

Recurrence of the \mathbb{Z}^d -valued infinite snake *via* unimodularity.

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Abstract

We use the concept of unimodular random graph to show that the branching simple random walk on \mathbb{Z}^d indexed by a critical geometric Galton-Watson tree conditioned to survive is recurrent if and only if $d \leq 4$.

Introduction

Consider a simple random walk on \mathbb{Z}^d indexed by some tree T . If T is a super-critical Galton-Watson tree, the process we obtain is a branching random walk on \mathbb{Z}^d . It has been shown by Biggins [10] that this branching random walk is almost surely recurrent *i.e.*, visits the origin of \mathbb{Z}^d infinitely often. When T is a critical Galton-Watson tree, the walk is closely related to the theory of superBrownian motion and the associated random snake of Le Gall, see [2, 17]. In this note we study the simple random walk on \mathbb{Z}^d indexed by the critical geometric Galton-Watson tree T_∞ *conditioned to survive* [15]. Specifically we prove:

Theorem 1. *The simple random walk on \mathbb{Z}^d indexed by T_∞ is recurrent if and only if $d \leq 4$.*

Recurrence of arbitrary Markov chains indexed by arbitrary trees was studied in [8], but the theorem above was not covered. Notice that the critical role of dimension 4 is reminiscent of the theory of superBrownian motion (see [17]) where the continuous analogue of Theorem 1 is known [22].

The proof of Theorem 1 is based on the use of the “Mass Transport Principle” (see [9, Section 3.2], and [3]) and the related concept of unimodular random graphs combined with simple geometric estimates regarding the tree T_∞ . In particular we do not use any calculations related to the theory of superBrownian motion. More important than the application to Theorem 1, we believe that our technique could be applied in a much wider setup, see Section 3.

The note is organized as follows. In the first section we recall the definition of the random infinite tree T_∞ and gather some simple geometric estimates. We also establish, using stationarity of the tree after re-rooting along a random walk, that the random graph T_∞ satisfies the Mass Transport Principle. Section 2 contains the proof of Theorem 1. We end the note with a few extensions, comments and open questions.

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1 Definition and properties of T_∞

1.1 Uniform plane trees

A rooted tree τ is a tree in the graph theoretic sense with a distinguished vertex ρ called the root vertex. The tree τ can thus be seen as a family tree with ancestor ρ . A rooted ordered tree (or plane tree) is a rooted tree for which we have specified an ordering for the children of each vertex. See [18] for a detailed definition. If u is a vertex of τ , we denote by $\deg(u)$ the *degree* of u in τ , that is its number of incident edges.

For $n \geq 1$, we denote the set of all plane trees with n edges by \mathcal{T}_n . For convenience, we associate with each tree $\tau \in \mathcal{T}_n$ a distinguished oriented edge \vec{e} going from the first child of ρ towards ρ . In the following T_n is a random variable uniformly distributed over \mathcal{T}_n and conditionally on T_n , X_1 is a one-step simple random walk on T_n starting at ρ : Equivalently X_1 is a uniform neighbor of the root ρ . We also keep track of the transition by indicating the oriented edge (ρ, X_1) . We denote by $T_n^{(1)}$ the plane tree obtained from the tree T_n by keeping the planar ordering and by changing the distinguished oriented edge from \vec{e} to (ρ, X_1) .

Proposition 2. *The random plane tree $T_n^{(1)}$ is uniformly distributed over \mathcal{T}_n .*

Proof. For every given tree $\tau \in \mathcal{T}_n$ with oriented edge $\vec{e} = (e_-, e_+)$, it is easy to see that exactly $\deg(e_-)$ plane trees can give rise to τ after changing the oriented edge by a one step random walk: They consist of all the trees obtained from τ after exchanging the oriented edge with an edge targeting e_- . Each of these trees has a probability $1/\deg(e_-)$ to be transformed into τ after a one-step simple random walk. Thus $T_n^{(1)}$ is a uniform plane tree with n edges. \square

If we denote E_- and $E_+ = \rho$ the origin and target vertices of the distinguished edge of T_n , we deduce that (T_n, E_-, E_+) and $(T_n^{(1)}, \rho, X_1)$ have the same distribution as random graphs with two distinguished neighboring vertices. Since the law of T_n is unchanged under reversion of the distinguished edge, we get

$$(T_n, \rho, X_1) \stackrel{(d)}{=} (T_n, X_1, \rho). \quad (1.1)$$

In other words, the random rooted graph obtained from T_n after forgetting the planar structure is *reversible* in the sense of [6, Definition 1].

1.2 The uniform infinite plane tree

If τ is a plane tree and $k \in \{0, 1, 2, \dots\}$, we denote by $[\tau]_k$ the plane tree obtained from τ by keeping the first k generations of τ only. Let T be a Galton-Watson (plane) tree with geometric offspring distribution of parameter $1/2$. It is classical that the distribution of T conditionally on T having n edges is uniform over \mathcal{T}_n , see [18]. Using this observation, the result of [13] (which has been folklore for a long time, see [1, 15] and [14] for the case of unordered trees) can be interpreted as follows:

Lemma 3. *Let T_n be uniformly distributed over \mathcal{T}_n . Then there exists a random infinite plane tree T_∞ such that for every $k \geq 0$ we have*

$$[T_n]_k \xrightarrow[n \rightarrow \infty]{(d)} [T_\infty]_k. \quad (1.2)$$

The random infinite plane tree T_∞ is called the uniform infinite plane tree or the geometric Galton-Watson tree conditioned to survive.

The random infinite plane tree T_∞ can be described as follows: Start with a semi-infinite line of vertices called the spine of the tree and graft to the left and to the right of each vertex of the spine an independent critical geometric Galton-Watson tree (with parameter $1/2$). The root vertex is the first vertex of the spine.

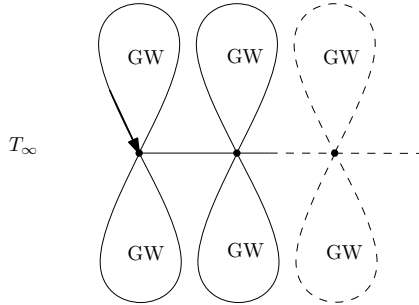


Figure 1: An illustration of T_∞ .

Remark 1. *There exists another equivalent way to define T_∞ , see [5, 19, 21]: In this description the vertices of the spine of T_∞ have an offspring distribution which is given by a size-biased version of the geometric distribution of parameter $1/2$, whereas all the other vertices have the standard geometric offspring distribution of parameter $1/2$. This construction has the advantage to be easily extended to a general critical offspring distribution. To recover the construction presented above from this one, note that if G_1 and G_2 are two independent geometric variables of parameter $1/2$ then $G_1 + G_2 + 1$ has the same law as a geometric variable of parameter $1/2$ biased by its size.*

1.3 The Mass Transport Principle

Before presenting the Mass Transport Principle, let us introduce some notation. A graph $G = (V(G), E(G))$ is a pair of sets, $V(G)$ representing the set of vertices and $E(G)$ the set of (unoriented) edges. In the following, all the graphs considered are countable, connected, locally finite and simple (no loop or multiple edges). For any pair $x, y \in G$, the *graph distance* $d_{\text{gr}}^G(x, y)$ is the minimal length of a path joining x and y in G . For every $r \in \mathbb{Z}_+$, the ball of radius r around x in G is the subgraph of G spanned by the vertices at distance less than or equal to r from x in G , and is denoted by $B_G(x, r)$. A *rooted graph* is a pair (G, ρ) where $\rho \in V(G)$ is called the root vertex. We will identify two rooted graphs if there is a graph isomorphism between them that maps their roots. The set of rooted graphs can be endowed with a metric, see [6, 9].

The Mass Transport Principle was introduced by Häggström to study percolation and was further developed by Benjamini, Lyons, Peres and Schramm [7]. It is extensively studied in [3]. We give here an informal definition and refer to [3, 6] for more details. A random rooted graph (G, ρ) satisfies the Mass Transport Principle if for every positive measurable function $F(g, x, y)$ that associates with each graph g given with two distinguished vertices x and y of g an “amount of mass” sent from x to y in g , we have

$$\mathbb{E} \left[\sum_{x \in V(G)} F(G, \rho, x) \right] = \mathbb{E} \left[\sum_{x \in V(G)} F(G, x, \rho) \right]. \quad (1.3)$$

Such a random graph is called *unimodular*.

Corollary 4. Let $(\tilde{T}_\infty, \tilde{\rho})$ be the random infinite rooted tree obtained from (T_∞, ρ) after biasing by the inverse of the degree of the root vertex ρ , that is

$$\mathbb{E} \left[f(\tilde{T}_\infty, \tilde{\rho}) \right] = \frac{\mathbb{E} [\deg(\rho)^{-1} f(T_\infty, \rho)]}{\mathbb{E} [\deg(\rho)^{-1}]},$$

for any positive Borel function f . Then $(\tilde{T}_\infty, \tilde{\rho})$ obeys the Mass Transport Principle.

Proof. By [6, Proposition 2.5] it is enough to check that the random rooted graph (T_∞, ρ) is reversible in the sense of [6, Definition 1]. This easily follows from equation (1.1) and Lemma 3. \square

1.4 Volume estimates

Let us introduce some notation. Recall the construction of the tree T_∞ . For $i, j \geq 0$ denote $L_i(j)$ (resp. $R_i(j)$) the number of vertices at the j -th generation in the tree grafted on the left (resp. right) of the i -th vertex of the spine of T_∞ . It is easy to see that for every $i \geq 0$ the process $(L_i(j))_{j \geq 0}$ is a martingale in its own filtration. By standard calculations on geometric distributions we have

$$\begin{aligned} \mathbb{E}[L_i(j)] &= 1 \\ \mathbb{E}[(L_i(j))^2] &= 1 + 2j. \end{aligned} \tag{1.4}$$

Furthermore, by the martingale property of $(L_i(j))_{j \geq 0}$, $\mathbb{E}[L_i(j)L_{i'}(j')]$ is equal to 1 if $i \neq i'$ and equals $1 + 2j$ if $i = i'$ and $j \leq j'$. In particular we can get estimates about the volume of the ball $B_{T_\infty}(\rho, r)$ of radius r around ρ in T_∞ ,

$$\begin{aligned} \#B_{T_\infty}(\rho, r) &= r + \sum_{i=0}^{r-1} \sum_{j=1}^{r-i} (L_i(j) + R_i(j)), \\ \mathbb{E}[\#B_{T_\infty}(\rho, r)] &\sim r^2. \end{aligned} \tag{1.5}$$

$$\mathbb{E}[(\#B_{T_\infty}(\rho, r))^2] \sim \frac{7r^4}{6}. \tag{1.6}$$

Furthermore, for every $i, h \in \{0, 1, 2, \dots\}$ the sum $\sum_{j=0}^h L_i(j)$ is the size of a critical geometric Galton-Watson tree cut at height h . By classical results on critical Galton-Watson trees with finite variance (see e.g. [4]), for every $A > 0$, there exists $\varepsilon > 0$ such that

$$\liminf_{h \rightarrow \infty} h \cdot \mathbb{P} \left(\sum_{j=0}^h L_i(j) > \varepsilon h^2 \right) \geq A.$$

It follows from the last display that the probability that one of the $\lfloor r/2 \rfloor$ first trees grafted on the left-hand side of the spine has a size larger than $\varepsilon r^2/4$ is asymptotically at least $1 - e^{-A}$ as $r \rightarrow \infty$. In particular on this event the number of vertices of $B_{T_\infty}(\rho, r)$ is larger than $\varepsilon r^2/4$. Combining this argument with a Markov inequality using equation (1.5) we deduce that

$$\liminf_{\lambda \rightarrow \infty} \inf_{r \geq 0} \mathbb{P}(\lambda^{-1}r^2 \leq \#B_{T_\infty}(\rho, r) \leq \lambda r^2) = 1. \tag{1.7}$$

2 Proof of Theorem 1

Let us define precisely what we mean by the simple random walk over \mathbb{Z}^d indexed by a random infinite rooted tree (T, ρ) : Conditionally on T we assign to each edge of T an independent variable uniformly distributed over the symmetric set of the standard basis elements and their inverses of \mathbb{Z}^d . For every vertex u in T , the sum of the assigned vectors along the only geodesic path from the root ρ to the vertex u is denoted by $\mathcal{S}_{(T, \rho)}(u)$ (note that $\mathcal{S}_{(T, \rho)}$ depends on T and on ρ).

This defines a random function $\mathcal{S}_{(T, \rho)} : T \rightarrow \mathbb{Z}^d$ from the vertices of T to the vertices of \mathbb{Z}^d such that $\mathcal{S}_{(T, \rho)}(\rho) = 0$. When $(T, \rho) = (T_\infty, \rho)$ this function is called *the critical simple random walk snake on \mathbb{Z}^d* . We say that the snake is recurrent if almost surely infinitely many vertices of the tree are mapped to 0 (the origin of \mathbb{Z}^d) *i.e.*

$$\#\mathcal{S}_{(T_\infty, \rho)}^{-1}(\{0\}) = \infty, \quad \text{a.s.},$$

where the almost surely is in the big probability space of trees and assignments. The snake is transient if $\#\mathcal{S}_{(T_\infty, \rho)}^{-1}(\{0\}) < \infty$ almost surely.

2.1 Transience for $d \geq 5$

For every $r \geq 1$, we denote the set of vertices of T_∞ at distance exactly r from the root vertex ρ by $\partial B_{T_\infty}(\rho, r)$. With the notation introduced in Section 1.4 the number of vertices of $\partial B_{T_\infty}(\rho, r)$ is

$$\#\partial B_{T_\infty}(\rho, r) = 1 + \sum_{i=0}^{r-1} (L_i(r-i) + R_i(r-i)). \quad (2.1)$$

Conditionally on T_∞ , for every $u \in \partial B_{T_\infty}(\rho, r)$, the probability that $\mathcal{S}_{(T_\infty, \rho)}(u) = 0$ is the probability that a simple random walk on \mathbb{Z}^d returns to the origin in exactly r steps, which is less than $\kappa r^{-d/2}$ for some $\kappa > 0$. Thus, for every $r \geq 1$ the probability that there exists $u \in \partial B_{T_\infty}(\rho, r)$ such that $\mathcal{S}_{(T_\infty, \rho)}(u) = 0$ is bounded above by

$$\begin{aligned} \mathbb{P}(\exists u \in \partial B_{T_\infty}(\rho, r) : \mathcal{S}_{(T_\infty, \rho)}(u) = 0) &\leq \mathbb{E} \left[\sum_{u \in \partial B_{T_\infty}(\rho, r)} \mathbf{1}_{\mathcal{S}_{(T_\infty, \rho)}(u)=0} \right] \\ &\leq \kappa r^{-d/2} \mathbb{E}[\#\partial B_{T_\infty}(\rho, r)] \\ &\leq 3\kappa r^{-d/2+1}, \end{aligned}$$

where we used (2.1) and (1.4) to compute $\mathbb{E}[\#\partial B_{T_\infty}(\rho, r)]$. Consequently, for $d \geq 5$ the preceding bound is summable over $r \in \mathbb{Z}^+$, and an application of Borel-Cantelli's lemma shows that $\mathcal{S}_{(T_\infty, \rho)}$ is transient.

2.2 Recurrence for $d \in \{1, 2, 3, 4\}$

Since the path indexed by the infinite line in T_∞ (the one starting from ρ) is distributed as a simple symmetric random walk, the snake is obviously recurrent when $d \leq 2$. We now suppose $d \in \{3, 4\}$. *We will argue by contradiction* and assume that the simple random walk snake on \mathbb{Z}^d with $d \in \{1, 2, 3, 4\}$ supported by T_∞ is not recurrent, in particular it is easy to see that we have

$$\mathbb{P}(\mathcal{S}_{(T_\infty, \rho)}^{-1}(\{0\}) = \{\rho\}) > 0, \quad (2.2)$$

in words, the root vertex ρ of T_∞ is the only vertex being mapped to the origin of \mathbb{Z}^d by the snake $\mathcal{S}_{(T_\infty, \rho)}$ with positive probability. The following key lemma then states that under this assumption, the range of the snake is somehow linear. This will be the only place where we use the Mass Transport Principle (1.3).

Lemma 5. *Assume (2.2), then there exists $c > 0$ such that for every $r \geq 0$,*

$$\mathbb{P}(\#\mathcal{S}_{(T_\infty, \rho)}(B_{T_\infty}(\rho, r)) > cr^2) > c.$$

Proof. Consider the tree $(\tilde{T}_\infty, \tilde{\rho})$ obtained after biasing (T_∞, ρ) by the inverse of the degree of ρ . By Corollary 4, this random rooted graph satisfies the Mass Transport Principle (1.3). Since we have

$$\begin{aligned} \mathbb{P}(\#\mathcal{S}_{(T_\infty, \rho)}(B_{T_\infty}(\rho, r)) > cr^2) &\geq \mathbb{E}\left[\deg(\rho)^{-1}\mathbf{1}_{\#\mathcal{S}_{(T_\infty, \rho)}(B_{T_\infty}(\rho, r)) > cr^2}\right] \\ &= \mathbb{E}[\deg(\rho)^{-1}]\mathbb{P}\left(\#\mathcal{S}_{(\tilde{T}_\infty, \tilde{\rho})}(B_{\tilde{T}_\infty}(\tilde{\rho}, r)) > cr^2\right), \end{aligned}$$

it is enough to prove the lemma for \tilde{T}_∞ instead of T_∞ . For every rooted tree (τ, ρ) , we denote by $\Psi(\tau, \rho)$ the probability that the simple random snake on \mathbb{Z}^d supported on (τ, ρ) reaches 0 only at ρ , that is

$$\Psi(\tau, \rho) = \mathbb{P}(\mathcal{S}_{(\tau, \rho)}^{-1}(\{0\}) = \{\rho\}).$$

The function Ψ is thus a positive Borel function over the set of all rooted trees. Notice that if (τ, ρ) is fixed and if $u \in \tau$ then $\mathcal{S}_{(\tau, \rho)} - \mathcal{S}_{(\tau, u)}$ has the same distribution as $\mathcal{S}_{(\tau, u)}$. Thus $\Psi(\tau, u)$ is the probability that the snake $\mathcal{S}_{(\tau, \rho)}$ is one-to-one at the point u that is

$$\Psi(\tau, u) = \mathbb{P}\left(\mathcal{S}_{(\tau, \rho)}^{-1}(\{\mathcal{S}_{(\tau, \rho)}(u)\}) = \{u\}\right).$$

Using the Mass Transport Principle (1.3) on $(\tilde{T}_\infty, \tilde{\rho})$ with the function

$$F(G, x, y) = \Psi(G, x)\mathbf{1}_{d_{\text{gr}}^G(x, y) \leq r},$$

we get

$$\begin{aligned} \mathbb{E}\left[\Psi(\tilde{T}_\infty, \tilde{\rho})\#B_{\tilde{T}_\infty}(\tilde{\rho}, r)\right] &= \mathbb{E}\left[\sum_{x \in B_{\tilde{T}_\infty}(\tilde{\rho}, r)} \Psi(\tilde{T}_\infty, x)\right] \\ &= \mathbb{E}\left[\sum_{x \in B_{\tilde{T}_\infty}(\tilde{\rho}, r)} \mathbf{1}_{\mathcal{S}_{(\tilde{T}_\infty, \tilde{\rho})}^{-1}(\{\mathcal{S}_{(\tilde{T}_\infty, \tilde{\rho})}(x)\}) = \{x\}}\right] = \mathbb{E}\left[\#I_r\right], \end{aligned}$$

where I_r is the set of vertices in $B_{\tilde{T}_\infty}(\tilde{\rho}, r)$ at which the snake $\mathcal{S}_{(\tilde{T}_\infty, \tilde{\rho})}$ is one-to-one. We obviously have $\#I_r = \#\mathcal{S}_{(\tilde{T}_\infty, \tilde{\rho})}(I_r) \leq \#\mathcal{S}_{(\tilde{T}_\infty, \tilde{\rho})}(B_{\tilde{T}_\infty}(\tilde{\rho}, r))$ yielding to

$$\mathbb{E}\left[\Psi(\tilde{T}_\infty, \tilde{\rho})\#B_{\tilde{T}_\infty}(\tilde{\rho}, r)\right] \leq \mathbb{E}\left[\#\mathcal{S}_{(\tilde{T}_\infty, \tilde{\rho})}(B_{\tilde{T}_\infty}(\tilde{\rho}, r))\right]. \quad (2.3)$$

By definition of \tilde{T}_∞ , (1.7) still holds if we replace T_∞ by \tilde{T}_∞ . Similarly our condition (2.2) which states that $\Psi(T_\infty, \rho) > 0$ with positive probability implies $\Psi(\tilde{T}_\infty, \tilde{\rho}) > 0$ with positive probability

as well. Using these remarks we deduce that the left hand side in (2.3) is always larger than some constant times r^2 :

$$\inf_{r \geq 1} r^{-2} \mathbb{E} \left[\Psi(\tilde{T}_\infty, \tilde{\rho}) \# B_{\tilde{T}_\infty}(\tilde{\rho}, r) \right] > 0. \quad (2.4)$$

Let us turn to the right-hand side $\mathbb{E}[\#\mathcal{S}_{(\tilde{T}_\infty, \tilde{\rho})}(B_{\tilde{T}_\infty}(\tilde{\rho}, r))]$. Note first that we have

$$\#\mathcal{S}_{(\tilde{T}_\infty, \tilde{\rho})}(B_{\tilde{T}_\infty}(\tilde{\rho}, r)) \leq \#B_{\tilde{T}_\infty}(\tilde{\rho}, r).$$

Let $\lambda > 0$ and set $f(x) = x \mathbf{1}_{x > \lambda r^2}$, we get

$$\mathbb{E} \left[f \left(\#\mathcal{S}_{(\tilde{T}_\infty, \tilde{\rho})}(B_{\tilde{T}_\infty}(\tilde{\rho}, r)) \right) \right] \leq \mathbb{E} [f(\#B_{\tilde{T}_\infty}(\tilde{\rho}, r))] \leq \mathbb{E}[\deg(\rho)^{-1}]^{-1} \mathbb{E} [f(\#B_{T_\infty}(\rho, r))].$$

Applying Cauchy-Schwarz inequality we obtain thanks to (1.5) and (1.6)

$$\mathbb{E} [f(\#B_{\tilde{T}_\infty}(\tilde{\rho}, r))] \leq \left(\mathbb{E} [(\#B_{T_\infty}(\rho, r))^2] \mathbb{P}(\#B_{T_\infty}(\rho, r) > \lambda r^2) \right)^{1/2} \leq Cr^2 \lambda^{-1/2},$$

for some positive constant $C > 0$. Hence

$$\sup_{r > 0} r^{-2} \mathbb{E} \left[\#\mathcal{S}_{(\tilde{T}_\infty, \tilde{\rho})}(B_{\tilde{T}_\infty}(\tilde{\rho}, r)) \mathbf{1}_{\#\mathcal{S}_{(\tilde{T}_\infty, \tilde{\rho})}(B_{\tilde{T}_\infty}(\tilde{\rho}, r)) > \lambda r^2} \right] \xrightarrow{\lambda \rightarrow \infty} 0. \quad (2.5)$$

Combining (2.3), (2.4) and (2.5), we deduce that there exists a constant $c > 0$ such that

$$\mathbb{P}(\#\mathcal{S}_{(\tilde{T}_\infty, \tilde{\rho})}(B_{\tilde{T}_\infty}(\tilde{\rho}, r)) > cr^2) > c,$$

which is the desired result. \square

Now consider $\eta > 0$ small enough so that

$$A_{\eta, r}(T_\infty) = \{u \in B_{T_\infty}(\rho, r) \text{ in a tree grafted on the spine of } T_\infty \text{ before } \lfloor \eta r \rfloor\}$$

has a cardinal less than $cr^2/2$ with probability at least $1 - c/2$, independently of $r \geq 1$. Then by Lemma 5, for every $r \geq 1$, the set $B_{T_\infty}(\rho, r) \setminus A_{\eta, r}(T_\infty)$ which consists of the vertices within distance r of ρ that are linked to the spine after the $\lfloor \eta r \rfloor$ -th vertex has an image by $\mathcal{S}_{(T_\infty, \rho)}$ of size larger than $cr^2/2$ with probability at least $c/2$,

$$\mathbb{P}(\#\mathcal{S}_{(T_\infty, \rho)}(B_{T_\infty}(\rho, r) \setminus A_{\eta, r}(T_\infty)) > cr^2/2) > c/2.$$

By diffusivity bounds on the simple random snake on T_∞ (see [11] for the case $d = 1$ which is easily extended to $d \geq 2$, see also [16]), one can find $M > 0$ large enough so that the image of $B_{T_\infty}(\rho, r)$ by the snake is contained in $B_{\mathbb{Z}^d}(0, M\sqrt{r})$ with probability larger than $1 - c/4$. If $d \in \{1, 2, 3\}$ we already reached a contradiction since $\#B_{\mathbb{Z}^d}(0, M\sqrt{r}) \leq 8M^3 r^{3/2}$ and thus $B_{\mathbb{Z}^d}(0, M\sqrt{r})$ cannot contain a set of size of order r^2 with positive probability for r large enough.

We now suppose $d = 4$. Summing-up, for every $r \geq 1$, with a probability at least $c/4$, the image of $B_{T_\infty}(\rho, r) \setminus A_{\eta, r}(T_\infty)$ by $\mathcal{S}_{(T_\infty, \rho)}$ is a random set in \mathbb{Z}^4 composed of at least $cr^2/2$ different vertices and whose diameter is less than $M\sqrt{r}$.

Lemma 6. *The point 0 (origin of \mathbb{Z}^4) is in the image of $B_{T_\infty}(\rho, r) \setminus A_{\eta, r}(T_\infty)$ by $\mathcal{S}_{(T_\infty, \rho)}$ with a probability bounded away from 0 independently of $r \geq 1$.*

Proof. All the vertices of $B_{T_\infty}(\rho, r) \setminus A_{\eta, r}(T_\infty)$ are linked to ρ by the same first $\lfloor \eta r \rfloor$ vertices of the spine of T_∞ . Besides, the increments along the first $\lfloor \eta r \rfloor$ edges of the spine are independent of the structure of $B_{T_\infty}(\rho, r) \setminus A_{\eta, r}(T_\infty)$ and also independent of the increments of the snake along the edges of $B_{T_\infty}(\rho, r) \setminus A_{\eta, r}(T_\infty)$.

Denote by S the image of $B_{T_\infty}(\rho, r) \setminus A_{\eta, r}(T_\infty)$ by $\mathcal{S}_{(T_\infty, \rho)}$ translated such that the image of the $\lfloor \eta r \rfloor$ -th vertex of the spine is 0. Let E_r be the event that S is of size larger than $cr^2/2$ and of diameter less than $M\sqrt{r}$. By the arguments developed before the lemma, we have $\mathbb{P}(E_r) > c/4$ for every $r \geq 1$. Besides, the image of $B_{T_\infty}(\rho, r) \setminus A_{\eta, r}(T_\infty)$ by $\mathcal{S}_{(T_\infty, \rho)}$ is a random translation of the set S by an independent $\lfloor \eta r \rfloor$ -steps random $X_{\lfloor \eta r \rfloor}$ walk on \mathbb{Z}^4 . We claim that the conditional probability $\mathbb{P}(\mathbf{1}_{0 \in S + X_{\lfloor \eta r \rfloor}} | E_r)$ is bounded away from 0 independently of $r \geq 1$. Indeed conditionally on S and E_r , the variable

$$I = \sum_{x \in S + X_{\lfloor \eta r \rfloor}} \mathbf{1}_{x=0}$$

takes values in $\{0, 1\}$ and its expectation is

$$\mathbb{E}[I | E_r] = \mathbb{E} \left[\sum_{x \in S} \mathbb{P}(X_{\lfloor \eta r \rfloor} = -x) \middle| E_r \right] \geq \frac{cr^2}{2} \cdot \kappa r^{-2},$$

for some constant $\kappa > 0$ depending on η and M only. Hence, on the event E_r , the expectation of I is bounded away from 0, thus $\mathbb{P}(I > 0 | E_r) = \mathbb{E}[I | E_r]$ is bounded away from 0. Since $\mathbb{P}(E_r) \geq c/4$ for all $r \geq 0$, the lemma is proved.

Proof. For $\eta > 0$ fixed, denote $\mathcal{B}_r = B_{T_\infty}(\rho, r) \setminus A_{\eta, r}(T_\infty)$ to simplify notation. Thanks to the preceding lemma, the probability that 0 belongs to the image of \mathcal{B}_r by the snake is bounded from below by some positive constant $c > 0$ independent of $r \geq 0$. On the other hand we have,

$$\mathbb{P}\left(0 \in \mathcal{S}_{(T_\infty, \rho)}(\mathcal{B}_r) \cap \mathcal{S}_{(T_\infty, \rho)}(\mathcal{B}_{r'})\right) \xrightarrow{r' \rightarrow \infty} \mathbb{P}\left(0 \in \mathcal{S}_{(T_\infty, \rho)}(\mathcal{B}_r)\right) \mathbb{P}\left(0 \in \mathcal{S}_{(T_\infty, \rho)}(\mathcal{B}_{r'})\right).$$

In words, the events $\{0 \in \mathcal{S}_{(T_\infty, \rho)}(\mathcal{B}_r)\}$ and $\{0 \in \mathcal{S}_{(T_\infty, \rho)}(\mathcal{B}_{r'})\}$ are asymptotically independent as $r' \rightarrow \infty$. Hence, by an adaptation of Borel-Cantelli's lemma we deduce that $\{0 \in \mathcal{S}_{(T_\infty, \rho)}(\mathcal{B}_r)\}$ occurs for infinitely many r 's. This implies recurrence, contradiction. \square

3 Extensions and comments

3.1 General trees

Theorem 1 still holds for more general critical Galton-Watson trees conditioned to survive. Namely, if ξ is a critical offspring distribution with finite variance, Lemma 3 is still true and the limiting tree called the ξ -Galton-Watson tree conditioned to survive can be described as in Remark 1. The geometric estimates of Section 1.4 can be adapted for this infinite tree. However the analogue of Proposition 2 and its Corollary 4 are not true in this general setting. To bypass this difficulty, one can rely on the trick introduced in [20] and use the so-called *augmented Galton-Watson* measure. To be precise, let T^ξ and T_∞^ξ be respectively a critical ξ -Galton-Watson tree and a critical ξ -Galton-Watson tree conditioned to survive. We suppose that T^ξ and T_∞^ξ are independent and we define the tree \overline{T}_∞^ξ obtained by joining the roots of the trees T^ξ and T_∞^ξ by an edge. The root ρ of this tree is with probability 1/2 the root vertex of T^ξ and with probability 1/2 the root vertex of T_∞^ξ .

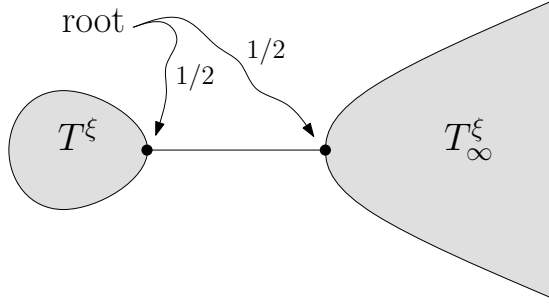


Figure 2: An illustration of the augmented critical Galton-Watson tree conditioned to survive.

Then one can show that $(\bar{T}_\infty^\xi, \rho)$ is stationary and reversible (in the sense of [6, Definition 1]) thus once biased by the inverse of the degree of the root it satisfies the Mass Transport Principle, see [20, 21] for closely related proofs.

3.2 Alternative proof and extension

It is likely that a proof of Theorem 1 would also follow from an adaptation of estimates for super-Brownian motion to discrete snakes (see [12, 17]), for example by showing

$$\mathbb{P}(\mathcal{S}_{(T,\rho)} \text{ reaches } x \in \mathbb{Z}^4) \asymp \frac{1}{\log(|x|)|x|^2},$$

where the snake $\mathcal{S}_{(T,\rho)}$ runs over a critical geometric Galton-Watson tree (T, ρ) not conditioned to be infinite. This approach has been carried out in an unpublished work of H. Kesten and Y. Peres (personal communication). In particular such estimates would be needed to answer the following questions:

Question 7. *What is the variance of the number of returns to 0 by the snake $\mathcal{S}_{(T_\infty, \rho)}$ restricted to the ball of radius r in T_∞ ? What is the range of the snake restricted to the ball of radius r in T_∞ ?*

However we believe that our more abstract argument relying on the Mass Transport Principle and rough volume estimates could be used in a more general setting including e.g. proving that a simple random snake on \mathbb{Z}^3 indexed by a critical Galton-Watson conditioned to be infinite whose offspring distribution is in the domain of attraction of a stable law of parameter $3/2$ is recurrent. We end this note with the following (possibly hard) question, generalizing intersecting probabilities for SRW in two dimensions:

Question 8. *Consider two independent snakes $\mathcal{S}_{(T_\infty, \rho)}$ and $\mathcal{S}'_{(T'_\infty, \rho')}$ on \mathbb{Z}^4 and starting from 0 (origin of \mathbb{Z}^4) at the roots ρ, ρ' of T_∞, T'_∞ . For $r \geq 0$, estimate the probability that the two snakes supported by the balls of radius r in the two trees intersect only at 0, that is estimate*

$$\mathbb{P}\left(\mathcal{S}_{(T_\infty, \rho)}^{-1}(B_{T_\infty}(\rho, r)) \cap \mathcal{S}'_{(T'_\infty, \rho')}^{-1}(B_{T'_\infty}(\rho', r)) = \{0\}\right).$$

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