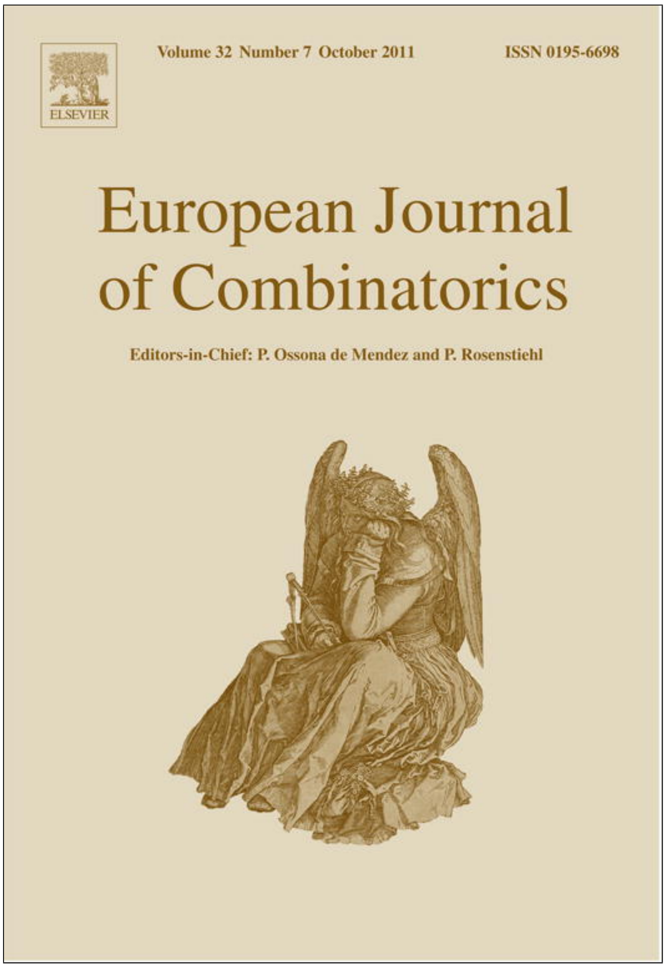
Volume 32 Number 7 October 2011
ISSN 0195-6698
ELSEVIER
European Journal of Combinatorics
Editors-in-Chief: P. Ossona de Mendez and P. Rosenstiehl

European Journal of Combinatorics 32 (2011) 975–984

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

On limits of graphs sphere packed in Euclidean space and applications

Itai Benjamini [a], Nicolas Curien [b]

[a] *The Weizmann institute of Science, 76100 Rehovot, Israel*

[b] *Équipe Probabilités, DMA - Ecole Normale Supérieure, 45, rue d'Ulm, 75230 Paris Cedex 05, France*

ARTICLE INFO

Article history:
Available online 16 April 2011

ABSTRACT

The core of this note is the observation that links between circle packings of graphs and potential theory developed in Benjamini and Schramm (2001) [4] and He and Schramm (1995) [11] can be extended to higher dimensions. In particular, it is shown that every limit of finite graphs sphere packed in $\mathbb{R}^d$ with a uniformly chosen root is d-parabolic. We then derive a few geometric corollaries. For example, every infinite graph packed in $\mathbb{R}^d$ has either strictly positive isoperimetric Cheeger constant or admits arbitrarily large finite sets W with boundary size which satisfies $|\partial W| \leqslant |W|^{\frac{d-1}{d}+o(1)}$. Some open problems and conjectures are gathered at the end.

1. Introduction

The theory of random planar graphs, also known as two-dimensional quantum gravity in the physics literature, has been rapidly developing for the last ten years; see [6] for a survey. The analogous theory in higher dimension is notoriously hard and not very well established so far. This is due in particular to the fact that enumeration techniques and bijective representations are lacking; see for instance [2].

However there are a couple of two-dimensional results that are not dependent on enumeration. For example, in [4], circle packing theory is used to show that limits (see Definition 2.3) of finite random *planar* graphs of bounded degree with a uniformly chosen root are almost surely recurrent. The goal of this note is to extend this result to higher dimensions and to derive some consequences and conjectures.

E-mail addresses: itai.benjamini@weizmann.ac.il (I. Benjamini), nicolas.curien@gmail.com (N. Curien).

0195-6698/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ejc.2011.03.016

We recall that recurrence means that the simple random walk on the graph returns to the origin almost surely, or in potential theory terminology, that the graph is parabolic. A graph is parabolic if and only if it supports no flow with one source of flux 1, no sinks, and with gradient in $\mathbb{L}^2$. Replacing 2 by $d \geqslant 3$ yields the concept of d-parabolicity; see [18] and Section 2.2.

The analogue of circle packing theory in dimension d is easy to describe. A graph is sphere packable in $\mathbb{R}^d$ if and only if it is the tangency graph of a collection of d-dimensional balls with disjoint interiors: the balls of the packing correspond to the vertices of the graph and the edges to tangent balls; see Section 2.1. The theory of circle packings of planar graphs is well developed and its relation to conformal geometry is well established; see the beautiful survey [15]. The higher dimensional version is not as neat. First, although all finite planar graphs (without loops and multiple edges) can be realized as the tangency graph of a circle packing in $\mathbb{R}^2$ (see below), yet there are no natural families of graphs packed in $\mathbb{R}^d$ for $d \geqslant 3$. Second, circle packings relate to $\mathbb{L}^2$ potential theory while in higher dimension the link is to d-potential theory; this is less natural and the probabilistic interpretation is lacking. Still, useful things can be proved and conjectured. Indeed the main observation of this note is that links between circle packings of graphs and potential theory over the graph (see [11]) can be extended to higher dimensions, leading in particular to a generalization of [4, Theorem 1.1], and suggests many problems for further research. For a precise formulation of our main theorem (Theorem 2.9) we must introduce several technical notions and definitions in the coming sections.

We hope that this minor contribution will open the doors to the theory in dimension 3 or higher for sphere packing and quantum gravity. The proofs essentially follow that of [4,11] with the appropriate modifications, and are followed by a report on some new geometric applications. For example we prove under a local bounded geometry assumption defined in the next section that a sequence of k-regular graphs with growing girth cannot all be packed in a fixed dimension and that every infinite graph packed in $\mathbb{R}^d$ either has strictly positive isoperimetric Cheeger constant or admits arbitrarily large finite sets W with boundary size which satisfies $|\partial W| \leqslant |W|^{\frac{d-1}{d}+o(1)}$.

Note that very recently the isoperimetric criterion of Proposition 4.1 was used in [12] to prove that acute triangulations of the space $\mathbb{R}^d$ do not exist for $d \geqslant 5$.

2. Notation and terminology

In the following, unless otherwise indicated, all graphs are locally finite and connected.

2.1. Packings

Definition 2.1. A d-dimensional sphere packing or, for short, *d-sphere packing* is a collection $P = (B_v, v \in V)$ of d-dimensional balls of centers C_v and radii $r_v > 0$ with disjoint interiors in $\mathbb{R}^d$. We associated with P an unoriented graph $G = (V, E)$ called a *tangency graph*, where we put an edge between two vertices u and v if and only if the balls B_u and B_v are tangent.

An *accumulation point* of a sphere packing P is an accumulation point of the centers of the balls of P. Note that the name "sphere packing" is unfortunate since it deals with balls. However this terminology is common and we will use it. The two-dimensional case is well-understood, thanks to the following theorem.

Theorem 2.2 (*Circle Packing Theorem*)**.** *A finite graph G is the tangency graph of a 2-sphere packing if and only if G is planar and contains no multiple edges or loops. Moreover if G is a triangulation then this packing is unique up to Möbius transformations.*

This beautiful result has a long history; we refer the reader to [20,15] for further information. For $d = 3$, very little is known. Although some necessary conditions for a finite graph to be the tangency graph of a 3-sphere packing are provided in [13] (for a related higher dimensional result see [1]), the characterization of 3-sphere packable graphs is still open (see the last section). For packing of infinite graphs see [5]. To bypass the lack of a result similar to the last theorem in dimension 3 or higher, we will restrict ourselves to packable graphs, which are graphs which admit a sphere packing representation. One useful lemma in circle packing theory is the so-called "Ring Lemma" that enables us to control the size of tangent circles under a bounded degree assumption.

Lemma 2.3 (*Ring Lemma [16]*). *There is a constant $r > 0$ depending only on $n \in \mathbb{Z}_+$ such that if n circles surround the unit disk then each circle has radius at least r.*

Here also, since we have no analogue of the Ring Lemma in high dimensions, we will required an additional property on the packings.

Definition 2.4. Suppose that $M > 0$. A d-sphere packing $P = (B_v, v \in V)$ is M*-uniform* if for any tangent balls B_u and B_v of radii r_u and r_v we have

$$\frac{r_u}{r_v} \leqslant M.$$

A graph G is M-uniform in dimension d if it is a tangency graph of an M-uniform sphere packing in $\mathbb{R}^d$.

Remark 2.5. Note that an M-uniform graph in dimension d has a maximal degree bounded by a constant depending only on M and d.

Remark 2.6. By the Ring Lemma, every planar graph of bounded degree without loops or multiple edges is M-uniform in dimension 2, where M only depends of the maximal degree of the graph. The same holds in dimension 3 provided that the complex generated by the centers of the spheres is a tetrahedrangulation (that is all simplexes of dimension 3 are tetrahedrons); see [21].

2.2. *d-parabolicity*

The classical theory of electrical networks and 2-potential theory is long studied and well-understood, in particular due to the connection with the simple random walk (see for example [9] for a nice introduction). On the other hand, non-linear potential theory is much more complicated and still developing; for background see [18]. A key concept for d-potential theory is the notion of extremal length and its relations with parabolicity (extremal length is common in complex analysis and was imported into the discrete setting by Duffin [10]). We present here the basic definitions that we use in the sequel.

Let $G = (V, E)$ be a locally finite connected graph. For $v \in V$ we let $\Gamma(v)$ be the set of all semi-infinite self-avoiding paths in G starting from v. If $m : V \to \mathbb{R}_+$ assigns length to vertices, the length of a path γ in G is

$$\mathrm{Length}_m(\gamma) := \sum_{v \in \gamma} m(v).$$

If $m \in \mathbb{L}^d(V)$, we denote by $\|m\|_d$ the usual $\mathbb{L}^d$ norm $(\sum_v m(v)^d)^{1/d}$. The graph G is d*-parabolic* if the d*-vertex extremal length* of $\Gamma(v)$,

$$d\text{-VEL}(\Gamma)(v) := \sup_{m \in \mathbb{L}^d} \inf_{\gamma \in \Gamma(v)} \frac{\mathrm{Length}_m(\gamma)^d}{\|m\|_d^d},$$

is infinite. It is easily seen that this definition does not depend upon the choice of $v \in V$. This natural extension of VEL parabolicity from [11] can be found earlier in [5].

Remark 2.7. In the context of bounded degree graphs, 2-parabolicity is equivalent to recurrence of the simple random walk on the graph; see [11] and the references therein. In general, 2-VEL is closely related to discrete conformal structures such as circle packings and square tilings; see [3,8,11,17].

2.3. *Limits of graphs*

A rooted graph $(G = (V, E), o \in V)$ is isomorphic to $(G' = (V', E'), o' \in V')$ if there is a graph-isomorphism of G onto G' which takes o to o'. We can define (as introduced in [4]) a distance Δ on the

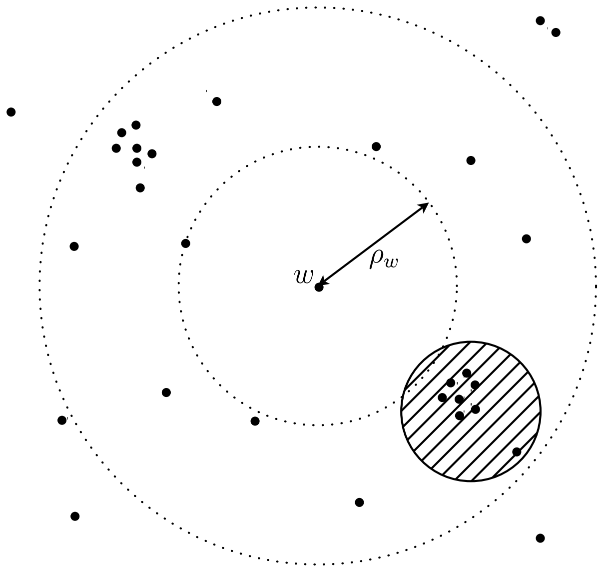

Fig. 1. Illustration of the definition of being (δ, s)-supported. Here, the point w is $(0.5, 10)$-supported.

space of isomorphism classes of locally finite rooted graphs by setting

$$\Delta\big((G, o), (G', o')\big) = \Big(1 + \sup\big\{k : \mathrm{Ball}_G(o, k) \text{ isomorphic to } \mathrm{Ball}_{G'}(o', k)\big\}\Big)^{-1},$$

where $\mathrm{Ball}_G(o, k)$ is the closed combinatorial ball of radius k around o in G for the graph distance. In this work, limits of a graph should be understood as referring to Δ. It is easy to see that the space of isomorphism classes of rooted graphs with maximal degree less than M is compact with respect to Δ. In particular every sequence of random rooted graphs of degree bounded by M admits weak limits.

Definition 2.8. A random rooted graph (G, o) is *unbiased* if (G, o) is almost surely finite and conditionally on G, the root o is uniform over all vertices of G.

We are now ready to state our main result. The case $d = 2$ is [4, Theorem 1].

Theorem 2.9. *Suppose that $M \geqslant 0$ and $d \in \{2, 3, \ldots\}$. Let $(G_n, o_n)_{n\geqslant 0}$ be a sequence of unbiased random rooted graphs such that, almost surely, for all $n \geqslant 0$, G_n is M-uniform in dimension d. If (G_n, o_n) converges in distribution towards (G, o) then G is almost surely d-parabolic.*

Applications of Theorem 2.9 will be discussed in Section 4.

3. Proof of Theorem 2.9

We follow the structure of the proof of [4, Theorem 1]:

1. We first construct a limiting random packing whose tangency graph *contains* the limit of the finite graphs.
2. The main step consists in showing that this packing has at most one accumulation point (for the centers) in $\mathbb{R}^d$, almost surely.
3. Finally we conclude by quoting a theorem relating packing in $\mathbb{R}^d$ and d-parabolicity.

Let $(G_n, o_n)_{n\geqslant 0}$ be a sequence of unbiased, M-uniform in dimension d, random rooted graphs converging to a random rooted graph (G, o). Given G_n, let P_n be a deterministic M-uniform packing of G_n in $\mathbb{R}^d$. We can assume that o_n is independent of P_n.

Suppose that $C \subset \mathbb{R}^d$ is a finite set of points (in the application below, C will be the set of centers of balls in P_n). When $w \in C$, we define its isolation radius as $\rho_w := \inf\big\{|v - w| : v \in C \setminus \{w\}\big\}$. Given $\delta \in (0, 1)$, $s > 0$ and $w \in C$, following [4] we say that w is (δ, s)-*supported* if in the ball of radius

$\delta^{-1}\rho_w$ centered at w, there are more than s points of C outside of every ball of radius $\delta\rho_w$, that is, if (see Fig. 1)

$$\inf_{p\in\mathbb{R}^d}\left|C\cap \mathrm{Ball}_{\mathbb{R}^d}(w,\delta^{-1}\rho_w)\setminus \mathrm{Ball}_{\mathbb{R}^d}(p,\delta\rho_w)\right|\geq s.$$

Lemma 3.1 (*[4]*). *Suppose that $d\geqslant 2$. For every $\delta\in(0,1)$ there is a constant $c(\delta,d)$ such that for every finite set $C\subset\mathbb{R}^d$ and every $s\geq 2$ the set of (δ,s)-supported points in C has cardinality at most $c(\delta,d)|C|/s$.*

Lemma 2.3 in [4] deals with the case $d=2$, but the proof when $d\geqslant 2$ is the same and is therefore omitted.

Now, thanks to this lemma and to the fact that the point o_n has been chosen independently of the packing P_n, for any $\delta>0$ and any $n\geqslant 0$, the probability that the center of the ball B_{o_n} is (δ,s)-supported at the centers of P_n goes to 0 as $s\to\infty$. Let $\tilde{P}_n$ be the image of P_n under a linear mapping such that the ball B_{o_n} is the unit ball in $\mathbb{R}^d$. Since the definition of being (δ,s)-supported is invariant under dilations and translations, we have

$$\mathbb{P}\big(0 \text{ is } (\delta,s)\text{-supported at the centers of } \tilde{P}_n\big)\underset{s\to\infty}{\longrightarrow}0. \tag{1}$$

Let $\tilde{\mathbf{P}}_n$ be the union of the spheres of the packing $\tilde{P}_n$ and $\tilde{\mathbf{C}}_n$ be the union of the centers of the spheres of $\tilde{P}_n$. By definition, $\tilde{\mathbf{P}}_n$ and $\tilde{\mathbf{C}}_n$ are random closed subsets of $\mathbb{R}^d$. The topology of Hausdorff convergence on every compact of $\mathbb{R}^d$ is a compact topology for closed subsets of $\mathbb{R}^d$. Hence, we can assume that along a subsequence we have the following convergence in distribution:

$$\big((G_n,o_n),\tilde{\mathbf{P}}_n,\tilde{\mathbf{C}}_n\big)\underset{n\to\infty}{\longrightarrow}\big((G,o),\mathbf{P},\mathbf{C}\big), \tag{2}$$

related to Δ for the first component and to the Hausdorff convergence on every compact of $\mathbb{R}^d$ for the second and third ones. Without loss of generality we can suppose that there is no need to pass to a subsequence and by Skorokhod representation theorem that the convergence (2) is almost sure.

Proposition 3.2. *The random closed set $\mathbf{P}$ is almost surely the closure of a sphere packing in $\mathbb{R}^d$ whose centers have at most one accumulation point in $\mathbb{R}^d$. Furthermore, the tangency graph associated with $\mathbf{P}$ almost surely contains (G,o) as a subgraph.*

Proof. We begin with the second claim of the proposition. By definition of $\tilde{P}_n$ we know that $\mathbf{P}$ contains the unit sphere of $\mathbb{R}^d$ that corresponds to $o\in G$. Since the packings $\tilde{P}_n$ are M-uniform, any vertex neighbor of o_n in G_n corresponds to ball in the packing whose radius is in $[M^{-1},M]$ and tangent to the unit ball of $\mathbb{R}^d$. This property passes to the limit and by (2) we deduce that any neighbor of o in G corresponds to a sphere of $\mathbf{P}$ of radius in $[M^{-1},M]$ and tangent to the unit sphere of $\mathbb{R}^d$. A similar argument shows that $\mathbf{P}$ almost surely contains tangent spheres whose tangency graph contains G. Note that in the set $\mathbf{P}$ new connections can occur (non-tangent spheres in $\tilde{P}_n$ can become tangent at the limit).

The first part of the proposition reduces to showing that $\mathbf{C}$ almost surely has at most one accumulation point in $\mathbb{R}^d$. We argue by contradiction and we suppose that with probability bigger than ε, there exist two accumulation points A_1 and A_2 in $\mathbf{C}$ such that $|A_1-A_2|\geqslant\varepsilon$ and $|A_1|,|A_2|\leqslant\varepsilon^{-1}$. This implies, by (2), that for any $s\geqslant 0$ with a probability asymptotically bigger than ε the point 0 is $(\varepsilon/2,s)$-supported in $\tilde{\mathbf{C}}_n$. Which contradicts (1). □

Since every subgraph of a d-parabolic graph is itself d-parabolic (obvious from the definition), the following extension of [11, Theorem 3.1(1)] together with the last proposition enables us to finish to proof of Theorem 2.9.

Theorem 3.3 (*[5, Theorem 7]*). *Let G be a graph of bounded degree. If G is packable in $\mathbb{R}^d$ and if the packing has finitely many accumulation points in $\mathbb{R}^d$, then G is d-parabolic.*

Remark 3.4. To be totally accurate, the d-parabolicity notion defined in [5] corresponds to the definitions of Section 2.2 when the function m is defined on the edges of the graph. But these two notions readily coincide in the bounded degree case.

4. Geometric applications

4.1. Isoperimetric inequalities and alternatives

If W is a subset of a graph G, we recall that ∂W is the set of vertices not in W but neighboring some vertex in W. We begin with an isoperimetric consequence of d-parabolicity which is an extension of [11, Theorem 9.1(1)]. The proof is similar.

Proposition 4.1. *Let $G = (V, E)$ be a locally finite, infinite, connected graph. Suppose that $o \in V$, and let $g : \mathbb{R}_+ \to \mathbb{R}_+^*$ be some non-decreasing function.*

(1) *Suppose that G is d-parabolic. If for every finite set W containing $o \in W$, we have $|\partial W| \geqslant g(|W|)$, then*

$$\sum_{n=1}^{\infty} g(n)^{-\frac{d}{d-1}} = \infty. \tag{3}$$

(2) *If g satisfies (3) and if $|\partial W_k| \leqslant g(|W_k|)$, for $(W_k)_{k\geqslant 0}$ defined recursively by*

$$W_0 = \{o\} \quad \text{and} \quad W_{k+1} = W_k \cup \partial W_k \quad \text{for } k \geqslant 0,$$

then G is d-parabolic.

Proof. We know by assumption that d-VEL$(\Gamma(o)) = \infty$. This implies that we can find functions $m_i : V \to \mathbb{R}_+$ such that $\|m_i\|_d = 2^{-i}$ and $\inf_{\gamma \in \Gamma(o)} \text{Length}_{m_i}(\gamma) \geqslant 1$. Hence $m := \sum_{i=0}^{\infty} m_i$ defines a function on V such that

$$\|m\|_d \leqslant 1 \quad \text{and} \quad \inf_{\gamma \in \Gamma(o)} \text{Length}_m(\gamma) = \infty.$$

Without loss of generality we will suppose that $m(v) > 0$ for all vertices $v \in V$. The function $m \in \mathbb{L}^d(V)$ defines a function $V \times V \to \mathbb{R}_+$, on setting

$$\mathrm{d}_m(v, v') := \inf\{\text{Length}_m(\gamma), \gamma : v \to v'\}.$$

The idea is to explore the graph G in a continuous manner according to d_m and to use the isoperimetric inequality provided by g. For each $v \in V$ suppose that

$$I_v := [\mathrm{d}_m(o, v) - m(v), \mathrm{d}_m(o, v)].$$

For $h \in \mathbb{R}_+$, we define $s_v(h) := \frac{\text{Leb}(I_v \cap [0,h])}{m(v)}$. Intuitively, water flows in the graph G starting from o; $m(v)$ is the time that water needs to wet v before flowing to its neighbors. A vertex $v \in V$ begins to get wet at $h = \min I_v$ and is completely wet at $h = \max I_v$. The function $s_v(h)$ represents the percentage of water in v. We set $s(h) := \sum_{v \in V} s_v(h)$. Since $\mathrm{d}_m(o, \infty) = \infty$, for every $h \in \mathbb{R}_+$ there are only finitely many $v \in V$ such that $s_v(h) \neq 0$ and then $s(h)$ is piecewise linear. We denote as $W_h := \{v \in V,\ h \geqslant \max I_v\}$ the set of vertices that are totally wet at time h and as $G_h := \{v \in V,\ \mathrm{d}_m(o, v) - m(v) \leqslant h < \mathrm{d}_m(o, v)\}$ the set of vertices that are getting wet at time h. Clearly $G_h = \partial W_h$. Suppose that

$$f(x) = \min\left(g\left(\frac{x}{2}\right), \frac{x}{2}\right).$$

If $|G_h| \geqslant s(h)/2$ then

$$|G_h| \geqslant f(s(h)), \tag{4}$$

and otherwise $|G_h| < s(h)/2$; then the number of completely wet vertices is at least $s(h)/2$ (because $s_v(h) \leqslant 1$) and consequently $|G_h| \geqslant g(s(h)/2)$. Thus (4) always holds.

At points where $h \mapsto s(h)$ is differentiable we have

$$\frac{ds}{dh}(h) = \sum_{v \in G_h} s_v'(h) = \sum_{v \in G_h} \frac{1}{m(v)}.$$

Writing $1 = m(v)^{(d-1)/d} m(v)^{-(d-1)/d}$ and using the Hölder inequality with $p = d$ we get

$$\left(\sum_{v \in G_h} 1 \right) \leqslant \left(\sum_{v \in G_h} \frac{1}{m(v)} \right)^{\frac{d-1}{d}} \left(\sum_{v \in G_h} m(v)^{d-1} \right)^{1/d},$$

and thus, using (4),

$$\frac{ds}{dh}(h) \geqslant \frac{|G_h|^{\frac{d}{d-1}}}{\left(\sum_{v \in G_h} m(v)^{d-1} \right)^{\frac{1}{d-1}}} \geqslant \frac{f(s(h))^{\frac{d}{d-1}}}{\left(\sum_{v \in G_h} m(v)^{d-1} \right)^{\frac{1}{d-1}}},$$

and therefore

$$\frac{ds}{f(s(h))^{\frac{d}{d-1}}} \geqslant \frac{dh}{\left(\sum_{v \in G_h} m(v)^{d-1} \right)^{\frac{1}{d-1}}}.$$

Integrating for $0 < a < h < b < \infty$ and using the Hölder inequality with $p = d$ we get

$$\int_{s(a)}^{s(b)} \frac{ds}{f(s)^{\frac{d}{d-1}}} \geqslant \int_a^b \frac{dh}{\left(\sum_{v \in G_h} m(v)^{d-1} \right)^{\frac{1}{d-1}}} \geqslant \frac{(b-a)^{d/(d-1)}}{\left(\int_a^b \left(\sum_{v \in G_h} m(v)^{d-1} \right) dh \right)^{1/(d-1)}}.$$

Remark that $\int_0^\infty \left(\sum_{v \in G_h} m(v)^{d-1} \right) dh = \sum_{v \in V} m(v)^d < \infty$, and that $s(b) \to \infty$ when $b \to \infty$. We conclude that the integral of $f(.)^{-\frac{d}{d-1}}$ diverges and the same conclusion holds for $g(.)^{-\frac{d}{d-1}}$. Since $g(.)$ is non-decreasing, a comparison series-integral ends the proof of the first part of the proposition.

For the second part, set $n_k = |W_k|$ and define for $N \in \mathbb{N}^*$ a function $m : V \to \mathbb{R}_+$ on G by

$$m(v) = \begin{cases} g(n_k)^{-\frac{1}{d-1}} & \text{for } v \in \partial W_k \text{ and } k \leqslant N, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $\inf\{\text{Length}_m(\gamma) : \gamma \in \Gamma(o)\} \geqslant \sum_{k=0}^N g(n_k)^{-\frac{1}{d-1}}$ and

$$\|m\|_d^d \leqslant \sum_{k=0}^N \frac{|\partial W_k|}{g(n_k)^{d/(d-1)}} \leqslant \sum_{k=0}^N g(n_k)^{-\frac{1}{d-1}}.$$

By the definition of the extremal length, it suffices to show that $\sum_{k=0}^\infty g(n_k)^{-\frac{1}{d-1}} = \infty$. Note that $n_{k+1} \leqslant n_k + g(n_k)$; thus by the monotonicity of g, we obtain

$$\frac{1}{g(n_k)^{\frac{1}{d-1}}} \geqslant \frac{1}{n_{k+1} - n_k} \sum_{n=n_k}^{n_{k+1}-1} \frac{1}{g(n)^{\frac{1}{d-1}}} \geqslant \sum_{n=n_k}^{n_{k+1}-1} \frac{1}{g(n_k)} \frac{1}{g(n)^{\frac{1}{d-1}}} \geqslant \sum_{n=n_k}^{n_{k+1}-1} \frac{1}{g(n)^{d/(d-1)}}$$

which implies $\sum_{k=0}^\infty g(n_k)^{-\frac{1}{d-1}} \geqslant \sum_{n_0}^\infty g(n)^{-d/(d-1)} = \infty$. □

Let us recall the definition of the Cheeger constant of a infinite graph G:

$$\mathrm{Cheeger}(G) := \inf\left\{\frac{|\partial W|}{|W|} : W \subset G, |W| < \infty\right\}.$$

The following corollary generalizes a theorem regarding planar graphs indicated by Gromov and proved by several authors. See Bowditch [7] for a very short proof and references for previous proofs.

Corollary 4.2. *Let G be an infinite locally finite connected graph which admits an M-uniform packing in $\mathbb{R}^d$. Then we have the following alternatives:*

- *either G has a positive Cheeger constant,*
- *or for any $\varepsilon > 0$, there are arbitrarily large subsets W of G such that*

$$|\partial W| \leqslant |W|^{\frac{d-1}{d}+\varepsilon}.$$

Proof. Let G be a infinite connected graph which is the tangency graph of an M-uniform packing in $\mathbb{R}^d$ (in particular G has bounded degree). If $\mathrm{Cheeger}(C) = 0$, then we can find a sequence of subsets $A_i \subset G$ such that

$$\frac{|\partial A_i|}{|A_i|} \xrightarrow[i\to\infty]{} 0.$$

Remark that the A_i's are not necessarily connected subgraphs. For each $i \geqslant 0$, we pick a vertex o_i uniformly at random among the vertices of A_i and denote as $\mathcal{C}(o_i, A_i)$ the connected component of A_i connecting o_i. By a compactness argument (see the discussion before Definition 2.8) we deduce that along a subsequence we have the weak convergence for Δ

$$\left(\mathcal{C}(A_i, o_i), o_i\right) \xrightarrow[i\to\infty]{(d)} (A, o),$$

where (A, o) is a random rooted graph. We assume that there is no need to pass to a subsequence. Therefore the sequence of rooted random graphs $(\mathcal{C}(A_i, o_i), o_i)_{i\geqslant 1}$ satisfies all the hypotheses of Theorem 2.9; in particular (A, o) is almost surely d-parabolic. By Proposition 4.1, for any $\delta, \varepsilon > 0$, there exists a.s. a random subset $W \subset A$ containing o and satisfying

$$|\partial W| \leqslant \delta |W|^{\frac{d-1}{d}+\varepsilon}.$$

In particular, $|W| \geqslant \delta^{-1/(\frac{d-1}{d}+\varepsilon)}$. We claim that there exists an isomorphic copy of W and its boundary already contained in G. Indeed for any $k \geqslant 0$, the bounded degree assumption combined with the fact that $\frac{|\partial A_i|}{|A_i|} \to 0$ implies that

$$\mathbb{P}\left(o_i \text{ is at a graph distance less than } k \text{ from } \partial A_i\right) \xrightarrow[i\to\infty]{} 0.$$

Hence, almost surely for any $k \geqslant 0$, the ball of radius k around o in A is a subgraph of some A_i's and thus of G. This finishes the proof of the corollary. □

4.2. The non-existence of M-uniform packing

As a consequence of the last corollary, the graph $\mathbb{Z}^{d+1}$ cannot be M-uniformly packed in $\mathbb{R}^d$ for some $M \geqslant 0$. This is a weaker result compared to that of [5], where it is shown that $\mathbb{Z}^{d+1}$ cannot be sphere packed in $\mathbb{R}^d$ using the non-existence of bounded non-constant d-harmonic functions on $\mathbb{Z}^d$.

The *parabolic index* of a graph G (see [19]) is the infimum of all $d \geqslant 0$ such that G is d-parabolic (with the convention that $\inf \varnothing = \infty$). For example, Maeda [14] proved that the parabolic index of $\mathbb{Z}^d$ is d. It is easy to see that the parabolic index of a regular tree is infinite, leading to the following consequence.

Corollary 4.3. *Let G_n be a deterministic sequence of finite graphs. If there exist $f(n) \underset{n\to\infty}{\longrightarrow} \infty$ and $k \in \{2, 3, \ldots\}$ such that*

$$\frac{\#\{v \in G_n, \mathrm{Ball}_{G_n}(v, f(n)) = \text{ k-regular tree up to level } f(n)\}}{|G_n|} \underset{n\to\infty}{\longrightarrow} 1,$$

then for all $M \geqslant 0$, G_n eventually cannot be M-uniformly packed in $\mathbb{R}^d$.

Proof. Note that any unbiased weak limit of G_n is the k-regular tree and apply Theorem 2.9. □

That is, if for a sequence of k-regular graphs, $k > 2$, the girth grows to infinity, then only finitely many of the graphs can be M-uniformly packed in any fixed dimension. The same holds if the limit is some other non-amenable graph.

5. Open problems

Several necessary conditions are provided in this paper for a graph to be (M-uniformly) packed in $\mathbb{R}^d$. The first two questions are related to the existence of packable graphs in $\mathbb{R}^d$.

Question 1. 1. *Find necessary and sufficient conditions for a graph to be (M-uniformly) packable in $\mathbb{R}^d$.*
2. *Exhibit a natural family of graphs which are (M-uniformly) packable in $\mathbb{R}^d$.*
3. *Show that the number of tetrahedrangulations in $\mathbb{R}^3$ with n vertices grows to infinity.*

Question 2. *It is of interest to understand what is the analogue of packing of a graph and the results above in the context of Riemannian manifolds. Is being packable in the discrete context of graphs analogous to being conformally flat?*

Question 3. *Show that the Cayley graph of Heisenberg group $\mathbf{H}_3(\mathbb{Z})$ generated by*

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

is not packable in $\mathbb{R}^d$ though it is known to be 4-parabolic; see e.g. [18].

The two following questions deal with the geometry of the accumulation points (of centers) of packing in $\mathbb{R}^d$.

Question 4. *Does there exist a graph G packable in $\mathbb{R}^d$ in two manners, P_1 and P_2, such that the set of accumulation points in $\mathbb{R}^d \cup \{\infty\}$ for P_1 is a point but that for P_2 is not?*

Question 5 ([5]). *Show that any packing of $\mathbb{Z}^3$ in $\mathbb{R}^3$ has at most one accumulation point in $\mathbb{R}^d \cup \{\infty\}$.*

Question 6 (*Parabolicity for Edges*). *What is left of Theorem 2.9 in the context of edge parabolicity (where the function m of Section 2.2 is defined on the edges of the graph) without the bounded degree assumption? For instance, is it the case that every limit of unbiased random planar graphs is 2-edge-parabolic (which means that SRW is recurrent)?*

Question 7 (*Diffusivity*). *Let G be a d-parabolic graph. Consider $(S_i)_{i\geqslant 0}$, a simple random walk on G. Do we have*

$$\liminf_{n\to\infty} \frac{\mathrm{d_{gr}}(S_0, S_n)}{\sqrt{n}} < \infty?$$

Question 8 (*Mixing Time*). *Let G be a finite graph packable in $\mathbb{R}^d$ with bounded degree. Show that the mixing time is bigger than $C_d\,\mathrm{diameter}(G)^2$. In particular, the planar $d = 2$ case is still open.*

Acknowledgements

Part of this work was done during a visit of the second author to the Weizmann Institute. The second author thanks his hosts for this visit. We are indebted to Steffen Rohde and to James T. Gill for pointing out several inaccuracies in a first version of this work. Thanks also go to the referee for a very useful report.

References

[1] N. Alon, Packings with large minimum kissing numbers, Discrete Math. 175 (1997) 249–251.
[2] B. Benedetti, G.M. Ziegler, On locally constructible spheres and balls, 2009, 31 pp. arXiv:0902.0436.
[3] I. Benjamini, O. Schramm, Random walks and harmonic functions on infinite planar graphs using square tilings, Ann. Probab. 24 (3) (1996) 1219–1238.
[4] I. Benjamini, O. Schramm, Recurrence of distributional limits of finite planar graphs, Electron. J. Probab. 6 (23) (2001) 1–13. arXiv:math/0011019.
[5] I. Benjamini, O. Schramm, Lack of sphere packings of graphs via non-linear potential theory, Preprint, 2009, 8 pp. arXiv:0910.3071.
[6] I. Benjamini, Random planar metrics, Preprint, 2010, 11 pp. ICM 2010 proceedings. http://www.wisdom.weizmann.ac.il/itai/randomplanar2.pdf.
[7] B.H. Bowditch, A short proof that a subquadratic isoperimetric inequality implies a linear one, Michigan Math. J. 42 (1) (1995) 103–107.
[8] R.L. Brooks, C.A.B. Smith, A.H. Stone, W.T. Tutte, The dissection of rectangles into squares, Duke Math. J. 7 (1940) 312–340.
[9] P.G. Doyle, J.L. Snell, Random Walks and Electric Networks, in: Carus Mathematical Monographs, vol. 22, Mathematical Association of America, Washington, DC, 1984.
[10] R.J. Duffin, The extremal length of a network, J. Math. Anal. Appl. 5 (1962) 200–215.
[11] Z.-X. He, O. Schramm, Hyperbolic and parabolic packings, Discrete Comput. Geom. 14 (1995) 123–149.
[12] E. Kopczynski, I. Pak, P. Przytycki, Acute triangulations of polyhedra and $\mathbb{R}^n$, Preprint, 2009, 22 pp. http://arxiv.org/abs/0909.3706.
[13] G. Kuperberg, O. Schramm, Average kissing numbers for non-congruent sphere packings, Math. Res. Lett. (1994).
[14] F.-Y. Maeda, A remark on parabolic index of infinite networks, Hiroshima Math. J. 7 (1) (1977) 147–152.
[15] S. Rohde, Oded Schramm: From Circle Packing to SLE, Preprint, 2010, 43 pp. http://arxiv.org/abs/1007.2007.
[16] B. Rodin, D. Sullivan, The convergence of circle packings to the Riemann mapping, J. Differential Geom. 26 (2) (1987) 349–360.
[17] O. Schramm, Square tilings with prescribed combinatorics, Israel J. Math. 84 (1–2) (1993) 97–118.
[18] P.M. Soardi, Potential Theory on Infinite Networks, in: Lecture Notes in Mathematics, vol. 1590, Springer-Verlag, Berlin, 1994.
[19] P.M. Soardi, M. Yamasaki, Parabolic index and rough isometries, Hiroshima Math. J. 23 (2) (1993) 333–342.
[20] K. Stephenson, Introduction to circle packing, in: The Theory of Discrete Analytic Functions, Cambridge University Press, Cambridge, 2005.
[21] J. Vasilis, On the ring lemma.