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Two results on the Hodge structure of complex tori

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Abstract

We prove two results regarding Hodge structures appearing in the cohomology of complex tori. First, we prove that if a polarizable Hodge structure appears in the cohomology of a complex torus T, it appears in the cohomology of an abelian variety isomorphic to a subquotient of T. Second, we prove a universality result for the Kuga–Satake construction applied to Hodge structures of $K3$ type that might not be polarized.

1 Introduction

This note is devoted to studying those Hodge structures that appear in the cohomology of complex tori. It is well-known that the functor which associates to a complex torus T its first Betti cohomology group $H^1(T, \mathbb{Q})$ together with its natural Hodge structure of weight 1 is an equivalence of categories between the category of complex tori up to isogeny and the category of rational Hodge structures of type $\{(1, 0), (0, 1)\}$. Furthermore, such a rational Hodge structure comes from an abelian variety if and only if it admits a polarization, namely, a rational bilinear form that satisfies the Hodge–Riemann positivity relations.

In general, we say that a Hodge structure V is abelian if it appears as a direct factor, up to a Tate twist, of the cohomology of an abelian variety. Concretely, this means that V appears as a direct factor of a Hodge structure of the form

$$H^1(A, \mathbb{Q})^{\otimes a} \otimes (H^1(A, \mathbb{Q})^\vee)^{\otimes b} \otimes \mathbb{Q}(c),$$

where a, b and c are integers with $a, b \geq 0$. Abelian Hodge structures are described in terms of their Mumford–Tate groups in [8, Section 1].

Our first result shows that for a polarizable Hodge structure V to be of abelian type, it suffices for V to appear in the cohomology of a complex torus T, without assuming that T is an abelian variety. More precisely, we prove the following (see Theorem 3.1):

Pour Olivier, avec amitié, un petit appendice à son beau livre "Tores et Variétés Abéliennes Complexes".

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Theorem 1.1 *Let V be a pure polarizable Hodge structure. Let W be a Hodge structure of type $\{(0,1), (1,0)\}$ such that V is isomorphic to a subquotient of the Hodge structure*

$$W^{\otimes a} \otimes (W^{\vee})^{\otimes b} \otimes \mathbb{Q}(c)$$

for some integers a, b, c with $a, b \geq 0$. Then there exists a polarizable Hodge structure W' which is isomorphic to a direct sum of subquotients of W, and an injection of Hodge structures

$$V \hookrightarrow W'^{\otimes a'} \otimes (W'^{\vee})^{\otimes b'} \otimes \mathbb{Q}(c')$$

for some integers a', b', c' with $a', b' \geq 0$.

In particular, those polarizable Hodge structures that appear in the cohomology of complex tori always come from algebraic varieties.

An instance of Hodge structures that appear in the cohomology of complex tori is given by the Kuga–Satake construction, first introduced in [4], that associates a Hodge structure W of type $\{(0,1), (1,0)\}$ to any Hodge structure V of $K3$ type endowed with a suitable quadratic form. The Hodge structure W is polarizable if the quadratic form is a polarization, and V is always a sub-Hodge structure of $\mathrm{End}(W)$. We prove the following in Theorem 4.8 below – which we refer to for the notation:

Theorem 1.2 *Let V be a Hodge structure of $K3$ type endowed with a Beauville–Bogomolov quadratic form. Let T be a complex torus. Assume that there exist integers a, b and c together with an injective morphism of Hodge structures*

$$V \hookrightarrow H^1(T, \mathbb{Q})^{\otimes a} \otimes (H^1(T, \mathbb{Q})^{\vee})^{\otimes b} \otimes \mathbb{Q}(c)$$

with a and b nonnegative.

Assume that the Mumford–Tate group of V is the full special orthogonal group $SO(V)$. Then T contains a simple factor of the Kuga–Satake variety of V as a subquotient up to isogeny.

In the situation where V is polarized, the universality theorem above is folklore and stated for instance in [15, Proposition 6], and essentially contained in [5, 1.3], following results of Satake [12], see also [3]. The result is also alluded to by Deligne in his paper [4].

Our arguments for the proof of Theorem 3.1 and Theorem 4.8 rely on the use of Mumford–Tate groups and basic elements of the theory of reductive groups. In Sect. 2, we gather notation and elementary or folklore results. Section 3 is devoted to the proof of Theorem 3.1, and Sect. 4 proves Theorem 4.8.

2 Notation and preliminary results

2.1 Hodge structures

2.1.1 We denote by $\mathbb{S}$ the *Deligne torus*, namely, the real algebraic group defined as the Weil restriction of the multiplicative group $\mathbb{G}_{m,\mathbb{R}}$ to $\mathbb{R}$. By definition, we have:

$$\mathbb{S}(\mathbb{R}) = \mathbb{C}^* \quad \text{and} \quad \mathbb{S}(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*.$$

We denote by

$$w\colon \mathbb{G}_{m,\mathbb{R}} \longrightarrow \mathbb{S}$$

the morphism corresponding at the level of real points to the inclusion of $\mathbb{R}^*$ in $\mathbb{C}^*$, and by

$$\mu\colon \mathbb{G}_{m,\mathbb{C}} \longrightarrow \mathbb{S}_{\mathbb{C}} \simeq \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$$

the cocharacter corresponding to the morphism $z \mapsto (z, 1)$.

2.1.2 Let V be a finite-dimensional vector space over $\mathbb{Q}$. A *Hodge structure* on V is the datum of a bigrading of the complex vector space $V_{\mathbb{C}}$:

$$V_{\mathbb{C}} = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q}$$

such that for all integers p, q, the spaces $V^{p,q}$ and $V^{q,p}$ are conjugate and the spaces

$$\bigoplus_{p+q=k} V^{p,q}$$

are defined over $\mathbb{Q}$ for all integers k. Given a subset S of $\mathbb{Z}^2$, we say that V is of type S if $V^{p,q} = 0$ when (p, q) does not lie in S.

The Hodge structure is pure of some weight k if $V^{p,q} = 0$ when $p + q \neq k$. If n is an integer, the Tate Hodge structure $\mathbb{Q}(n)$ is the unique pure Hodge structure of type $(-n, -n)$ on the rational vector space $\mathbb{Q}$.[1]

The datum of an action of the torus $\mathbb{S}$ on the real vector space $V_{\mathbb{R}}$ is equivalent to a bigrading of the complex vector space $V_{\mathbb{C}}$:

$$V_{\mathbb{C}} = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q}$$

such that for all integers p, q, the spaces $V^{p,q}$ and $V^{q,p}$ are conjugate – here $V^{p,q}$ is the subspace of $V_{\mathbb{C}}$ on which $\mathbb{C}^*$ acts through $z^{-p}\overline{z}^{-q}$. Via the morphism $w\colon \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$, an action of $\mathbb{S}$ on $V_{\mathbb{R}}$ induces an action of $\mathbb{G}_{m,\mathbb{R}}$ on $V_{\mathbb{R}}$. This latter action is defined over $\mathbb{Q}$ if and only if the spaces

$$\bigoplus_{p+q=k} V^{p,q}$$

are defined over $\mathbb{Q}$ for all integers k, i.e., if and only if the bigrading above defines a Hodge structure on V. Given an action $h\colon \mathbb{S} \to GL(V_{\mathbb{R}})$, the corresponding Hodge structure on V is pure if and only if the morphism

$$h \circ w\colon \mathbb{G}_{m,\mathbb{R}} \longrightarrow GL(V_{\mathbb{R}})$$

factors through the center of $GL(V)_{\mathbb{R}}$.

If V is a pure Hodge structure of weight k, a *polarization* of V is a morphism of Hodge structures

$$\phi\colon V \otimes_{\mathbb{Q}} V \longrightarrow \mathbb{Q}(-k)$$

such that the bilinear form

$$V_{\mathbb{R}} \otimes V_{\mathbb{R}} \longrightarrow \mathbb{R},\ \ (x, y) \mapsto \phi_{\mathbb{R}}(x, h(i)y)$$

is positive definite.

[1] Since we will need to use Tate Hodge structures of fractional weight, we find it more convenient to drop the factor $(2\pi i)^n$ in the definition of Tate Hodge structures. Of course, this does not change the isomorphism class of these Hodge structures.

2.1.3 We denote by $MT(V)$ the smallest algebraic subgroup G of the algebraic group $GL(V)$ over $\mathbb{Q}$ such that $G_{\mathbb{R}}$ contains the image of h. The group $MT(V)$ is called the *Mumford–Tate group* of the Hodge structure V. Let

$$h\colon \mathbb{S} \longrightarrow MT(V)_{\mathbb{R}}$$

be the morphism that defines the Hodge structure on V. We denote by

$$w_h\colon \mathbb{G}_{m,\mathbb{Q}} \longrightarrow MT(V)$$

the morphism induced by $h \circ w$. It is a central cocharacter if and only if V is pure.

2.1.4 We will need to consider *fractional Hodge structures*. A fractional Hodge structure on a finite-dimensional vector space V over $\mathbb{Q}$ is defined as the datum of a bigrading of the complex vector space $V_{\mathbb{C}}$:

$$V_{\mathbb{C}} = \bigoplus_{p,q \in \mathbb{Q}} V^{p,q}$$

such that for all rational numbers p, q, the spaces $V^{p,q}$ and $V^{q,p}$ are conjugate and the spaces

$$\bigoplus_{p+q=k} V^{p,q}$$

are defined over $\mathbb{Q}$ for all rational numbers k. Given a subset S of $\mathbb{Q}^2$, we say that V is of type S if $V^{p,q} = 0$ when (p, q) does not lie in S.

If a is a rational number, the fractional Tate Hodge structure $\mathbb{Q}(a)$ is the unique pure Hodge structure of type $(-a, -a)$ on the rational vector space $\mathbb{Q}$.

Fractional Hodge structures may be understood in terms of the Deligne torus as follows. Let T be an algebraic torus. We denote by $\widetilde{T}$ the *universal covering* of T, namely, the projective system $(T_n)_{n\in\mathbb{N}\setminus\{0\}}$ where $T_n = T$ for all positive n, $\mathbb{N} \setminus \{0\}$ is ordered by divisibility, and the transition maps are

$$T_{mn} \longrightarrow T_n,\ x \mapsto x^m.$$

A *fractional morphism* from T to an algebraic group G is a morphism

$$\phi\colon \widetilde{T} \longrightarrow G.$$

Such a morphism may be represented by an actual morphism

$$\phi_n\colon T_n \longrightarrow G.$$

We denote by

$$\widetilde{w}\colon \widetilde{\mathbb{G}_{m,\mathbb{R}}} \longrightarrow \widetilde{\mathbb{S}}$$

the morphism induced by w.

As in the case of usual Hodge structures, fractional Hodge structures on V correspond bijectively to those fractional morphisms

$$\widetilde{h}\colon \widetilde{\mathbb{S}} \longrightarrow GL(V)_{\mathbb{R}}$$

such that $\widetilde{h} \circ \widetilde{w}$ is defined over $\mathbb{Q}$.

Given a fractional morphism $\widetilde{h}$ as above, we may find a positive integer n and a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathbb{S}} & \xrightarrow{x \mapsto x^n} & \widetilde{\mathbb{S}} \\ & \searrow^{\widetilde{h}_n} & \downarrow \widetilde{h} \\ & & GL(V)_{\mathbb{R}} \end{array}$$

in which $\widetilde{h}_n$ comes from an actual morphism $h_n \colon \mathbb{S} \to GL(V)_{\mathbb{R}}$. The Hodge structure defined by h_n is the one obtained by replacing the indices (p, q) in the bigrading of $V_{\mathbb{C}}$ with indices (np, nq). Indeed, choosing n allows us to turn all the p and q into integers.

The smallest algebraic subgroup G of $GL(V)$ over $\mathbb{Q}$ such that $G_{\mathbb{R}}$ contains the image of h_n does not depend on the choice of n, we will denote it again by $MT(V)$ and call it the Mumford–Tate group of V.

2.1.5 Fractional morphisms and fractional Hodge structures appear through the following elementary result.

Proposition 2.1 *Let k be a field of characteristic zero, let T be an algebraic torus over k, and let*

$$h \colon T \longrightarrow H$$

be a morphism of algebraic groups over k. Let

$$p \colon G \longrightarrow H$$

be an isogeny. Then h lifts uniquely to a fractional morphism

$$\widetilde{h} \colon \widetilde{T} \longrightarrow G.$$

Proof The unicity statement is clear. To prove existence, we may replace both T and H with the image of T in H, and G with the identity component of the preimage of $h(T)$ in G.

Since p has finite kernel, there exists a positive integer N such that the kernel K of p is contained in the N-torsion subgroup of G. In particular, the morphism

$$G \longrightarrow G, \ x \mapsto x^N$$

factors through a morphism $\phi \colon T \to G$: the composition

$$G \xrightarrow{p} T \xrightarrow{\phi} G$$

is $x \mapsto x^N$. Consider the composition

$$\psi \colon T \xrightarrow{\phi} G \xrightarrow{p} T.$$

Then

$$\psi \circ p \colon G \longrightarrow T$$

is the morphism $p \circ (\phi \circ p)$, i.e.

$$G \xrightarrow{x \mapsto x^N} G \xrightarrow{p} T.$$

Consider the morphism

$$\psi_N \colon T \longrightarrow T, \ x \mapsto x^N.$$

Then $\psi \circ p = \psi_N \circ p$, so that $\psi = \psi_N$ as p is an epimorphism by e.g. [2, Lemma 2.1].

In particular, we obtain a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\phi} & G \\ \big\downarrow{\scriptstyle \psi_N} & & \big\downarrow{\scriptstyle p} \\ T & \xrightarrow{\mathrm{Id}_T} & T, \end{array}$$

and ϕ defines a fractional lift of Id_T to G. □

In the remainder of this text, a Hodge structure is always meant in the classical sense, namely, with an integral bigrading. We will always use the adjective "fractional" to refer to fractional Hodge structures.

2.2 The reductive quotient of a Mumford–Tate group and the semisimplification of a Hodge structure

Let V be a Hodge structure. Recall that V is said to be simple if all sub-Hodge structures of V are isomorphic to 0 or V itself. We say that V is semisimple if V is isomorphic to a direct sum of simple Hodge structures. It is well-known that polarizable Hodge structures are semisimple and that their Mumford–Tate group is reductive.

In general, there exists an increasing filtration

$$0 = V_0 \subset \cdots \subset V_n = V$$

of V by sub-Hodge structures such that, for all i between 0 and $n-1$, the quotient Hodge structure

$$V_{i+1}/V_i$$

is simple. The direct sum

$$V^{\mathrm{ss}} := \bigoplus_{i=0}^{n-1} V_{i+1}/V_i$$

is a semisimple Hodge structure by construction, and it is readily checked that its isomorphism class does not depend on the choice of the filtration $V_\bullet$.

Let G be the Mumford–Tate group of V, and let $h\colon \mathbb{S} \to G_{\mathbb{R}}$ be the morphism defining the Hodge structure on V.

Proposition 2.2 *The Hodge structure V is semisimple if and only if the Mumford–Tate group G is reductive.*

Proof The sub-Hodge structures of V are exactly the G-invariant subspaces of V. As a consequence, V is semisimple as a Hodge structure if and only if V is semisimple as a representation of G.

If V is semisimple, then the unipotent radical of G acts trivially on V. Since G acts faithfully on V, this implies that G is reductive.

Conversely, if G is reductive, then any representation of G is semisimple, so that V is semisimple. □

We can describe the Mumford–Tate group of the semisimplification of V in general.

Proposition 2.3 *The Mumford–Tate group of V^{ss} may be identified with the reductive quotient G^{red} of G. Under this identification, the morphism $\mathbb{S} \to G^{red}_{\mathbb{R}}$ defining the Hodge structure on V^{ss} is the composition of h with the quotient map $G_{\mathbb{R}} \to G^{red}_{\mathbb{R}}$.*

Proof Consider an increasing filtration

$$0 = V_0 \subset \cdots \subset V_n = V$$

of V by sub-Hodge structures such that, for all i between 0 and $n - 1$, the quotient Hodge structure

$$V_{i+1}/V_i$$

is simple. Then the spaces V_i are invariant under the action of G on V.

Let U be the unipotent radical of G. By assumption, the representation of G on V_{i+1}/V_i is simple for all i, so that U acts trivially on V_{i+1}/V_i. As a consequence, the action of G on V induces an action of the reductive quotient $G/U = G^{red}$ on V^{ss}. By functoriality, the Hodge structure on V^{ss} is defined by this action and the morphism

$$h^{ss} : \mathbb{S} \to G^{red}_{\mathbb{R}}$$

defined as the composition of h with the quotient map $G_{\mathbb{R}} \to G^{red}_{\mathbb{R}}$.

Since h has dense image in G, h^{ss} has dense image in $G^{red}_{\mathbb{R}}$. To prove that G^{red} is the Mumford–Tate group of V, it remains to prove that the representation of G^{red} on V^{ss} is faithful, i.e., that the kernel of the action of G on $\bigoplus_{i=0}^{n-1} V_{i+1}/V_i$ is the unipotent radical U. Clearly, this kernel is normal and unipotent, so that it is contained in U, which finishes the proof. □

3 Polarizable Hodge structures coming from complex tori are abelian

3.1 Statement of the theorem

The main theorem of this section is the following.

Theorem 3.1 *Let V be a pure polarizable Hodge structure. Let W be a Hodge structure of type $\{(0, 1), (1, 0)\}$ such that V is isomorphic to a subquotient of the Hodge structure*

$$W^{\otimes a} \otimes (W^{\vee})^{\otimes b} \otimes \mathbb{Q}(c)$$

for some integers a, b, c with $a, b \geq 0$. Then there exists a polarizable direct factor W' of the Hodge structure W^{ss} and an injection of Hodge structures

$$V \hookrightarrow W'^{\otimes a'} \otimes (W'^{\vee})^{\otimes b'} \otimes \mathbb{Q}(c')$$

for some integers a', b', c' with $a', b' \geq 0$.

In particular, V appears in the cohomology of an abelian variety, so we obtain the following corollary.

Corollary 3.2 *Let V be a polarizable Hodge structure. Let W be a Hodge structure of type $\{(0, 1), (1, 0)\}$ such that V is isomorphic to a subquotient of the Hodge structure*

$$W^{\otimes a} \otimes (W^{\vee})^{\otimes b} \otimes \mathbb{Q}(c)$$

for some integers a, b, c with $a, b \geq 0$. Then V is isomorphic to a direct factor of a Hodge structure arising from the cohomology of an algebraic variety.

3.2 Mumford–Tate groups of Hodge structures of weight 1

3.2.1 We start with two elementary results. The first is well-known, and we include it for the sake of reference.

Proposition 3.3 *Let G_1 and G_2 be two reductive algebraic groups over a field k of characteristic zero, and let V be a representation of $G_1 \times G_2$ on a finite-dimensional k-vector space. Then there exist finite-dimensional representations V_1 and V_2 of G_1 and G_2 respectively on k-vector spaces, and a $G_1 \times G_2$-equivariant surjective morphism*

$$V_1 \otimes V_2 \longrightarrow V.$$

Proof Let V_1 be an irreducible factor of V considered as a representation of G_1. Define

$$V_2 = \mathrm{Hom}_{G_1}(V_1, V).$$

Then V_2 may be considered as a representation of G_2. The natural morphism

$$V_1 \otimes_k V_2 \longrightarrow V$$

is $G_1 \times G_2$-equivariant and nonzero, so that it is surjective. □

A fractional representation of $\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$ on a complex vector space V is equivalent to the datum of a bigrading

$$V = \bigoplus_{p,q \in \mathbb{Q}} V^{p,q}$$

of V. The *type* of the fractional representation V is the set of those pairs (p, q) such that $V^{p,q} \neq 0$.

Proposition 3.4 *Let $W_1, \ldots, W_k$ be fractional, finite-dimensional complex representations of $\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$. Assume that the representation*

$$W := W_1 \otimes \ldots \otimes W_k$$

of $\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$ has type $\{(0, 1), (1, 0)\}$.

Then there exists a unique integer i between 1 and k with the following property: for all $j \neq i$, there exists a rational number a_j such that W_j has type $\{(a_j, a_j)\}$. Furthermore, there exists a rational number a_i such that W_i has type $\{(a_i + 1, a_i), (a_i, a_i + 1)\}$.

Proof If H is a complex representations of $\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$, let $l(H)$ denote the number of those pairs of rational numbers (p, q) such that $H^{p,q} \neq 0$. It is readily checked that, for any two nonzero fractional representations H and H', we have:

$$l(H \otimes_{\mathbb{Q}} H') \geq l(H) + l(H') - 1.$$

In particular, we obtain:

$$2 = l(W) = 1 + \sum_{i=1}^{k} (l(W_i) - 1),$$

i.e.:

$$\sum_{i=1}^{k} (l(W_i) - 1) = 1.$$

This implies that there exists a unique integer i between 1 and k such that $l(W_i) = 2$, and that $l(W_j) = 1$ for $j \neq k$, which is equivalent to our assertion. □

3.2.2 Let W be a Hodge structure of type $\{(0, 1), (1, 0)\}$, and let G be the Mumford–Tate group of W. We assume that G is reductive. By Proposition 2.2, this is equivalent to W being semisimple.

Consider connected normal subgroups $T, G_1, \dots, G_r$ of G that commute pairwise such that the following two conditions are satisfied:

(1) The multiplication map induces a central isogeny:

$$p\colon T \times G_1 \times \cdots \times G_r \longrightarrow G.$$

Namely, p has finite, central kernel.

(2) For any i between 1 and r, there are no nontrivial characters $G_i \to \mathbb{G}_m$.

Note that the second condition above is satisfied if G_i is simple or $G_i(\mathbb{R})$ is compact.

Let

$$h\colon \mathbb{S} \longrightarrow G$$

be the morphism corresponding to the Hodge structure W, and let

$$\widetilde{h}\colon \widetilde{\mathbb{S}} \longrightarrow T \times G_1 \times \cdots \times G_r$$

be the unique fractional lift of h to $T \times G_1 \times \cdots \times G_r$ provided by Proposition 2.1. We write

$$\widetilde{h} = \widetilde{h}_0 \widetilde{h}_1 \dots \widetilde{h}_r,$$

where $\widetilde{h}_0\colon \widetilde{\mathbb{S}} \to T$ and $\widetilde{h}_i\colon \widetilde{\mathbb{S}} \to G_i$ for $1 \leq i \leq r$ are the components of $\widetilde{h}$.

Proposition 3.5 *Let W' be a simple factor of the Hodge structure W. Then the following statements hold*:

(i) *there exists at most one integer i in $\{1, \dots, r\}$ such that G_i acts nontrivially on W'.*

Assume that G_i acts nontrivially on W'.

(ii) *There exists a rational number a such that the fractional Hodge structure on W' induced by $\widetilde{h}_i$ and the action of G_i on W' is of type*

$$(\{(a+1, a),\ (a, a+1)\}.$$

(iii) *If G_i is simple, the natural surjection*

$$\pi : G = MT(W) \longrightarrow MT(W')$$

restricts to a central isogeny of G_i onto its image, and it induces an isomorphism

$$\mathbb{G}_m \pi(G_i) \simeq MT(W'),$$

where $\mathbb{G}_m$ is identified with the group of homotheties in $GL(W') \supset MT(W')$.

Note that as G is reductive, W is a direct sum of simple Hodge structures, so that W' is a direct summand of W.

Proof Via the central isogeny p, we may consider W' as a simple representation of $T \times G_1 \times \ldots \times G_r$. Proposition 3.3 guarantees that we may find an equivariant surjection

$$W_0 \otimes \cdots \otimes W_r \longrightarrow W', \tag{1}$$

where W_0 is a simple representation of T, and W_i is a simple representation of G_i for all i between 1 and r.

For simplicity of notation, write $T = G_0$. For any i between 0 and r, let V_i denote a simple factor of the representation $W_{i,\mathbb{C}}$ of $G_{i,\mathbb{C}}$. Then $W_{i,\mathbb{C}}$ is a direct sum of conjugates $\sigma(V_i) = V_i \otimes_{\mathbb{C},\sigma} \mathbb{C}$ of the representation V_i under automorphisms σ of $\mathbb{C}$. For any automorphisms $\sigma_0, \ldots, \sigma_r$ of $\mathbb{C}$, the representation

$$\sigma_0(V_0) \otimes \cdots \otimes \sigma_r(V_r) \tag{2}$$

is a simple representation of $G_{0,\mathbb{C}} \times \cdots \times G_{r,\mathbb{C}}$, see e.g. [13, Corollary of Lemma 68], so that the representation $W'_{\mathbb{C}}$ is a direct sum of representations of the form (2).

Any space $\sigma(V_i)$ carries via the fractional morphism $\widetilde{h}_i$ a representation of $\mathbb{S}_{\mathbb{C}} = \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$. It is readily checked that this representation is independent of the choice of the automorphism σ of $\mathbb{C}$.

Applying Proposition 3.4 and the irreducibility of the V_i, we see that there exists a unique integer i between 0 and r such that for all $j \neq i$, V_j has type $\{(a_j, a_j)\}$ for some rational number a_j, and V_i has type $\{(a_i + 1, a_i), (a_i, a_i + 1)\}$ for some rational number a_i.

Let $j \neq i$ be an integer between 1 and r. Since $W_{j,\mathbb{C}}$ is isomorphic to a direct sum of the $\sigma(V_j)$, it has type $\{(a_j, a_j)\}$ as a representation of $\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$. As a consequence, the fractional Hodge structure on W_i induced by $\widetilde{h}_i$ is isomorphic to a sum of copies of $\mathbb{Q}(a_j)$, and G_j acts on W_j through a character. By assumption (2), this character is trivial, so that $a_j = 0$ and G_j acts trivially on W_j. This proves (i) and (ii).

With the notation of (iii), since G_i is simple, the restriction of the surjection π to G_i is a central isogeny. The surjection

$$\pi : G = MT(W) \longrightarrow MT(W')$$

maps all the G_j to 1 for $j \neq i$, and maps T to $\mathbb{G}_m$ by Proposition 3.4. This proves (iii). □

For any integer i between 1 and r, let W_i denote the sum of those simple direct factors of W on which G_i acts nontrivially. For any $j \neq i$, the group G_j acts trivially on W_i.

Proposition 3.6 *Let $i_1, \ldots, i_k$ be k distinct integers between* 1 *and* r, *and let $W_{i_1,\ldots,i_k}$ be the direct factor*

$$W_{i_1,\ldots,i_k} = W_{i_1} \oplus \cdots \oplus W_{i_k}$$

of the Hodge structure W. Let

$$\pi : G = MT(W) \longrightarrow MT(W_{i_1,\ldots,i_k})$$

be the natural surjection. Then the restriction of π to the subgroup $G_{i_1} \ldots G_{i_k}$ of G is injective.

Proof Since W is semisimple, we may write

$$W = W_{i_1,\ldots,i_k} \oplus W',$$

as a direct sum of Hodge structures. By assumption, the group $G_{i_1} \ldots G_{i_k}$ acts trivially on W'. Since the representation of $G_{i_1} \ldots G_{i_k}$ on W is faithful by construction, this proves that $G_{i_1} \ldots G_{i_k}$ acts faithfully on $W_{i_1,\ldots,i_k}$, which proves the result. □

3.3 Beginning of the proof

We keep the notation of Theorem 3.1 and start its proof.

3.3.1 After replacing V with $V \otimes \mathbb{Q}(-c)$, we may assume that $c = 0$, i.e., that V is isomorphic to a subquotient of the Hodge structure

$$W^{\otimes a} \otimes (W^{\vee})^{\otimes b}.$$

Denote by G (resp. H) the Mumford–Tate group of W (resp. V), and by

$$h_G : \mathbb{S} \longrightarrow G_{\mathbb{R}}$$

and

$$h_H : \mathbb{S} \longrightarrow H_{\mathbb{R}}$$

the corresponding morphisms from the Deligne torus $\mathbb{S}$.

The group G acts naturally on the space $W^{\otimes a} \otimes (W^{\vee})^{\otimes b}$. Together with h_G, this action defines the Hodge structure on $W^{\otimes a} \otimes (W^{\vee})^{\otimes b}$. Since V is a subquotient of $W^{\otimes a} \otimes (W^{\vee})^{\otimes b}$, the group G acts on V in such a way that the action of $\mathbb{S}$ on $V_{\mathbb{R}}$ induced by h_G defines the Hodge structure on V. As a consequence, the action of G on V defines a morphism of Mumford–Tate group $p : G \to H$, and the diagram

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{h_g} & G_{\mathbb{R}} \\ & \searrow^{h_H} & \downarrow p_{\mathbb{R}} \\ & & H_{\mathbb{R}} \end{array}$$

is commutative.

By construction, the image of $h_H(\mathbb{C}^*)$ in $H(\mathbb{R})$ is Zariski-dense in G. As a consequence, p is surjective.

3.3.2 Since V is polarizable, its Mumford–Tate group H is reductive. Let U be the unipotent radical of G. Then $p(U)$ is a connected, unipotent subgroup of H. It is normal since p is surjective. This proves that U is contained in the kernel of p and that p factors through the reductive quotient G^{red} of G.

Consider the faithful representation W^{ss} of G^{red}. It is semisimple of type $\{(0, 1), (1, 0)\}$. Via the surjection $G^{\text{red}} \to H$, consider V as a representation of G^{red}. By [6, Chapter I, Proposition 3.1(a)], the representation V of G^{red} is isomorphic to a direct factor of $(W^{\text{ss}})^{\otimes a'} \otimes (W^{ss,\vee})^{\otimes b'}$ for some nonnegative integers a' and b'. In particular, the Hodge structure V is isomorphic to a direct factor of $(W^{\text{ss}})^{\otimes a'} \otimes (W^{\text{ss},\vee})^{\otimes b'}$.

By the previous two paragraphs, we may replace G by its reductive quotient G^{red} and W by its semisimplification W^{ss}. From now on, we assume that G is reductive and W is semisimple.

3.3.3 We apply the results of Sect. 3.2. Let Z be the identity component of the center of G and let G^{der} be the derived subgroup of G. The multiplication map induces a central isogeny

$$Z \times G^{\text{der}} \longrightarrow G.$$

Additionally, the group G^{der} is semisimple.

Let Z' be the identity component of the center of H and consider the diagram

$$\begin{array}{ccc} Z \times G^{\mathrm{der}} & \longrightarrow & G \\ \downarrow & & \downarrow p \\ Z' \times H^{\mathrm{der}} & \longrightarrow & H, \end{array}$$

where the leftmost vertical map respects the product structure. The horizontal maps are surjective with finite kernel, and the rightmost vertical map is surjective. This implies that the surjection $G \to H$ induces a surjection $Z \to Z'$. As a consequence, we may find connected subgroups T and G_1 of Z such that the multiplication map

$$T \times G_1 \longrightarrow Z$$

is an isogeny, and the composition

$$G_1 \longrightarrow Z \longrightarrow Z'/w_{h_H}(\mathbb{G}_m)$$

is an isogeny.

By [5, Proposition 1.1.14], the group

$$(Z'/w_{h_H}(\mathbb{G}_m))(\mathbb{R})$$

is compact. As a consequence, the group $G_1(\mathbb{R})$ is compact.

Let $(G_i)_{2 \leq i \leq r}$ denote the minimal connected normal subgroups of G^{der}. Then the G_i are simple normal subgroups of G that commute pairwise, and the multiplication map induces a central isogeny

$$q : T \times G_1 \cdots \times G_r \longrightarrow G.$$

After possibly reordering the groups G_i, we may find an integer $k \geq 1$ such that, for any i between 1 and k, the morphism

$$p_{|G_i} : G_i \longrightarrow H$$

is an isogeny onto its image, and, for any i between $k+1$ and r, p maps G_i to the identity element of H. In particular, the restriction of $p \circ q$ to $G_1 \times \cdots \times G_k$ is surjective onto $H/w_{h_H}(\mathbb{G}_m)$.

Lemma 3.7 *For any integer j between 2 and k, conjugation by $h_G(i)$ on $G_{\mathbb{R}}$ induces a Cartan involution of $G_{j,\mathbb{R}}$, namely, the group*

$$\{g \in G_j(\mathbb{C}),\ g = h(i)\overline{g}h(i)^{-1}\}$$

is compact.

Proof Let C (resp. C') denote the involution of $G_{\mathbb{R}}$ (resp. $H_{\mathbb{R}}$) given by conjugation by $h_G(i)$ (resp. $h_H(i)$). Since G_j is a normal subgroup of G, $G_{j,\mathbb{R}}$ is stable under C.

By construction, the restriction of $p \circ q$ to G_j defines an isogeny of G_j onto a closed subgroup of the adjoint group H^{ad} of H. In particular, it defines an isogeny from the group

$$G_j^{(C)}(\mathbb{R}) := \{g \in G_j(\mathbb{C}),\ g = C(\overline{g})\}$$

onto a closed subgroup of

$$H^{\mathrm{ad},(C')}(\mathbb{R}) := \{h \in H^{\mathrm{ad}}(\mathbb{C}),\ h = C'(\overline{h})\}.$$

Since V is polarizable, [5, Proposition 1.1.14] shows that C' is a Cartan involution of $H_{\mathbb{R}}^{\mathrm{ad}}$. This proves that $G_j^{(C)}(\mathbb{R})$ is compact. □

Proposition 3.8 *Let j be an integer between 1 and k, and let W' be a simple direct factor of the Hodge structure W such that G_j acts nontrivially on W'. Then W' is polarizable.*

Proof Let

$$\pi : G = MT(W) \longrightarrow MT(W')$$

denote the natural surjection. Let

$$h' : \mathbb{S} \longrightarrow MT(W')_{\mathbb{R}}$$

be the morphism defining the Hodge structure on W', so that $h' = \pi \circ h$.

Proposition 3.5 shows that $MT(W')$ is equal to $\mathbb{G}_m\pi(G_j)$, where $\mathbb{G}_m$ is identified with the group of homotheties of W'. Together with Lemma 3.7 in case $j \geq 2$ and by construction if $j = 1$, this implies that $h'(i)$ is a Cartan involution of $MT(W')/w_{h'}(\mathbb{G}_m)$, so that W' is polarizable, see [4, Proposition 2.11] or [10, Proposition 3.2]. □

For any integer j between 1 and r, let W_j denote the sum of those simple direct factors of the Hodge structure W on which G_j acts nontrivially. Let W_p be the Hodge structure

$$W_p = \bigoplus_{j=1}^{k} W_j.$$

Note that Proposition 3.5 shows that the W_i do not intersect, so that W_p is a sub-Hodge structure of W. Note also that the groups G_j act trivially on W_p for any $j > k$.

Corollary 3.9 *The Hodge structure W_p is polarizable.*

Proof This is a formal consequence of Proposition 3.8. □

Note that the surjections

$$G = MT(W) \longrightarrow MT(W_p)$$

and

$$G \longrightarrow H = MT(V)$$

endow both V and W_p with actions of G.

3.4 Proof of Theorem 3.1

Proposition 3.10 *There exists nonnegative integers a' and b', and a linear injection*

$$\phi : V \hookrightarrow W_p^{\otimes a'} \otimes (W_p^{\vee})^{\otimes b'}$$

that is equivariant with respect to the action of the subgroup $G_1 \ldots G_k$ of G.

Proof Proposition 3.6 shows that W_p is a faithful representation of the reductive group $G_1 \ldots G_k$. The result follows from [6, Chapter I, Proposition 3.1(a)]. □

Proof of Theorem 3.1 To prove Theorem 3.1, it suffices to show that there exists an integer c' such that the injection ϕ of Proposition 3.10 induces an injection of Hodge structures

$$\psi : V \hookrightarrow W_p^{\otimes a'} \otimes (W_p^{\vee})^{\otimes b'} \otimes \mathbb{Q}(c').$$

As both V and $W_p^{\otimes a'} \otimes (W_p^{\vee})^{\otimes b'}$ are pure Hodge structures, this is in turn equivalent to proving that for any simple sub-Hodge structure V' of V, there exists an integer c' such that ϕ induces an injection of Hodge structures

$$V' \longrightarrow W_p^{\otimes a'} \otimes (W_p^{\vee})^{\otimes b'} \otimes \mathbb{Q}(c').$$

Indeed, if this holds, then c' is independent of V' by purity. We pick such a V'.

Let H' be the image of the subgroup $G_1 \dots G_r$ in H. Then the multiplication map induces an isogeny

$$w_h(\mathbb{G}_m) \times H' \longrightarrow H,$$

and we may lift $h_H : \mathbb{S} \to H_{\mathbb{R}}$ to a fractional morphism

$$\widetilde{h}' : \widetilde{\mathbb{S}} \longrightarrow w_h(\mathbb{G}_m)_{\mathbb{R}} \times H'_{\mathbb{R}}.$$

We denote by $\widetilde{h}_{H'} : \widetilde{\mathbb{S}} \to H'_{\mathbb{R}}$ the component of $\widetilde{h}'$ mapping to $H'_{\mathbb{R}}$.

Since V' is simple, the group $w_{h_H}(\mathbb{G}_m)$ acts on V' by a homothety. This proves that the Hodge structure on V' coincides up to a twist by some $\mathbb{Q}(c')$ with the fractional Hodge structure on V' induced by $\widetilde{h}_{H'}$.

Similarly, consider the lift of h_G to a fractional morphism

$$\widetilde{h} : \widetilde{\mathbb{S}} \longrightarrow (Z \times G_1 \times \dots \times G_r)_{\mathbb{R}},$$

as well as its component $\widetilde{h}_{1,\dots,k} : \widetilde{\mathbb{S}} \longrightarrow (G_1 \times \dots \times G_k)_{\mathbb{R}}$.

By construction, Z acts through a character on W_p, and $G_{k+1}, \dots, G_r$ all act trivially on W_p. Consequently, the same argument as above shows that if W'_p is a simple sub-Hodge structure of W_p, the Hodge structure on W'_p coincides up to a twist by some $\mathbb{Q}(d')$ with the fractional Hodge structure on W'_p induced by $\widetilde{h}_{1,\dots,k}$.

Finally, the previous paragraph implies that if W'' is a simple sub-Hodge structure of $W_p^{\otimes a'} \otimes (W_p^{\vee})^{\otimes b'}$, the Hodge structure on W'' coincides up to a twist by some $\mathbb{Q}(e')$ with the fractional Hodge structure on W'' induced by $\widetilde{h}_{1,\dots,r}$.

The equivariance of ϕ with respect to the action of $G_1 \dots G_k$ proves that ϕ defines a morphism between the fractional Hodge structures on V and $W_p^{\otimes a'} \otimes (W_p^{\vee})^{\otimes b'}$ defined by $\widetilde{h}_{H'}$ and $\widetilde{h}_{1,\dots,k}$ respectively. This finishes the proof of Theorem 3.1. □

4 Universality of the Kuga–Satake construction

4.1 The Kuga–Satake construction

4.1.1 We recall briefly the Kuga–Satake construction in the context of Hodge structures that are not necessarily polarizable. We refer to [4] and [7, Chapter 4] for more details, see in particular [7, Chapter 4, Remark 2.3] for the non polarized case.

Definition 4.1 A *Hodge structure of* $K3$ *type* is a pure Hodge structure V of weight 0, of type $\{(1, -1), (0, 0), (-1, 1)\}$ with $\dim V^{1,-1} = 1$.

A *Beauville–Bogomolov* form on V is a nondegenerate quadratic form q on V that induces a morphism of Hodge structures

$$q : V \otimes V \longrightarrow \mathbb{Q}$$

and is definite positive on the real part of $V^{1,-1} \oplus V^{-1,1}$.

Note that a Beauville–Bogomolov form does not define a polarization on V in general. If X is a compact hyperkähler manifold, the Hodge structure $H^2(X, \mathbb{Q})$ is of $K3$ type, endowed with a natural Beauville–Bogomolov form, see [1].

Let V be a Hodge structure of $K3$ type endowed with a Beauville–Bogomolov form. The natural morphism

$$h : \mathbb{S} \longrightarrow GL(V)_{\mathbb{R}}$$

defining the Hodge structure on V factors through the special orthogonal group $SO(V)_{\mathbb{R}}$.

Let $C(V)$ denote the Clifford algebra of V, $C^+(V)$ its even part. We regard V as a subspace of $C(V)$. Let $\mathrm{CSpin}(V)$ denote the Clifford group of V, defined as the algebraic group of invertible elements g of $C^+(V)$ such that

$$gVg^{-1} = V.$$

Conjugation by elements of $\mathrm{CSpin}(V)$ defines a surjection

$$\mathrm{CSpin}(V) \longrightarrow SO(V)$$

with kernel reduced to the scalars. The computation of [7, Chapter 4, 2.1] shows that the morphism h lifts uniquely to a morphism

$$h' : \mathbb{S} \longrightarrow \mathrm{CSpin}(V)_{\mathbb{R}}.$$

Consider the representation of the group $\mathrm{CSpin}(V)$ on $C^+(V)$ by multiplication on the left. By [4, Proposition 4.5], the Hodge structure induced by h' and this representation on $C^+(V)$ has type $\{(0, 1), (1, 0)\}$. This is the *Kuga–Satake Hodge structure* associated to (V, q). It is known to be polarizable if q defines a polarization on V. We denote it by H_{KS}.

4.1.2 Let n be the dimension of V. The dimension of the algebra $C^+(V)$ is 2^{n-1}. By e.g. [4, 3.4], we may understand the representation of $\mathrm{CSpin}(V_{\mathbb{C}})$ on $C^+(V_{\mathbb{C}})$ by multiplication on the left as follows.

If n is odd, then $C^+(V_{\mathbb{C}})$ is a direct sum of $2^{(n-1)/2}$ copies of the spin representation, which is irreducible of dimension $2^{(n-1)/2}$.

If n is even, then $C^+(V_{\mathbb{C}})$ is a direct sum of $2^{n/2-1}$ copies of the direct sum of the two half-spin representations, which are both irreducible of dimension $2^{n/2-1}$.

Assume that the Mumford–Tate group of the Hodge structure V is the full special orthogonal group $SO(V)$. Considering the commutative diagram

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{h'} & \mathrm{CSpin}(V)_{\mathbb{R}} \\ & {\scriptstyle h} \searrow & \downarrow \\ & & SO(V)_{\mathbb{R}} \end{array}$$

in which the vertical map is surjective shows that the Mumford–Tate group of H_{KS} is the group $\mathrm{CSpin}(V)$. In particular, it is reductive, so that the Hodge structure H_{KS} is semisimple. We want to give a description of the simple factors of H_{KS}. The simple factors of H_{KS} are exactly the simple factors of the representation of $\mathrm{Cspin}(V)$ on H_{KS}.

Lemma 4.2 *Let H and H' be two nonisomorphic simple factors of the Hodge structure H_{KS}.*

Then n is even, then the representation of $\mathrm{CSpin}(V_{\mathbb{C}})$ on $H_{\mathbb{C}}$ is a direct sum of copies of one of the half-spin representations, and the representation of $\mathrm{CSpin}(V_{\mathbb{C}})$ on $H'_{\mathbb{C}}$ is a direct sum of copies of the other half-spin representation.

Proof Since H and H' are nonisomorphic and simple, there are no nonzero $\mathrm{CSpin}(V)$-equivariant morphisms from H to H'. As a consequence, there are no nonzero $\mathrm{CSpin}(V_{\mathbb{C}})$-equivariant morphisms from $H_{\mathbb{C}}$ to $H'_{\mathbb{C}}$. This immediately implies the result by the above description of the representation $H_{KS,\mathbb{C}}$ of $\mathrm{CSpin}(\mathrm{V}_{\mathbb{C}})$. □

The following theorem is an elaboration on [14, Section 8].

Theorem 4.3 *Let δ be the discriminant of the quadratic form q. Then the following holds.*

(i) *If n is odd, then there exists a simple Hodge structure H and a positive integer such that*

$$H_{KS} \simeq H^{\oplus N}.$$

There exists $r \in \{1, 2\}$ such that the representation of $\mathrm{CSpin}(V_{\mathbb{C}})$ on $H_{\mathbb{C}}$ is a direct sum of r copies of the spin representation. If $r = 1$, then H has dimension $2^{(n-1)/2}$ and $N = 2^{(n-1)/2}$. If $r = 2$, then H has dimension $2^{(n+1)/2}$ and $N = 2^{(n-3)/2}$.

(ii) *If n is even and $(-1)^{n/2}\delta$ is not a square in $\mathbb{Q}$, then there exist a simple Hodge structure H and a positive integer such that*

$$H_{KS} \simeq H^{\oplus N}.$$

There exists $r \in \{1, 2\}$ such that the representation of $\mathrm{CSpin}(V_{\mathbb{C}})$ on $H_{\mathbb{C}}$ is a direct sum of r copies of the direct sum of the two half-spin representations. If $r = 1$, then H has dimension $2^{n/2}$ and $N = 2^{n/2-1}$. If $r = 2$, then H has dimension $2^{n/2+1}$ and $N = 2^{n/2-2}$.

(iii) *If n is even and $(-1)^{n/2}\delta$ is a square in $\mathbb{Q}$, then there exist two nonisomorphic simple Hodge structures H and H', and a positive integer N such that*

$$H_{KS} \simeq (H \oplus H')^{\oplus N}.$$

There exists a positive integer $r \in \{1, 2\}$ such that the representation of $\mathrm{CSpin}(V_{\mathbb{C}})$ on $H_{\mathbb{C}}$ is a direct sum of r copies of one of the half-spin representations, and the representation of $\mathrm{CSpin}(V_{\mathbb{C}})$ on $H'_{\mathbb{C}}$ is a direct sum of r copies of the other half-spin representation. If $r = 1$, then H and H' have dimension $2^{n/2-1}$ and $N = 2^{n/2-1}$. If $r = 2$, then H and H' have dimension $2^{n/2}$ and $N = 2^{n/2-2}$.

In [14], it is shown that all the situations above do occur.

Proof Lemma 4.2 shows that there are at most two nonisomorphic simple factors of the Hodge structure H_{KS}, and that if n is odd, there is only one simple factor.

We proceed to investigate the structure of the endomorphism algebra of the Hodge structure H_{KS}. Since the Mumford–Tate group of H_{KS} is $\mathrm{CSpin}(V)$, we have:

$$\mathrm{End}_{\mathrm{Hodge}}(H_{KS}) = \mathrm{End}_{\mathrm{CSpin}(V)}(C^+(V)),$$

where $\mathrm{CSpin}(V)$ acts on the Clifford algebra $C^+(V)$ by multiplication on the left. This latter algebra is computed in [14, Theorem 7.7].

Assume that n is odd. Then

$$\mathrm{End}_{\mathrm{Hodge}}(H_{KS}) \simeq M_{2^{(n-3)/2}}(D),$$

where D is a quaternion algebra over $\mathbb{Q}$. This proves that, in case (i), we have

$$H_{KS} \simeq H^{\oplus 2^{(n-3)/2}} \quad \text{or} \quad H_{KS} \simeq H^{\oplus 2^{(n-1)/2}}$$

depending on whether or not D is split. Accordingly, the dimension of H is either $2^{(n+1)/2}$ or $2^{(n-1)/2}$, so that H is the sum of 1 or 2 copies of the spin representation. This is the case described in (i).

Assume that n is even and $(-1)^{n/2}\delta$ is not a square in $\mathbb{Q}$. Then

$$\mathrm{End}_{\mathrm{Hodge}}(H_{KS}) \simeq M_{2^{n/2-1}}(D),$$

where D is a quaternion algebra over $\mathbb{Q}(\sqrt{(-1)^{n/2}\delta})$.

If D is split, then we argue as above to show that

$$H_{KS} \simeq H^{\oplus 2^{n/2}},$$

where H is simple and $H_{\mathbb{C}}$ is isomorphic to the sum of two copies of the sum of the two half-spin representations.

If D is nonsplit, then

$$H_{KS} \simeq H^{\oplus 2^{n/2}-2},$$

where H is simple and $H_{\mathbb{C}}$ is isomorphic to the sum of the two half-spin representations. This is the case described in (ii).

Assume that n is even and $(-1)^{n/2}\delta$ is a square in $\mathbb{Q}$. Then

$$\mathrm{End}_{\mathrm{Hodge}}(H_{KS}) \simeq M_{2^{n/2-2}}(D) \times M_{2^{n/2-2}}(D),$$

where D is a quaternion algebra over $\mathbb{Q}$.

If D is split, then

$$\mathrm{End}_{\mathrm{Hodge}}(H_{KS}) \simeq M_{2^{n/2-1}}(\mathbb{Q}) \times M_{2^{n/2-1}}(\mathbb{Q}),$$

so that

$$H_{KS} \simeq (H \oplus H')^{\oplus 2^{n/2-1}},$$

where H and H' are two nonisomorphic simple factors of H_{KS}. Lemma 4.2 shows that $H_{\mathbb{C}}$ is a sum of copies of one of the half-spin representation, and $H'_{\mathbb{C}}$ a sum of copies of the other. The structure of the representation $H_{KS,\mathbb{C}}$ implies that $H_{\mathbb{C}}$ is actually isomorphic to one of the half-spin representations, and $H'_{\mathbb{C}}$ to the other.

If D is nonsplit, the same argument shows that

$$H_{KS} \simeq (H \oplus H')^{\oplus 2^{n/2-2}},$$

where H and H' are two nonisomorphic simple factors of H_{KS}, $H_{\mathbb{C}}$ is isomorphic to the sum of two copies of one of the half-spin representation, and $H'_{\mathbb{C}}$ to the sum of two copies of the other. This is the case described in (iii). □

Example 4.4 Let T be a very general complex torus of dimension 2, and let

$$V = H^2(T, \mathbb{Q})$$

endowed with the quadratic form q given by cup-product. Then (V, q) is isomorphic to the sum of three copies of the hyperbolic plane, so that the discriminant δ of q is -1. Since V is 6-dimensional, this proves that we are in the situation of (iii) above. It is possible to show that the simple factors of H_{KS} are $H^1(T, \mathbb{Q})$ and its dual, which are nonisomorphic—note that T is not an abelian variety. The reader may consult [11] for the related case of abelian surfaces.

Example 4.5 Let X be a holomorphic symplectic variety. Then $H^2(X, \mathbb{Q})$ endowed with its Beauville–Bogomolov quadratic form q is a Hodge structure of $K3$ type. The quadratic form q has signature $(3, b_2(X) - 3)$, where $b_2(X)$ is the dimension of $H^2(X, \mathbb{Q})$. Let δ be the discriminant of q.

Assume that $b_2(X)$ is odd, and $V = H^2(X, \mathbb{Q})$. Then we are in case (i) and the unique simple factor of H_{KS} has dimension $2^{(b_2(X)+1)/2}$ or $2^{(b_2(X)-1)/2}$.

Assume that $b_2(X)$ is divisible by 4, and $V = H^2(X, \mathbb{Q})$. Then δ is negative, and $(-1)^{b_2(X)/2}\delta$ is negative, so that we are in case (ii) and the unique simple factor of H_{KS} has dimension $2^{b_2(X)/2}$ or $2^{b_2(X)/2+1}$.

Assume that $b_2(X)$ is even, X is equipped with an ample line bundle, and V is the orthogonal of the polarization in $H^2(X, \mathbb{Q})$. Then we are in case (i) and the unique simple factor of H_{KS} has dimension $2^{b_2(X)/2-1}$ or $2^{b_2(X)/2}$.

Assume that $b_2(X)$ is congruent to 3 modulo 4, X is equipped with an ample line bundle, and V is the orthogonal of the polarization in $H^2(X, \mathbb{Q})$. Then the discriminant of the restriction of q to V is positive, so that we are in case (ii) again, and the unique simple factor of H_{KS} has dimension $2^{(b_2(X)-1)/2}$ or $2^{(b_2(X)+1)/2}$.

Two special cases in even dimension are as follows.

4.2 A preliminary group-theoretic result

Let $V_{\mathbb{C}}$ be a finite-dimensional complex vector space endowed with a nondegenerate quadratic form $q_{\mathbb{C}}$. Let u and v be two elements of V with

$$q(u) = q(v) = 1$$

and

$$q(u, v) = 0.$$

Let

$$\nu : \mathbb{G}_{m,\mathbb{C}} \longrightarrow SO(V_{\mathbb{C}}),\ z \mapsto \nu(z)$$

be the cocharacter that acts on u (resp. v) by multiplication by z (resp. z^{-1}) and acts trivially on the orthogonal of the 2-plane generated by u and v.

It is readily checked that the action of $\mathbb{G}_{m,\mathbb{C}}$ on Lie $SO(V_{\mathbb{C}})$ through ν and the adjoint action factors through the characters z, z^{-1} and 1. As a consequence of [5, 1.2.5], see also [9, 2.3], we may attach to the conjugacy class of ν a *special vertex* of the Dynkin diagram of $SO(V_{\mathbb{C}})$ as follows. Choose a maximal torus T of $SO(V_{\mathbb{C}})$ through which ν factors, S a basis for the root system of T, R^+ the corresponding set of positive roots. After conjugating ν, we may assume that the integers $\langle \alpha, \nu \rangle$ are nonnegative for $\alpha \in R^+$. Then there exists a unique simple root α for which

$$\langle \alpha, \nu \rangle = 1.$$

The corresponding vertex of the Dynkin diagram of $SO(V_{\mathbb{C}})$ is the special vertex associated to ν.

Lemma 4.6 *The special vertex associated to ν is the leftmost vertex of the Dynkin diagram of $SO(V_{\mathbb{C}})$.*

Proof We assume that the dimension of $V_{\mathbb{C}}$ is an even number $2m$, the odd case being similar. Consider a basis $e_1, \ldots, e_m, f_1, \ldots, f_m$ of $V_{\mathbb{C}}$ with $e_1 = u$, $f_1 = v$, and, for all i, j,

$$q(e_i, e_j) = q(f_i, f_j) = 0$$

and

$$q(e_i, f_j) = \delta^i_j.$$

Let T be the maximal torus of $SO(V_{\mathbb{C}})$ consisting of diagonal matrices with diagonal coefficients

$$(t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1})$$

and identify the t_i with characters of T. Then we may take for R^+ the simple roots

$$\alpha_1 = t_1/t_2,\ \ldots,\ \alpha_{n-1} = t_{n-1}/t_n,\ \alpha_n = t_{n-1}t_n.$$

We obtain

$$\langle \alpha_1, \nu \rangle = 1$$

and

$$\langle \alpha_i, \nu \rangle = 0$$

for $i > 1$. This proves that the special vertex corresponds to the simple root α_1. □

We keep the notation above.

Proposition 4.7 *Let*

$$G_{1,\mathbb{C}} \longrightarrow SO(V_{\mathbb{C}})$$

be an isogeny of complex algebraic groups and let

$$\widetilde{\nu} \colon \widetilde{\mathbb{G}_{m,\mathbb{C}}} \longrightarrow G_{1,\mathbb{C}}$$

be the fractional cocharacter of $G_{1,\mathbb{C}}$ lifting ν. Let W be a simple representation of $G_{1,\mathbb{C}}$ such that the action of $\widetilde{\mathbb{G}_{m,\mathbb{C}}}$ on W via $\widetilde{\nu}$ has only two weights a and $a+1$ for some rational number a. Then $G_{1,\mathbb{C}}$ is the universal cover of $SO(V_{\mathbb{C}})$. Furthermore, if $\dim V_{\mathbb{C}}$ *is odd, then W is the spin representation of G_1, and, if* $\dim V_{\mathbb{C}}$ *is even, W is one of the two half-spin representations of G_1.*

Proof The classification of all complex representations of G_1 as in the statement of the proposition is due to Satake [12], and is explained in [5, 1.3.5–1.3.9]. In the end, table 1.3.9 there, or the last table in [12] proves the result—indeed, Lemma 4.6 proves that, with the notation of Deligne, we are only considering the diagrams B_n and $D_n^{\mathbb{R}}$. The reader may also consult [9, Section 2 and Section 10] for a more detailed discussion, where the Dynkin diagram equipped with a special vertex is denoted by $D_n(1)$. □

4.3 The goal of this section is to prove the following result.

Theorem 4.8 *Let V be a Hodge structure of $K3$ type, of dimension n, let q be a Beauville–Bogomolov form for V and let T be a complex torus. Assume that there exist integers a, b and c together with an injective morphism of Hodge structures*

$$V \hookrightarrow H^1(T, \mathbb{Q})^{\otimes a} \otimes (H^1(T, \mathbb{Q})^\vee)^{\otimes b} \otimes \mathbb{Q}(c)$$

with a and b nonnegative.

Assume that the Mumford–Tate group of V is the full special orthogonal group $SO(V)$. Then T contains a simple factor of the Kuga–Satake variety of V as a subquotient up to isogeny.

In particular, the dimension of T is at least $2^{n/2-2}$ if n is even, and $2^{(n-3)/2}$ if n is odd. If n is even and $(-1)^{n/2}\delta$ is not a square in $\mathbb{Q}$, where δ is the discriminant of q, then the dimension of T is at least $2^{n/2-1}$.

Proof Let $H = H^1(T, \mathbb{Q})$, and let G be the Mumford–Tate group of H. By assumption, the Mumford–Tate group of V is reductive. Arguing as in Sect. 3.3.2, we may replace H with its semisimplification and assume that G is reductive.

Let $\mathfrak{g}$ be the Lie algebra of G. As in Sect. 3.3.1, we obtain a surjection

$$p\colon G \longrightarrow SO(V) = MT(V)$$

and a commutative diagram

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{h_G} & G_{\mathbb{R}} \\ & \searrow^{h} & \downarrow{\scriptstyle p_{\mathbb{R}}} \\ & & SO(V)_{\mathbb{R}} \end{array}$$

where h and h_G are the morphisms defining the Hodge structures on V and H respectively. Let

$$\nu\colon \mathbb{G}_{m,\mathbb{C}} \longrightarrow SO(V)_{\mathbb{C}}$$

be the cocharacter defined as the composition

$$\mathbb{G}_{m,\mathbb{C}} \xrightarrow{\mu} \mathbb{S}_{\mathbb{C}} \xrightarrow{h_{\mathbb{C}}} SO(V)_{\mathbb{C}}.$$

We may find connected normal subgroups G_1 and G_2 that commute such that the multiplication map $G_1 \times G_2 \to G$ is an isogeny and the restriction of p to G_1 is an isogeny onto $SO(V)$. Let

$$\widetilde{h}\colon \mathbb{S} \longrightarrow (G_1 \times G_2)_{\mathbb{R}}$$

be the fractional lift of h_G, and let $\widetilde{h}_1$ be the component of $\widetilde{h}$ mapping to $G_{1,\mathbb{R}}$.

By Proposition 3.5, we may find a nonzero sub-Hodge structure H_1 of H such that the fractional Hodge structure induced by $\widetilde{h}_1$ on H_1 has type $\{(a+1, a), (a, a+1)\}$ for some rational number a. In particular, the fractional cocharacter

$$\widetilde{\nu}\colon \mathbb{G}_{m,\mathbb{C}} \longrightarrow G_{1,\mathbb{C}}$$

obtained by lifting ν has only two weights a and $a+1$ on the representation $H_{1,\mathbb{C}}$ of $G_{1,\mathbb{C}}$.

If the dimension n of V is odd (resp. even), Proposition 4.7 shows that H_1 contains the spin representation (resp. one of the half-spin representations). In particular, there is a nonzero

CSpin($V_{\mathbb{C}}$)-equivariant morphism from $H_{1,\mathbb{C}}$ to $H_{KS,\mathbb{C}}$, so that is a nonzero CSpin(V)-equivariant morphism from H_1 to H_{KS}. In particular, these two Hodge structures have a nonzero simple factor in common.

The statement on the dimension on T is now a direct consequence of Theorem 4.3. □

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