Local limits of random graphs

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[a]full of mistakes and typos, please contact us if you found one...

1 Definitions

1.1 The space $\mathscr{G}_\bullet$

A *graph* G is a couple $G = (V, E)$ where V denotes the set of vertices of G and V the set of undirected edges. We will assume that the graphs considered are *simple*, that is they do not have multiple edge nor loop. The degree $\deg(v)$ of a vertex v of G is the number of edges attached to v. In the following except explicitly mentioned all the graphs considered are locally finite (no vertex of infinite degree), countable and connected. The graph distance in G is denoted by $\mathrm{d}_{\mathrm{gr}}^G(.,.)$.
A *rooted* graph (G,ρ) is a graph G together with a distinguished vertex ρ of G. Two rooted graphs (G,ρ) and (G',ρ') are said to be equivalent $(G,\rho) \simeq (G',\rho')$ if there is a graph homomorphism $G \to G'$ that maps ρ to ρ'. Following [4], we define a pseudo-metric on the set of all locally finite connected rooted graphs by

$$\mathrm{d}_{\mathrm{loc}}\left((G,\rho),(G',\rho')\right) \;=\; \Big(1+\sup\left\{r \geqslant 0 : (\mathrm{B}_G(\rho,r),\rho) \simeq (\mathrm{B}_{G'}(\rho',r),\rho')\right\}\Big)^{-1}, \qquad (1)$$

where $B_G(\rho, r)$ stands for the combinatorial ball of radius r around ρ in G for $\mathrm{d}_{\mathrm{gr}}^G$ and $\simeq$ is the graph equivalence. Hence $\mathrm{d}_{\mathrm{loc}}$ induces a metric (still denoted $\mathrm{d}_{\mathrm{loc}}$) on the set $\mathscr{G}_\bullet$ of equivalence classes of (locally finite connected) rooted graphs.

Proposition 1.1. *The metric space $(\mathscr{G}_\bullet, \mathrm{d}_{\mathrm{loc}})$ is Polish (separable and complete). Furthermore the subspace $\mathscr{G}_\bullet^M \subset \mathscr{G}_\bullet$ of (isometry classes of) graphs of degree bounded by $M \geqslant 0$ is compact.*

Remark 1.2. *Formally, elements of $\mathscr{G}_\bullet$ are equivalence classes of rooted graphs, but we will not distinguish between graphs and their equivalence classes and we use the same terminology and notation. One way to bypass this identification is to choose once for all a canonical representant in each class, see [1, Section 2].*

Proof. The countable set of finite rooted graphs is dense in $(\mathscr{G}, \mathrm{d}_{\mathrm{loc}})$. If (G_n, ρ_n) is a Cauchy sequence for $\mathrm{d}_{\mathrm{loc}}$ it is easy to see that the combinatorial balls $\mathrm{B}_{G_n}(\rho_n, r)$ rooted at ρ_n are stationary in n for all r and thus converge to some $\mathrm{B}_G(\rho, r)$ for some (possibly infinite) graph (G, ρ) which is checked to be the limit of the G_n's.

For the second assertion, notice that there are only finitely many rooted graphs in $\mathscr{G}_\bullet^M$ of radius r. Hence a diagonal procedure yields the result. □

1.2 Unbiased random graphs

Let (G,ρ) be a random (rooted) graph which is almost surely finite. Conditionally on (G,ρ) choose independently a new root $\tilde{\rho}$ uniformly over the vertices of G. Equivalently, the distribution ν of $(G,\tilde{\rho})$ is given by

$$\int \mathrm{d}\nu(G,\tilde{\rho}) f(G,\tilde{\rho}) = \int \mathrm{d}\nu(G,\rho) \frac{1}{|G|} \sum_{x\in G} f(G,x),$$

for any Borel positive function f. If the random graph $(G,\tilde{\rho})$ has the same distribution as (G,ρ), that is $\mu=\nu$, we say that (G,ρ) is *unbiased* or *uniformly rooted*.

The class of unbiased random graphs is particulary interesting and generalizes Cayley graphs. Let us introduce a fundamental tool : *The Mass-Transport-Principle*. Suppose we are given a function f that takes as parameters a graph G and two vertices $x,y\in G$ and returns a non-negative number. Suppose also that f is homomorphism invariant that is, if $G\mapsto G'$ is a graph homomorphism that takes x to x' and y to y' then $f(G,x,y)=f(G',x',y')$. Finally assume that f is measurable in the space $\mathscr{G}_{\bullet\bullet}$ of isometry classes of bi-rooted graphs (with an easy extension of $\mathrm{d}_{\mathrm{loc}}$). Then if (G,ρ) is a Cayley graph we have

$$\sum_{x\in G} f(G,\rho,x) \quad = \quad \sum_{x\in G} f(G,\rho,x^{-1}) \quad = \quad \sum_{x\in G} f(G,x,\rho),$$

since $x\mapsto x^{-1}$ is an involution of G. We can generalize the last equation to random graphs as follows. A random rooted graph (G,ρ) of distribution μ satisfies the Mass-Transport-Principle (MTP) [1, 3, 4, 6] iff

$$\int_{\mathscr{G}_\bullet} \mathrm{d}\mu(G,\rho) \sum_{x\in G} f(G,\rho,x) \quad = \quad \int_{\mathscr{G}_\bullet} \mathrm{d}\mu(G,\rho) \sum_{x\in G} f(G,x,\rho), \tag{2}$$

such a graph is also called *unimodular* (coming from the terminology of transitive graphs). Later such functions f will be called *a mass-transport function* and will be interpreted as a quantity of mass that x sends to y. Thus the mean quantity of mass that ρ sends is equal to the mean quantity it receives.

Remark 1.3. *The sum over $x\in G$ in (2) has no meaning because (G,ρ) is formally an equivalence class of rooted graphs. But it is easily checked that the quantity we are interested in do not depend on a representative of (G,ρ).*

Example 1. *Uniformly rooted random graphs satisfy the MTP.*

Proof. Let (G,ρ) distributed according to μ be a uniformly rooted random graph. We have

$$\begin{aligned}\int_{\mathscr{G}_\bullet} \mathrm{d}\mu(G,\rho) \sum_{y\in G} f(G,\rho,y) \quad &= \quad \int_{\mathscr{G}_\bullet} \mathrm{d}\mu(G,\rho) \frac{1}{|G|} \sum_{x,y\in G} f(G,x,y) \\ &= \quad \int_{\mathscr{G}_\bullet} \mathrm{d}\mu(G,\rho) \frac{1}{|G|} \sum_{x,y\in G} f(G,y,x) \;=\; \int_{\mathscr{G}_\bullet} \mathrm{d}\mu(G,\rho) \sum_{y\in G} f(G,y,\rho).\end{aligned}$$

□

Exercice 1. *Show that a random rooted finite graph which satisfies the MTP is uniformly rooted.*

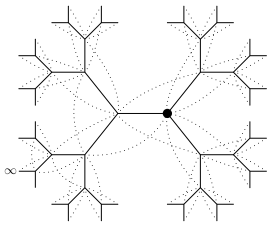

Figure 1: The grand father graph. Consider the transport function $f(G,x,y)=1$ if x is the grand son of y and 0 otherwise. Then the mass transport does not hold.

Counter-Example: Consider the grand father graph. Obtained from the three-regular tree by choosing a direction orienting the tree and adding the edges linking grand fathers to grand sons. Root the tree anywhere you want (it is vertex-transitive).

The definition of uniformly rooted graph cannot be made precise for infinite random graphs. However the MTP still has a sense in that setting. And we have:

Proposition 1.4 ([4]). *Let (G_n,ρ_n) be a sequence of unimodular random graphs that converge in distribution for $\mathrm{d}_{\mathrm{loc}}$ towards (a possibly infinite) random rooted graph (G,ρ). Then (G,ρ) satisfies the MTP.*

Proof. Suppose first that f is a mass-transport-function that only depends on a finite neighborhood around the root vertex that is $f(G,x,y)=\mathbf{1}_{\mathrm{d_{gr}}(x,y)\leqslant k} f(B_G(x,k),x,y)$ for some $k\geqslant 0$. For $l\in\{1,2,\dots\}\cup\{\infty\}$ we let

$$f_l(G,x,y)=\mathbf{1}_{\#\mathrm{B}_G(x,k)<l}\mathbf{1}_{\#\mathrm{B}_G(y,k)<l}\big(f(G,x,y)\wedge l\big),$$

for any bi-rooted graph (G,x,y). Such that the functionals

$$\psi_l:(G,\rho)\in\mathscr{G}_\bullet\longmapsto\sum_{x\in G}f_l(G,\rho,x)\quad\text{and}\quad\phi_l:(G,\rho)\in\mathscr{G}_\bullet\longmapsto\sum_{x\in G}f_k(G,x,\rho),$$

are bounded (thanks to the l) and continuous (thanks to the fact that f only depends on a finite neighborhood). Since (G_n,ρ_n) is unimodular we have $\mathbf{E}\left[\psi_l(G_n,\rho_n)\right]=\mathbf{E}\left[\phi_l(G_n,\rho_n)\right]$ and thus $\mathbf{E}\left[\psi_l(G,\rho)\right]=\mathbf{E}\left[\phi_l(G,\rho)\right]$. By monotone convergence we get $\mathbf{E}\left[\psi_\infty(G,\rho)\right]=\mathbf{E}\left[\phi_\infty(G,\rho)\right]$ which is the MTP with the function f. Going from functions satisfying the above hypothesis to general ones is a standard (but a bit technical) argument. □

Question 1. *Let (G,ρ) be an (infinite otherwise it is easy) unimodular random graph. Does there exists a sequence (G_n,ρ_n) of finite uniformly rooted graphs such that $(G_n,\rho_n)\to(G,\rho)$ in distribution for $\mathrm{d}_{\mathrm{loc}}$? (Very open, but known for trees [**?**]).*

1.3 Links

1.3.1 Stationarity

Consider a mass-transport function of the form $f(G,x,y)=f(G,x,y)\mathbf{1}_{x\sim y}$. Applying the MTP to a unimodular random graph (G,ρ) of law μ with the function f we get

$$\int_{\mathscr{G}_\bullet}\mathrm{d}\mu(G,\rho)\sum_{x\sim\rho}f(G,\rho,x)\;=\;\int_{\mathscr{G}_\bullet}\mathrm{d}\mu(G,\rho)\sum_{x\sim\rho}f(G,x,\rho),$$

or equivalently

$$\int_{\mathscr{G}_\bullet} \mathrm{d}\mu(G,\rho)\ \deg(\rho)\frac{1}{\deg(\rho)}\sum_{x\sim\rho} f(G,\rho,x) \ \ = \ \ \int_{\mathscr{G}_\bullet} \mathrm{d}\mu(G,\rho)\ \deg(\rho)\frac{1}{\deg(\rho)}\sum_{x\sim\rho} f(G,x,\rho).$$

In other words, if $(\bar{G},\bar{\rho})$ is distributed according to (G,ρ) biased by $\deg(\rho)$ (assuming that $\int \mathrm{d}\mu\ \deg(\rho) < \infty$) and if conditionally on $(\bar{G},\bar{\rho})$, X_1 is a one-step simple random walk starting on $\bar{\rho}$ in $\bar{G}$ then we have the following equality in distribution

$$(\bar{G},\bar{\rho},X_1) \ \stackrel{(d)}{=} \ (\bar{G},X_1,\bar{\rho}). \tag{3}$$

Such a graph $(\bar{G},\bar{\rho})$ is called *stationary and reversible*. It is also possible to go from a stationary and reversible random graph towards a unimodular random one by biasing by $\deg(\rho)^{-1}$, see [2].
Another way to introduce the same concept is to say that if we consider a random graph (G,ρ) and let run over it a be-infinite simple random walk $(X_n)_{n\in\mathbb{Z}}$ such that $X_0=\rho$. Then we have a probability distribution over the set of (equivalence classes of) graphs with a be -infinite path on them. The stationarity and reversibility assumption exactly tells us that this probability distribution is invariant under the shift operations which consist in translating the root point of the path by 1 or -1. Hence a lot of ergodic theory can be applied.

1.3.2 Measured equivalence relations

Let (B,μ) be a standard Borel space with a probability measure μ and let $E\subset B^2$ be a symmetric Borel set. We denote the smallest equivalence relation containing E by $\mathscr{R}$. Under mild assumptions the triple (B,μ,E) is called *a measured graphed equivalence relation*. The set E induces a graph structure on B by setting $x\sim y\in B$ if $(x,y)\in E$ or $(y,x)\in E$. For $x\in B$, one can interpret the equivalence class of x as a graph with the edge set given by E, which we root at the point x. If x is sampled according to μ, any measured graphed equivalence relation can be seen as a random rooted graph. More precisely, if $\mathscr{C}(x)$ is the connected component of x is the graph (B,E) we define a random graph (G,ρ) by the formula

$$\mathbb{E}[f(G,\rho)] = \int_B \mathrm{d}\mu(x) f(\mathscr{C}(x),x),$$

for any $f:\mathscr{G}_\bullet\to\mathbb{R}_+$ Borel.

Reciprocally, the set $\mathscr{G}_\bullet$ can be equipped with a natural symmetric Borel set E where $((G,\rho),(G',\rho'))\in E$ if (G,ρ) and (G',ρ') represent the same isomorphism class of non-rooted graph but rooted at two different neighbor vertices. Denote $\mathscr{R}$ the smallest equivalence relation on $\mathscr{G}_\bullet$ that contains E. Thus a random rooted graph (G,ρ) of distribution $\mathbf{P}$ gives rise to $(\mathscr{G}_\bullet,\mathbf{P},E)$ which, under slight assumptions on (G,ρ) is a MGEQ.

In particular we have the following dictionary between the notions of harmonic MGEQ [9, Definition 1.11], totally invariant MGEQ [9, Definition 1.12], measure preserving MGEQ [5, Section 8] or [1, Example 9.9] and the corresponding analogous for random rooted graphs.

measured graphed equivalence relation	random rooted graph
harmonic	stationary
totally invariant	reversible
measure preserving	unimodular

2 Some Applications

We give here as illustrations and motivations some known results about unimodular random graphs.

Theorem 2.1 ([3]). *Let (G,ρ) be unimodular random graph a.s. infinite. Then the expected degree of the root is bigger than 2, $\mathbf{E}\left[\deg(\rho)\right] \geqslant 2$.*

Proof. Let us consider the mass-transport function $f(G,x,y)$ which sends a unit of mass from x to y if there is exactly one edge between x and y (they must be neighbors) and the removal of this edge leaves x in a finite component. Then applying the MTP with his function shows that the expected mass that ρ sends is equal to the expected mass it receives. Let us examine the different cases.

- If $\deg(\rho) = 1$ then ρ sends mass 1 to its only neighbor and receives nothing.
- If $\deg(\rho) \geqslant 2$ and if x sends his mass to somebody then the mass it receives is less than $\deg(\rho)-1$.
- If $\deg(\rho) \geqslant 2$ and if x does not send his mass then the mass it receives is less than $\deg(\rho)-2$.

In any case $D+S-R \geqslant 2$ with obvious notation. Taking expectation yields the result.

□

Let G be a infinite graph, and consider $K_1 \subset K_2 \subset \ldots$ an exhaustion of G by finite sets of vertices of G. Then an *end* of G is a sequence $U_1 \subset U_2 \subset \ldots$ where U_i is a infinite connected component of $G \backslash K_i$. The number of ends does not depend on the sequence $(K_i)_{i\geqslant 1}$.

Theorem 2.2 ([8]). *Let (G,ρ) be a unimodular random graph then the numbers of ends of G is either* $0,1,2$ *or* ∞.

Let D_n be the set of vertices v of (G,ρ) such that the removal of $B_G(v,n)$ disconnects G in at least three infinite components. We consider the mass-transport function $f(G,x,y)$ which sends a unit of mass from x and spread it to all the z such that x is in an infinite connected component of $G\backslash B_G(y,n)$ where y is a closest point of D_n to x for the graph metric $\mathrm{d}_{\mathrm{gr}}^G$ and $\mathrm{d}_{\mathrm{gr}}^G(y,z) \leqslant l$. And $f = 0$ otherwise. Applying the MTP we get

$$\int_{\mathscr{G}_\bullet} \mathrm{d}\mu(G,\rho) \sum_{x\in G} f(G,\rho,x) \quad = \quad \int_{\mathscr{G}_\bullet} \mathrm{d}\mu(G,\rho) \sum_{x\in G} f(G,x,\rho). \tag{4}$$

The left-hand side is the quantity of mass that ρ sends, it is thus less than 1. Hence the right-hand side, which is the mean quantity of mass that ρ receives, is finite as well. Imagine now that G has an isolated end and more than three ends, with positive probability. Then there exists a finite set that disconnects this end from the other ones and for some n and l the root itself will get an infinite mass.

Theorem 2.3 ([2]). *Let (G,ρ) be a unimodular random graph satisfying*

- $\mathbf{E}\left[\deg(\rho)\right] < \infty$,
- $\mathbf{E}\left[\log(\mathrm{B}_G(\rho,r))\right] = o(r)$.

Then (G,ρ) is almost surely Liouville (that is admits no non-constant bounded harmonic functions).

Theorem 2.4 ([4]). *Let (G_n,ρ_n) be a sequence of uniformly rooted finite random graphs such that*

(i) there exists some $M > 0$ such that $\sup_n \sup_{v\in G_n} \deg(v) < M$,

(ii) the random graphs (G_n, ρ_n) are a.s. simple planar graphs.

If $(G_n, \rho_n) \to (G, \rho)$ in distribution for $\mathrm{d}_{\mathrm{loc}}$ then (G, ρ) is almost surely recurrent.

Example 2. *Consider (G_n, ρ_n) be the full binary tree up to level n with a uniformly root vertex ρ_n. Then, the limit of (G_n, ρ_n) **is not** the full binary tree seen from the root, but rather seen from the top : This is the canopy tree rooted at a point of level n with probability $2^{-(n+1)}$.*

3 Examples

We will present different examples of unimodular random graphs. Some of them coming from a "slight" random modification of transitive graphs. Others ones purely random.

3.1 Construction from existing (random) graphs

Let (G, ρ) be a unimodular random graph. Conditionally on (G, ρ), consider a sequence $(B_e)_{e \in E}$ of independent Bernoulli variables of parameter $p \in (0, 1)$ indexed by, say, the edges of G. We define a random graph from G and (B_e) by deleting the edges carrying the value 0 and keeping those carrying the value 1. Of course many connected components can occur and we focus on $\mathscr{C}(\rho)$, the one containing the origin ρ.

Proposition 3.1. *The random rooted graph $(\mathscr{C}(\rho), \rho)$ is unimodular.*

Proof. We directly verify the MTP. Denote μ the distribution of (G, ρ) and let f be a transport function. To simplify notation, we write $\mathscr{C}$ instead of $\mathscr{C}(\rho)$. We have

$$\int \mathrm{d}\mathbb{P}(\mathscr{C}, \rho) \sum_{x \in \mathscr{C}} f(\mathscr{C}, \rho, x) = \int \mathrm{d}\mu(G, \rho) \sum_{x \in G} \underbrace{\int \mathrm{d}\mathbb{P}(B_e)_{e \in \mathrm{E}(G)} f(\mathscr{C}, \rho, x) \mathbf{1}_{x \in \mathscr{C}}}_{F(G, \rho, x)}.$$

Thus the function $F(.,.,.)$ is a transport function and applying the MTP yields the result. □

Remark 3.2. *We just used the fact that (B_e) is a Bernoulli percolation in order to say that F is a transport function. This reasoning is valid for a much wider class of percolation processes.*

Exercice 2. *Let (G, ρ) be a unimodular random graph. Delete all the vertices of degree bigger than $M \geqslant 0$ for some M. Show that the component of the root vertex rooted at ρ is still a unimodular random graph.*

3.2 Augmented Galton-Watson trees (AGW)

Let $\mathbf{p} = (p_k)_{k \geqslant 0}$ a probability distribution over $\mathbb{Z}_+$. A Galton-Watson tree is a random rooted tree (in fact it bears an additionnal planar structure that we forget in our setting) defined informally as follows. Start with the root ρ of the tree and sample its number of children according to $\mathbf{p}$, then iterate this device independently of each child obtained. The random rooted tree obtained is not homogeneous because the root has stochastically one neighbord less than the other vertices. To cope up with this phenomenon, we define the *Augmented* Galton-Watson measure as the measure obtained when we force the root ρ to have one more child that is when its offspring distribution over $\mathbb{Z}_+$ is given by $\mathbb{P}(\rho \text{ has } k \text{ children}) = p_{k-1}$ for $k \geqslant 1$. Then the random rooted tree (T, ρ) obtained by the above device is not unimodular, indeed consider the transport function $f(T,x,y) = \frac{1}{\deg(x)}$ if $x \sim y$ and 0 otherwise. If it held, the MTP would give

$$
\begin{aligned}
1 &= \int_{\mathcal{G}_\bullet} d\mu(T,\rho) \sum_{x\sim\rho} \frac{1}{\deg(x)} \\
&= \mathbf{E}[1+X]\,\mathbf{E}\left[\frac{1}{1+X}\right]
\end{aligned}
$$

where X is a random variable over $\mathbb{Z}_+$ distributed according to $\mathbf{p}$. The last inequality is not always fulfilled.

Theorem 3.3 ([**?**])**.** *The random variable (T,ρ) biaised by* $\deg^{-1}$ *is unimodular.*

Proof. We will not give all the details of this proof but only indicated why we have to bias by $\deg(\rho)^{-1}$. In fact the random variable (T,ρ) is a stationary and reversible random graph. This can be seen heuristically as follows. Imagine that we start at the root ρ of T and that we take a one step random walk. The scenery that we see is an edge that we just come from, and two independent GW trees grafted on that edge, which is the same as (T,ρ) in distribution. By the discussing of Section **??**, (T,ρ) biaised by $\deg(\rho)^{-1}$ is unimodular. □

3.3 UIPT/Q

A *planar map* is a embedding of a finite connected planar graph into the two-dimensional sphere seen up to continuous deformations that preserve the orientation. We deal with planar maps because the little additional structure they bear compared to planar graphs enable us to do combinatorics with them easily. A planar map is a quadrangulation if all its faces have degree four and is rooted if it has a distinguished oriented edge. We denote $\mathcal{Q}_n$ the set of all rooted quadrangulations with n faces.

Theorem 3.4 ([**?**], see also [**?**] for triangulations)**.** *Let $\mathfrak{Q}_n$ be a random quadrangulation in $\mathcal{Q}_n$ and let (Q_n,ρ) be its associated graph rooted at the origin of the roote edge of $\mathfrak{Q}_n$ then we have the following convergence in distribution for* $\mathrm{d}_{\mathrm{loc}}$

$$
(Q_n,\rho) \underset{n\to\infty}{\longrightarrow} (Q_\infty,\rho), \tag{5}
$$

where (Q_∞,ρ) is a random infinite rooted planar graph called the Uniform Infinite Planar Quadrangulation (UIPQ)[1].

The UIPQ is in fact a stationary and reversible random graph, hence its biased version by $\deg(\rho)^{-1}$ is unimodular. This random graph (and its family) has been extensively studied over the last ten years, see the work of Angel and Schramm, Chassaing and Durhuus, Krikun, Le Gall and Ménard... In particular the geometry of the UIPQ is very weird as the typical volume of a ball of radius r is r^4. Unforntunately (or fortunately ?) basic questions such as recurrence or transience of simple random walk on this graph are still open.

[1]the real theorem directly deals with maps and does use the "projection" on random graphs so that the UIPQ is a random infinite planar map, see [**?**]

References

[1] D. Aldous and R. Lyons. Processes on unimodular random networks. *Electron. J. Probab.*, 12:no. 54, 1454–1508 (electronic), 2007.

[2] I. Benjamini and N. Curien. Ergodic theory on stationary random graphs. *arxiv:1011.2526*, 2010.

[3] I. Benjamini, R. Lyons, Y. Peres, and O. Schramm. Group-invariant percolation on graphs. *Geom. Funct. Anal.*, 9(1):29–66, 1999.

[4] I. Benjamini and O. Schramm. Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.*, 6(23), 2001.

[5] D. Gaboriau. Invariant percolation and harmonic Dirichlet functions. *Geom. Funct. Anal.*, 15(5):1004–1051, 2005.

[6] O. Häggström. Infinite clusters in dependent automorphism invariant percolation on trees. *Ann. Probab.*, 25(3):1423–1436, 1997.

[7] V. A. Kaimanovich and F. Sobieczky. Stochastic homogenization of horospheric tree products. In *Probabilistic approach to geometry*, volume 57 of *Adv. Stud. Pure Math.*, pages 199–229. Math. Soc. Japan, Tokyo, 2010.

[8] R. Lyons and O. Schramm. Indistinguishability of percolation clusters. *Ann. Probab.*, 27(4):1809–1836, 1999.

[9] F. Paulin. Propriétés asymptotiques des relations d'équivalences mesurées discrètes. *Markov Process. Related Fields*, 5(2):163–200, 1999.