formula [12, (6.3.2)], we have

$$\overline{\mathcal{L}}.\overline{\mathcal{M}} = h_{\overline{\mathcal{L}}}(D) - \int_{\mathcal{X}(\mathbb{C})} \log ||s_{\mathbb{C}}|| c_1(\overline{\mathcal{L}}),$$

where $h_{\overline{\mathcal{L}}}$ denotes the height with respect to D. The first term is nonnegative since $\overline{\mathcal{L}}$ is ample, and vanishes if and only if $D = 0$. Since s is effective and the curvature form of $\overline{\mathcal{L}}$ is semipositive—and positive on a Zariski-dense open subset of $\mathcal{X}(\mathbb{C})$ as $\mathcal{L}_{\mathbb{C}}$ is ample—the second term is nonnegative as well, and vanishes if and only if the norm of s is identically 1.

As a consequence, for $\overline{\mathcal{L}}.\overline{\mathcal{M}}$ to vanish, it is necessary for $\overline{\mathcal{M}}$ to have a nowhere vanishing section of norm identically 1, i.e., to be isomorphic to $\overline{\mathcal{O}}_{\mathcal{X}}$. ☐

PROPOSITION 4.13. – *Let $\mathcal{X}$ be a projective arithmetic surface, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Let ω be a semipositive real form of type $(1,1)$ on $\mathcal{X}(\mathbb{C})$ with $\int_{\mathcal{X}(\mathbb{C})} \omega \neq 0$. Let N be a subgroup of the group $\widehat{\mathrm{Pic}}_\omega(\mathcal{X})$ of ω-admissible hermitian line bundles on $\mathcal{X}$. Assume that the intersection of N with $\mathrm{Ker}(\widehat{\mathrm{Pic}}_\omega(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X})) \simeq \mathbb{R}$ is discrete. Then N is a group of finite type. Let ρ be the rank of N, and let N_{eff} denote the subspace of N consisting of effective line bundles. As n tends to ∞, we have*

$$\left|\{\overline{\mathcal{M}} \in N_{\mathrm{eff}} | \, \overline{\mathcal{L}}.\overline{\mathcal{M}} \leq n\}\right| = O(n^\rho).$$

Proof. – The abelian group $\mathrm{Pic}(\mathcal{X})$ is finitely generated by [29]—see [20] for a modern proof—so that the image of N in $\mathrm{Pic}(\mathcal{X})$ is a group of finite type. Since the intersection of N with $\mathrm{Ker}(\widehat{\mathrm{Pic}}_\omega(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}))$ is discrete, it is of finite type as well, which proves that N is a group of finite type.

The linear form on N

$$\overline{\mathcal{M}} \mapsto \overline{\mathcal{L}}.\overline{\mathcal{M}}$$

extends to a linear form on $N_{\mathbb{R}} := N \otimes \mathbb{R}$ which we still denote by

$$\alpha \mapsto \overline{\mathcal{L}}.\alpha.$$

Let $\overline{N}_{\mathrm{eff}}$ be the closure of N_{eff} in $N \otimes \mathbb{R}$. Lemma 4.12 shows that the linear form above is nonnegative on N_{eff}, so it is nonnegative on $\overline{N}_{\mathrm{eff}}$.

Our assumption on N guarantees that the first chern class map

$$c_1 : \widehat{\mathrm{Pic}}_\omega(\mathcal{X}) \to \widehat{\mathrm{CH}}^1(\mathcal{X})$$

extends to an injection

$$c_{1,\mathbb{R}} : N \otimes \mathbb{R} \to \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}),$$

where $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})$ is the arithmetic Chow group with real coefficients defined in [7, 5.5]. Indeed, we have an exact sequence

$$0 \to (N \cap \mathrm{Ker}(\widehat{\mathrm{Pic}}_\omega(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}))) \otimes \mathbb{R} \to N \otimes \mathbb{R} \to \mathrm{Pic}(\mathcal{X}) \otimes \mathbb{R},$$

and the first term can be identified with $\mathbb{R}$ by assumption.

By the Hodge index theorem of Faltings [14] and Hriljac [18] as stated in [7, Theorem 5.5, (2)], the intersection pairing on $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})$ is non-degenerate: it has signature $(+,-,-,\ldots)$. Since ω is semipositive and $\int_{\mathcal{X}(\mathbb{C})} \omega \neq 0$, there exists an ample line bundle $\overline{\mathcal{H}}$ in $\widehat{\mathrm{Pic}}_\omega(\mathcal{X})$. Then $\overline{\mathcal{H}}.\overline{\mathcal{H}} > 0$, so that the intersection pairing on $\widehat{\mathrm{Pic}}_\omega(\mathcal{X})$ is non-degenerate as well.

In particular, if x is a nonzero element of $\overline{N}_{\mathrm{eff}}$, we can find a hermitian line bundle $\overline{\mathcal{M}} \in \widehat{\mathrm{Pic}}_\omega(\mathcal{X})$ with $\overline{\mathcal{M}}.x < 0$. If n is a large enough integer, Corollary 2.5 shows that $\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}$ is ample, so that the discussion above guarantees the inequality

$$(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}).x = n\overline{\mathcal{L}}.x + \overline{\mathcal{M}}.x \geq 0.$$

This shows that $\overline{\mathcal{L}}.x$ is positive.

The linear form $x \mapsto \overline{\mathcal{L}}.x$ is positive on the complement of the origin in the closed cone $\overline{N}_{\mathrm{eff}}$. As a consequence, the number of integral points x of $\overline{N}_{\mathrm{eff}}$ with $\overline{\mathcal{L}}.x \leq n$ is bounded above by a quantity of the form $O(n^{\rho})$, where ρ is the rank of N. ☐

5. Irreducible ample divisors on arithmetic surfaces

5.1. Setup and an easy estimate

In this section, we prove Theorem 1.6 for arithmetic surfaces.

Let $f : \mathcal{X} \to \operatorname{Spec}\mathbb{Z}$ be a projective arithmetic surface, and $\overline{\mathcal{L}}$ an ample line bundle on $\mathcal{X}$. If n is a large enough integer, we want to give an upper bound for the number of sections of $\overline{\mathcal{L}}^{\otimes n}$ that define a divisor which is not irreducible. We will give three different bounds that depend on the geometry and the arithmetic of the irreducible components of that divisor.

In the statement below, $\mathcal{X}$ is not assumed to be regular, but heights are still well-defined, see [39, (1.2)].

Proposition 5.1. – *Let α be a real number with $0 < \alpha < \frac{1}{2}$. If n is an integer, the proportion of those elements s of $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ that vanish on some Weil divisor D of $\mathcal{X}$ with $h_{\overline{\mathcal{L}}}(D) \leq n^{\alpha}$ goes to zero as n goes to infinity.*

Proof. – Assume that n is large enough. By [25, Theorem B], the number of divisors D on $\mathcal{X}$ with $h_{\overline{\mathcal{L}}}(D) \leq n^{\alpha}$ is bounded above by $e^{Cn^{2\alpha}}$ for some positive constant C. By Theorem 2.21, we can find positive constants C' and η such that for any D as above, the proportion of those elements s of $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ that vanish on D is bounded above by $C'e^{-n\eta}$.

As a consequence, the proportion of those s that vanish on any D with $h_{\overline{\mathcal{L}}}(\mathcal{X}) \leq n^{\alpha}$ is bounded above by

$$C'e^{Cn^{2\alpha}-n\eta},$$

which goes to zero as n goes to infinity. ☐

5.2. Degree bounds and reduction modulo p

Let $f : \mathcal{X} \to \operatorname{Spec}\mathbb{Z}$ be as above. We want to investigate irreducible divisors on the fibers of f above closed points and derive global consequences. Our goal here is to prove Proposition 5.6.

Since $\mathcal{X}$ is reduced, we can find a non-empty open subset S of $\operatorname{Spec}\mathbb{Z}$ such that the restriction $f_S : \mathcal{X}_S \to S$ has reduced fibers.

Let r be the number of irreducible components of the geometric generic fiber of f. Up to shrinking S, we may assume that if $\overline{s}$ is any geometric point of S, then the number of irreducible components of $\mathcal{X}_{\overline{s}}$ is exactly r. Since $\mathcal{X}_s$ is reduced by assumption, this is equivalent to the fact that the specialization map induces a bijection between the components of $\mathcal{X}_{\overline{\mathbb{Q}}}$ and those of $\mathcal{X}_{\overline{s}}$.

The degree of $\mathcal{L}_{\mathbb{Q}}$ equals rd, where d is the degree of the restriction of $\mathcal{L}$ to a component of $\mathcal{X}_{\overline{\mathbb{Q}}}$. Write $\mathcal{L}_p$ for the restriction of $\mathcal{L}$ to $\mathcal{X}_p$.

If X is a reduced scheme, and C is an irreducible component of X, we will always consider C as a closed subscheme of X, endowed with its reduced structure.

Lemma 5.2. – *Let C be an integral projective curve over a perfect field, with arithmetic genus $p_a(C)$. Let $\mathcal{L}$ be a line bundle on C. Then*

$$h^0(C,\mathcal{L}) \geq 1 - p_a(C) + \deg(\mathcal{L})$$

and equality holds if the degree of $\mathcal{L}$ is strictly bigger than $p_a(C)$.

Proof. – The first statement follows directly from the Riemann-Roch theorem. To prove the second one, consider the normalization $\pi : \widetilde{C} \to C$ of C. Then $\widetilde{C}$ is smooth over the base field k, and its genus is bounded above by the arithmetic genus $p_a(C)$ of C.

Since C is reduced, it is Cohen-Macaulay, so that the dualizing sheaf $\omega_{C/k}$ of C is Cohen-Macaulay by [32, Tag 0BS2]. In particular, it is torsion-free, so that the morphism $\pi^*\omega_{C/k} \to \omega_{\widetilde{C}/k}$ is injective. Now assume that the degree of $\mathcal{L}$ is strictly bigger than $p_a(C)$. In particular, we have

$$\deg(\pi^*\mathcal{L}) > \deg(\omega_{\widetilde{C}/k})$$

and

$$h^1(C,\mathcal{L}) = h^0(C,\mathcal{L}^\vee \otimes_{\mathcal{O}_C} \omega_{C/k}) \leq h^0(\widetilde{C},\pi^*\mathcal{L}^\vee \otimes_{\mathcal{O}_{\widetilde{C}}} \omega_{\widetilde{C}/k}) = 0.$$

By Riemann-Roch, we have

$$h^0(C,\mathcal{L}) = \chi(\mathcal{L}) = 1 - p_a(C) + \deg(\mathcal{L}). \qquad \square$$

Lemma 5.3. – *Let p be a prime number corresponding to a point in S, and let $\overline{\mathbb{F}_p}$ be an algebraic closure of $\mathbb{F}_p$. If C is an irreducible component of $\mathcal{X}_p$, let r_C be the number of irreducible components of $C_{\overline{\mathbb{F}_p}}$, and if k is a positive integer, let $N_k(C)$ be the number of irreducible divisors of degree k on C. Then the following holds as k tends to ∞:*

$$\left|N_{r_C k}(C) - \frac{1}{k}p^{r_C k}\right| = O(p^{\frac{r_C k}{2}}),$$

where the implied constants only depend on $f_S : \mathcal{X}_S \to S$.

Proof. – The r_C irreducible components of $C_{\overline{\mathbb{F}_p}}$ are all defined over $\mathbb{F}_{p^{r_C}}$, and they form a single orbit under Galois. Denote them by $C_1, \dots, C_{r_C}$. The Lang-Weil estimates of [22] give us the inequality, for any positive integer k:

$$\left||C_1(\mathbb{F}_{p^{r_C k}})| - p^{r_C k}\right| = O(p^{\frac{r_C k}{2}}),$$

where the implied constants only depend on the degree of an embedding of C_1 into some projective space—in particular, it only depends on f_S. As a consequence, if M_k is the number of elements in $C_1(\mathbb{F}_{p^{r_C k}})$ with residue field exactly $\mathbb{F}_{p^{r_C k}}$, we have:

$$\left|M_k - p^{r_C k}\right| \leq \textstyle\sum_{i|k, i\neq k} p^{r_C i} + O(\sum_{i|k} p^{\frac{r_C i}{2}}) = O(kp^{\frac{r_C k}{2}}).$$

Now assume that $r_C k$ is strictly larger than the degree of the residue field of any singular point of C—this degree can be bounded independently of C as f_S is generically smooth. Irreducible divisors of degree $r_C k$ on C are in one-to-one correspondence irreducible divisors of degree k on $C_1/\mathbb{F}_{p^{r_C}}$, which in turn are in one-to-one correspondence with Galois orbits over $\mathbb{F}_{p^{r_C}}$ of elements of $C_1(\mathbb{F}_{p^{k r_C}})$ with residue field exactly $\mathbb{F}_{p^{r_C k}}$. As a consequence, we have

$$N_k(C) = \frac{1}{k}M_k,$$

which proves the lemma. $\square$

Lemma 5.4. – *There exists a positive integer N with the following property: for any prime number p corresponding to a point in S, any irreducible component C of $\mathcal{X}_p$, and any $n \geq N$, the restriction map*

$$H^0(\mathcal{X}_p, \mathcal{L}_p^{\otimes n}) \to H^0(C, \mathcal{L}_p^{\otimes n})$$

is surjective.

Proof. – Since the result certainly holds if N is allowed to depend on p by general vanishing results for ample line bundle, we may replace S by any nonempty open subset, which we will do along the proof.

Choose a finite flat map $S' \to S$ such that the irreducible components of the generic fiber of $\mathcal{X}_{S'} \to S$ are geometrically ireducible. In particular, our assumption on s guarantees that the irreducible components of the fiber of $\mathcal{X}_{S'} \to S'$ over any closed point s' are geometrically irreducible, and are the intersection of an irreducible component of $\mathcal{X}_{S'}$ with $\mathcal{X}_{s'}$.

Let s' be a point of S' over p, and let $C_{s'}$ be the union of irreducible components of $\mathcal{X}_{s'}$ corresponding to C. Up to shrinking S, we may assume that $C_{s'}$, as a reduced scheme, is the intersection of $\mathcal{X}_{s'}$ and some union $\mathcal{C}$ of irreducible components of $\mathcal{X}_{S'}$. Let $\mathcal{I}_{\mathcal{C}}$ be the sheaf of ideals on $\mathcal{X}$ defining $\mathcal{C}$. Then the sheaf of ideals defining $C_{s'}$ is $\mathcal{I}_{\mathcal{C}} \otimes_{\mathcal{O}_{\mathcal{X}_{S'}}} \mathcal{O}_{\mathcal{X}_{s'}}$. Note that there are only finitely many possibilities for $\mathcal{C}$.

Let k be a positive integer such that $\mathcal{L}^{\otimes k}$ has a nonzero section. Up to shrinking S, we may assume that this section does not vanish along any component of a fiber of $\mathcal{X}_{S'} \to S'$. Consider the map

$$\pi : \mathcal{X}_{S'} \to S'$$

If n is large enough and since $\mathcal{L}$ is relatively ample, relative vanishing guarantees that the coherent sheaf on S

$$R^1\pi_*(\mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_{\mathcal{X}_{S'}}} \mathcal{I}_{\mathcal{C}})$$

is zero. Pick a positive N once and for all such that the vanishing above holds for $n = N, \dots, N+k-1$. Then after shrinking S once again, we may assume that the vanishing above implies

$$H^1(\mathcal{X}_{s'}, \mathcal{L}^{\otimes N+i} \otimes_{\mathcal{O}_{\mathcal{X}_{S'}}} \mathcal{I}_{C'}) = 0$$

for $i = 0, \dots, k-1$.

Now since $\mathcal{L}^{\otimes k}$ has a nonzero section over $\mathcal{X}_{s'}$, we have an exact sequence, for any integer n,

$$0 \to \mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_{\mathcal{X}_{S'}}} \mathcal{I}_{C'} \to \mathcal{L}^{\otimes n+k} \otimes_{\mathcal{O}_{\mathcal{X}_{S'}}} \mathcal{I}_{C'} \to \mathcal{K} \to 0,$$

where $\mathcal{K}$ is a coherent sheaf supported on a zero-dimensional subscheme of $\mathcal{X}_{s'}$. In particular, the map

$$H^1(\mathcal{X}_{s'}, \mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_{\mathcal{X}_{S'}}} \mathcal{I}_{C'}) \to H^1(\mathcal{X}_{s'}, \mathcal{L}^{\otimes n+k} \otimes_{\mathcal{O}_{\mathcal{X}_{S'}}} \mathcal{I}_{C'})$$

is onto and the right-hand term vanishes as soon as the left-hand one does. Finally, we have found N, independent of C and p, such that for all $n \geq N$, we have

$$H^1(\mathcal{X}_{s'}, \mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_{\mathcal{X}_{S'}}} \mathcal{I}_{C'}) = 0,$$

which implies that the map

$$H^0(\mathcal{X}_p, \mathcal{L}_p^{\otimes n}) \to H^0(C, \mathcal{L}_p^{\otimes n})$$

is surjective. □

Proposition 5.5. – *Let p be a prime number corresponding to a point in S, and let $\mathcal{L}_p$ be the restriction of $\mathcal{L}$ to $\mathcal{X}_p$. Let C be an irreducible component of $\mathcal{X}_p$, and let r_C be the number of irreducible components of $C_{\overline{\mathbb{F}_p}}$.*

Let β be a real number with $0 < \beta < 1$. There exist positive constants A and B, depending only on β and $\mathcal{X}_S \to S$ but not on p, such that for any $n \geq A$, the proportion of those sections s of $H^0(\mathcal{X}_p, \mathcal{L}_p^{\otimes n})$ that do not vanish identically on C and such that $\operatorname{div}(s)$ has an irreducible component of degree at least $r_C(nd - n^{\beta})$ lying on C is at least $Bn^{\beta-1}$.

Proof. – Our assumption on p guarantees that C is reduced. The degree of $\mathcal{L}_p$ on C equals $r_C d$. Let n be a large enough positive integer. Let k be an integer such that $nr_C d \geq r_C k$. Let D be an irreducible divisor of degree $r_C k$ on C. Then the number of sections of $\mathcal{L}_p^{\otimes n}$ over C that vanish on D is equal to the number of sections of $\mathcal{L}_p^{\otimes n}(-D)$ over C, which, according to Lemma 5.2, is bounded below by

$$p^{1-p_a(C)+nr_C d-r_C k}.$$

Assume that $r_C k > \frac{1}{2}nr_C d$. Then a nonzero section of $\mathcal{L}_p^{\otimes n}$ over C vanishes on at most one irreducible divisor of degree $r_C k$. Applying Lemma 5.3, it follows that the number of nonzero sections of $\mathcal{L}_p^{\otimes n}$ over C that vanish on some irreducible divisor of degree $r_C k$ is bounded below by

$$\frac{1}{k}p^{1-p_a(C)+nr_C d}(1 - O(p^{-\frac{1}{2}r_C k})) - \frac{1}{k}p^{r_C k},$$

the last term taking care of the zero section being counted multiple times.

Assume now that

$$r_C k \leq nr_C d - p_a(C).$$

Then the term above is bounded below by

$$\frac{1}{2k}p^{1-p_a(C)+nr_C d}$$

for large enough n.

Summing over all those k such that $r_C k \geq nr_C d - r_C n^{\beta}$, we find that the number of those elements s of $H^0(\mathcal{X}_p, \mathcal{L}_p^{\otimes n})$ such that $\operatorname{div}(s)$ has an irreducible component of degree at least $nr_C d - n^{\beta}$ is at least

$$n^{\beta}\frac{1}{2nd}p^{1-p_a(C)+nr_C d}(1 + o(1)).$$

as n goes to infinity, the implied constants depending only on β, $p_a(C)$ and the ones occurring in Lemma 5.3. Since $p_a(C)$ is the genus of some reunion of irreducible components of the geometric generic fiber of $\mathcal{X}$, the implied constants only depend on β and $\mathcal{X}$.

By Lemma 5.2, if $nr_C d > p_a(C)$, we have

$$h^0(C, \mathcal{L}_p^{\otimes n}) = 1 - p_a(C) + nr_C d.$$

This shows that the proportion of those sections s of $\mathcal{L}_p^{\otimes n}$ over C such that $\operatorname{div}(s)$ has an irreducible component of degree at least $nr_C d - r_C n^{\beta}$ is at least $Bn^{\beta-1}$ for some constant B as in the statement of the proposition.

By Lemma 5.4, after choosing n large enough, this implies the desired statement. □

We can now prove the main result of 5.2.

PROPOSITION 5.6. – *In the situation of 5.1, let β be a real number with $0 < \beta < 1$. Then the proportion of those elements σ of $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that $\mathrm{div}(\sigma)_{\mathbb{Q}}$ has an irreducible component on $\mathcal{X}_{\mathbb{Q}}$ of degree at least $n \deg \mathcal{L}_{\mathbb{Q}} - rn^{\beta}$ goes to 1 as n goes to infinity.*

Proof. – Let γ be a real number with $1 - \beta < \gamma < 1$. Let n be a large enough integer. Letting t be the largest integer smaller than n^{γ}, let $p_1, \dots, p_t$ be the t smallest primes corresponding to points of S, and let N be their product. By the prime number theorem, we have $p_i \sim i \log i$ as $t \to \infty$, so that $p_i \leq 2i \log i$ for large enough i, and, when t is large enough:

$$N \leq (2t \log t)^t = O(e^{n^{\gamma'}}), \tag{5.1}$$

where γ' is any real number with $\gamma < \gamma' < 1$.

Write Λ_n for $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$ and let $\mathcal{X}_N \to \operatorname{Spec} \mathbb{Z}/N\mathbb{Z}$ be the reduction of $\mathcal{X}$ modulo N. The exact sequence defining $\mathcal{X}_N$ is

$$0 \to N\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}_N} \to 0,$$

hence the exact sequence

$$0 \to \Lambda_n/N\Lambda_n \to H^0(\mathcal{X}_N, \mathcal{L}^{\otimes n}) \to H^1(\mathcal{X}, \mathcal{L}^{\otimes n})[N] \to 0.$$

If n is large enough, then $H^1(\mathcal{X}, \mathcal{L}^{\otimes n}) = 0$ and we have

$$H^0(\mathcal{X}_N, \mathcal{L}^{\otimes n}) = \Lambda_n/N\Lambda_n. \tag{5.2}$$

The scheme $\mathcal{X}_N$ is the disjoint union of the $\mathcal{X}_{p_i}$, $1 \leq i \leq t$. As a consequence, we have

$$H^0(\mathcal{X}_N, \mathcal{L}^{\otimes n}) = \prod_{1 \leq i \leq t} H^0(\mathcal{X}_{p_i}, \mathcal{L}^{\otimes n}).$$

Given a prime number p that corresponds to a point of S, let E_p be the subset of $H^0(\mathcal{X}_p, \mathcal{L}^{\otimes n})$ described by Proposition 5.5: E_p is the set of sections s of $H^0(\mathcal{X}_p, \mathcal{L}^{\otimes n})$ such that there exists an irreducible component C of $\mathcal{X}_p$, such that $C_{\overline{\mathbb{F}_p}}$ has r_C irreducible components, the restriction of s to C is not identically zero and vanishes along an irreducible divisor D_p of degree at least $r_C(nd - n^{\beta})$.

By Proposition 5.5, if n is greater than A, the proportion of those elements s of $H^0(\mathcal{X}_N, \mathcal{L}^{\otimes n})$ such that s does not project to E_{p_i} for any $i \in \{1, \dots, t\}$ is bounded above by

$$(1 - Bn^{\beta-1})^t \geq (1 - Bn^{\beta-1})^{n^{\gamma}} = \exp(-Bn^{\gamma+\beta-1} + o(n^{\gamma+\beta-1})) = o(1)$$

since $\gamma + \beta - 1 > 0$, so that as n goes to infinity, the proportion of those elements of $H^0(\mathcal{X}_N, \mathcal{L}^{\otimes n})$ that project to at least one of the E_{p_i} goes to 1.

By Proposition 2.15 which we may apply thanks to (5.1), and by (5.2), the proportion of those elements of $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ that restrict to E_{p_i} for some $i \in \{1, \dots, t\}$ goes to 1 as n goes to infinity. We claim that these elements satisfy the condition of the proposition we are proving.

Let p be a prime number that corresponds to a point of S. Let σ be a section of $\mathcal{L}^{\otimes n}$ over $\mathcal{X}$ such that the restriction of σ to $\mathcal{X}_p$ belongs to E_p. Let C be a component of $\mathcal{X}_p$ such that $C_{\overline{\mathbb{F}_p}}$ has r_C irreducible components, and let D_p be an irreducible divisor of degree at least $r_C(nd - n^{\beta})$ on C such that σ vanishes on D_p.

We can find an irreducible component D of $\mathrm{div}(\sigma)$ with $D_{|\mathcal{X}_p} = D_p$. If n is large enough, we can assume that no component of $(D_p)_{\overline{\mathbb{F}_p}}$ lies on two distinct irreducible components of $C_{\overline{\mathbb{F}_p}}$—it is enough to require that n is large enough compared to the degree of the residue fields of the intersection points of any two components of $C_{\overline{\mathbb{F}_p}}$. As a consequence, the degree of the restriction of D_p to any of the r_C components of $C_{\overline{\mathbb{F}_p}}$ is at least $nd - n^{\beta}$. Since $\mathcal{X}_{\mathbb{Q}}$ is irreducible, the degree of the restriction of $D_{\mathbb{Q}}$ to any component of $\mathcal{X}_{\overline{\mathbb{Q}}}$ is at least $nd - n^{\beta}$ as well, so that the degree of $D_{\mathbb{Q}}$ is at least

$$r(nd - n^{\beta}) = n \deg \mathcal{L}_{\mathbb{Q}} - rn^{\beta}.$$

This is what we needed to prove. □

5.3. End of the proof

We can finish the proof of Theorem 1.6 in the case where $\mathcal{X}$ is an arithmetic surface. We will state this intermediate result in Proposition 5.10 below. The strategy follows roughly the outline of the proof of [11, Proposition 4.1] which deals with the corresponding result over finite fields.

Let $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a resolution of singularities of $\mathcal{X}$. Recall that we denoted by r the number of irreducible components of $\mathcal{X}_{\mathbb{C}}$. Then the complex curve $\widetilde{\mathcal{X}}_{\mathbb{C}}$ is the disjoint union of r smooth, connected components.

Define $\overline{\mathcal{B}} = \pi^*\overline{\mathcal{L}}$. Let ω be the first Chern class of $\overline{\mathcal{B}}$. Then ω is semipositive. We say that a hermitian line bundle on $\widetilde{\mathcal{X}}$ is admissible if it is ω-admissible, and we write $\widehat{\mathrm{Pic}}_{\omega}(\widetilde{\mathcal{X}})$ for the group of isomorphism classes of ω-admissible hermitian line bundles on $\widetilde{\mathcal{X}}$.

We have an exact sequence

$$0 \to \mathbb{R} \to \widehat{\mathrm{Pic}}_{\omega}(\widetilde{\mathcal{X}}) \to \mathrm{Pic}(\widetilde{\mathcal{X}}) \to 0.$$

We fix once and for all a subgroup N of $\widehat{\mathrm{Pic}}_{\omega}(\widetilde{\mathcal{X}})$ such that the following conditions hold:

(i) N is a group of finite type;

(ii) N surjects onto $\mathrm{Pic}(\widetilde{\mathcal{X}})$ and contains the class of $\overline{\mathcal{B}}$;

(iii) $N \cap \mathrm{Ker}(\widehat{\mathrm{Pic}}_{\omega}(\widetilde{\mathcal{X}}) \to \mathrm{Pic}(\widetilde{\mathcal{X}}))$ has rank 1.

Such a group N certainly exists since $\mathrm{Pic}(\widetilde{\mathcal{X}})$ is a group of finite type. Note that these conditions mean that N is a discrete cocompact subgroup of $\widehat{\mathrm{Pic}}_{\omega}(\widetilde{\mathcal{X}})$. In particular, there exists a positive constant C such that for any admissible hermitian line bundle $\overline{\mathcal{M}} = (\mathcal{M}, ||.||)$ on $\widetilde{\mathcal{X}}$, there exists a hermitian metric $||.||'$ on $\mathcal{M}$ such that $(\mathcal{M}, ||.||')$ belongs to N and the norms $||.||$ and $||.||'$ satisfy the inequality

$$C^{-1}||.|| \leq ||.||' \leq C||.||. \tag{5.3}$$

The following result is classical in the geometric setting: big divisors are (rationally) the sum of ample divisors and effective divisors.

LEMMA 5.7. – *The hermitian line bundle $\overline{\mathcal{B}}$ satisfies the conditions of Proposition 4.7. Furthermore, there exists a positive integer k, and line bundles $\overline{\mathcal{A}}$ and $\overline{\mathcal{E}}$ on $\widetilde{\mathcal{X}}$ which are ample and effective respectively, such that*

$$\overline{\mathcal{B}}^{\otimes k} \simeq \overline{\mathcal{A}} \otimes \overline{\mathcal{E}}.$$

Proof. – Since $\overline{\mathcal{L}}$ is ample, some power of $\overline{\mathcal{L}}$ is effective, and so is the same power of $\overline{\mathcal{B}}$. We also have $\overline{\mathcal{B}}.\overline{\mathcal{B}} = \overline{\mathcal{L}}.\overline{\mathcal{L}} > 0$. Finally, let $\overline{\mathcal{M}}$ be an effective line bundle on $\widetilde{\mathcal{X}}$, let s be a nonzero effective section of $\overline{\mathcal{M}}$, and let D be the divisor of s. Then

$$\overline{\mathcal{B}}.\overline{\mathcal{M}} = h_{\overline{\mathcal{B}}}(D) - \int_{\widetilde{\mathcal{X}}(\mathbb{C})} \log ||s_{\mathbb{C}}|| \pi^* c_1(\overline{\mathcal{L}}).$$

Considering an effective section of some power of $\overline{\mathcal{B}}$ that does not vanish along any component of D—which exists since large powers of $\overline{\mathcal{L}}$ are generated by their effective sections—we see that first term is nonnegative. The second one is nonnegative as well since $c_1(\overline{\mathcal{L}})$ is semipositive on $\mathcal{X}$. This shows the first statement of the proposition.

Let $\overline{\mathcal{A}}$ be an effective ample line bundle on $\widetilde{\mathcal{X}}$, let σ be a section of $\overline{\mathcal{A}}$ and let H be the divisor of σ. Let H' be the schematic image $\pi(H)$. Since $\mathcal{L}$ is ample, we can find an integer k_1 and a nonzero section s_1 of $\mathcal{L}^{\otimes k_1}$ that vanishes on H'. We can write

$$\pi^* s_1 = \sigma \sigma_1,$$

where σ_1 is a section of $\mathcal{B}^{\otimes k_1} \otimes \mathcal{A}^{\otimes -1}$. Choose a large enough integer k_2, and let s_2 be a nonzero section of $\mathcal{L}^{\otimes k_2}$ with small enough norm. Writing $\sigma_2 = \pi^* s_2$, we have

$$\pi^*(s_1 s_2) = \sigma \sigma_1 \sigma_2,$$

and $\sigma_1\sigma_2$ is an effective section of the hermitian line bundle $\overline{\mathcal{B}}^{\otimes(k_1+k_2)} \otimes \overline{\mathcal{A}}^{\otimes -1}$, which proves the result. □

Let α and β be real numbers with $0 < \beta < \alpha < \frac{1}{2}$. If n is a positive integer, let H'_n be the subset of $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ consisting of those effective sections σ of $\overline{\mathcal{L}}^{\otimes n}$ such that:

(i) σ does not vanish on any Weil divisor D of $\mathcal{X}$ with $h_{\overline{\mathcal{L}}}(D) \leq n^{\alpha}$;

(ii) there exists an irreducible component D of $\mathrm{div}(\sigma)$ such that

$$\deg(D_{\mathbb{Q}}) \geq n \deg \mathcal{L}_{\mathbb{Q}} - r n^{\beta}.$$

Use Lemma 5.7 to find a positive integer k with

$$\overline{\mathcal{B}}^{\otimes k} \simeq \overline{\mathcal{A}} \otimes \overline{\mathcal{E}},$$

where $\overline{\mathcal{A}}$ is ample and $\overline{\mathcal{E}}$ is effective.

LEMMA 5.8. – *The set* $\bigcup_{n>0} H'_n$ *has density* 1 *in* $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$.

Proof. – This is a direct consequence of Proposition 5.1 and Proposition 5.6. □

LEMMA 5.9. – *Let* δ, γ *be any real numbers with* $0 < \beta < \gamma < \delta < \alpha$. *Let n be a large enough integer, and let σ be an element of H'_n such that* $\mathrm{div}(\sigma)$ *is not irreducible. Then we can find hermitian line bundles $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$ on $\widetilde{\mathcal{X}}$, and sections*

$$\sigma_i \in H^0(\widetilde{\mathcal{X}}, \mathcal{L}_i),$$

$i = 1, 2$*, with the following properties:*

(i) $\overline{\mathcal{L}}_1$ *and* $\overline{\mathcal{L}}_2$ *belong to* N*;*

(ii) $||\sigma_i||_{\infty} \leq e^{n^{\gamma}}, i = 1, 2$*;*

(iii) $n^{\delta} \leq \overline{\mathcal{L}}_1.\overline{\mathcal{B}} \leq n\overline{\mathcal{B}}.\overline{\mathcal{B}} - n^{\delta}$*;*

(iv) $\overline{\mathcal{L}}_1.\overline{\mathcal{A}} \leq kn\,\overline{\mathcal{B}}.\overline{\mathcal{B}}$;

(v) $\overline{\mathcal{L}}_1 \otimes \overline{\mathcal{L}}_2 \simeq \overline{\mathcal{B}}^{\otimes n}$;

(vi) *up to the isomorphism above,* $\sigma_1\sigma_2 = \sigma$.

Proof. – Let D be the divisor of $\pi^*\sigma$. Since the divisor of σ is not irreducible, D is not irreducible either and we can write

$$D = D_1 + D_2,$$

where the D_i are nonzero effective divisors on the regular scheme $\widetilde{\mathcal{X}}$ such that both Weil divisors $\pi_*(D_1)$ and $\pi_*(D_2)$ are nonzero. Since $\operatorname{div}(\sigma)$ has an irreducible component of generic degree bounded below by $n \deg \mathcal{L}_{\mathbb{Q}} - rn^{\beta}$, and since

$$\deg D_{1,\mathbb{Q}} + \deg D_{2,\mathbb{Q}} = n \deg \mathcal{L}_{\mathbb{Q}}, \tag{5.4}$$

we can assume, up to exchanging D_1 and D_2,

$$n \deg \mathcal{L}_{\mathbb{Q}} - rn^{\beta} \leq \deg D_{1,\mathbb{Q}} \leq n \deg \mathcal{L}_{\mathbb{Q}}, \tag{5.5}$$

$$0 \leq \deg D_{2,\mathbb{Q}} \leq rn^{\beta}. \tag{5.6}$$

We can also assume that no component of D_1 is contracted by the morphism $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$—simply by replacing D_2 by the sum of D_2 and all those contracted components of D_1, which are all supported above closed points of $\operatorname{Spec}\mathbb{Z}$. Let $\mathcal{L}_i$ be the line bundle $\mathcal{O}_{\widetilde{\mathcal{X}}}(D_i)$ for $i = 1, 2$. Then we can identify $\mathcal{L}_1 \otimes \mathcal{L}_2$ with $\mathcal{B}^{\otimes n}$, and we can find sections σ_i of $\mathcal{L}_i$ with $\operatorname{div}(\sigma_i) = D_i$ such that $\sigma = \sigma_1\sigma_2$.

Recall that we defined ω as $c_1(\overline{\mathcal{B}})$. We consider the norms $||.||_0$ with respect to ω. Consider the unique hermitian metric $||.||_{\sigma_2}$ on $\mathcal{L}_2$ which is admissible with respect to ω, scaled so that

$$||\sigma_2||_{\sigma_2,0} = 1.$$

By (5.3), we can find a metric $||.||$ on $\mathcal{L}_2$ such that $\overline{\mathcal{L}}_2 := (\mathcal{L}_2, ||.||)$ belongs to N and

$$C^{-1} \leq ||\sigma_2||_0 \leq C. \tag{5.7}$$

Endow $\mathcal{L}_1$ with the unique hermitian metric such that

$$\overline{\mathcal{B}}^{\otimes n} = \overline{\mathcal{L}}_1 \otimes \overline{\mathcal{L}}_2$$

as hermitian line bundles on $\widetilde{\mathcal{X}}$, where we write $\overline{\mathcal{L}}_1$ for the induced hermitian line bundles. Since $\overline{\mathcal{B}}$ belongs to N by assumption, so do $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$. This makes sure that conditions (i), (v) and (vi) of the lemma are satisfied.

Since $||\sigma||_\infty \leq 1$, we have

$$||\sigma_2||_0 \leq C||\sigma_1|||_0||\sigma_2||_0 = C||\sigma||_0 \leq C. \tag{5.8}$$

The inequalities (5.4), (5.6) imply, via Proposition 4.4 the following estimate, since $||\sigma_2||_0 \geq C^{-1}$ and $||\sigma||_\infty \leq 1$:

$$||\sigma_1||_\infty \leq C^{-1}(nC_2 \deg \mathcal{L}_{\mathbb{Q}})^{rn\beta}$$

for some constant C_2 depending only on $\mathcal{X}$ and $\overline{\mathcal{L}}$. Similarly, (5.6) and Proposition 4.2 give us, for some constant C_2 depending only on $\mathcal{X}$ and $\overline{\mathcal{L}}$:

$$||\sigma_2||_\infty \leq CC_1^{rn^{\beta}}.$$

For any $\gamma > \beta$, and any n large enough, this ensures that condition (ii) is satisfied.

We now turn to condition (iii). For $i = 1, 2$, choose a nonzero effective section s_i of some power $\overline{\mathcal{L}}^{\otimes \ell}$ of $\overline{\mathcal{L}}$ such that the divisor of $\pi^* s_i$ has no common component with D_i. Computing the height $h_{\overline{\mathcal{B}}^{\otimes \ell}}(D_i)$ using the section $\pi^* s_i$ of $\overline{\mathcal{B}}^{\otimes \ell}$, we get:

$$h_{\overline{\mathcal{B}}^{\otimes \ell}}(D_i) = h_{\overline{\mathcal{L}}^{\otimes \ell}}(\pi_*(D_i)).$$

and

$$h_{\overline{\mathcal{B}}^{\otimes \ell}}(D_i) = \ell\, h_{\overline{\mathcal{L}}^{\otimes \ell}}(\pi_*(D_i)) \geq \ell n^{\alpha}.$$

Write

$$\ell \overline{\mathcal{L}}_i.\overline{\mathcal{B}} = h_{\overline{\mathcal{B}}^{\otimes \ell}}(D_i) - \ell \int_{\mathcal{X}(\mathbb{C})} \log ||\sigma_i|| \omega \geq \ell n^{\alpha} - \ell \int_{\mathcal{X}(\mathbb{C})} \log ||\sigma_i|| \omega$$

and use $\log ||\sigma_i||_{\infty} \leq n^{\gamma}$. We find

$$\overline{\mathcal{L}}_i.\overline{\mathcal{B}} \geq n^{\alpha} - n^{\gamma} \deg \mathcal{L}_{\mathbb{Q}} \geq n^{\delta} \tag{5.9}$$

for any large enough n since $\delta, \gamma < \alpha$. Since

$$\overline{\mathcal{L}}_1.\overline{\mathcal{B}} + \overline{\mathcal{L}}_2.\overline{\mathcal{B}} = n \overline{\mathcal{B}}.\overline{\mathcal{B}},$$

this proves that (iii) holds.

Let us prove condition (iv). Since $\overline{\mathcal{B}}^{\otimes k}$ is isomorphic to $\overline{\mathcal{A}} \otimes \overline{\mathcal{E}}$, we have

$$\overline{\mathcal{L}}_i.\overline{\mathcal{A}} = k \overline{\mathcal{L}}_i.\overline{\mathcal{B}} - \overline{\mathcal{L}}_i.\overline{\mathcal{E}}$$

for $i = 1, 2$, so that

$$\overline{\mathcal{L}}_1.\overline{\mathcal{A}} = k \overline{\mathcal{L}}_1.\overline{\mathcal{B}} - \overline{\mathcal{L}}_1.\overline{\mathcal{E}} = kn \overline{\mathcal{B}}.\overline{\mathcal{B}} - k \overline{\mathcal{L}}_2.\overline{\mathcal{B}} - \overline{\mathcal{L}}_1.\overline{\mathcal{E}}. \tag{5.10}$$

Let τ be a nonzero effective section of $\overline{\mathcal{E}}$, with divisor D_{τ}. Then we have

$$\overline{\mathcal{L}}_1.\overline{\mathcal{E}} = h_{\overline{\mathcal{L}}_1}(D_{\tau}) - \int_{\widetilde{\mathcal{X}}(\mathbb{C})} \log ||\tau|| c_1(\overline{\mathcal{L}}_1).$$

Since the degree of $\overline{\mathcal{L}}_1$ is nonnegative, the form $c_1(\overline{\mathcal{L}}_1)$ is a nonnegative multiple of ω, and since τ is effective, we have

$$-\int_{\widetilde{\mathcal{X}}(\mathbb{C})} \log ||\tau|| c_1(\overline{\mathcal{L}}_1) \geq 0.$$

By assumption, no component of the divisor D_1 of σ_1 is contracted by the resolution π. Furthermore, the definition of the set H'_n guarantees that if C is any component of D_1, then the height of $\pi_*(C)$ with respect to $\overline{\mathcal{L}}$ is bounded below by n^{α}. This implies that if n is large enough, the divisors D_1 and D_{τ} have no component in common, so that

$$h_{\overline{\mathcal{L}}_1}(D_{\tau}) \geq -\deg D_{\tau,\mathbb{Q}} \log ||\sigma_1|| \geq -\deg D_{\tau,\mathbb{Q}} n^{\gamma}$$

and, as a consequence,

$$\overline{\mathcal{L}}_1.\overline{\mathcal{E}} \geq -n^{\gamma} \deg \mathcal{E}_{\mathbb{Q}}. \tag{5.11}$$

Putting the inequalities (5.11) and (5.9) together with (5.10), we obtain

$$\overline{\mathcal{L}}_1.\overline{\mathcal{A}} \leq kn\, \overline{\mathcal{B}}.\overline{\mathcal{B}} + n^{\gamma} \deg \mathcal{E}_{\mathbb{Q}} - k\, n^{\delta}.$$

Since $\overline{\mathcal{L}}$ is ample, $\overline{\mathcal{B}}.\overline{\mathcal{B}} = \overline{\mathcal{L}}.\overline{\mathcal{L}}$ is positive, and since $\gamma < \delta$, this shows that condition (iv) of the lemma is satisfied as soon as n is large enough. □

We can finally prove the key result of this paper via a counting argument.

Proposition 5.10. – *Let $\mathcal{X}$ be an integral projective arithmetic surface, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Then the set*

$$\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}),\ \mathrm{div}(\sigma) \text{ is irreducible}\}$$

has density 1.

Proof. – Choose δ and γ with $\beta < \gamma < \delta < \alpha$. Lemma 5.8 shows that the set $\bigcup_{n>0} H'_n$ has density 1 in $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$, so that we only have to prove that the set of those σ in $\bigcup_{n>0} H'_n$ with reducible divisor has density 0 in $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$. Let Z_n be this set.

Let n be large enough so that Lemma 5.9 applies. To any σ in Z_n, we can associate hermitian line bundles $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$, together with respective sections σ_1 and σ_2, so that the conditions $(i)-(vi)$ of the lemma hold. Since $\sigma = \sigma_1\sigma_2$, the data of the $\overline{\mathcal{L}}_i$ and σ_i for $i = 1, 2$ determine σ.

We will give an upper bound for the number of elements σ in Z_n by estimating the number of possible $\overline{\mathcal{L}}_i$ and σ_i. In other words, we will count the number of triples $(\overline{\mathcal{L}}_1, \sigma_1, \sigma_2)$, where $\overline{\mathcal{L}}_1$ is a hermitian line bundle on $\widetilde{\mathcal{X}}$, σ_1 is a section of $\mathcal{L}_1$, and, setting $\overline{\mathcal{L}}_2 := \overline{\mathcal{B}}^{\otimes n} \otimes \overline{\mathcal{L}}_1^{\otimes -1}$, σ_2 is a section of $\overline{\mathcal{L}}_2$, so that

(i) $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$ belong to N;

(ii) $||\sigma_i|| \leq e^{n^\gamma}, i = 1, 2$;

(iii) $n^\delta \leq \overline{\mathcal{L}}_1.\overline{\mathcal{B}} \leq n\overline{\mathcal{B}}.\overline{\mathcal{B}} - n^\delta$;

(iv) $\overline{\mathcal{L}}_1.\overline{\mathcal{A}} \leq kn\overline{\mathcal{B}}.\overline{\mathcal{B}}$.

Below, when using the O notations, implied constants only depend on $\widetilde{\mathcal{X}} \to \mathcal{X}, \overline{\mathcal{L}}, \overline{\mathcal{A}}, \alpha, \beta, \delta, \gamma$.

Let $\overline{\mathcal{L}}_1$ be a hermitian line bundle as above, and write $i := \overline{\mathcal{L}}_1.\overline{\mathcal{B}}$, so that

$$n^\delta \leq i \leq n\overline{\mathcal{B}}.\overline{\mathcal{B}} - n^\delta.$$

We want to bound the number of sections of $\mathcal{L}_1$ that have norm at most e^{n^γ}, that is, the number of effective sections of $\overline{\mathcal{L}}_1(n^\gamma)$. First remark that $\deg \mathcal{L}_{1,\mathbb{Q}} \leq n \deg \mathcal{B}_{\mathbb{Q}}$ as the degree of $\mathcal{L}_{1,\mathbb{Q}}$ and $\mathcal{L}_{2,\mathbb{Q}}$ are both nonnegative and have sum $n \deg \mathcal{B}_{\mathbb{Q}}$. Furthermore, we have

$$\overline{\mathcal{L}}_1(n^\gamma).\overline{\mathcal{B}} = i + n^\gamma \overline{\mathcal{O}}_{\widetilde{\mathcal{X}}}(1).\overline{\mathcal{B}} = O(n)$$

since $\gamma < \delta < 1$.

Corollary 4.10 gives us

$$h^0_{\mathrm{Ar}}(\widetilde{\mathcal{X}}, \overline{\mathcal{L}}_1(n^\gamma)) \leq \frac{(i + Kn^\gamma)^2}{2\overline{\mathcal{B}}.\overline{\mathcal{B}}} + O(n \log n), \tag{5.12}$$

where K is the constant $\overline{\mathcal{O}}_{\widetilde{\mathcal{X}}}(1).\overline{\mathcal{B}}$.

Similarly, we have

$$h^0_{\mathrm{Ar}}(\widetilde{\mathcal{X}}, \overline{\mathcal{L}}_2(n^\gamma)) \leq \frac{(n\overline{\mathcal{B}}.\overline{\mathcal{B}} - i + Kn^\gamma)^2}{2\overline{\mathcal{B}}.\overline{\mathcal{B}}} + O(n \log n). \tag{5.13}$$

Adding (5.12) and (5.13), we find, recalling that $0 < \gamma < 1$:

$$\begin{aligned} h^0_{\mathrm{Ar}}(\widetilde{\mathcal{X}}, \overline{\mathcal{L}}_1(n^\gamma)) + h^0_{\mathrm{Ar}}(\widetilde{\mathcal{X}}, \overline{\mathcal{L}}_2(n^\gamma)) &\leq \frac{1}{2}n^2\overline{\mathcal{B}}.\overline{\mathcal{B}} - \frac{2i(n\overline{\mathcal{B}}.\overline{\mathcal{B}} - i)}{2\overline{\mathcal{B}}.\overline{\mathcal{B}}} \\ &\quad + \frac{2K^2n^{2\gamma} + 2K\overline{\mathcal{B}}.\overline{\mathcal{B}}n^{1+\gamma}}{2\overline{\mathcal{B}}.\overline{\mathcal{B}}} + O(n\log n) \\ &\leq \frac{1}{2}n^2\overline{\mathcal{B}}.\overline{\mathcal{B}} - \frac{i(n\overline{\mathcal{B}}.\overline{\mathcal{B}} - i)}{\overline{\mathcal{B}}.\overline{\mathcal{B}}} + O(n^{1+\gamma}). \end{aligned}$$

Since $n^\delta \leq i \leq n\overline{\mathcal{B}}.\overline{\mathcal{B}} - n^\delta$, we have

$$\frac{i(n\overline{\mathcal{B}}.\overline{\mathcal{B}} - i)}{\overline{\mathcal{B}}.\overline{\mathcal{B}}} \geq n^{1+\delta} - \frac{1}{\overline{\mathcal{B}}.\overline{\mathcal{B}}}n^{2\delta}$$

and, since $2\delta < 1 < 1 + \gamma$,

$$h^0_{\mathrm{Ar}}(\widetilde{\mathcal{X}}, \overline{\mathcal{L}}_1(n^\gamma)) + h^0_{\mathrm{Ar}}(\widetilde{\mathcal{X}}, \overline{\mathcal{L}}_2(n^\gamma)) \leq \frac{1}{2}n^2\overline{\mathcal{B}}.\overline{\mathcal{B}} - n^{1+\delta} + O(n^{1+\gamma}).$$

We now count the number of possible $\overline{\mathcal{L}}_1$. Let $t > 0$ be such that $\overline{\mathcal{O}}(t)$ belong to N. Let $k(n)$ be the smallest positive integer such that $k(n)t \geq n^\gamma$. Then the hermitian line bundle $\overline{\mathcal{L}}_1(k(n)t)$ is effective, belongs to N, and we have

$$\overline{\mathcal{L}}_1(k(n)t).\overline{\mathcal{A}} = O(n)$$

since $\gamma < 1$. As a consequence of Proposition 4.13, this shows that the number of possible $\overline{\mathcal{L}}_1$—or equivalently, $\overline{\mathcal{L}}_1(k(n)t)$—appearing in the triples above is bounded by $O(n^\rho)$, where ρ is the rank of N.

The estimates above show that we have the following inequality:

$$\log|Z_n| \leq O(\rho\log n) + \frac{1}{2}n^2\overline{\mathcal{B}}.\overline{\mathcal{B}} - n^{1+\delta} + O(n^{1+\gamma}) = \frac{1}{2}n^2\overline{\mathcal{B}}.\overline{\mathcal{B}} - n^{1+\delta} + O(n^{1+\gamma}).$$

However, Theorem 2.11, (iii) shows that we have

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}) \geq \frac{1}{2}n^2\overline{\mathcal{L}}.\overline{\mathcal{L}} + O(n\log n) = \frac{1}{2}n^2\overline{\mathcal{B}}.\overline{\mathcal{B}} + O(n\log n).$$

Since $\delta > \gamma$, these two inequalities prove that $\bigcup_{n>0} Z_n$ has density 0 in $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$, which proves the proposition. □

6. Proofs of the main results

The goal of this section is to give a proof of Theorem 1.1. We will deduce it from its special case Theorem 1.6

6.1. Proof of Theorem 1.6

We first state the Bertini irreducibility theorem of [11] in the form that we will need.

THEOREM 6.1. – *Let k be a finite field, and let X be a projective variety over k. Let L be an ample line bundle over X. Let Y be an integral scheme of finite type over k, and let $f : Y \to X$ be a morphism which is generically smooth onto its image. Assume that the dimension of the closure of $f(Y)$ is at least 2. Then the set of those $\sigma \in \bigcup_{n>0} H^0(X, \mathcal{L}^{\otimes n})$ such that $\operatorname{div}(f^*\sigma)_{\mathrm{horiz}}$ is an irreducible Cartier divisor has density 1.*

Proof. – This is almost a special case of [11, Theorem 1.6]. There, the result is given when X is a projective space and $L = \mathcal{O}(1)$. This means that—unfortunately—[11] can formally only be applied to the situation where L is very ample. However, the proofs of [11] apply with no change when projective space is replaced by an arbitrary projective scheme with a distinguished ample line bundle.

A second difference between our statement and that of [11, Corollary 1.4] is that we claim that we can require $\operatorname{div}(f^*\sigma)$ to be irreducible as a Cartier divisor: the underlying scheme is irreducible and has no multiple component, whereas the statement in [11] only states irreducibility.

The fact that for a density 1 of σ, the divisor $\operatorname{div}(\sigma)$ has no multiple component follows from arguments in [11]. Indeed, since Y is reduced and k is perfect, there is a dense open subset U of Y that is smooth over k and such that $f_{|U}$ is smooth onto its image. By [11, Lemma 3.3], for a density 1 of sections σ, all the components of $\operatorname{div}(f^*\sigma)_{\mathrm{horiz}}$ intersect U, and by [11, Lemma 3.5], for a density 1 of σ, the intersection $\operatorname{div}(f^*\sigma) \cap U$ is smooth outside a finite number of points, so that it does not have any multiple component. □

LEMMA 6.2. – *Let $\mathcal{X}$ be a projective arithmetic variety of dimension at least 2, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Then the set*

$$\left\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}), \operatorname{div}(\sigma) \text{ has no vertical component}\right\}$$

has density 1.

Proof. – If $\mathcal{X}$ is an arithmetic surface, the result follows from Proposition 5.10. Let d be the relative dimension of $\mathcal{X}$ over $\operatorname{Spec}\mathbb{Z}$, and assume that $d \geq 2$.

Apply Theorem 2.21 where Y runs through the irreducible components of the fibers of $\mathcal{X}$ over closed points of $\operatorname{Spec}\mathbb{Z}$. Since these components have dimension d, we find that for any small enough $\varepsilon > 0$, the proportion of these elements σ of $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that $\operatorname{div}(\sigma)$ has a vertical component over some prime p with $p \leq \exp(\varepsilon n^2)$ is bounded above by a quantity of the form

$$O(\exp(\varepsilon n^2 - \eta n^d)) = o(1),$$

as n goes to infinity.

We now show that for most $\sigma \in H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$, $\operatorname{div}(\sigma)$ does not have any vertical component above a large prime.

Let $C \subset \mathcal{X}$ be a closed arithmetic curve, flat over $\operatorname{Spec}\mathbb{Z}$, such that for any large enough prime p, the intersection of C with any irreducible component of the fiber $\mathcal{X}_p$ of $\mathcal{X}$ above p is nonempty. Let n be a positive integer, and let σ be an element of $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$. If $\operatorname{div}(\sigma)$

does not contain C, and if it has a vertical component above a prime p, then $\operatorname{div}(\sigma)$ and C intersect at a point above p, so that

$$nh_{\overline{\mathcal{L}}}(C) = h_{\overline{\mathcal{L}}^{\otimes n}}(C) \geq \log p.$$

In particular, for such a σ, we have $p \leq \exp(nh_{\overline{\mathcal{L}}}(C))$.

By Theorem 2.21, the proportion of those $\sigma \in H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ that vanish on C tends to 0 as n tends to infinity. In particular, the proportion of those $\sigma \in H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that $\operatorname{div}(\sigma)$ has a vertical component above a prime $p > \exp(nh_{\overline{\mathcal{L}}}(C))$ goes to 0 as n goes to infinity.

Together with the above estimate, this shows the result. □

Proof of Theorem 1.6. – If $\mathcal{X}$ is an arithmetic surface, then the result was proved in Proposition 5.10. Assume that $\mathcal{X}$ has dimension at least 3. Let p be a prime number large enough so that $\mathcal{X}_p$ is reduced, and specialization indices a bijection between the irreducible components of $\mathcal{X}_{\overline{\mathbb{Q}}}$ and those of $\mathcal{X}_{\overline{\mathbb{F}_p}}$. Let $\mathcal{X}_{0,p}$ be an irreducible component of $\mathcal{X}_p$, endowed with the reduced structure.

Let n be a positive integer, and let σ be a global section of $\mathcal{L}^{\otimes n}$. If D is a horizontal component of $\operatorname{div}(\sigma)$, then D intersects all components of $\mathcal{X}_{\overline{\mathbb{Q}}}$, so that D intersects $\mathcal{X}_{0,p}$. This shows that for any section σ of $\mathcal{L}^{\otimes n}$, if $\operatorname{div}(\sigma_{|\mathcal{X}_{0,p}})$ is irreducible as a Weil divisor, then $\operatorname{div}(\sigma)$ has a single component that is flat over $\mathbb{Z}$.

Now we have the following results:

(i) the density of those $\sigma_p \in \bigcup_{n>0} H^0(\mathcal{X}_{0,p}, \mathcal{L}^{\otimes n})$ such that $\operatorname{div}(\sigma_p)$ is an irreducible Cartier divisor is 1;

(ii) the density of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that $\operatorname{div}(\sigma)$ does not have a vertical component is 1.

Indeed, (i) follows from Theorem 6.1 with $X = Y$, and (ii) is Lemma 6.2. By the discussion above, if σ satisfies (i) and (ii), then $\operatorname{div}(\sigma)$ is irreducible. Finally, Corollary 2.18 shows that the density of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that the restriction of σ to $\mathcal{X}_{0,p}$ satisfies (i) is 1. This proves the result. □

6.2. Proof of Theorem 1.1

In this section, we deduce Theorem 1.4 from Theorem 1.6, following the arguments of [11, Section 5]. We then prove Theorem 1.1 as a consequence.

In the following, fix a projective arithmetic variety $\mathcal{X}$, together with an ample hermitian line bundle $\overline{\mathcal{L}}$.

LEMMA 6.3. – *Let $\mathcal{Y}$ be an irreducible scheme of finite type over* $\operatorname{Spec}\mathbb{Z}$*, together with a morphism $f : \mathcal{Y} \to \mathcal{X}$. Let U be an open dense subscheme of $\mathcal{Y}$. Then for all σ in a density* 1 *subset of $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$, we have the equivalence*

$$\operatorname{div}(f^*\sigma)_{\mathrm{horiz}} \text{ is irreducible} \Leftrightarrow (\operatorname{div}(f^*\sigma) \cap U)_{\mathrm{horiz}} \text{ is irreducible.}$$

Proof. – This is analogous to [11, Lemma 3.3]. The implication

$$\mathrm{div}(f^*\sigma)_{\mathrm{horiz}} \text{ is irreducible} \implies (\mathrm{div}(f^*\sigma) \cap U)_{\mathrm{horiz}} \text{ is irreducible}$$

always holds. We prove the reverse implication.

Let D be an irreducible component of $\mathcal{Y} \setminus U$ whose image under f is positive-dimensional—meaning by definition that D is a component of $(\mathcal{Y}\setminus U)_{\mathrm{horiz}}$. By Theorem 2.21, the density of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ that vanish identically on $f(D)$ is zero.

Now assume that σ does not vanish identically along any component of $(\mathcal{Y} \setminus U)_{\mathrm{horiz}}$—this is a condition satisfied by a density 1 set of sections by the paragraph above. Then any horizontal component of $\mathrm{div}(f^*\sigma)_{\mathrm{horiz}}$ meets U, which implies that the Zariski closure of $(\mathrm{div}(f^*\sigma) \cap U)_{\mathrm{horiz}}$ is $\mathrm{div}(f^*\sigma)_{\mathrm{horiz}}$.

In particular, for those σ, the implication

$$(\mathrm{div}(f^*\sigma) \cap U)_{\mathrm{horiz}} \text{ is irreducible} \implies \mathrm{div}(f^*\sigma)_{\mathrm{horiz}} \text{ is irreducible}$$

holds. □

LEMMA 6.4. – *Let $\mathcal{Y}$ and $\mathcal{Z}$ be two irreducible schemes that are flat, of finite type over* Spec $\mathbb{Z}$. *Let*

$$\pi : \mathcal{Y} \to \mathcal{Z}$$

be a finite étale morphism, and let

$$\psi : \mathcal{Z} \to \mathcal{X}$$

be a morphism that has relative dimension s at all points of $\mathcal{Z}$. Assume that the dimension of the closure of $\psi(\mathcal{Z})$ in $\mathcal{X}$ is at least 2. *Then for all σ in a density* 1 *subset of $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$, we have the implication*

$$\mathrm{div}(\psi^*\sigma) \text{ is irreducible} \implies \mathrm{div}(\pi^*\psi^*\sigma) \text{ is irreducible.}$$

Proof. – We follow the argument of [11, Lemma 5.1]. Irreducibility is more difficult to achieve if we replace $\mathcal{Y}$ by a finite cover. As a consequence, we may assume that π is a Galois étale cover. Let G be the corresponding Galois group. Let m be the dimension of $\overline{\psi(\mathcal{Z})}$.

If z is a closed point of $\mathcal{Z}$, let $|z|$ be the cardinality of the residue field of z and let F_z denote the conjugacy class in G associated to the Frobenius. We claim that for a density 1 set of σ, the conjugacy classes F_z cover all conjugacy classes of G as z runs through the closed points of $\mathrm{div}(\psi^*\sigma)$.

Indeed, let C be such a conjugacy class. Let U be a normal, dense affine open subset of $\mathcal{Z}$. By the Chebotarev density theorem of [31, Theorem 9.11] applied to $\pi^{-1}(U) \to U$, the number of closed points z of U with $|z| \leq t$ and $F_z = C$ is equivalent to

$$\frac{|C|}{|G|}\frac{t^{s+m}}{(s+m)\log t}$$

as t tends to ∞. Let $E_{C,t}$ be the set of those z.

By the Lang-Weil estimates, since the fibers of ψ have all dimension s, the number of points z with $|z| \leq t$ in a given fiber of ψ above a closed point is bounded above by a quantity of the form

$$\alpha t^s,$$

for some positive α, so that $|\psi(E_{C,t})|$ is bounded below by a quantity of the form

$$\beta\frac{t^m}{\log t}$$

for some positive β. Note that if $x \in \psi(E_{C,t})$, then $|x| \leq t$.

Fix t large enough. Theorem 2.17 shows that the density of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ that do not vanish on any element of $\psi(E_{C,t})$ is equal to

$$\Pi_{x\in\psi(E_{C,t})}(1-|x|^{-1}) \leq (1-t^{-1})^{\beta t^m/\log t} = \exp\Big(-\beta\frac{t^{m-1}}{\log t}(1+o(1))\Big),$$

which tends to zero as t tends to ∞ since $m \geq 2$. As a consequence, the density of those σ such that $\psi^*\sigma$ vanishes at a closed point z with $F_z = C$ is 1, which proves the claim.

Now let $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that $\mathrm{div}(\psi^*\sigma)$ is irreducible and contains closed points z such that the F_z cover all conjugacy classes of G. Then $\pi^{-1}\mathrm{div}(\psi^*\sigma) = \mathrm{div}(\pi^*\psi^*\sigma)$ is irreducible. This proves the lemma. □

Proof of Theorem 1.4. – We follow the argument of [11, Lemma 5.2]. By Lemma 6.3, we can replace $\mathcal{Y}$ by any dense open subscheme. As a consequence, we can assume that f factors as

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\pi"] & \mathcal{Z} \arrow[r, "\psi"] & \mathcal{X}, \end{tikzcd}$$

where π is finite étale, $\mathcal{Z}$ is an open subset of some affine space $\mathbb{A}^s_{\mathcal{X}}$ and ψ is the projection onto $\mathcal{X}$—indeed, the function field of $\mathcal{Y}$ is a finite separable extension of a purely transcendental extension of the function field of $\mathcal{X}$.

By Lemma 6.3 and Lemma 6.4, for σ in a density 1 subset of $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$, the implication

$$\mathrm{div}(\sigma) \text{ is irreducible } \Longrightarrow \mathrm{div}(f^*\sigma)_{\mathrm{horiz}} \text{ is irreducible}$$

holds. By Theorem 1.6, the divisor $\mathrm{div}(\sigma)$ is irreducible for σ in a density 1 subset of $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$, which proves the result. □

Proof of Theorem 1.1. – We first assume that $\mathcal{Y}$ is not flat over $\mathrm{Spec}\,\mathbb{Z}$. Then $f : \mathcal{Y} \to \mathcal{X}$ factors as

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "f_p"] & \mathcal{X}_p \arrow[r] & \mathcal{X} \end{tikzcd}$$

for some prime number p. By Theorem 6.1, the density of those $s \in \bigcup_{n>0} H^0(\mathcal{X}_p, \mathcal{L}^{\otimes n})$ such that $\mathrm{div}(f_p^*s)_{\mathrm{horiz}}$ is irreducible is equal to 1. Applying Corollary 2.18 to $\overline{\mathcal{L}}(\varepsilon)$ proves the theorem.

We now assume that $\mathcal{Y}$ is flat over $\mathrm{Spec}\,\mathbb{Z}$. Let $\mathcal{Y}'$ be the Zariski closure of $f(\mathcal{Y})$ in $\mathcal{X}$. Then $\mathcal{Y}'$ is a projective arithmetic variety, and the restriction of $\overline{\mathcal{L}}$ to $\mathcal{Y}'$ is ample by Corollary 2.7. Furthermore, the map $f_{\mathcal{Y}'} : \mathcal{Y} \to \mathcal{Y}'$ is dominant by assumption. Theorem 1.4 guarantees that the density of the set E consisting of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{Y}', \overline{\mathcal{L}}^{\otimes n})$ such that $\mathrm{div}(f^*_{\mathcal{Y}'}\sigma)_{\mathrm{horiz}}$ is irreducible is equal to 1. Applying Corollary 2.18 to $\overline{\mathcal{L}}(\varepsilon)$ proves the theorem. □

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