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A remark on uniform boundedness for Brauer groups

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Abstract

The ℓ-adic Tate conjecture for divisors on varieties over number fields is equivalent to the finiteness of ℓ-primary torsion in the Brauer group. We show that this finiteness is actually uniform in 1-dimensional families for varieties that satisfy the ℓ-adic Tate conjecture for divisors, for example abelian varieties and $K3$ surfaces.

1. Introduction

1.1. Let k be a field of characteristic $p \geqslant 0$, and let $\overline{k}$ be a separable closure of k. Let X be a smooth proper variety over k. For any prime number $\ell \neq p$, consider the cycle class map

$$c_1 \colon \mathrm{Pic}(X_{\overline{k}}) \otimes \mathbb{Q}_\ell \to H^2(X_{\overline{k}}, \mathbb{Q}_\ell(1)) \,.$$

The absolute Galois group $\mathrm{Gal}\big(\overline{k}/k\big)$ of k acts naturally on the étale cohomology group $H^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))$, and the image of the cycle class map c_1 is contained in the group

$$\bigcup_U H^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))^U \,,$$

where U runs through the open subgroups of $\mathrm{Gal}\big(\overline{k}/k\big)$. We say that X satisfies the ℓ-adic Tate conjecture for divisors if the image of c_1 is equal to the group above.

As is well known, the Tate conjecture for divisors is related to finiteness results for the Brauer group. Indeed, given X and ℓ as above, X satisfies the ℓ-adic Tate conjecture for divisors if and only if for every subgroup U of finite index in $\mathrm{Gal}\big(\overline{k}/k\big)$, the group

$$\mathrm{Br}(X_{\overline{k}})^U[\ell^\infty]$$

is finite.

1.2. In this paper, we investigate whether one might be able to find uniform bounds on the Brauer groups above when the variety X varies in a family. Following Várilly-Alvarado's conjectures on $K3$ surfaces [Vár17, § 4], such a question might be regarded as an analogue of results of Manin [Man69], Faltings–Frey [Fre94], Mazur [Maz77] and Merel [Mer96] on uniform bounds for the torsion of elliptic curves. Related boundedness results for families of smooth proper varieties have been developed by Cadoret–Tamagawa [CT12, CT13, CT19, CT16]. It should be noted that

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all the aforementioned results can only treat the case of 1-dimensional bases. Dealing with the higher-dimensional case seems to be much harder, as rational points are not well understood on varieties of dimension at least 2.

The main result of this paper is as follows.

Theorem 1.2.1. *Let k be a finitely generated field of characteristic zero. Let S be a curve over k, and let $\pi\colon X \to S$ be a smooth proper morphism. Let ℓ be a prime number. Then for every positive integer d, there exists a constant C_d satisfying the following property: for every field extension $k \subset K \subset \overline{k}$ with degree $[K : k] \leqslant d$ and every K-point s such that X_s satisfies the ℓ-adic Tate conjecture for divisors,*

$$\left|\mathrm{Br}(X_{s,\overline{k}})^{\mathrm{Gal}(\overline{k}/K)}[\ell^\infty]\right| \leqslant C_d\,.$$

Examples of projective varieties for which the Tate conjecture for divisors is known—and so for which Theorem 1.2.1 applies—include abelian varieties and $K3$ surfaces [Fal83, Tat94].

In the case of certain 1-parameter families of abelian and $K3$ surfaces—which are actually parametrized by Shimura curves—the theorem above was proved by Várilly-Alvarado–Viray [VV17, Theorem 1.5, Corollary 1.6]. While their proof relies on deep arithmetic properties of elliptic curves, Theorem 1.2.1 is derived as a consequence of the results of Cadoret–Tamagawa in [CT13]. We will also prove a version of Theorem 1.2.1 that does not involve the Tate conjecture; see Theorem 3.5.1.

Outline of the paper. Section 2 is devoted to some general statements and questions about Brauer groups and their behavior in families. In Section 3, we prove Theorem 1.2.1 and its unconditional variant.

Notation. If A is an abelian group and n is an integer, we write $A[n]$ for the n-torsion subgroup of A. If ℓ is a prime number, we write

$$A[\ell^\infty] := \varinjlim A[\ell^n]$$

for the ℓ-primary torsion subgroup of A and

$$T_\ell(A) := \varprojlim A[\ell^n]$$

for its ℓ-adic Tate module.

Let X be a smooth proper variety over a field k. We write $\mathrm{Br}(X)$ for the Brauer group of X, that is, the étale cohomology group $H^2(X, \mathbb{G}_m)$, and $\mathrm{NS}(X)$ for the Néron–Severi group of X, that is, the image of $\mathrm{Pic}(X)$ in $H^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))$, where $\overline{k}$ is a separable closure of k.

2. A question on uniform finiteness for Brauer groups

In this section, we recall the relationship between the Tate conjecture and finiteness of the Brauer group. While the Tate conjecture is a qualitative statement, finiteness problems can be made quantitative, which leads us to Question 2.2.1.

2.1. The following proposition provides the basic relationship between the ℓ-adic Tate conjecture and finiteness for Brauer groups.

Let k be a field of characteristic $p \geqslant 0$. Let X be a smooth proper variety over k. Choose a prime ℓ different from p. Let $B := \mathrm{Br}(X_{\overline{k}})$ be the Brauer group of $X_{\overline{k}}$, and set

$$T_\ell := T_\ell(B)\,, \quad V_\ell := T_\ell \otimes \mathbb{Q}_\ell\,, \quad M_\ell := T_\ell \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \simeq V_\ell/T_\ell\,.$$

For every integer $n \geqslant 0$, we can identify $M_\ell[\ell^n]$ with T/ℓ^n.

The following proposition is well known to experts and can be found in various forms in the literature; see, in particular, [Tan03], [SZ08, Lemma 2.3], [CS13, Proposition 4.1].

Proposition 2.1.1. *The following assertions are equivalent:*

(1) *The ℓ-adic Tate conjecture holds for divisors on X.*
(2) *The group $B^U[\ell^\infty]$ is finite for every open subgroup $U \subset \mathrm{Gal}(\overline{k}/k)$.*
(3) *The group M_ℓ^U is finite for every open subgroup $U \subset \mathrm{Gal}(\overline{k}/k)$.*
(4) *One has $T_\ell^U = 0$ for every open subgroup $U \subset \mathrm{Gal}(\overline{k}/k)$.*

If these hold, then for every open subgroup $U \subset \mathrm{Gal}(\overline{k}/k)$, one has

(5) $\big|B^U[\ell^\infty]\big| \leqslant \big|M_\ell^U\big| + \big|\mathrm{H}^3(X_{\overline{k}}, \mathbb{Z}_\ell(1))[\ell^\infty]\big|$.

Proof. We first prove the equivalence of (2), (3) and (4). We can restrict ourselves to the case where U is normal. More precisely, we show that for any given normal subgroup $U \subset \mathrm{Gal}(\overline{k}/k)$, the following are equivalent:

(2) The group $B^U[\ell^\infty]$ is finite.
(3) The group M_ℓ^U is finite.
(4) $T_\ell^U = 0$.

If these hold, one has

(5) $\big|B^U[\ell^\infty]\big| \leqslant \big|M_\ell^U\big| + \big|\mathrm{H}^3(X_{\overline{k}}, \mathbb{Z}_\ell(1))[\ell^\infty]\big|$.

For every positive integer n, the cohomology of the Kummer short exact sequence

$$1 \to \mu_{\ell^n} \to \mathbb{G}_m \to \mathbb{G}_m \to 1$$

induces an exact sequence

$$0 \to \mathrm{NS}(X_{\overline{k}}) \otimes \mathbb{Z}/\ell^n \to \mathrm{H}^2(X_{\overline{k}}, \mu_{\ell^n}) \to B[\ell^n] \to 0\,.$$

Let K_ℓ and C denote the kernel and cokernel of the natural map $T_\ell \to B[\ell^n]$. We get a commutative diagram with exact lines and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ell^n\mathrm{NS}(X_{\overline{k}}) \otimes \mathbb{Z}_\ell & \longrightarrow & \mathrm{NS}(X_{\overline{k}}) \otimes \mathbb{Z}_\ell & \longrightarrow & \mathrm{NS}(X_{\overline{k}}) \otimes \mathbb{Z}/\ell^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \ell^n\mathrm{H}^2(X_{\overline{k}}, \mathbb{Z}_\ell(1)) & \longrightarrow & \mathrm{H}^2(X_{\overline{k}}, \mathbb{Z}_\ell(1)) & \longrightarrow & \mathrm{H}^2(X_{\overline{k}}, \mu_{\ell^n}) & \longrightarrow & \mathrm{H}^3(X_{\overline{k}}, \mathbb{Z}_\ell(1))[\ell^n] \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K_\ell & \longrightarrow & T_\ell & \longrightarrow & B[\ell^n] & \longrightarrow & C \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & 0 & & 0 & & 0
\end{array}$$

from which the snake lemma ensures that $K_\ell = \ell^n T_\ell$, whence a short exact sequence

$$0 \to T_\ell \otimes \mathbb{Z}/\ell^n \to B[\ell^n] \to \mathrm{H}^3(X_{\overline{k}}, \mathbb{Z}_\ell(1))[\ell^n] \to 0\,.$$

Taking inductive limits and identifying $T_\ell \otimes \mathbb{Z}/\ell^n \simeq M[\ell^n]$, we obtain a short exact sequence

$$0 \to M_\ell \to B[\ell^\infty] \to \mathrm{H}^3(X_{\overline{k}}, \mathbb{Z}_\ell(1))[\ell^\infty] \to 0\,,$$

and taking U-invariants, we obtain an exact sequence

$$0 \to M_\ell^U \to B^U[\ell^\infty] \to \mathrm{H}^3(X_{\overline{k}}, \mathbb{Z}_\ell(1))^U[\ell^\infty]\,. \tag{2.1}$$

This shows the equivalence (2) $\Leftrightarrow$ (3) and that, in that case, statement (5) holds. For the equivalence (3) $\Leftrightarrow$ (4), let $U_\ell \subset \mathrm{GL}(T_\ell)$ denote the image of $U \subset \mathrm{Gal}\big(\overline{k}/k\big)$ acting on T_ℓ, and consider the exact sequence

$$0 \to T_\ell^U \to V_\ell^U \to M_\ell^U \xrightarrow{\delta} \mathrm{H}^1(U_\ell, T_\ell)\,.$$

Since U_ℓ is a compact ℓ-adic Lie group, $\mathrm{H}^1(U_\ell, T_\ell)$ is a topologically finitely generated $\mathbb{Z}_\ell$-module [Ser64, Proposition 9]. In particular, $\mathrm{im}(\delta)$ is finite. The conclusion follows from the fact that either $T_\ell^U = V_\ell^U = 0$ or V_ℓ^U/T_ℓ^U is infinite.

Now, assume that X satisfies the ℓ-adic Tate conjecture for divisors. Then, since $\mathrm{NS}(X_{\overline{k}})$ is a finitely generated abelian group ([Nér52, Théorème 2, p. 145], [BJG+71, Exposé XIII, 5.1]), we can find an open subgroup U_0 of $\mathrm{Gal}\big(\overline{k}/k\big)$ such that the action of U_0 on $\mathrm{NS}(X_{\overline{k}})$ is trivial, and the map

$$\mathrm{NS}(X_{\overline{k}}) \otimes \mathbb{Q}_\ell \to \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))^U$$

is onto for every open subgroup U of U_0. Furthermore, [Tat94, Proposition (5.1)] shows that as a consequence of the ℓ-adic Tate conjecture for X, the exact sequence of U-modules

$$0 \to \mathrm{NS}(X_{\overline{k}}) \otimes \mathbb{Q}_\ell \to \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_\ell(1)) \to V_\ell \to 0$$

is split, so that the surjectivity above implies $V_\ell^U = 0$. Since T_ℓ is a torsion-free $\mathbb{Z}_\ell$-module, this implies $T_\ell^U = 0$. Since this vanishing holds for any small enough open subgroup U of $\mathrm{Gal}\big(\overline{k}/k\big)$, it holds for any open subgroup U of $\mathrm{Gal}\big(\overline{k}/k\big)$. This shows the implication (1) $\Rightarrow$ (4). For the converse implication, let $U_0 \subset \mathrm{Gal}\big(\overline{k}/k\big)$ be an open subgroup such that

$$H^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))^{U_0} = \bigcup_{[\mathrm{Gal}(\overline{k}/k):U]<\infty} H^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))^U\,.$$

Since $T_\ell^{U_0} = 0$ implies $V_\ell^{U_0} = 0$, we obtain an isomorphism

$$(\mathrm{NS}(X_{\overline{k}}) \otimes \mathbb{Q}_\ell)^{U_0} \xrightarrow{\sim} H^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))^{U_0}\,,$$

hence, a fortiori, an isomorphism $\mathrm{NS}(X_{\overline{k}}) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} H^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))^{U_0}$. □

2.2. With Proposition 2.1.1 in mind, we offer the following question on uniform boundedness for Brauer groups, which we view as an enhancement of the ℓ-adic Tate conjecture for divisors. Theorem 1.2.1 settles the case where S is 1-dimensional for those points that satisfy the ℓ-adic Tate conjecture.

Question 2.2.1. Let k be a finitely generated field of characteristic zero, and let ℓ be a prime. Let S be a quasi-projective variety over k. Let $\pi\colon X \to S$ be a smooth projective morphism. Then for any positive integer d, there exists a constant C_d satisfying the following property: for

any field extension $k \subset K \subset \overline{k}$ of degree $[K : k] \leqslant d$ and any K-point $s \in S(K)$, we have

$$\left|\mathrm{Br}(X_{s,\overline{k}})^{\mathrm{Gal}(\overline{k}/k)}[\ell^\infty]\right| \leqslant C_d\,.$$

For $d = 1$, Question 2.2.1 is closely related to (the ℓ-adic Tate conjecture for divisors and) the Bombieri–Lang conjecture (see, for example, [Bru20, Conjecture 1.8]). More precisely, assuming that the X_s, for $s \in S(k)$, satisfy the ℓ-adic Tate conjecture for divisors, Question 2.2.1 would follow from the following Galois-theoretic conjecture. For every integer $C \geqslant 1$, let $S^{\leqslant C}(k)$ be the set of all $s \in S(k)$ such that the image of $\mathrm{Gal}(\overline{k}/k)$ acting on $H^2(X_{\overline{s}}, \mathbb{Q}_\ell)$ is open of index at most C in the image of $\pi_1(S)$, and let $S^{\leqslant C,\mathrm{ex}}(k)$ be the complement of $S^{\leqslant C}(k)$ in $S(k)$. Then we have the following.

Conjecture 2.2.2. There exists an integer $C \geqslant 1$ such that $S^{\leqslant C,\mathrm{ex}}(k)$ is not Zariski dense in S.

When S is a curve, Conjecture 2.2.2 is a special case of the main result of [CT12]. We explain the relationship between the general case of Conjecture 2.2.2 and the Bombieri–Lang conjecture. One can always construct a strictly increasing sequence of integers C_n, for $n \geqslant 0$, and a projective system

$$\cdots \to S_{n+1} \to S_n \to S_{n-1} \to \cdots$$

of étale covers of S with the property that the image of $S_n(k)$ in S is $S^{\leqslant C_n,\mathrm{ex}}(k)$ (see [CT12, § 3.1.2]). So proving that $S^{\leqslant C,\mathrm{ex}}(k)$ is not Zariski dense in S for C large enough amounts to proving that $S_n(k)$ is not Zariski dense in S_n for $n \gg 0$. According to the Bombieri–Lang conjecture, this would follow from the fact that S_n is of general type for $n \gg 0$. While the Bombieri–Lang conjecture is still completely open, proving that S_n is of general type for $n \gg 0$ seems to be a more tractable problem. The projective system (S_n) can indeed be regarded as a generalization of towers of Shimura varieties. In this setting, Brunebarbe recently obtained an asymptotic hyperbolicity result (implying the general type property); see [Bru20] and the references therein, as well as [AV18].

As a concrete example, we now explain how a positive answer to the question above in the case of families of $K3$ surfaces—for which the Tate conjecture is known—yields a positive answer to the ℓ-primary version of Conjecture 4.6 of Várilly-Alvarado in [Vár17].

Proposition 2.2.3. *Assume that Question 2.2.1 has a positive answer for families of $K3$ surfaces. Let d be a positive integer, and let Λ be a lattice. Then there exists a positive constant C_d such that for any degree d number field k with algebraic closure $\overline{k}$ and any $K3$ surface X over k with $\mathrm{NS}(X_{\overline{k}}) \simeq \Lambda$, the following inequality holds:*

$$\left|\mathrm{Br}(X_{\overline{k}})^{\mathrm{Gal}\left(\overline{k}/k\right)}[\ell^\infty]\right| \leqslant C_d\,.$$

Using Theorem 1.2.1, we see that Proposition 2.2.3 is a consequence of the following result.

Proposition 2.2.4. *There exists a positive integer N with the following property: Let Λ be a lattice. There exists a smooth projective family of $K3$ surfaces $\pi_\Lambda\colon \mathcal{X} \to S$ over a quasi-projective base S over $\mathbb{Q}$, depending only on Λ, such that given any field k of characteristic zero with algebraic closure $\overline{k}$ and any $K3$ surface X over k with $\mathrm{NS}(X_{\overline{k}}) \simeq \Lambda$, there exist an extension K/k of degree N and a K-point s of S with $\mathcal{X}_s \simeq X_K$.*

We start with a lemma. If L is a line bundle on a $K3$ surface X, write $c_1(L)$ for its class in $\mathrm{NS}(X) \simeq \mathrm{Pic}(X)$.

Lemma 2.2.5. *There exists a positive integer N with the following property: Let Λ be a lattice. There exists a positive integer δ_Λ, depending only on Λ, such that for any field k with algebraic closure $\overline{k}$ and any $K3$ surface X over k with $\mathrm{NS}(X_{\overline{k}}) \simeq \Lambda$, there exist an extension K/k of degree N and an ample line bundle L on X_K with $c_1(L)^2 = \delta_\Lambda$.*

Proof. Given any integer n, there exist only finitely many finite groups that can occur as subgroups of $\mathrm{GL}_n(\mathbb{Z})$. Since the continuous action of any profinite group on a discrete finitely generated abelian group factors through a finite quotient, and since the Néron–Severi group of any $K3$ surface is free of rank at most 22, this shows that there exist finitely many finite groups that can occur as the image of $\mathrm{Gal}\big(\overline{k}/k\big) \to \mathrm{NS}(X_{\overline{k}})$ for a $K3$ surface X over a field k with algebraic closure $\overline{k}$.

In particular, there exists an integer N, independent of X and k, with the following property: for any k, $\overline{k}$ and X as above, there exists a Galois extension $k \subset K \subset \overline{k}$ of degree at most N such that the action of $\mathrm{Gal}\big(\overline{k}/k\big)$ on $\mathrm{NS}(X_{\overline{k}})$ factors through $\mathrm{Gal}(K/k)$.

Let X be a $K3$ surface over a field k with algebraic closure $\overline{k}$ such that $\mathrm{NS}(X_{\overline{k}}) \simeq \Lambda$. By [LMS14, Lemma 2.3.2], there exists a constant C_Λ, depending only on Λ, such that there exists an ample line bundle $\overline{L}$ on $X_{\overline{k}}$ with $c_1(\overline{L})^2 = C_\Lambda$. Write $\alpha = c_1(\overline{L})$. Then, N being as above, we can find a Galois extension K of k of degree N such that α is $\mathrm{Gal}\big(\overline{k}/K\big)$-invariant.

Consider the exact sequence coming from the Hochschild–Serre spectral sequence and Hilbert's Theorem 90

$$0 \to \mathrm{NS}(X_K) \to \mathrm{NS}(X_{\overline{k}})^{\mathrm{Gal}(\overline{k}/K)} \to \mathrm{Br}(K) \to \mathrm{Br}(X_K)\,.$$

The zero-cycle $c_2(X)$ on X has degree 24 and induces a morphism $\mathrm{Br}(X_K) \to \mathrm{Br}(K)$ such that the composition $\mathrm{Br}(K) \to \mathrm{Br}(X_K) \to \mathrm{Br}(K)$ is multiplication by 24. This shows that the cokernel of the map $\mathrm{NS}(X_K) \to \mathrm{NS}(X_{\overline{k}})^{\mathrm{Gal}(\overline{k}/K)}$ is killed by 24. As a consequence, we can find a line bundle L on X_K with $c_1(L) = 24\alpha$.

The pullback of L to $X_{\overline{k}}$ is $\overline{L}^{\otimes 24}$, which is ample. Since ampleness is a geometric property, L itself is ample. We have $c_1(L)^2 = 24^2 C_\Lambda =: \delta_\Lambda$, which depends only on Λ. □

Proof of Proposition 2.2.4. Apply Lemma 2.2.5 to find an integer N such that for X as in the proposition, there exist an extension K of k of degree N and an ample line bundle L on X_K with $c_1(L)^2 = \delta$, where δ depends only on Λ.

By Kollár–Matsusaka's refinement of Matsusaka's big theorem [KM83], and since the canonical bundle of a $K3$ surface is trivial by definition, we can find integers i and r, depending only on Λ, such that $L^{\otimes i}$ is very ample and induces an embedding of X into a projective space of dimension at most r as a subvariety of degree at most δi^2. The existence of the Hilbert scheme (or the theory of Chow forms) allows us to conclude. □

3. Proof of Theorem 1.2.1 and its unconditional variant

Let k be a finitely generated field of characteristic zero.

3.1. Let S be a smooth, separated and geometrically connected variety over k, with generic point η. If s is a point of S, we will always write $\overline{s}$ for a geometric point lying above s.

Let $\pi\colon X \to S$ be a smooth proper morphism. For every prime ℓ, let

$$\rho_\ell\colon \pi_1(S) \to \mathrm{GL}\big(\mathrm{H}^2(X_{\overline{\eta}}, \mathbb{Q}_\ell(1))\big)$$

denote the corresponding representation.[1] Let K be a field, and let s be a K-point of S. We write $\pi(s)$ for the absolute Galois group of K considered with its natural map to $\pi_1(S)$, and we also write ρ_ℓ for the representation of $\pi_1(s)$ obtained by composing the above with $\pi_1(s) \to \pi_1(S)$.

Let S_ℓ^{gen} denote the set of all points s of S with value in a field such that $\rho_\ell(\pi_1(s))$ is an open subgroup of $\rho_\ell(\pi_1(S))$. Let S_ℓ^{ex} be the complement of S_ℓ^{gen}. In other words, S_ℓ^{gen} is the set of points s such that the Zariski closures of $\rho_\ell(\pi_1(s))$ and $\rho_\ell(\pi_1(S))$ have the same neutral component. Note that the generic point η always belongs to S_ℓ^{gen}.

3.2. We start with the following proposition, which will allow us to compare the Galois-invariant part of Brauer groups of points in S_ℓ^{gen}.

Proposition 3.2.1. *For every $s \in S_\ell^{\mathrm{gen}}$, the natural specialization maps $\mathrm{NS}(X_{\overline{\eta}}) \to \mathrm{NS}(X_{\overline{s}})$ and $\mathrm{Br}(X_{\overline{\eta}}) \to \mathrm{Br}(X_{\overline{s}})$ are isomorphisms. In particular, if X_s satisfies the ℓ-adic Tate conjecture for divisors, then $X_{s'}$ satisfies the ℓ-adic Tate conjecture for divisors for all $s' \in S_\ell^{\mathrm{gen}}$.*

Proof. For every prime ℓ', consider the commutative specialization diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{NS}(X_{\overline{\eta}}) \otimes \mathbb{Z}_{\ell'} & \longrightarrow & \mathrm{H}^2(X_{\overline{\eta}}, \mathbb{Z}_{\ell'}(1)) & \longrightarrow & T_{\ell'}(\mathrm{Br}(X_{\overline{\eta}})) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{NS}(X_{\overline{s}}) \otimes \mathbb{Z}_{\ell'} & \longrightarrow & \mathrm{H}^2(X_{\overline{s}}, \mathbb{Z}_{\ell'}(1)) & \longrightarrow & T_{\ell'}(\mathrm{Br}(X_{\overline{s}})) & \longrightarrow & 0\,. \end{array}$$

By the smooth base change theorem, the middle vertical map is an isomorphism, so that the leftmost vertical map is injective and the rightmost vertical map is surjective. Since this holds for every ℓ', the specialization map $\mathrm{NS}(X_{\overline{\eta}}) \to \mathrm{NS}(X_{\overline{s}})$ is injective.

Let U be an open subgroup of $\rho_\ell(\pi_1(s))$ acting trivially on $\mathrm{NS}(X_{\overline{s}})$ and such that $\mathrm{NS}(X_{\overline{s}}) \otimes \mathbb{Z}_\ell$ injects into $\mathrm{H}^2(X_{\overline{s}}, \mathbb{Z}_\ell(1))^U$. Since $s \in S_\ell^{\mathrm{gen}}$, the subgroup U is open in $\rho_\ell(\pi_1(S))$; hence,

$$U \cap \rho_\ell(\pi_1(S_{\overline{k}})) \text{ is open in } \rho_\ell(\pi_1(S_{\overline{k}}))\,. \tag{3.1}$$

Choose an embedding $\overline{k} \subset \mathbb{C}$. We may assume $\overline{s} \in S(\mathbb{C})$. Let $[L] \in \mathrm{NS}(X_{\overline{s}})$, where L is a line bundle on $X_{\overline{s}}$. From (3.1) and comparison between the analytic and étale sites, we see that the image $c_1(L)$ of $[L]$ in $H^2(X_{\overline{s}}, \mathbb{Q}(1))$ is invariant under a finite index subgroup of $\pi_1(S_{\mathbb{C}})$ corresponding to a finite étale cover $S'_{\mathbb{C}}$ of $S_{\mathbb{C}}$. Fix a smooth compactification $\overline{X}_{S'_{\mathbb{C}}}$ of $X_{S'_{\mathbb{C}}}$. The theorem of the fixed part and the degeneration of the Leray spectral sequence show that $H^2(X_{\overline{s}}, \mathbb{Q}(1))^{\pi_1(S'_{\mathbb{C}})}$ is a sub-Hodge structure of $H^2(X_{\overline{s}}, \mathbb{Q}(1))$ onto which the Hodge structure $H^2(\overline{X}_{S'_{\mathbb{C}}}, \mathbb{Q}(1))$ surjects. The semisimplicity of the category of polarized Hodge structures ensures that $c_1(L)$ can be lifted to a Hodge class in $H^2(\overline{X}_{S'_{\mathbb{C}}}, \mathbb{Q}(1))$. By the Lefschetz $(1,1)$ theorem, this implies that some multiple of $[L]$ can be lifted to an element of $\mathrm{NS}(X_{\overline{\eta}})$. As a consequence, the specialization map

$$\mathrm{NS}(X_{\overline{\eta}}) \to \mathrm{NS}(X_{\overline{s}})$$

has torsion cokernel. Since for any prime number ℓ', both cycle class maps to ℓ'-adic cohomology have torsion-free cokernel (as the ℓ-adic Tate modules are torsion-free), the specialization map is surjective, so it is an isomorphism.

[1] The choice of base points for étale fundamental groups will play no part in what follows; we ignore such choices systematically and do not mention base points in the notation.

Finally, for every prime ℓ' and positive integer n, consider the specialization diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{NS}(X_{\overline{\eta}}) \otimes \mathbb{Z}/\ell'^n\mathbb{Z} & \longrightarrow & \mathrm{H}^2(X_{\overline{\eta}}, \mu_{\ell'^n}) & \longrightarrow & \mathrm{Br}(X_{\overline{\eta}})[\ell' s^n] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{NS}(X_{\overline{s}}) \otimes \mathbb{Z}/\ell'^n\mathbb{Z} & \longrightarrow & \mathrm{H}^2(X_{\overline{s}}, \mu_{\ell'^n}) & \longrightarrow & \mathrm{Br}(X_{\overline{s}})[\ell'^n] & \longrightarrow & 0\,. \end{array}$$

Both the leftmost and the middle vertical maps are isomorphisms, so the rightmost one is an isomorphism as well. Since the Brauer groups are torsion, this proves that the specialization map on Brauer groups is an isomorphism. This concludes the proof of the first part of the assertion in Proposition 3.2.1. The second part of the assertion is a formal consequence of the first. □

3.3. For every open normal subgroup $U \subset \rho_\ell(\pi_1(S))$, let $S_U \subset S$ denote the set of all points s with value in a field such that $\rho_\ell(\pi_1(s)) \supset U$. The set S_U is well defined since $\rho_\ell(\pi_1(s))$ is well defined up to conjugation by an element of $\rho_\ell(\pi_1(S))$ and U is normal in $\rho_\ell(\pi_1(S))$. By definition, $S_U \subset S_\ell^{\mathrm{gen}}$ and S_ℓ^{gen} is the union of the S_U as U runs through all normal open subgroups of $\rho_\ell(\pi_1(S))$.

Proposition 3.3.1. *Let $U \subset \rho_\ell(\pi_1(S))$ be a normal open subgroup.*

(1) *If there exists an $s_0 \in S_\ell^{\mathrm{gen}}$ such that the variety X_{s_0} satisfies the ℓ-adic Tate conjecture for divisors, then for every $s \in S_U$, there exists a constant $C_U \geqslant 1$ such that*

$$\left|\mathrm{Br}(X_{\overline{s}})^{\pi_1(s)}[\ell^\infty]\right| \leqslant C_U\,.$$

(2) *Assume that the Zariski closure G_ℓ of $\rho_\ell(\pi_1(S))$ is connected. Then there exists a constant $C_U \geqslant 1$ such that for every $s \in S_U$,*

$$\left[\mathrm{Br}(X_{\overline{s}})^{\pi_1(s)}[\ell^\infty] : \mathrm{Br}(X_{\overline{\eta}})^{\pi_1(S)}[\ell^\infty]\right] \leqslant C_U\,.$$

Proof. Write $B = \mathrm{Br}(X_{\overline{\eta}})$, $T_\ell = T_\ell(B)$, $V_\ell = T_\ell \otimes \mathbb{Q}_\ell$ and $M_\ell = V_\ell/T_\ell$.

By the second part of Proposition 3.2.1 and Proposition 2.1.1, the group $B^U[\ell^\infty]$ is finite. By the first part of Proposition 3.2.1, the specialization map $B \to \mathrm{Br}(X_{\overline{s}})$ is an isomorphism, which is obviously Galois equivariant. In particular, since $\rho_\ell(\pi_1(s))$ contains U, we have a natural injection

$$\mathrm{Br}(X_{\overline{s}})^{\pi_1(s)}[\ell^\infty] \hookrightarrow B^U[\ell^\infty]\,.$$

This shows statement (1).

Let $s \in S_U$. From the exact sequence (2.1), we find

$$0 \to M_\ell^{\pi_1(s)}/M_\ell^{\pi_1(S)} \to B^{\pi_1(s)}[\ell^\infty]/B^{\pi_1(S)}[\ell^\infty] \to Q \to 0\,,$$

where Q is a subquotient of $\mathrm{H}^3(X_{\overline{\eta}}, \mathbb{Z}_\ell(1))[\ell^\infty]$ and $M_\ell^{\pi_1(s)}/M_\ell^{\pi_1(S)}$ injects into $M_\ell^U/M_\ell^{\pi_1(S)}$ by the definition of S_U. In particular, statement (2) follows from Lemma 3.3.2 below. □

Let T be a free $\mathbb{Z}_\ell$-module of rank r, and let Π be a closed subgroup of $\mathrm{GL}(T)$. Write again $V = T \otimes \mathbb{Q}_\ell$ and $M = T \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \simeq V/T$. For any positive integer n, we can identify $M[\ell^n]$ with $T \otimes \mathbb{Z}/\ell^n\mathbb{Z}$. Let G_ℓ denote the Zariski closure of Π in $\mathrm{GL}(V)$.

Lemma 3.3.2. *Assume that G_ℓ is connected. Then for every normal open subgroup $U \subset \Pi$, the index of M^Π in M^U is finite.*

Note that the assumption that G_ℓ is connected is necessary, as can be shown by considering the $r = 1$ case with Π acting through a finite quotient and U being trivial.

Proof. Since G_ℓ is connected, we have $V^U = V^\Pi$ and $T^U = T \cap V^U = T \cap V^\Pi = T^\Pi$. The commutative diagram with exact lines

$$\begin{array}{ccccccc} 0 & \longrightarrow & V^\Pi/T^\Pi & \longrightarrow & M^\Pi & \xrightarrow{\delta_\Pi} & \mathrm{H}^1(\Pi, T) \\ & & \| & & \downarrow & & \downarrow \mathrm{res} \\ 0 & \longrightarrow & V^U/T^U & \longrightarrow & M^U & \xrightarrow{\delta_U} & \mathrm{H}^1(U, T) \end{array}$$

yields an isomorphism

$$M^U/M^\Pi \simeq \mathrm{im}(\delta_U)/\mathrm{im}(\delta_\Pi \circ \mathrm{res})\,.$$

In particular, the group M^U/M^Π is a quotient of $\mathrm{im}(\delta_U)$, and the conclusion follows from the fact that $\mathrm{im}(\delta_U)$ is finite as a torsion submodule of $\mathrm{H}^1(U, T)$. □

3.4. For every positive integer d, define

$$S_{\leqslant d} = \bigcup_{[K:k]\leqslant d} S(K)\,.$$

Write $S^{\mathrm{gen}}_{\ell,\leqslant d}$ and $S^{\mathrm{ex}}_{\ell,\leqslant d}$ accordingly. The following key result is a reformulation of [CT13, Theorem 1.1 and Remark 3.1.2(1)].

Fact 3.4.1. Assume that S is a curve. Then for every positive integer d, the image of $S^{\mathrm{ex}}_{\ell,\leqslant d}$ in S is finite, and there exists an open subgroup U_d of $\rho_\ell(\pi_1(S))$ such that $S^{\mathrm{gen}}_{\ell,\leqslant d} \subset S_{U_d}$.

Proof of Theorem 1.2.1. Without loss of generality, we may replace k with a finite field extension and S with its reduced subscheme. Also, by Proposition 2.1.1, we may replace S with an open subscheme. Working separately on each connected component, we may assume that S is a smooth, separated and geometrically connected variety over k. Let U_d be an open subgroup of $\rho_\ell(\pi_1(S))$ as in Fact 3.4.1. Let $s \in S_{\leqslant d}$ be such that X_s satisfies the ℓ-adic Tate conjecture for divisors.

If $s \in S^{\mathrm{gen}}_\ell$, then Fact 3.4.1 and Proposition 3.3.1(1) yield $|\mathrm{Br}(X_{\overline{s}})^{\pi_1(s)}[\ell^\infty]| \leqslant C_{U_d}$ for some constant C_{U_d} depending only on U_d.

If $s \in S^{\mathrm{ex}}_\ell$, then, by Fact 3.4.1, there are only finitely many possibilities for the image s_0 of s in S. Since $\rho_\ell(\pi_1(s_0))$ is an ℓ-adic Lie group, it has only finitely many open subgroups of index at most d. The intersection of all these open subgroups is an open subgroup $U_d(s_0)$ contained in $\rho_\ell(\pi_1(s))$. By construction,

$$\mathrm{Br}(X_{\overline{s}})^{\pi_1(s)}[\ell^\infty] \subset \mathrm{Br}(X_{\overline{s}})^{U_d(s_0)}[\ell^\infty]\,.$$

From Proposition 2.1.1 applied to X_{s_0}, it follows that $\mathrm{Br}(X_{\overline{s}})^{U_d(s_0)}[\ell^\infty]$ is finite. □

3.5. Using statement (2) of Proposition 3.3.1 instead of statement (1) in the proof above, one obtains the following unconditional variant of Theorem 1.2.1.

Theorem 3.5.1. *Let k be a finitely generated field of characteristic zero, and let S be a quasi-projective curve over k. Let $\pi\colon X \to S$ be a smooth proper morphism. Then for every prime ℓ such that the Zariski closure of the image of $\pi_1(S)$ acting on $\mathrm{H}^2(X_{\overline{\eta}}, \mathbb{Q}_\ell(1))$ is connected and for every positive integer d, there exists a constant C_d satisfying the following property: for every $s \in S^{\mathrm{gen}}_{\ell,\leqslant d}$,*

$$\left[\mathrm{Br}(X_{\overline{s}})^{\pi_1(s)}[\ell^\infty] : \mathrm{Br}(X_{\overline{\eta}})^{\pi_1(S)}[\ell^\infty]\right] \leqslant C_d\,.$$

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