

# Iterating Brownian motions, *ad libitum*

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## Abstract

Let  $B_1, B_2, \dots$  be independent one-dimensional Brownian motions defined over the whole real line such that  $B_i(0) = 0$  for every  $i \geq 1$ . We consider the  $n$ th iterated Brownian motion  $W_n(t) = B_n(B_{n-1}(\dots(B_2(B_1(t))))\dots)$ . Although the sequences of processes  $(W_n)_{n \geq 1}$  do not converge in a functional sense, we prove that the finite-dimensional marginals converge. As a consequence, we deduce that the random occupation measures of  $W_n$  converge towards a random probability measure  $\mu_\infty$ . We then prove that  $\mu_\infty$  almost surely has a continuous density which must be thought of as the local time process of the infinite iteration of independent Brownian motions.

*Keywords and phrases.* BROWNIAN MOTION, ITERATED BROWNIAN MOTION, HARRIS CHAIN, RANDOM MEASURE, EXCHANGEABILITY, WEAK CONVERGENCE, LOCAL TIME

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## 1 Introduction

Let  $B_+ = (B_+(t), t \geq 0)$  and  $B_- = (B_-(t), t \geq 0)$  be independent standard one-dimensional Brownian motions starting from 0. The process  $B(t) := B_+(t)$  if  $t \geq 0$  and  $B(t) := B_-(-t)$  if  $t \leq 0$  is called a two-sided Brownian motion. In this paper we study the iterations of independent (two-sided) Brownian motions. Formally, let  $B_1, B_2, \dots$  be a sequence of independent two-sided Brownian motions and, for every  $n \geq 1$  and every  $t \in \mathbb{R}$ , set

$$W_n(t) := B_n \left( B_{n-1} \left( \dots \left( B_2(B_1(t)) \right) \dots \right) \right).$$

Burdzy [Bur93] studied sample path properties of the random function  $t \mapsto W_2(t)$  and coined the terminology of (second) “iterated Brownian motion” for this object. A motivation for this study is that iterated Brownian motion can be used to construct a solution to the 4-th order PDE  $\partial u / \partial t = \frac{1}{8} \partial^4 u / \partial x^4$ ; see [Fun79]. This model has triggered a lot of work, see [Ber96, Bur93, BK95, BK95, ES99, KL99] and the references therein.

Another motivation is that the process  $W_n$  is not a semimartingale (unless  $n = 1$ ). Indeed, for  $n = 2$ , a simple calculation shows that the quadratic variation of  $W_2 = B_2 \circ B_1$  does not exist, but its quartic variation does. Similarly the  $2^n$ -variation of  $W_n$  is finite for  $n \geq 1$ . Hence, as  $n$  increases, the process  $W_n$  becomes wilder and wilder. Also,  $W_n$  is self-similar with index  $2^{-n}$ , i.e.,

$$(W_n(\alpha t), t \in \mathbb{R}) \stackrel{(d)}{=} (\alpha^{2^{-n}} W_n(t), t \in \mathbb{R}),$$

for all  $\alpha > 0$ . See Fig. 1 for a comparison of sample paths of  $W_n$ ,  $n = 1, 2, 3$ . All these reasons make one suspect that convergence of the laws of the  $W_n$ 's in a “nice” function space (e.g., the space of continuous functions) is impossible (see Remark 1).

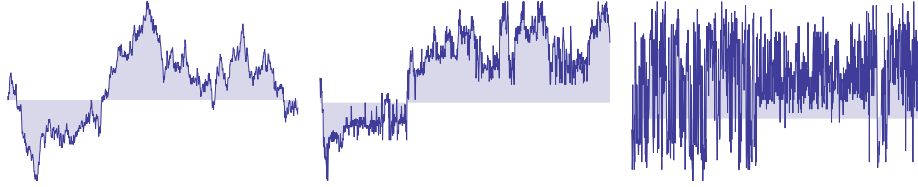


Figure 1: First, second and third iterations of Brownian motions.

However, following the principle of Berman saying that wild functions must have smooth local times (see the nice survey [GH80]), we prove that the occupation measures  $\mu_n$  of  $W_n$  over  $[0, 1]$  converge in distribution towards a random measure  $\mu_\infty$  which can be interpreted as the occupation measure of the infinite iteration “ $W_\infty$ ”. More precisely, let  $\mu_n$  be the random probability measure defined by

$$\int_{\mathbb{R}} d\mu_n f = \int_0^1 dt f(W_n(t)), \quad (1)$$

for every Borel  $f : \mathbb{R} \rightarrow \mathbb{R}_+$ . Our main result is a limit theorem for the sequence  $(\mu_n)_{n \geq 1}$ . Restricting the integration to the unit interval is convenient and poses no loss of generality due to the self-similarity property of  $W_n$ .

Let  $\mathcal{M}$  be the set of all positive Radon measures on  $\mathbb{R}$ . Although we focus on the real line, the interested reader should consult [Kal76] for the general theory of random measures. Endow  $\mathcal{M}$  with the topology  $\mathcal{T}$  of vague convergence, that is, the weakest topology which makes the mappings

$$\mu \in \mathcal{M} \mapsto \mu f := \int_{\mathbb{R}} d\mu f, \quad f \in C_K,$$

continuous. (Here,  $C_K$  is the set of continuous functions with compact support.) A *random measure* is a random element of the space  $(\mathcal{M}, \mathcal{T})$ , viewed as a measurable space with  $\sigma$ -algebra generated by the sets in  $\mathcal{T}$ . A sequence  $\lambda_1, \lambda_2, \dots$  of random measures converges in distribution towards a random measure  $\lambda$  if for any bounded continuous mapping  $F : (\mathcal{M}, \mathcal{T}) \rightarrow \mathbb{R}$  we have  $E[F(\lambda_i)] \rightarrow E[F(\lambda)]$  as  $i \rightarrow \infty$ . We write

$$\lambda_n \xrightarrow{(d)} \lambda,$$

to denote this notion. Convergence of  $\lambda_n$  to  $\lambda$  in distribution is equivalent to:  $\lambda_n f \xrightarrow{(d)} \lambda f$ , for any continuous  $f \in C_K$  (see Theorem 4.2 in [Kal76]). The latter convergence is convergence in distribution of real-valued random variables.

**Theorem 1.** *There exists a random measure  $\mu_\infty$  such that  $\mu_n \xrightarrow{(d)} \mu_\infty$ . Moreover  $\mu_\infty(\mathbb{R}) = 1$ , a.s. The random probability measure  $\mu_\infty$  almost surely admits a density  $(L_a)_{a \in \mathbb{R}}$  with respect to the Lebesgue measure such that  $a \mapsto L_a$  has compact support and is Hölder continuous with exponent  $1/2 - \varepsilon$  for every  $\varepsilon > 0$ .*

We can think of  $\mu_\infty$  as the occupation measure of the infinite iteration of i.i.d. Brownian motions. Thus the random function  $(L_a)_{a \in \mathbb{R}}$  must be thought of as the local time of this infinite iteration. The convergence of the last theorem will be obtained by proving convergence of finite dimensional marginals of the iterated Brownian motions, namely:

**Theorem 2.** *For any integer  $p \geq 1$ , there exists a random vector  $(X_i)_{1 \leq i \leq p}$  such that for any pairwise distinct nonzero real numbers  $x_1, x_2, \dots, x_p$  we have the following convergence in distribution*

$$(W_n(x_i))_{1 \leq i \leq p} \xrightarrow[n \rightarrow \infty]{(d)} (X_i)_{1 \leq i \leq p}. \quad (2)$$

The random variables  $X_1, \dots, X_p$  and the differences  $X_i - X_j$ ,  $i \neq j$ , have all identical distribution which is that of a “signed exponential” with parameter 2, i.e. a distribution with density  $e^{-2|x|}$ ,  $x \in \mathbb{R}$ .

The case  $p = 1$  has already been noticed in the Physics literature [Tur04] in the context of infinite iteration of i.i.d. random walks. We are unfortunately unable to give an explicit formula for the distribution of  $(X_1, \dots, X_p)$  when  $p \geq 2$ , see Section 4 for a discussion and simulations of this intriguing probability distribution. It is quite interesting to observe that the marginals and differences have all the same distribution but are, of course, dependent. In a certain sense, the infinite iteration is both self-similar at all scales and long-range dependent.

The paper is organized as follows. The first section is devoted to the proof of Theorem 2 which is the cornerstone in the proof of Theorem 1. We then turn to the study of occupation measures in Section 2. The last section presents some open questions and comments.

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## 2 Finite dimensional marginals

In this section, we prove Theorem 2. The convergence of one-dimensional marginal is a special case because we can explicitly give its limiting distribution, which will be of great use throughout this paper. In the general case, the convergence of finite dimensional marginals comes from ergodic property of random iterations of independent Brownian motions.

### 2.1 One-dimensional marginals

Let  $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ . For  $\lambda > 0$ , we denote by  $\pm\mathcal{E}(\lambda)$  a signed exponential distribution with parameter  $\lambda$ , i.e., one which has density proportional to  $e^{-\lambda|x|}$ ,  $x \in \mathbb{R}$ .

**Proposition 3.** *For any  $t \in \mathbb{R}^*$  we have the following convergence in distribution*

$$W_n(t) \xrightarrow[n \rightarrow \infty]{(d)} \pm\mathcal{E}(2), \quad (3)$$

**Remark 1.** *Proposition 3 already implies that the sequence of processes  $(W_n)$  is not tight for the topology of uniform convergence on compact intervals. Indeed, if this were the case, and since  $W_n(0) = 0$ , we would have that  $\sup_{n \geq 1} P(W_n(\varepsilon) > \eta) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , for every  $\eta > 0$ , and this contradicts Proposition 3.*

*Proof.* By standard properties of Gaussian variables we have the following chain of equalities in distribution

$$\begin{aligned} W_n(t) &= B_n \left( \dots (B_3(B_2(B_1(t)))) \dots \right) \\ &\stackrel{(d)}{=} B_n \left( \dots (B_3(B_2(\sqrt{|t|}B_1(1)))) \dots \right) \\ &\stackrel{(d)}{=} B_n \left( \dots (B_3(|t|^{1/4} \sqrt{|B_1(1)|} B_2(1))) \dots \right) \\ &\stackrel{(d)}{=} \pm t^{2^{-n}} \prod_{i=0}^{n-1} |\mathcal{N}_i|^{2^{-i}}, \end{aligned}$$

where  $\mathcal{N}_i$  are i.i.d. standard normal variables and  $\pm$  is an independent fair random sign. It is then easy to see that the modulus of the right hand side of the last display actually converges

almost surely as  $n \rightarrow \infty$ . Indeed the series of  $P(|\log(|\mathcal{N}_i|^{2^{-i}})| > i^{-2})$  is summable and an application of the Borel-Cantelli lemma proves the claim. Thus, if we set

$$\mathcal{X} := \lim_{n \rightarrow \infty} \prod_{i=0}^{n-1} |\mathcal{N}_i|^{2^{-i}} \quad a.s.,$$

we have the convergence  $W_n(t) \rightarrow \pm \mathcal{X}$  in distribution as  $n \rightarrow \infty$ , where  $\pm$  is fair random sign independent of  $\mathcal{X}$ . Notice that the limit does not depend on  $t$  as long as  $t \neq 0$ . To identify the limit distribution, we note that  $\mathcal{X}$  satisfies the following recursive distributional equation

$$\mathcal{X} \stackrel{(d)}{=} \sqrt{\mathcal{X}} \cdot |\mathcal{N}|. \quad (\text{E})$$

Iterating this equation and applying the same arguments as above, it is easy to see that it admits a unique fixed point as long as  $\mathcal{X} > 0$  almost surely. Guided by the result of [Tur04], an easy computation shows that the exponential variable of density  $2e^{-2x}\mathbf{1}_{x>0}$  also satisfies (E) thus completing the proof of the proposition.  $\square$

## 2.2 General case

The goal of this section is to prove Theorem 2 in the case  $p \geq 2$ . The convergence (2) will be achieved by applying arguments from the theory of Harris chains. The idea is to consider the random transformation which associates with any  $p$  points  $x_1, \dots, x_p \in \mathbb{R}$  the images  $B(x_1), \dots, B(x_p)$  of these  $p$  points under a two-sided Brownian motion  $B$  and to show that independent applications of this map possess an ergodic property. That is, for any initial state  $(x_1, \dots, x_p)$ , the distribution of  $(W_n(x_1), \dots, W_n(x_p))$  converges weakly to a unique invariant probability measure.

Let  $p \geq 2$ . Denote by  $\mathcal{R}^p$  the set of  $p$ -uplets  $(x_1, \dots, x_p)$  of pairwise distinct nonzero real numbers. Note that if  $\mathbf{x} = (x_1, \dots, x_p) \in \mathcal{R}^p$  then its image  $B(\mathbf{x}) := (B(x_1), \dots, B(x_p))$  under a two-sided Brownian motion  $B$  almost surely belongs to  $\mathcal{R}^p$ .

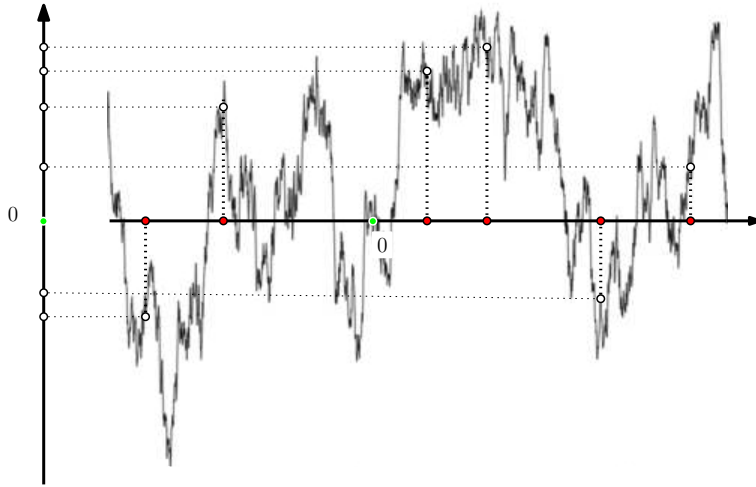


Figure 2: A 6-uplet and its image after applying an independent Brownian motion.

**Proposition 4.** *For any  $p \geq 2$ , there is a unique probability measure  $\nu_p$  on  $\mathcal{R}^p$  such that if  $(X_1, \dots, X_p)$  is distributed according to  $\nu_p$  and if  $B$  is an independent two-sided Brownian*

motion then we have the equality in distribution

$$(B(X_i))_{1 \leq i \leq p} \stackrel{(d)}{=} (X_i)_{1 \leq i \leq p}. \quad (4)$$

Furthermore, for any  $\mathbf{x} = (x_1, \dots, x_p) \in \mathcal{R}^p$  we have the following convergence in distribution

$$(W_n(x_i))_{1 \leq i \leq p} \xrightarrow[n \rightarrow \infty]{(d)} (X_i)_{1 \leq i \leq p}.$$

Notice that this argument does not give an explicit expression for the stationary probability measures  $\nu_p$  but characterizes them uniquely by (4). Let  $p \geq 2$  be a fixed integer. Fix also a  $p$ -uplet  $\mathbf{x} = (x_1, \dots, x_p) \in \mathcal{R}^p$ . For every  $n \geq 1$  we set

$$\mathcal{W}_n := (W_n(x_1), \dots, W_n(x_p)).$$

Thus the process  $(\mathcal{W}_n)_{n \geq 0}$  is a Markov chain with state space  $\mathcal{R}^p$ , starting from  $\mathcal{W}_0 := \mathbf{x}$ , with transition probability kernel given by

$$\mathbb{P}(\mathbf{y}; A) = E[(B(y_1), \dots, B(y_p)) \in A],$$

for any  $\mathbf{y} \in \mathcal{R}^p$  and  $A \subset \mathcal{R}^p$ , where  $B$  is a two-sided Brownian motion. We denote by  $E_{\mathbf{x}}[\cdot]$  the expectation of this chain started from  $\mathbf{x} \in \mathcal{R}^p$ . Since we restricted ourselves to  $\mathcal{R}^p$ , the chain  $(\mathcal{W}_n)$  is easily seen to be irreducible with respect to the  $p$ -dimensional Lebesgue measure on  $\mathcal{R}^p$  and aperiodic. We will show that the chain is in fact positive Harris recurrent, which will imply that it admits a unique invariant probability measure, and Proposition 4 will directly flow from it, see [MT09, Theorem 13.0.1]. The key to prove this is to consider the following sets:

**Definition 1.** Fix  $M > 1$  a (large) real number. We say that the  $p$ -uplet  $\mathbf{x} = (x_1, \dots, x_p) \in \mathcal{R}^p$  is  $M$ -sparse if we have

$$M^{-1} \leq \sup_{1 \leq i \leq p} |x_i| \leq M, \quad \text{and} \quad \min_{1 \leq i \neq j \leq p} |x_i - x_j| \geq M^{-1}.$$

The set of all  $M$ -sparse  $p$ -uplets in  $\mathcal{R}^p$  is denoted by  $S_M$ .

The basic observation is that if  $\mathbf{x} = (x_1, \dots, x_p)$  and  $\mathbf{y} = (y_1, \dots, y_p)$  are two  $M$ -sparse  $p$ -uplets and if  $B$  is a two-sided Brownian motion then the two random  $p$ -uplets  $(B(x_1), \dots, B(x_p))$  and  $(B(y_1), \dots, B(y_p))$  are mutually absolutely continuous with Radon-Nikodým derivative bounded from below by a positive constant  $c_{p,M}$  depending on  $M$  and  $p$  only. The reason is that the ratio of the two densities  $f(x_1, \dots, x_p)/f(y_1, \dots, y_p)$  is a continuous function on the compact set  $S_M$ . In other words, there exists two (dependent) Brownian motions  $B$  and  $\tilde{B}$  such that  $B(x_1) = \tilde{B}(y_1), \dots, B(x_p) = \tilde{B}(y_p)$  with probability at least  $c_{p,M}$ . Put it otherwise, if we fix  $\mathbf{x} \in S_M$ , then

$$\mathbb{P}(\mathbf{y}, A) \geq c_{p,M} \mathbb{P}(\mathbf{y}, A),$$

for all  $\mathbf{y} \in S_M$  and all measurable  $A \subset \mathcal{R}^p$ . Thus Ney's minorization condition holds and, therefore, the set  $S_M$  is a petite set in the sense of [MT09, Chapter 5].

*Proof of Proposition 4.* In order to prove that  $(\mathcal{W}_n)$  is positive Harris recurrent, we will show that, for some  $M > 0$ , the expected return time to the petite set  $S_M$ , by the Markov chain  $(\mathcal{W}_n)$  started from  $\mathbf{x}$ , is bounded above by a finite constant, *uniformly* over all  $\mathbf{x} \in S_M$ .

The technical tool that we use here is the so-called drift condition, see [MT09], for a Lyapunov or "potential" function  $V : \mathcal{R}^p \rightarrow \mathbb{R}_+$ . It is required that  $V$  be unbounded on  $\mathcal{R}^p$ . The drift of such a function is defined by

$$DV(x) := E_{\mathbf{x}}[V(\mathcal{W}_1)] - V(\mathbf{x}), \quad \mathbf{x} \in \mathcal{R}^p.$$

Thus,  $D$  is the generator of the Markov chain. We will show that there exists a Lyapunov function  $V$  such that, for some  $0 < a, b < \infty$ ,

$$DV(x) \leq -a + b\mathbf{1}_{\mathbf{x} \in S_M}, \quad (5)$$

for every  $\mathbf{x} \in \mathcal{R}^p$ . More precisely, we show that the preceding condition is satisfied for the function  $V$  defined by

$$V(x_1, \dots, x_p) := \max_{1 \leq i \leq p} |x_i| + \sum_{0 \leq i < j \leq p} |x_i - x_j|^{-1/2},$$

and for some  $a, b, M > 0$ . In the last display, we use the notation  $x_0 := 0$  to avoid extra terms in the definition of  $V$ . It is convenient to consider the terms  $U(\mathbf{x}) := \max_{1 \leq i \leq p} |x_i|$ , and  $G(\mathbf{x}) := \sum_{0 \leq i < j \leq p} |x_i - x_j|^{-1/2}$  comprising  $V$ , separately. For any  $\lambda > 0$ ,

$$E_{\mathbf{x}}[U(W_1)] = E\left[\max_{1 \leq i \leq p} |B(x_i)|\right] = \sqrt{\lambda} E\left[\max_{1 \leq i \leq p} |B(x_i/\lambda)|\right]$$

Letting  $\lambda = U(\mathbf{x})$ , we thus have

$$E_{\mathbf{x}}[U(W_1)] \leq C_1 \sqrt{U(\mathbf{x})},$$

where  $C_1 = E[\max\{|B(t)| : -1 \leq t \leq 1\}]$ . For the other term, we have, with  $\mathcal{N}$  a standard normal variable,

$$\begin{aligned} E_{\mathbf{x}}[G(W_1)] &= \sum_{0 \leq i < j \leq p} E[|B(x_i) - B(x_j)|^{-1/2}] = \sum_{0 \leq i < j \leq p} |x_i - x_j|^{-1/4} E[|\mathcal{N}|^{-1/2}] \\ &\leq C_2 \sqrt{G(\mathbf{x})}, \end{aligned}$$

where  $C_2 = pE[|\mathcal{N}|^{-1/2}]$ . Putting the terms together, we find

$$E_{\mathbf{x}}[V(W_1)] \leq C_3 \sqrt{V(\mathbf{x})},$$

where  $C_3 = 2 \max(C_1, C_2)$ . Thus, for all  $x \in \mathcal{R}^p$ ,

$$DV(\mathbf{x}) \leq C_3 \sqrt{V(\mathbf{x})} - V(\mathbf{x}).$$

If  $\mathbf{x} \in S_M$  then  $V(\mathbf{x}) \leq M + (p+1)^2 \sqrt{M} < \infty$ . If  $\mathbf{x} \notin S_M$  then  $|x_i| > M$  for some  $i$  or  $|x_i - x_j| < 1/M$  for some  $0 \leq i < j \leq p$ . In the first instance,  $V(\mathbf{x}) > M$ ; in the second instance,  $V(\mathbf{x}) > \sqrt{M}$ . So, for  $M > 1$ , we have  $V(\mathbf{x}) > \sqrt{M}$ , for all  $\mathbf{x} \notin S_M$ . Let  $M = \max(16C_3^4, 1)$ . Then for  $\mathbf{x} \notin S_M$ , we have  $\sqrt{V(\mathbf{x})} > 2C_3$ , implying that  $DV(\mathbf{x}) < -C_3 \sqrt{V(\mathbf{x})} < -C_2 M^{1/4}$ . Thus (5) holds. Theorem 13.0.1 in [MT09] now applies, showing positive Harris recurrence. This finishes the proof of the proposition.  $\square$

**Remark 2.** *The proof actually shows that there are constants  $C_4, C_5 > 0$  such that*

$$DV(\mathbf{x}) \leq -C_4 \sqrt{V(\mathbf{x})},$$

*when  $V(\mathbf{x}) > C_5$ , enabling one to establish a rate of convergence towards the stationary distribution. We shall not pursue this further herein.*

### 2.3 Properties of the $\nu_p$ 's

We can think of  $\nu_p$  as a measure on the product space  $\mathbb{R}^p$  (by letting it have mass 0 outside  $\mathcal{R}^p$ ). The probability measures  $(\nu_p)_{p \geq 1}$  are consistent. This follows from the fact that  $\nu_p$  is the limit in distribution of  $(W_n(x_1), \dots, W_n(x_p))$ , whereas  $\nu_{p+1}$  is the limit of  $(W_n(x_1), \dots, W_n(x_p), W_n(x_{p+1}))$ , for any  $(x_1, \dots, x_p, x_{p+1}) \in \mathcal{R}^p$ . Therefore  $\nu_p$  is the projection of  $\nu_{p+1}$  on the first  $p$  coordinates. By Kolmogorov's extension theorem, there is a probability measure  $\nu$  on  $\mathbb{R}^{\mathbb{N}}$  such that  $\nu_p$  is obtained from  $\nu$  by projecting on the first  $p$  coordinates. The family  $((\nu_p)_{p \geq 1}, \nu)$  has some further properties.

### 2.3.1 Exchangeability

The last statement of Proposition 4 actually shows that  $\nu$  is exchangeable, that is, invariant under permutations of finitely many coordinates. Let  $(X_1, X_2, \dots)$  be a random element of  $\mathbb{R}^{\mathbb{N}}$  with law  $\nu$ . By the classical de Finetti/Ryll-Nardzewski/Hewitt-Savage theorem (Theorem 11.10 [Kal02]) we know that there exists a random element of  $\mathcal{M}$  (random probability measure)  $\mu_\infty$ , such that the law of  $(X_1, X_2, \dots)$  conditionally on  $\mu_\infty$  is a product measure (the law of i.i.d. random variables). The random measure  $\mu_\infty$  will be identified and interpreted in the next section as the occupation measure of the infinitely iterated Brownian motion.

### 2.3.2 Stationarity of the increments

The family of measures  $(\nu_p)_{p \geq 1}$  also possesses another property which can be described as stationarity of the increments.

**Proposition 5.** *Let  $p \geq 2$  and let  $(X_1, X_2, \dots, X_p)$  be distributed according to  $\nu_p$ . Then for every  $1 \leq \ell \leq p$  we have*

$$(X_1 - X_\ell, \dots, X_{\ell-1} - X_\ell, X_{\ell+1} - X_\ell, \dots, X_p - X_\ell) \stackrel{(d)}{=} \nu_{p-1}.$$

*Proof.* Let  $(x_1, x_2, \dots, x_p) \in \mathcal{R}^p$ . It is easy to prove by induction on  $n \geq 1$  and using elementary manipulation of the Gaussian distribution that the random vector  $(W_n(x_i) - W_n(x_\ell))_{i \neq \ell}$  has the same distribution as the vector  $(W_n(x_i - x_\ell))_{i \neq \ell}$ . Notice that the vector  $(x_i - x_\ell)_{i \neq \ell} \in \mathcal{R}^{p-1}$ , thus we can apply Proposition 4 and this finishes the proof of the proposition.  $\square$

## 3 Occupation measure of the infinitely iterated Brownian motion

### 3.1 Existence of $\mu_\infty$

Recall the definition of the random measures  $\mu_n$  given by formula (1) in the Introduction, as well as the notion of convergence in distribution of random elements of  $(\mathcal{M}, \mathcal{T})$ . With Theorem 2 in our hands, it is now easy to prove convergence in distribution of the random measures  $\mu_n$ . We first need a lemma, characterizing convergence in distribution of random elements of  $(\mathcal{M}, \mathcal{T})$ , tailor-made for our case. The lemma can be of independent interest.

**Lemma 6.** *Let  $\lambda_1, \lambda_2, \dots$  be random probability measures on  $\mathbb{R}$ . The following are equivalent:*

- (i) *The sequence  $\lambda_n$  converges in distribution to some random probability measure.*
- (ii) *For each  $n \geq 1$ , and conditionally on  $\lambda_n$ , let  $X_1^n, X_2^n, \dots$  be i.i.d. real-valued random variables each with (conditional) law  $\lambda_n$ :*

$$P(X_1^n \in A_1, \dots, X_p^n \in A_p \mid \lambda_n) = \lambda_n(A_1) \cdots \lambda_n(A_p), \quad a.s.,$$

*for all  $p \geq 1$ , and Borel sets  $A_1, \dots, A_p$ . The random vector  $(X_1^n, \dots, X_p^n)$  converges in distribution, as  $n \rightarrow \infty$ , to some probability measure on  $\mathbb{R}^p$ .*

*Proof.* The implication (i)  $\Rightarrow$  (ii) is an easy consequence of [Kal76, Theorem 4.2]. For the other direction, fix  $f \in C_K$ . We will show that the random variable  $\lambda_n f = \int_{\mathbb{R}} f d\lambda_n$  converges in distribution as  $n \rightarrow \infty$ . Consider the random probability measure

$$\xi_n^{(p)} := p^{-1} \sum_{i=1}^p \delta_{X_i^n}$$

(i.e., the empirical distribution of  $(X_1^n, \dots, X_p^n)$ ). We compare  $\lambda_n f$  to  $\xi_n^{(p)} f = p^{-1} \sum_{i=1}^p f(X_i^n)$ . We have

$$P(|\lambda_n f - \xi_n f| \geq \varepsilon) = E[P(|\lambda_n f - \xi_n f| \geq \varepsilon \mid \lambda_n)] \leq \frac{\|f\|_\infty^2}{\varepsilon^2 p},$$

by Chebyshev's inequality. On the other hand,  $\xi_n^{(p)} f$  converges in distribution as  $n \rightarrow \infty$ . Hence  $\lambda_n f$  converges in distribution, and thus  $\lambda_n$  converges in distribution to some random measure  $\lambda$ . It remains to prove that  $\lambda$  almost surely has mass one. Since  $X_1^n$  converges in distribution, it follows that the sequence of random variables  $\{X_1^n\}_{n \geq 1}$  is tight. Thus for every  $\varepsilon > 0$  there exists  $M > 0$  such that  $P(|X_1^n| > M) \leq \varepsilon$  for every  $n \geq 1$ . Conditionally on  $\lambda_n$  we have  $P(|X_1^n| > M \mid \lambda_n) = 1 - \lambda_n([-M, M])$ . Taking expectation we deduce that  $E[\lambda_n([-M, M])] \geq 1 - \varepsilon$  for every  $n \geq 1$ . This is sufficient to apply Theorem 4.9 in [Kal86] and deduce that  $\lambda$  is almost surely a random probability measure.  $\square$

Let us go back to our setting and show that the occupation measures  $\mu_n$  converge to a random probability measure  $\mu_\infty$ . Let  $p \geq 1$  and, conditionally on  $\mu_n$ , let  $X_1^n, \dots, X_p^n$  be i.i.d. random variables with common distribution  $\mu_n$ . We show that  $(X_1^n, \dots, X_p^n)$  converges (unconditionally) towards the random vector  $(X_1, \dots, X_p)$  of law  $\nu_p$  identified in Proposition 4. Indeed, by the definition of  $\mu_n$ , for any Borel bounded function  $f : \mathbb{R}^p \rightarrow \mathbb{R}$  we have, by Fubini's theorem,

$$\begin{aligned} E[f(X_1^{(n)}, \dots, X_p^{(n)})] &= E\left[\int_0^1 du_1 \dots du_p f(W_n(u_1), \dots, W_n(u_p))\right] \\ &= \int_0^1 du_1 \dots du_p E[f(W_n(u_1), \dots, W_n(u_p))]. \end{aligned}$$

The last integral converges towards  $E[f(X_1, \dots, X_p)]$  as  $n \rightarrow \infty$  because of Theorem 2 and dominated convergence. Applying Lemma 6 we get the existence of a random probability measure  $\mu_\infty$  such that  $\mu_n \rightarrow \mu_\infty$  in distribution as  $n \rightarrow \infty$ .

### 3.2 Support of $\mu_\infty$

**Proposition 7.** *Almost surely, the random probability measure  $\mu_\infty$  has a bounded support.*

In order to prove the last proposition we use a very general fact:

**Lemma 8.** *Let  $\lambda_n$  be a sequence of random probability measures converging in distribution towards a random probability measure  $\lambda_\infty$ . Suppose that the support of  $\lambda_n$  is contained in  $[A_n, B_n]$  and that the sequences of random variables  $(A_n)$  and  $(B_n)$  are tight. Then  $\lambda_\infty$  has a compact support, almost surely.*

*Proof.* Let us argue by contradiction and suppose that  $\lambda_\infty$  has probability at least  $\varepsilon > 0$  of having an unbounded support. By the assumption made on  $A_n$  and  $B_n$  there exists  $M > 0$  such that  $P(|A_n| \geq M) \leq \varepsilon/10$  and  $P(|B_n| \geq M) \leq \varepsilon/10$  for every  $n \geq 1$ . For the  $M > 0$  chosen above, there exists  $\delta > 0$  such that we have  $\lambda_\infty([-M, M]^c) > \delta$  with probability at least  $\varepsilon/2$ . Now choose  $p \geq \delta^{-1}$ . Suppose that, conditionally on  $\lambda_\infty$ , the random variables  $X_1^\infty, \dots, X_p^\infty$  are i.i.d. with common law  $\lambda_\infty$ . We then have

$$\begin{aligned} P\left(\sup_{1 \leq i \leq p} |X_i^\infty| \geq M\right) &= E\left[P\left(\sup_{1 \leq i \leq p} |X_i^\infty| \geq M \mid \lambda_\infty([-M, M]^c) > \delta\right)\right] \\ &\geq \frac{\varepsilon}{2} (1 - (1 - \delta)^p) \geq \varepsilon(1 - e^{-1})/2. \end{aligned} \tag{6}$$

On the other hand, if  $X_1^n, \dots, X_p^n$  are i.i.d., conditionally on  $\lambda_n$ , we have

$$P\left(\sup_{1 \leq i \leq p} |X_i^n| \geq M\right) \leq P(|A_n| \geq M) + P(|B_n| \geq M) \leq \varepsilon/5. \quad (7)$$

Since for  $p$  fixed we have  $(X_i^n)_{1 \leq i \leq p} \rightarrow (X_i^\infty)_{1 \leq i \leq p}$  in distribution as  $n \rightarrow \infty$ , comparing (6) and (7) leads to a contradiction.  $\square$

We further establish a result on the limit of the oscillation of  $W_n$  on an interval, as  $n \rightarrow \infty$ . Recall, that the oscillation of a function  $f$  on an interval  $J$  is defined by  $\text{osc}(f; J) := \sup_{s, t \in J} |f(t) - f(s)|$ , and let, for  $t > 0$ ,

$$\Delta_n(t) := \text{osc}(W_n; [0, t]).$$

**Lemma 9.** *Let  $D := \text{osc}(B; [0, 1])$  be the oscillation of a Brownian motion on a unit interval and let  $D_1, D_2, \dots$  be i.i.d. copies of  $D$ . Then, for all  $t > 0$ ,*

$$\Delta_n(t) \xrightarrow[n \rightarrow \infty]{(d)} \prod_{i=0}^{\infty} D_i^{2^{-i}}.$$

*Proof.* Let

$$I_n(t) := \inf_{x \in [0, t]} W_n(x), \quad S_n(t) := \sup_{x \in [0, x]} W_n(x).$$

Then  $\Delta_n(t) = S_n(t) - I_n(t)$ , and

$$\begin{aligned} \Delta_{n+1}(t) &= \sup_{0 \leq x \leq t} B_{n+1}(W_n(x)) - \inf_{0 \leq x \leq t} B_{n+1}(W_n(x)) \\ &= \sup_{I_n(t) \leq u \leq S_n(t)} B_{n+1}(u) - \inf_{I_n(t) \leq u \leq S_n(t)} B_{n+1}(u) \\ &= \text{osc}(B_{n+1}; [I_n(t), S_n(t)]) \\ &\stackrel{(d)}{=} \text{osc}(B_{n+1}; [0, \Delta_n(t)]) \\ &= \sqrt{\Delta_n(t)} \text{osc}(B_{n+1}; [0, 1]). \end{aligned}$$

Thus iterating this equation, we get in the spirit of the proof of Proposition 3 the following equality in distribution

$$\Delta_n(t) \stackrel{(d)}{=} t^{2^{-n}} \prod_{i=1}^n D_i^{2^{-(i-1)}}.$$

An argument similar to the one used in the proof of Proposition 3 shows that the right-hand side of the last display actually converges almost surely as  $n \rightarrow \infty$ , to the infinite product  $\prod_{i=0}^{\infty} D_i^{2^{-i}}$ . Hence  $\Delta_n(t)$  converges in distribution to the same random variable.  $\square$

**Remark 3.** *Roughly speaking, the oscillation of the infinitely iterated Brownian motion is the same, in distribution, over any interval of any length.*

*Proof of Proposition 7.* By Lemma 9,  $\Delta_n(1)$  converges in distribution, it is tight. Thus the support of  $\mu_n$  is contained in a compact interval whose endpoints are tight. Applying Lemma 8, we deduce that, almost surely,  $\mu_\infty$  has bounded support, as required.  $\square$

### 3.3 Density of $\mu_\infty$

This section is devoted to the analysis of the properties of the density of  $\mu_\infty$ . We will proceed in two steps. First, using standard technique of Fourier analysis for occupation densities (see e.g. [Ber69]), we will prove that  $\mu_\infty$  almost surely has a density which is in  $\mathbb{L}^2$ . At the same time, we obtain some estimates about this Fourier transform. We will then use a very general result of Pitt [Pit78] on local times to prove that this density is in fact continuous and even Hölder continuous with exponent  $1/2 - \varepsilon$  for every  $\varepsilon > 0$ .

### 3.3.1 Harmonic analysis of $\mu_\infty$

For  $\xi \in \mathbb{R}$ , let

$$\Phi(\xi) := \int_{\mathbb{R}} d\mu_\infty(x) \exp(i\xi x),$$

be the Fourier transform of the random probability measure  $\mu_\infty$ .

**Proposition 10.** *For any  $\xi \in \mathbb{R}$  we have*

$$E[|\Phi(\xi)|^2] = \frac{4}{4 + \xi^2}. \quad (8)$$

*Proof.* By the definition of the Fourier transform  $\Phi(\xi)$ , we have

$$E[|\Phi(\xi)|^2] = E \left[ \int_{\mathbb{R}^2} d\mu_\infty(x) d\mu_\infty(y) e^{i\xi(x-y)} \right].$$

By the convergences already established we have

$$E \left[ \int_{\mathbb{R}^2} d\mu_\infty(x) d\mu_\infty(y) e^{i\xi(x-y)} \right] = \lim_{n \rightarrow \infty} E \left[ \exp(i\xi(W_n(U_1) - W_n(U_2))) \right],$$

where  $U_1, U_2$  are two independent random variables uniformly distributed over  $[0, 1]$  and also independent of the sequence of Brownian motions  $B_1, B_2, \dots$ . By stationarity of the increments (see the proof of Proposition 5), for any  $s, t \in \mathbb{R}$ , we have  $W_n(t) - W_n(s) = W_n(t - s)$  in distribution, and thus,

$$\begin{aligned} E \left[ \int_{\mathbb{R}^2} d\mu_\infty(x) d\mu_\infty(y) e^{i\xi(x-y)} \right] &= \lim_{n \rightarrow \infty} E \left[ \exp(i\xi(W_n(U_1 - U_2))) \right] \\ &= \lim_{n \rightarrow \infty} \int \int_0^1 du dv E[\exp(i\xi W_n(u - v))]. \end{aligned}$$

Applying Proposition 3 and the dominated convergence theorem we get that

$$E \left[ \int_{\mathbb{R}^2} d\mu_\infty(x) d\mu_\infty(y) e^{i\xi(x-y)} \right] = E[\exp(i\xi[\pm\mathcal{E}(2)])] = \frac{4}{4 + \xi^2}.$$

□

In particular, applying Fubini's theorem we deduce from (8) that

$$E[\|\Phi\|_2^2] = \int_{\mathbb{R}} \left( \frac{4}{4 + \xi^2} \right)^2 d\xi < \infty,$$

and thus that  $\Phi \in \mathbb{L}^2$  almost surely. By standard results on Fourier transforms, this implies that, almost surely,  $\mu_\infty$  has density  $(L_a)_{a \in \mathbb{R}}$  with respect to the Lebesgue measure which is in  $\mathbb{L}^2$ . Notice that the estimates (8) in fact give us a bit more than that, namely, for every  $0 < s < 1/2$  we have

$$\|\Phi\|_{H^s} := \sqrt{\int_{\mathbb{R}} d\xi (1 + \xi^2)^s |\Phi(\xi)|^2} < \infty, \quad \text{a.s.}$$

Applying the standard Sobolev inequality, we deduce that for every  $0 < s < 1/2$  we have

$$\|L\|_{\mathbb{L}^{2/(1-2s)}} \leq C \|\Phi\|_{H^s},$$

where  $C$  is a universal constant; see Theorem 1.38 in [BCD11]. Since  $s$  can be arbitrarily close to  $1/2$  and since  $L$  is itself a density (thus  $L \in \mathbb{L}^1$  a.s.) we get that  $L \in \mathbb{L}^q$  for any  $1 \leq q < \infty$  almost surely.

### 3.3.2 Continuity of the density

The idea to prove continuity of the density  $(L_a)_{a \in \mathbb{R}}$  is to apply once again a Brownian motion “on top of  $\mu_\infty$ ” and to use the general theory developed in [Pit78]. Formally, if  $\mu_\infty$  has density  $(L_a)_{a \in \mathbb{R}}$  (which is in every  $\mathbb{L}^q$ ,  $1 \leq q < \infty$ ) we define the random measure  $\tilde{\mu}_\infty$  by

$$\tilde{\mu}_\infty f := \int d\mu_\infty(x) f(B(x)) = \int da L_a f(B(a)),$$

where  $B$  is an independent two-sided Brownian motion. Clearly, we have

$$\tilde{\mu}_\infty \stackrel{(d)}{=} \mu_\infty$$

(equality in distribution). So it suffices to show that  $\tilde{\mu}_\infty$  has a continuous density. This follows from the lemma:

**Lemma 11.** *Let  $g : \mathbb{R} \rightarrow \mathbb{R}_+$  be the density of a probability measure supported on a compact interval  $I \subset \mathbb{R}$  such that  $g \in \mathbb{L}^q$  for every  $1 \leq q < \infty$ . Let also  $B$  be a two-sided Brownian motion and consider the occupation measure of  $B$  with respect to the measure  $g(t)dt$  defined as follows*

$$\mu f = \int_{\mathbb{R}} g(t)dt f(B(t)),$$

for any Borel positive function  $f$ . Then the random measure  $\mu$  almost surely has a density which is locally Hölder continuous of exponent  $1/2 - \varepsilon$  for any  $\varepsilon > 0$ .

*Proof.* Let  $0 < \delta < 1$  and choose  $q \geq 1$  such that  $(1 + \delta)q^* < 2$  where  $q^*$  is defined by  $1/q + 1/q^* = 1$ . Applying Hölder’s inequality we get that

$$\sup_{s \in I} \int_I dt \frac{g(t)}{\sqrt{|t-s|^{1+\delta}}} \leq \left( \int_I dt |g(t)|^q \right)^{1/q} \sup_{s \in I} \left( \int_I dt |t-s|^{-(1+\delta)q^*/2} \right)^{1/q^*} < \infty.$$

We can thus apply Theorem 4 of [Pit78] and get that  $\mu$  a continuous density  $L$  over  $\mathbb{R}$  which satisfies a local Hölder condition  $|L(x) - L(y)| \leq K|x-y|^{\delta'}$ , for any  $\delta' < \delta/2$ .  $\square$

## 4 Comments

**Occupation over other measures.** In this work we considered the occupations measures  $\mu_n$  of  $W_n$  over the time interval  $[0, 1]$  but the proofs apply in the more general setting of occupation measures of  $W_n$  over any probability measure on  $\mathbb{R}$  which is not atomic. More precisely if  $\lambda$  is a probability measure on  $\mathbb{R}$  which has no atom, we define the occupation of  $W_n$  over  $\lambda$  by

$$\mu_n^{(\lambda)} f := \int_{\mathbb{R}} \lambda(dt) f(W_n(t)),$$

for any Borel  $f : \mathbb{R} \rightarrow \mathbb{R}_+$ . Then the random probability measures  $\mu_n^{(\lambda)}$  converge towards  $\mu_\infty$  in distribution as well.

**Explicit finite dimensional marginals.** The consistent family distributions  $\{\nu_p, p \geq 1\}$  introduced in Proposition 4 are the limiting distributions of the finite-dimensional marginals of the  $W_n$ ’s. They are characterized by (4). Although they arise naturally in the study of iteration of Brownian motions, to the best of our knowledge, they have not been investigated so far for  $p \geq 2$  and in particular no explicit formulae are known for the density of  $\nu_p$ ,  $p \geq 2$ .

In Fig. 3 many independent points have been sampled according to  $\nu_2$ . A clear shape “sea star” emerges and we conjecture that the level-lines of the density of  $\nu_2$  would be dilatation of this unique shape. However we do not have any candidate for this density.

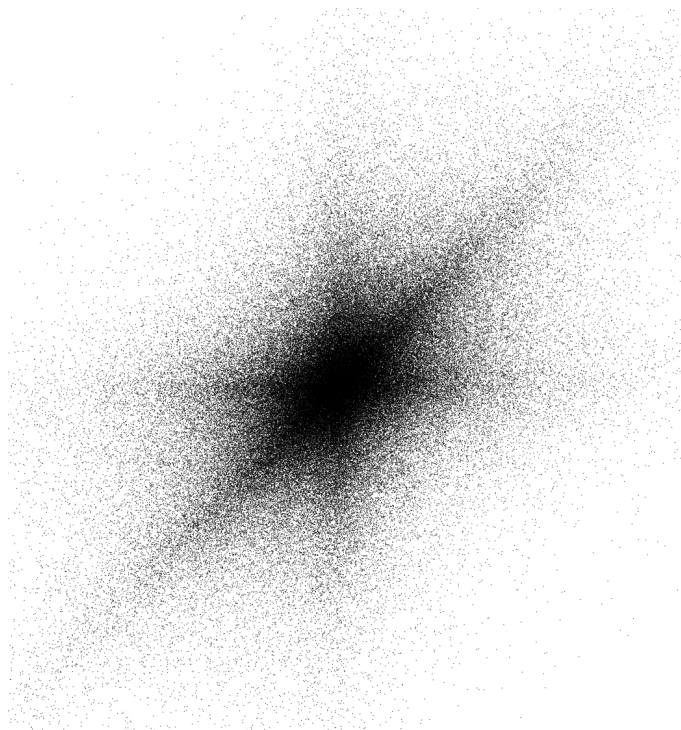


Figure 3: About 300000 independent samples of the distribution  $\nu_2$ . Simulation realized by Alexander Holroyd.

**Funaki's PDE.** Following Funaki [Fun79], we can show that  $W_n$  solves the PDE  $\partial u / \partial t = \partial^k u / \partial x^k$ , where  $k = 2^n$ , with initial data  $u(t = 0, x) = u_0(x)$ ,  $x \in \mathbb{R}$ , where  $u_0$  is a function satisfying growth conditions [Fun79]. The solution is given by  $u_n(t, x) = E[u_0(x + W_n(2^{2^n-1}t))]$ . Although it is probably not possible to give a PDE meaning to the exchangeable collection of random variables  $(W_\infty(x))_{x \in \mathbb{R}}$ , we remark that the limit to the signed exponential translates into  $\lim_{n \rightarrow \infty} u_n(t/2^{2^n-1}, x) = \int_{\mathbb{R}} u_0(x - y) e^{-2|y|} dy$ .

**Ray-Knight theorem.** In the spirit of the famous Ray-Knight theorems, do we have another way to describe the density of  $\mu_\infty$  as a diffusion process?

**Reflected Brownian motion.** It is possible to extend part of the previous work to iterations of reflected Brownian motions. Namely, it is likely that Proposition 4 goes through but Proposition 5 fails, and thus the analysis of the Fourier transform of the limiting measure would be more challenging.

**Fractional Brownian motion.** Fractional Brownian motion generalizes Brownian motion in that it is a Gaussian  $H$ -self-similar process with stationary increments, where  $0 < H < 1$ . Again, it is likely that Proposition 4 go through, but the limit is even harder to describe. Of course, this raises the following interesting question: what kind of processes can be iterated *ad libitum* and result into some kind of limit?

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