

## TRISECANT LINES AND JACOBIANS

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Let  $X$  be a principally polarized complex abelian variety. Symmetric representatives  $\Theta$  of the polarization differ by translations by points of order 2, hence the linear system  $|2\Theta|$  is well defined. It is base point free and defines a morphism  $K: X \rightarrow |2\Theta|^*$ , whose image is the *Kummer variety* of  $X$ . When  $X$  is the Jacobian of an algebraic curve, there are infinitely many *triseccants* to  $K(X)$ , i.e., lines in the projective space  $|2\Theta|^*$  meeting  $K(X)$  in at least three points.

Welters conjectured in [9] that the existence of *one* triseccant line to the Kummer variety should characterize Jacobians among all indecomposable principally polarized abelian varieties. This conjecture should be thought of as a discrete analog of Novikov's conjecture (now proved by Shiota in [8]). It is known to hold if  $X$  is a Prym variety [6] and, in particular, *it holds in dimension  $\leq 5$* .

The existence of a line meeting  $K(X)$  at three distinct points  $K(a)$ ,  $K(b)$ ,  $K(c)$  is equivalent to the scheme-theoretic inclusion:

$$(0.1) \quad \Theta_a \cdot \Theta_b \subset \Theta_c \cup \Theta_{-c},$$

where, for a point  $x$  of  $X$ , we write  $\Theta_x$  for  $\Theta + x$ . This forces the intersection  $\Theta_a \cdot \Theta_b$  to be reducible.

We prove here a weak version of Welters' conjecture: *an indecomposable principally polarized complex abelian variety  $X$  is a Jacobian if and only if there exist points  $a, b, c$  of  $X$  with  $2a \neq 2b$ , such that  $K(a)$ ,  $K(b)$ ,  $K(c)$  are distinct and collinear and  $\Theta_a \cdot \Theta_b$  is reduced and has two irreducible components* (Theorem 3.1). The assumptions on  $\Theta_a \cdot \Theta_b$  can be replaced by: *the points  $b - a, c - b, a - c$  generate  $X$  and  $\dim \text{Sing } \Theta \leq \dim X - 4$*  (Theorem 3.5).

Our method consists in proving first that, under the above geometrical assumptions, inclusion (0.1) implies:

$$(0.2) \quad \Theta_a \cdot \Theta_b \subset \Theta_c \cup \Theta_{a+b-c}.$$

This inclusion is equivalent to the existence of a line tangent to  $K(X)$  at some point and meeting  $K(X)$  at some other point: a degenerate trisecant! It is also equivalent to the fact that the scheme

$$V_{a,b,c} = 2\{\zeta \in X \mid K(\zeta + a), K(\zeta + b), K(\zeta + c) \text{ are collinear}\}$$

has length  $\geq 2$  at  $(-a - b)$ .

As in [1] and [2], our aim is to prove that  $\dim_{-a-b} V_{a,b,c} > 0$  and to apply Welters' criterion [9] to conclude that  $X$  is a Jacobian. This is done in §2, assuming now that (0.2) holds and that  $\Theta_a \cdot \Theta_b$  is only reduced (Theorems 2.1 and 2.2). The method is one that Arbarello used in [1]: first translate Welters' criterion into an infinite set of differential equations (this is done in §1), then show that each of these equations follows from the first one, which is itself equivalent to (0.2).

### 1. A reformulation of Welters' criterion

Let  $X$  be a complex indecomposable principally polarized abelian variety, let  $\Theta$  be a symmetric representative of the polarization, and let  $K: X \rightarrow |2\Theta|^*$  be the Kummer map. Let  $a, b, c$  be distinct points of  $X$ .

Riemann's addition formula [7, p. 336] tells us that there exist a basis  $\{\theta\}$  for  $H^0(X, \Theta)$  and a basis  $\{\theta_n\}_n$  for  $H^0(X, 2\Theta)$  such that

$$(1.1) \quad \forall z, \zeta \in X \quad \theta(z + \zeta)\theta(z - \zeta) = \sum_n \theta_n(z)\theta_n(\zeta).$$

The subscheme  $V_{a,b,c}$  of  $X$  defined in the introduction is the set of points  $2\zeta$  of  $X$  such that

$$\exists \alpha, \beta, \gamma \in \mathbb{C} \quad \forall n \quad \alpha\theta_n(\zeta + a) + \beta\theta_n(\zeta + b) + \gamma\theta_n(\zeta + c) = 0.$$

Formula (1.1) implies that these equations are equivalent to:  $\exists \alpha, \beta, \gamma \in \mathbb{C} \quad \forall z \in X$

$$(1.2) \quad \alpha\theta(z + \zeta + a)\theta(z - \zeta - a) + \beta\theta(z + \zeta + b)\theta(z - \zeta - b) \\ + \gamma\theta(z + \zeta + c)\theta(z - \zeta - c) = 0.$$

Welters' criterion [9] states that  $X$  is a Jacobian if and only if there exist distinct points  $a, b, c$  of  $X$  such that  $\dim V_{a,b,c} > 0$ .

Since  $K$  takes the same value at opposite points,  $(-a - b)$  always belongs to  $V_{a,b,c}$ . The aim of this section is to translate the condition  $\dim_{-a-b} V_{a,b,c} > 0$  into an infinite set of equations.

This condition is equivalent to the existence of a formal curve:

$$2\zeta(\varepsilon) = -a - b + D(\varepsilon), \quad \text{with } D(\varepsilon) = \sum_{i \geq 1} D_i \varepsilon^i,$$

contained in  $V_{a,b,c}$ . The  $D_i$ 's are tangent vectors to  $X$  at the origin, or constant vector fields. We will look for a smooth germ ( $D_1 \neq 0$ ).

This is in turn equivalent to a relation of the type:

$$(1.3) \quad \begin{aligned} \forall z \in X \quad & \alpha(\varepsilon)\theta(z + \zeta(\varepsilon) + a)\theta(z - \zeta(\varepsilon) - a) \\ & + \beta(\varepsilon)\theta(z + \zeta(\varepsilon) + b)\theta(z - \zeta(\varepsilon) - b) \\ & + \gamma(\varepsilon)\theta(z + \zeta(\varepsilon) + c)\theta(z - \zeta(\varepsilon) - c) = 0, \end{aligned}$$

where  $\alpha(\varepsilon)$ ,  $\beta(\varepsilon)$ , and  $\gamma(\varepsilon)$  are relatively prime elements of  $\mathbb{C}[[\varepsilon]]$ . It is convenient to set  $u = a - \frac{1}{2}(a+b)$  and  $v = c - \frac{1}{2}(a+b)$  in (1.3), which becomes

$$(1.4) \quad \begin{aligned} & \alpha(\varepsilon)\theta(z + u + \frac{1}{2}D(\varepsilon))\theta(z - u - \frac{1}{2}D(\varepsilon)) \\ & + \beta(\varepsilon)\theta(z - u + \frac{1}{2}D(\varepsilon))\theta(z + u - \frac{1}{2}D(\varepsilon)) \\ & + \gamma(\varepsilon)\theta(z + v + \frac{1}{2}D(\varepsilon))\theta(z - v - \frac{1}{2}D(\varepsilon)) = 0. \end{aligned}$$

Write  $P(z, \varepsilon) = \sum_{s \geq 0} P_s(z)\varepsilon^s$  for the left-hand side of (1.4). Formula (1.1) implies, as in [2], that  $P_s$  is a section of  $\mathcal{O}_X(2\Theta)$ .

For any point  $x$  of  $X$ , write  $\theta_x$  for the translate of  $\theta$  by  $x$ . We have

$$P_0 = (\alpha(0) + \beta(0))\theta_{-u} \cdot \theta_u + \gamma(0)\theta_{-v} \cdot \theta_v.$$

Since  $a, b, c$  are distinct and  $\Theta$  is irreducible,  $P_0$  vanishes if and only if  $\alpha(0) + \beta(0) = \gamma(0) = 0$ . It follows that  $\alpha(\varepsilon)$  and  $\beta(\varepsilon)$  are units and that, by dividing by  $-\beta(\varepsilon)$ , we may assume:

$$\alpha(\varepsilon) = 1 + \sum_{i \geq 1} \alpha_i \varepsilon^i, \quad \beta(\varepsilon) = -1, \quad \gamma(\varepsilon) = \sum_{i \geq 1} \gamma_i \varepsilon^i.$$

One has

$$(1.5) \quad P_1 = \alpha_1 \theta_{-u} \cdot \theta_u + D_1 \theta_{-u} \cdot \theta_u - \theta_{-u} \cdot D_1 \theta_u + \gamma_1 \theta_{-v} \cdot \theta_v.$$

Since  $D_1$  is assumed to be nonzero and  $2u$  is nonzero,  $\gamma_1$  has to be nonzero if  $P_1$  vanishes. Allowing linear changes of the  $D_i$ 's, we may finally assume  $\gamma(\varepsilon) = \varepsilon$ .

We can state the following translation of Welters' criterion:

**Theorem 1.6.** *The abelian variety  $X$  is a Jacobian if and only if there exist complex numbers  $\alpha_1, \alpha_2, \dots$  and constant vector fields  $D_1, D_2, \dots$  on  $X$  with  $D_1 \neq 0$ , such that the sections  $P_s$  vanish for all positive integers  $s$ .*

Following [2], we can go further. Notice that  $P_s$  only depends on  $\alpha_1, \dots, \alpha_s$  and  $D_1, \dots, D_s$ . Moreover, one can write  $P_s$  as

$$(1.7) \quad P_s = Q_s + \alpha_s \theta_{-u} \cdot \theta_u + D_s \theta_{-u} \cdot \theta_u - \theta_{-u} \cdot D_s \theta_u,$$

where  $Q_s$  does not depend on  $\alpha_s$  nor on  $D_s$ . Notice that  $Q_s$ , which corresponds to vanishing  $\alpha_s$  and  $D_s$ , is also a section of  $\mathcal{O}_X(2\Theta)$ .

It follows from (1.7) that the restriction of  $P_s$  to  $\Theta_u \cdot \Theta_{-u}$  is independent of  $\alpha_s$  and  $D_s$ . Moreover, one has:

**Lemma 1.8.** *The section  $P_s$  of  $\mathcal{O}_X(2\Theta)$  vanishes for some choice of  $\alpha_s$  and  $D_s$  if and only if the restriction of  $Q_s$  to  $\Theta_u \cdot \Theta_{-u}$  does.*

*Proof of Lemma 1.8.* The cohomology sequence of the exact sequence

$$0 \rightarrow \mathcal{O}_{\Theta_u}(\Theta_u) \xrightarrow{\cdot\theta_{-u}} \mathcal{O}_{\Theta_u}(2\Theta) \rightarrow \mathcal{O}_{\Theta_u \cdot \Theta_{-u}}(2\Theta) \rightarrow 0$$

implies that  $Q_s$  vanishes on  $\Theta_u \cdot \Theta_{-u}$  if and only if there exists a vector field  $D$  such that

$$Q_s|_{\Theta_u} = D\theta_u \cdot \theta_{-u}.$$

The section  $(Q_s - D\theta_u \cdot \theta_{-u} + \theta_u \cdot D\theta_{-u})$  of  $\mathcal{O}_X(2\Theta)$  then vanishes on  $\Theta_u$  and one concludes with the cohomology sequence of the exact sequence

$$0 \rightarrow \mathcal{O}_X(\Theta_{-u}) \xrightarrow{\cdot\theta_u} \mathcal{O}_X(2\Theta) \rightarrow \mathcal{O}_{\Theta_u}(2\Theta) \rightarrow 0. \quad \text{q.e.d.}$$

In particular,  $P_1$ , given by (1.5), will vanish for some choice of  $\alpha_1$  and  $D_1$  if and only if

$$(1.9) \quad \Theta_u \cdot \Theta_{-u} \subset \Theta_v \cup \Theta_{-v}.$$

## 2. Characterizations of Jacobians by a degenerate trisecant

We start from an indecomposable principally polarized abelian variety  $X$  and points  $u$  and  $v$  of  $X$  such that (1.9) holds. In geometric terms, this is equivalent to the existence of a line passing through  $K(v)$  and tangent to the Kummer variety  $K(X)$  at  $K(u)$ : a degenerate trisecant!

In the notation of the last section, (1.9) translates into:  $P_1 = 0$  for some choice of  $\alpha_1$  and  $D_1$ . Our aim is to prove that this forces  $X$  to be a Jacobian by showing, by induction, that the sections  $P_s$  all vanish. Unfortunately, we need an extra hypothesis:

**Theorem 2.1.** *Let  $X$  be a complex indecomposable principally polarized abelian variety and let  $\Theta$  be a symmetric representative of the polarization. Then  $X$  is a Jacobian if and only if there exist points  $u$  and  $v$  of  $X$  with  $u, -u, v, -v$  all distinct, such that  $\Theta_u \cdot \Theta_{-u} \subset \Theta_v \cup \Theta_{-v}$  and  $\Theta_u \cdot \Theta_{-u}$  is reduced.*

*Proof of Theorem 2.1.* Suppose  $X$  is the Jacobian of a curve  $C$ . For distinct points  $p, q, r, s$  on  $C$ , one has

$$\Theta \cdot \Theta_{p-q} = V_p \cup W_q \subset \Theta_{p-r} \cup \Theta_{s-q},$$

where  $V_q$  and  $W_q$  are integral, depend only on  $p$  and  $q$  respectively, and are distinct one from another for general choices of  $p$  and  $q$ . Taking

$r = s$ ,  $u = \frac{1}{2}(p + q) - q$ , and  $v = \frac{1}{2}(p + q) - r$  proves one direction of the theorem.

To prove the other direction, it is enough to show, by (1.6) and (1.8), that for any integer  $s \geq 2$ , one has

$$P_1 = \dots = P_{s-1} = 0 \Rightarrow P_s \text{ (or } Q_s) \text{ vanishes on } \Theta_u \cdot \Theta_{-u}.$$

It is convenient to slightly change the setting. Write

$$R(z, \varepsilon) = P(z + \frac{1}{2}D(\varepsilon), \varepsilon) = \sum_{s \geq 1} R_s(z) \varepsilon^s.$$

Assume  $P_1 = \dots = P_{s-1} = 0$ . As in [1], it is elementary to check that  $R_1 = \dots = R_{s-1} = 0$  and that  $P_s = R_s$ . Therefore, it is enough to prove that  $R_s$  vanishes on  $\Theta_u \cdot \Theta_{-u}$ . Notice that

$$R(z, \varepsilon) = \alpha(\varepsilon)\theta(z + D(\varepsilon) + u)\theta(z - u) - \theta(z + D(\varepsilon) - u)\theta(z + u) + \varepsilon\theta(z + D(\varepsilon) + v)\theta(z - v),$$

whose restriction to  $\Theta_u \cdot \Theta_{-u}$  is just  $\varepsilon\theta(z + D(\varepsilon) + v)\theta(z - v)$ .

Writing  $e^{D(\varepsilon)} = \sum_{s \geq 0} \Delta_s \varepsilon^s$ , we get

$$R_s|_{\Theta_u \cdot \Theta_{-u}} = (\Delta_{s-1} \theta_{-v}) \cdot \theta_v.$$

Now set

$$T(z, \varepsilon) = P(z - \frac{1}{2}D(\varepsilon), \varepsilon) = \sum_{s \geq 1} T_s(z) \varepsilon^s$$

and  $e^{-D(\varepsilon)} = \sum_{s \geq 0} \Delta_s^- \varepsilon^s$ . Again, we get

$$T_s|_{\Theta_u \cdot \Theta_{-u}} = \theta_{-v} \cdot (\Delta_{s-1}^- \theta_v).$$

Moreover,

$$T(z, \varepsilon) = R(z - D(\varepsilon), \varepsilon) = e^{-D(\varepsilon)} R(z, \varepsilon),$$

We now give another version of Theorem 2.1, in which some assumptions on  $\Theta$  and  $u$  will automatically insure that  $\Theta_u \cdot \Theta_{-u}$  is reduced.

**Theorem 2.2.** *Let  $X$  be a principally polarized complex abelian variety and let  $\Theta$  be a symmetric representative of the polarization. Suppose that there exist points  $u$  and  $v$  of  $X$  with  $u, -u, v, -v$ , all distinct, such that:*

$$(1.9) \quad \Theta_u \cdot \Theta_{-u} \subset \Theta_v \cup \Theta_{-v},$$

$$(2.3) \quad \text{the closed subgroup of } X \text{ generated by } u \text{ has dimension } \geq 2,$$

$$(2.4) \quad \dim \text{Sing } \Theta \leq \dim X - 4.$$

*Then  $X$  is the Jacobian of a nonhyperelliptic curve.*

**Remarks 2.5.** (1) Notice the similarity between the hypothesis of this theorem and those of the theorem on p. 60 of [1].

(2) Hypothesis (2.4) is quite strong, since (1.9) implies  $\dim \text{Sing } \Theta \geq \dim X - 4$  [3, Théorème 2.3].

Theorem 2.2 follows immediately from Theorem 2.1 and the following proposition.

**Proposition 2.6.** *Let  $X$  be a principally polarized complex abelian variety. Let  $\Theta$  be a representative of the polarization and suppose (2.4) holds. Let  $w$  be a point of  $X$  such that the closed subgroup it generates has dimension  $\geq 2$ . Then  $\Theta \cdot \Theta_w$  is reduced.*

*Proof of Proposition 2.6.* Suppose  $\Theta \cdot \Theta_w$  is not reduced and pick a nonreduced component  $Z$ . Let  $n: \tilde{Z} \rightarrow Z_{\text{red}}$  be the normalization. The argument of the proof of Proposition 3 of [4] shows that (2.4) implies that  $w$  is in the kernel of the composed homomorphism

$$X \xrightarrow{\varphi} \text{Pic}^0 X \xrightarrow{n^*} \text{Pic}^0 \tilde{Z},$$

where  $\varphi$  is the isomorphism associated with the principal polarization of  $X$ . The transpose of  $n^*$  is the Albanese morphism  $\text{Alb}(\tilde{Z}) \rightarrow X$ , whose image contains a translate of  $Z_{\text{red}}$ , hence has codimension  $\leq 2$ . Using (2.3), this implies that the abelian variety  $A$  generated by  $w$  (i.e., the neutral component of the closed subgroup generated by  $w$ ) has dimension 2,  $Z_{\text{red}}$  is a translate of an abelian subvariety  $B$  of  $X$ , and the addition map  $\pi: A \times B \rightarrow X$  is an isogeny. According to Proposition (9.1) of [5], the polarization  $\pi^* \Theta$  splits as  $L_A \boxtimes L_B$ , and there exists a basis  $\{s_i\}_{1 \leq i \leq N}$  (resp.  $\{t_i\}_{1 \leq i \leq N}$ ) for  $H^0(A, L_A)$  (resp.  $H^0(B, L_B)$ ) such that  $\sum_{i=1}^N s_i t_i$  is an equation for  $\pi^* \Theta$ . Write  $Z_{\text{red}} = z_0 + B$  with  $z_0 \in A$ . The kernel of

$n^*: \text{Pic}^0(X) \rightarrow \text{Pic}^0(B)$  is connected; hence the kernel of  $n^* \circ \varphi$  is  $A$  and  $w \in A$ . We get

$$\forall z \in B \quad \sum_{i=1}^N s_i(z_0)t_i(z) = \sum_{i=1}^N s_i(z_0 - w)t_i(z) = 0.$$

This implies that both  $z_0$  and  $z_0 - w$  are in the base locus  $F_A$  of  $L_A$ .  
 But  $F$  is stable by translation by a subgroup  $H(L_A)$  of  $A$  of order  $N^2$ .

points of  $F_A$ . But  $L_A^2 = 2N$ ; hence either  $N = 1$  and  $X$  is decomposable, which would contradict (2.4), or  $F_A$  is contained in a translate of an elliptic curve, which would contradict the fact that  $w$ , which is the difference of two points of  $F_A$ , generates  $A$ .

Hence  $\Theta \cdot \Theta_w$  is reduced. q.e.d.

assume that  $V = a + b - V$  and  $W = a + b - W$ . On the other hand, by (1.2) again, (3.2) also implies

$$\Theta_a \cdot \Theta_c \subset \Theta_b \cup \Theta_{-b}.$$

The scheme  $\Theta_a \cdot \Theta_c$  contains  $V$ , hence  $a + c - V$ , which is equal to  $a + c - (a + b - V) = V + c - b$ , which is not contained in  $\Theta_{-b}$  by (3.4). It follows that  $a + c - V$  is contained in  $\Theta_b \cdot \Theta_a \cdot \Theta_c$ , whose only component of this dimension is  $V$ . Therefore  $V = a + c - V = V + c - b$ . By switching the roles of  $a$  and  $b$ , we get  $V = V + c - a$ ; hence  $V = V + a - b$ . By switching  $c$  and  $-c$ , one also gets  $W = W + a - b$ , from which it follows that  $\Theta_a \cdot \Theta_b$ , hence also  $\Theta \cdot \Theta_{a-b}$ , is invariant by translation by  $(a - b)$ . This is possible only if  $2(a - b) = 0$ , which we assumed did not hold. Therefore, one cannot have  $V = a + b - V$  and the proof is complete. q.e.d.

We conclude with another characterization of nonhyperelliptic Jacobians in the spirit of theorem 2.2 and the theorem on p. 60 of [1].

**Theorem 3.5.** *Let  $X$  be a principally polarized complex abelian variety and let  $\Theta$  be a representative of the polarization. Then  $X$  is a nonhyperelliptic Jacobian if and only if there exist points  $a, b, c$  on  $X$  such that:*

(3.6) *The points  $K(a), K(b)$ , and  $K(c)$  of the Kummer variety are distinct and collinear.*

(3.7) *The points  $a - c, b - a, c - b$  generate closed subgroups of  $X$  of dimensions at least 2, 2, and  $(\dim X - 1)$ , respectively.*

(3.8)  $\dim \text{Sing } \Theta \leq \dim X - 4.$

By working a bit harder, it is possible to weaken the hypothesis on  $(c - b)$  by asking only that the subgroup it generates have codimension  $\leq 2$ .

*Proof of Theorem 3.5.* We may assume  $\Theta$  is symmetric and  $\dim X \geq 4$ . It follows from Proposition 2.6 that  $\Theta_a \cdot \Theta_b$ ,  $\Theta_b \cdot \Theta_c$ , and  $\Theta_c \cdot \Theta_a$  are reduced. We begin with:

**Lemma 3.9.** *Under the hypothesis of the theorem, the components of  $\Theta_a \cdot \Theta_b$  which are contained in  $\Theta_c$  and  $\Theta_{-c}$  are also contained in  $\Theta_{-a}$  and  $\Theta_{-b}$  (same statement with  $a, b, c$  permuted).*

*Proof of Lemma 3.9.* Let  $Z$  be a component of  $\Theta_a \cdot \Theta_b$  contained in  $\Theta_c \cdot \Theta_{-c}$ . By differentiating the trisecant relation

$$\alpha\theta_a \cdot \theta_{-a} + \beta\theta_b \cdot \theta_{-b} + \gamma\theta_c \cdot \theta_{-c} = 0,$$



we get that the section  $(\alpha D\theta_a \cdot \theta_{-a} + \beta D\theta_b \cdot \theta_{-b})$  vanishes on  $Z$  for any constant vector field  $D$ .

Since  $\Theta_a \cdot \Theta_b$  is reduced,  $Z$  is not contained in its singular locus, from which it follows that  $\theta_{-a}$  and  $\theta_{-b}$  vanish on  $Z$ . q.e.d.

Now let  $\mathcal{V}$  be the set of components  $Z$  of  $\Theta_a \cdot \Theta_b$  such that neither  $Z$  nor  $(a+b-Z)$  is contained in  $\Theta_{-c}$ .

**Lemma 3.10.** *Assume  $\mathcal{V}$  is nonempty and let  $V$  be the union of the components of  $\Theta_a \cdot \Theta_b$  which belong to  $\mathcal{V}$ . Then  $V = V + b - c$ .*

*Proof of Lemma 3.10.* It is enough to show that, for any  $Z$  in  $\mathcal{V}$ ,  $Z + b - c$  is also in  $\mathcal{V}$ . Since  $Z$  is contained in  $\Theta_a \cdot \Theta_b$  but not in  $\Theta_{-c}$ , it is contained in  $\Theta_c$ , hence in  $\Theta_a \cdot \Theta_c$ . Therefore  $(a+c-Z) \subset \Theta_a \cdot \Theta_c \subset \Theta_b \cup \Theta_{-b}$ . Since  $(a+b-Z) \not\subset \Theta_{-c}$ , one has

$$(3.11) \quad (a+c-Z) \not\subset \Theta_{-b},$$

and therefore  $(a+c-Z) \subset \Theta_b$ , i.e.,  $(Z+b-c) \subset \Theta_a$ .

Furthermore,  $Z \subset \Theta_c$  implies  $(Z+b-c) \subset \Theta_b$ .

Since  $Z \not\subset \Theta_{-c}$ , Lemma 3.9 implies  $Z \not\subset \Theta_{-b}$ , that is,  $(Z+b-c) \not\subset$

is,  $(a+b-(Z+b-c)) \not\subset \Theta_{-c}$ . This proves that  $(Z+b-c) \in \mathcal{V}$ . q.e.d.

The abelian subvariety  $A$  of  $X$  generated by  $(b-c)$  has codimension

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