

RÉSUMÉ – Le degré d’une courbe C contenue dans une variété abélienne polarisée (X, λ) est l’entier $d = C \cdot \lambda$. Lorsque C engendre X , on obtient une minoration de d en fonction de n et du degré de la polarisation λ . Le plus petit degré possible est $d = n$ et n’est atteint que pour une courbe lisse dans sa jacobienne avec sa polarisation principale canonique (Ran, Collino). On étudie les cas $d = n + 1$ et $d = n + 2$. Lorsque X est simple, on montre de plus, en utilisant des résultats de Smyth sur la trace des entiers algébriques totalement positifs, que si $d \leq 1,7719n$, alors C est lisse et X est isomorphe à sa jacobienne. Nous obtenons aussi une borne supérieure pour le genre géométrique de C en fonction de son degré.

ABSTRACT – The degree of a curve C in a polarized abelian variety (X, λ) is the integer $d = C \cdot \lambda$. When C generates X , we find a lower bound on d which depends on n and the degree of the polarization λ . The smallest possible degree is $d = n$ and is obtained only for a smooth curve in its Jacobian with its principal polarization (Ran, Collino). The cases $d = n + 1$ and $d = n + 2$ are studied. Moreover, when X is simple, it is shown, using results of Smyth on the trace of totally positive algebraic integers, that if $d \leq 1.7719n$, then C is smooth and X is isomorphic to its Jacobian. We also get an upper bound on the geometric genus of C in terms of its degree.

1. Introduction

Although curves in projective spaces have attracted a lot of attention for a long time, very little is known in comparison about curves in abelian varieties. We try in this article to partially fill this gap.

Let (X, λ) be a principally polarized abelian variety of dimension n defined over an algebraically closed field k . The degree of a curve C contained in X is $d = C \cdot \lambda$.

The first question we are interested in is to find what numbers can be degrees of curves C in X . When C generates X , we prove that $d \geq n(\lambda^n/n!)^{1/n} \geq n$. It is known ([C], [R]) that $d = n$ if and only if C is smooth and X is isomorphic to its Jacobian J_C with its canonical principal polarization. What about the next cases? We get partial characterizations for $d = n + 1$ and $d = n + 2$, and we show (example 6.11) that all degrees $\geq n + 2$ actually occur when $\text{char}(k) = 0$. However, it seems necessary to assume X simple to go further. We prove, using results of Smyth ([S]), that *if C is an irreducible curve of degree $< 2n$ if $n \leq 7$, and $\leq 1.7719n$ if $n > 7$, on a simple principally polarized abelian variety X of dimension n , then C is smooth, has degree $2n - m$ for some divisor m of n , the abelian variety X is isomorphic to J_C (with a non-canonical principal polarization)*

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and C is canonically embedded in X . We conjecture this result to hold for any n under the assumption that C has degree $< 2n$. This would be a consequence of our conjecture 6.2, which holds for $n \leq 7$: *the trace of a totally positive algebraic integer σ of degree n is at least $2n - 1$ and equality can hold only if σ has norm 1*. Smooth curves of genus n and degree $2n - 1$ in their Jacobians have been constructed by Mestre for any n in [Me].

The second question is the Castelnuovo problem: bound the geometric genus $p_g(C)$ of a curve C in a polarized abelian variety X of dimension n in terms of its degree d . We prove, using the original Castelnuovo bound for curves in projective spaces, the inequality $p_g(C) < \frac{(2d-1)^2}{2(n-1)}$, which is far from being sharp (better bounds are obtained for small degrees). This in turn yields a lower bound in $O(n^{3/2})$ on the degree of a curve in a *generic* principally polarized abelian variety of dimension n .

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2. Endomorphisms and polarizations of abelian varieties

Let X be an abelian variety of dimension n defined over an algebraically closed field k and let $\text{End}(X)$ be its ring of endomorphisms. The degree $\text{deg}(u)$ of an endomorphism u is defined to be 0 if u is not surjective, and the degree of u as a map otherwise. For any prime l different from the characteristic of k , the Tate module $T_l(X)$ is a free \mathbf{Z}_l -module of rank $2n$ ([Mu], p.171) and the l -adic representation $\rho_l : \text{End}(X) \rightarrow \text{End}(T_l(X))$ is injective. For any endomorphism u of X , the characteristic polynomial of $\rho_l(u)$ has coefficients in \mathbf{Z} and is independent of l . It is called the characteristic polynomial of u and is denoted by P_u . It satisfies $P_u(t) = \text{deg}(t\text{Id}_X - u)$ for any integer t ([Mu], theorem 4, p.180). The opposite $\text{Tr}(u)$ of the coefficient of t^{2n-1} is called the trace of u .

The Néron-Severi group of X is the group of algebraic equivalence classes of invertible sheaves on X . Any element μ of $\text{NS}(X)$ defines a morphism $\phi_\mu : X \rightarrow \text{Pic}^0(X)$ ([Mu], p.60) whose scheme-theoretic kernel is denoted by $K(\mu)$. The Riemann-Roch theorem gives $\chi(X, \mu) = \mu^n/n!$, a number which will be called the degree of μ . One has $\text{deg } \phi_\mu = (\text{deg } \mu)^2$ ([Mu], p.150). A polarization λ on X is the algebraic equivalence class of an ample invertible sheaf on X ; it is said to be *separable* if its degree is prime to $\text{char}(k)$. In that case, ϕ_λ is a separable isogeny and its kernel is isomorphic to a group $(\mathbf{Z}/\delta_1\mathbf{Z})^2 \times \cdots \times (\mathbf{Z}/\delta_n\mathbf{Z})^2$, where $\delta_1 | \cdots | \delta_n$ and $\delta_1 \cdots \delta_n = \text{deg}(\lambda)$. We will say that λ is of type $(\delta_1 | \cdots | \delta_n)$.

We will need the following result:

THEOREM 2.1.(Kempf, Mumford, Ramanujan)– *Let X be an abelian variety of dimension n , and let λ and μ be two elements of $\text{NS}(X)$. Assume that λ is a polarization. Then:*

- (i) *The roots of the polynomial $P(t) = (t\lambda - \mu)^n$ are all real.*
- (ii) *If μ is a polarization, the roots of P are all positive.*

- (iii) If P has no negative roots and r positive roots, there exist a polarized abelian variety (X', μ') of dimension r and a surjective morphism $f : X \rightarrow X'$ with connected kernel such that $\mu = f^* \mu'$.

Proof. The first point is part of [MK], theorem 2, p. 98. The second point follows from the same theorem and the fact that if M is an ample line bundle on X with class μ , one has $H^i(X, M) = 0$ for $i > 0$ ([Mu], § 16). For the last point, the same theorem from [MK] yields that the neutral component K of the group $K(\mu)$ has dimension $n - r$. The restriction of M to K is algebraically equivalent to 0 (*loc.cit.*, lemma 1, p.95) hence, since the restriction $\text{Pic}^0(X) \rightarrow \text{Pic}^0(K)$ is surjective, there exists a line bundle N on X algebraically equivalent to 0 such that the restriction of $M \otimes N$ to K is trivial. It follows from theorem 1, p.95 of *loc.cit.* that $M \otimes N$ is the pull-back of a line bundle on $X' = X/K$. ■

(2.2) Suppose now that θ is a principal polarization on X , i.e. a polarization of degree 1. It defines a morphism of \mathbf{Z} -modules $\beta_\theta : \text{NS}(X) \rightarrow \text{End}(X)$ by the formula $\beta_\theta(\mu) = \phi_\theta^{-1} \circ \phi_\mu$. Its image consists of all endomorphisms invariant under the *Rosati involution*, which sends an endomorphism u to $\phi_\theta^{-1} \circ \text{Pic}^0(u) \circ \phi_\theta$ ([Mu], (3) p.190). Moreover, one has, for any integer t :

$$\left(\frac{(t\theta - \mu)^n}{n!} \right)^2 = \deg(t\phi_\theta - \phi_\mu) = \deg(t \text{Id}_X - \beta_\theta(\mu)) = P_{\beta_\theta(\mu)}(t).$$

(2.3) Let (X, λ) be a polarized abelian variety. For $0 < r \leq n$, we set $\lambda_{\min}^r = \lambda^r / (r! \delta_1 \cdots \delta_r)$. If $k = \mathbf{C}$, the class of λ_{\min}^r is minimal (i.e. non-divisible) in $H^{2r}(X, \mathbf{Z})$. If k is any algebraically closed field, and if l is a prime number different from the characteristic of k , the group $H_{\text{ét}}^1(X, \mathbf{Z}_l)$ is a free \mathbf{Z}_l -module of rank n ([Mi], theorem 15.1) and the algebras $H_{\text{ét}}^*(X, \mathbf{Z}_l)$ with its cup-product structure and $\wedge^* H_{\text{ét}}^1(X, \mathbf{Z}_l)$ with its wedge-product structure, are isomorphic ([Mi], remark 15.4). In particular, the class $[\lambda]_l$ in $H_{\text{ét}}^2(X, \mathbf{Z}_l)$ of the polarization λ can be viewed as an alternating form on a free \mathbf{Z}_l -module, and as such has elementary divisors. If λ is *separable*, (X, λ) lifts in characteristic 0 to a polarized abelian variety of the same type $(\delta_1 | \cdots | \delta_n)$. The elementary divisors of $[\lambda]_l$ are then the maximal powers of l that divide $\delta_1, \dots, \delta_n$. Since intersection corresponds to cup-product in étale cohomology, the class of λ_{\min}^r is in $H_{\text{ét}}^{2r}(X, \mathbf{Z}_l)$ and is not divisible by l .

Throughout this article, all schemes we consider will be defined over an algebraically closed field k . We will denote numerical equivalence by \sim . If C is a smooth curve, J_C will be its Jacobian and θ_C its canonical principal polarization.

3. Curves and endomorphisms

We summarize here some results from [Ma] and [Mo]. Let C be a curve on a polarized abelian variety (X, λ) and let D be an effective divisor that represents λ . Morikawa proves that the following diagram, where d is the degree of C and S is the sum morphism, defines an endomorphism $\alpha(C, \lambda)$ of X which is independent on the choice of D :

$$\begin{array}{ccccccc} \alpha(C, \lambda) : X & \dashrightarrow & C^{(d)} & \xrightarrow{S} & X & \xrightarrow{\text{translation}} & X \\ x & \longmapsto & (D + x) \cap C & & & & \end{array}$$

(3.1) Let N be the normalization of C . The morphism $\iota : N \rightarrow X$ factorizes through a morphism $p : JN \rightarrow X$. Set $q = \iota^* \circ \phi_\lambda : X \rightarrow JN$; Matsusaka proves that $\alpha(C, \lambda) = p \circ q$ ([Ma], lemma 3).

(3.2) He also proves (*loc.cit.*, theorem 2) that $\alpha(C, \lambda) = \alpha(C', \lambda)$ if and only if $C \sim C'$. Since $\alpha(\lambda^{n-1}, \lambda) = (\lambda^n/n)\text{Id}_X$, it follows that:

$$\alpha(C, \lambda) = m \text{Id}_X \iff C \sim \frac{m}{(n-1)! \deg \lambda} \lambda^{n-1} .$$

If moreover λ is separable of type $(\delta_1 | \dots | \delta_n)$ and if l is a prime distinct from $\text{char}(k)$, the discussion of (2.3) yields that there exists a class ϵ in $H_{\text{ét}}^2(X, \mathbf{Z}_l)$ such that $\lambda^{n-1} \cdot \epsilon$ is $(n-1)! \delta_1 \dots \delta_{n-1}$ times a generator of $H_{\text{ét}}^{2n}(X, \mathbf{Z}_l)$. It follows that $c = m/\delta_n$ must be in \mathbf{Z}_l . But δ_n is prime to $\text{char}(k)$, hence c is an integer and $C \sim c \lambda_{\min}^{n-1}$.

Let θ_N be the canonical principal polarization on JN . One has:

$$(3.3) \quad \phi_{q^* \theta_N} = \text{Pic}^0(q) \circ \phi_{\theta_N} \circ q = \phi_\lambda \circ p \circ \phi_{\theta_N}^{-1} \circ \iota^* \circ \phi_\lambda = \phi_\lambda \circ \alpha(C, \lambda) .$$

Similarly:

$$\phi_{p^* \lambda} = \text{Pic}^0(p) \circ \phi_\lambda \circ p = \phi_{\theta_N} \circ q \circ \phi_\lambda \circ \phi_\lambda^{-1} \circ p = \phi_{\theta_N} \circ q \circ p .$$

Note that, if g is the genus of N , one has $C \cdot \lambda = N \cdot p^* \lambda = \theta_N^{g-1} / (g-1)! \cdot p^* \lambda$. In particular, $-2(C \cdot \lambda)$ is the coefficient of t^{2g-1} in the polynomial:

$$\deg(t \theta_N - p^* \lambda)^2 = \deg(t \phi_{\theta_N} - \phi_{p^* \lambda}) = \deg(t \text{Id}_{JN} - q \circ p) .$$

Since $\text{Tr}(q \circ p) = \text{Tr}(p \circ q) = \text{Tr}(\alpha(C, \lambda))$, the following equality, originally proved by Matsusaka ([Ma], corollary, p.8), holds:

$$(3.4) \quad \text{Tr}(\alpha(C, \lambda)) = 2(C \cdot \lambda) .$$

(3.5) If the Néron-Severi group of X has rank 1 (this holds for a generic principally polarized X by [M], theorem 6.5, hence for a generic X with any polarization by [Mu], corollary 1, p.234), and ample generator l' , we can write $q^* \theta_N = r l'$ and $l = s l'$ with r and s integers. We get $r \phi_{l'} = s \phi_{l'} \circ \alpha(C, l)$ hence $\alpha(C, l)(sx) = rx$ for all x in X . By taking degrees, one sees that s divides r and $\alpha(C, l) = (r/s)\text{Id}_X$. By (3.2), the curve C is numerically equivalent to a rational multiple of λ^{n-1} and its degree is a multiple of n . If l' is separable of type $(\delta_1 | \dots | \delta_n)$, the curve C is numerically equivalent to an integral multiple of λ_{\min}^{n-1} and its degree is a multiple of $n \delta_n$.

LEMMA 3.6.— Let C be an irreducible curve that generates a polarized abelian variety (X, λ) of dimension n . Then, the polynomial $P_{\alpha(C, \lambda)}$ is the square of a polynomial whose roots are all real and positive.

Proof. Let $\alpha = \alpha(C, \lambda)$. By (3.3), one has $\phi_{q^* \theta_N} = \phi_\lambda \circ \alpha$, hence, for any integer t :

$$\begin{aligned} P_\alpha(t) \deg \phi_\lambda &= \deg(t \text{Id}_X - \alpha) \deg \phi_\lambda \\ &= \deg(t \phi_\lambda - \phi_\lambda \circ \alpha) \\ &= \deg(t \phi_\lambda - \phi_{q^* \theta_N}) \\ &= \deg(\phi_{t\lambda - q^* \theta_N}) = \left[\frac{1}{n!} (t\lambda - q^* \theta_N)^n \right]^2. \end{aligned}$$

The lemma then follows from theorem 2.1. ■

We end this section with a proof of Matsusaka's celebrated criterion:

THEOREM 3.7.(Matsusaka)— Let C be an irreducible curve in a polarized abelian variety (X, λ) of dimension n . Assume that $\alpha(C, \lambda) = \text{Id}_X$. Then C is smooth and (X, λ) is isomorphic to (JC, θ_C) .

Proof. Let N be the normalization of C . The morphism $\alpha(C, \lambda)$ is the identity and factors as:

$$X \dashrightarrow N^{(n)} \longrightarrow W_n(N) \longrightarrow \text{JN} \longrightarrow X.$$

It follows that $\dim \text{JN} = g(N) \geq n$. Moreover, the image of X in JN has dimension n , hence is the entire $W_n(N)$, which is therefore an abelian variety. This is possible only if $g(N) \leq n$. Hence N has genus n . It follows that the morphism $q : X \rightarrow \text{JN}$ is an isogeny, which is in fact an isomorphism since $p \circ q = \alpha(C, \lambda) = \text{Id}_X$. By (3.3), the polarizations $q^* \theta_N$ and λ are equal, hence q induces an isomorphism of the polarizations. ■

4. Degrees of curves

Let C be a curve that generates a polarized abelian variety (X, λ) of dimension n . We want to study its degree $d = C \cdot \lambda$. First, by description (3.1), the dimension of the image of $\alpha(C, \lambda)$ is the dimension of the abelian subvariety $\langle C \rangle$ generated by C . This and the definition of $\alpha(C, \lambda)$ imply:

$$C \cdot \lambda \geq n.$$

It was proved by Ran ([R]) for $k = \mathbf{C}$ and by Collino ([C]) in general, that if $C \cdot \lambda = n$, the minimal value, then C is smooth and (X, λ) is isomorphic to its Jacobian (JC, θ_C) . This suggests that there should be a better lower bound on the degree that involves the type of the polarization λ . The following proposition provides such a bound.

PROPOSITION 4.1.— *Let C be an irreducible curve that generates a polarized abelian variety (X, λ) of dimension n . Then:*

$$C \cdot \lambda \geq n(\deg \lambda)^{\frac{1}{n}} .$$

If λ is separable, there is equality if and only if C is smooth and (X, λ) is isomorphic to $(JC, \delta\theta_C)$, for some integer δ prime to $\text{char}(k)$.

Recall that by (3.5), the degree of any curve on a *generic* polarized abelian variety (X, λ) is a multiple of n . When λ is separable of type $(\delta_1 | \cdots | \delta_n)$, this degree is even a multiple of $n\delta_n$.

Proof of the proposition. We know by lemma 3.6 that $P_{\alpha(C, \lambda)}$ is the square of a polynomial Q whose roots β_1, \dots, β_n are real and positive. We have:

$$\begin{aligned} C \cdot \lambda &= \frac{1}{2} \text{Tr } \alpha(C, \lambda) = \frac{1}{2} (2\beta_1 + \cdots + 2\beta_n) \\ &\geq n (\beta_1 \cdots \beta_n)^{\frac{1}{n}} = n Q(0)^{\frac{1}{n}} = n P_{\alpha(C, \lambda)}(0)^{\frac{1}{2n}} \\ &= n (\deg \alpha(C, \lambda))^{\frac{1}{2n}} \geq n (\deg \phi_\lambda)^{\frac{1}{2n}} = n (\deg \lambda)^{\frac{1}{n}} . \end{aligned}$$

This proves the inequality in the proposition. If there is equality, β_1, \dots, β_n must be all equal to the same number m , which must be an integer since $P_{\alpha(C, \lambda)}$ has integral coefficients. It follows from the proof of lemma 3.6 that:

$$\left[\frac{1}{n!} (t\lambda - q^*\theta_N)^n \right]^2 = P_\alpha(t) \deg \phi_\lambda = (t - m)^{2n} \deg \phi_\lambda .$$

Theorem 2.1.(iii) yields $m\lambda = q^*\theta_N$. It follows from (3.3) that $\alpha(C, \lambda) = m \text{Id}_X$.

If λ is separable of type $(\delta_1 | \cdots | \delta_n)$, by (3.2), the number $c = m/\delta_n$ is an integer and C is numerically equivalent to $c\lambda_{\min}^{n-1}$. We get:

$$cn\delta_n = C \cdot \lambda = n(\deg \lambda)^{\frac{1}{n}} = n(\delta_1 \cdots \delta_n)^{\frac{1}{n}} \leq n\delta_n .$$

This implies $c = 1$ and $\delta_1 = \cdots = \delta_n = \delta$. But then λ is δ times a principal polarization θ ([Mu], theorem 3, p.231) and $C \sim \theta_{\min}^{n-1}$. The conclusion now follows from Matsusaka's criterion 3.7. ■

COROLLARY 4.2.(Ran, Collino)— *Let C be an irreducible curve that generates a polarized abelian variety (X, λ) of dimension n . Assume that $C \cdot \lambda = n$. Then C is smooth and (X, λ) is isomorphic to its Jacobian (JC, θ_C) .*

Proof. Although the converse of the proposition was proved only for λ separable, we still get from its proof that $\alpha(C, \lambda)$ is the identity of X and we may then apply Matsusaka's criterion 2.7. This is the same proof as Collino's. ■

More generally, if $C \cdot \lambda = \dim \langle C \rangle$, the same reasoning can be applied on $\langle C \rangle$ with the induced polarization to prove that C is smooth and that (X, λ) is isomorphic to the product of (JC, θ_C) with a polarized abelian variety.

COROLLARY 4.3.— *Let X be an abelian variety with a separable polarization λ of type $(\delta_1 | \cdots | \delta_n)$. Let C be an irreducible curve in X and let m be the dimension of the abelian subvariety that it generates. Then:*

$$C \cdot \lambda \geq m(\delta_1 \cdots \delta_m)^{\frac{1}{m}} .$$

Proof. Apply the proposition on the abelian subvariety Y generated by C . All there is to show is that the degree $Y \cdot \lambda^m / m!$ of the restriction λ' of λ to Y is at least $\delta_1 \cdots \delta_m$. We will prove that it is actually *divisible* by $\delta_1 \cdots \delta_m$. When $k = \mathbf{C}$, this follows from the fact that the class λ_{\min}^m is integral. The following argument for the general case was kindly communicated to the author by Kempf. Let ι be the inclusion of Y in X . Then $\phi_{\lambda'} = \text{Pic}^0(\iota) \circ \phi_{\lambda} \circ \iota$, hence $\deg(\lambda')^2$, which is the order of the kernel of $\phi_{\lambda'}$, is a multiple of the order of its subgroup $K(\lambda) \cap Y$, hence a fortiori a multiple of the order of its (r, r) part K' . In other words, since $K' \simeq (\mathbf{Z}/\delta'_1 \mathbf{Z})^2 \times \cdots \times (\mathbf{Z}/\delta'_m \mathbf{Z})^2$ for some integers $\delta'_1 | \delta'_2 | \cdots | \delta'_m$ prime to $\text{char}(k)$, it is enough to show that $\delta'_1 \delta'_2 \cdots \delta'_m$ is a multiple of $\delta_1 \delta_2 \cdots \delta_m$.

Let l be a prime number distinct from $\text{char}(k)$ and let \mathbf{F}_l be the field with l elements. For any integer s , let X_s be the kernel of the multiplication by l^s on X . Then X_s/X_{s-1} is a \mathbf{F}_l -vector space of dimension $2n$ of which Y_s/Y_{s-1} is a subspace of dimension $2m$. Since $K(\lambda)$ is isomorphic to $(\mathbf{Z}/\delta_1 \mathbf{Z})^2 \times \cdots \times (\mathbf{Z}/\delta_n \mathbf{Z})^2$, the rank over \mathbf{F}_l of $(K(\lambda) \cap X_s)/(K(\lambda) \cap X_{s-1})$ is twice the cardinality of the set $\{i \in \{1, \dots, n\} \mid l^s \text{ divides } \delta_i\}$. The dimension formula yields:

$$\text{rank}(K(\lambda) \cap Y_s)/(K(\lambda) \cap Y_{s-1}) \geq 2 \text{Card} \{i \mid l^s \text{ divides } \delta_i\} - 2n + 2m .$$

But the rank of $(K(\lambda) \cap Y_s)/(K(\lambda) \cap Y_{s-1}) = (K' \cap X_s)/(K' \cap X_{s-1})$ is also twice the cardinality of $\{i \in \{1, \dots, m\} \mid l^s \text{ divides } \delta'_i\}$. It follows that:

$$\text{Card} \{i \in \{1, \dots, n\} \mid l^s \nmid \delta_i\} \geq \text{Card} \{i \in \{1, \dots, m\} \mid l^s \nmid \delta'_i\} .$$

This implies what we need. ■

COROLLARY 4.4.— *Let C be an irreducible curve that generates a principally polarized abelian variety (X, θ) of dimension n . Assume that C is invariant by translation by an element ϵ of X of order m . Then $C \cdot \theta \geq n m^{1 - \frac{1}{n}}$.*

Proof. Let H be the subgroup scheme generated by ϵ . The abelian variety $X' = X/H$ has a polarization λ of degree m^{n-1} whose pull-back on X is $m\theta$ ([Mu], corollary, p.231). If C' is the image of C in X' , the proposition yields $C \cdot \theta = C' \cdot \lambda \geq n m^{1 - \frac{1}{n}}$. ■

Note that in the situation of corollary 4.4, if (X, θ) is a *generic* principally polarized abelian variety of dimension n , and m is prime to $\text{char}(k)$, then mn divides $C \cdot \theta$. With the notation of the proof above, this follows from the fact that any curve on X' is numerically equivalent to an integral multiple of λ_{\min}^{n-1} (see (3.5)).

5. Bounds on the genus

We keep the same setting: C is a curve that generates a polarized abelian variety (X, λ) , its normalization is N , and its degree is $d = C \cdot \lambda$. The composition:

$$X \dashrightarrow N^{(d)} \longrightarrow W_d(N) \longrightarrow JN$$

is a morphism with finite kernel (since $\alpha(C, \lambda)$ is an isogeny), hence $W_d(N)$ contains an abelian variety of dimension n . We can apply the ideas of [AH] to get a bound of Castelnuovo type on the genus of N . Note that if C does not generate X , the same bound holds with n replaced by the dimension of $\langle C \rangle$.

THEOREM 5.1.— *Let C be an irreducible curve that generates a separably polarized abelian variety (X, λ) of dimension $n > 1$. Let N be the normalization of C and let $d = C \cdot \lambda$. Then:*

$$g(N) < \frac{(2d - 1)^2}{2(n - 1)} .$$

The inequality in the second part of lemma 8 in [AH] would improve this bound when $\text{char}(k) = 0$, but its proof is incorrect.

Proof. Let A be the image of X in $W_d(N)$ and let A_2 be the image of $A \times A$ in $W_{2d}(N)$ under the addition map. We want to show that the morphism associated with a generic point of A_2 is generically injective on N . The linear systems corresponding to points of A_2 are of the form $|\mathcal{O}_N(2D_x)|$, where x varies in X , where D is an effective divisor that represents λ and $D_x = D + x$. It is therefore enough to show that the restriction to $C - x$ of the morphism ϕ_{2D} associated with $|2D|$ is generically injective for x generic. If not, for x generic in X and a generic in $C - x$, there exists b in $C - x$ with $a \neq b$ and $\phi_{2D}(a) = \phi_{2D}(b)$. The same holds for a generic in X and x generic in $C - a$. Since ϕ_{2D} is finite ([Mu], p.60), b does not depend on x , hence $C - a = C - b$. Since C generates X , this implies that $\epsilon = a - b$ is torsion, hence does not depend on a . Letting a vary, we see that any divisor in $|2D|$ is invariant by translation by ϵ . The argument in [Mu], p.164, yields a contradiction.

It follows that the image of the morphism $N \rightarrow \mathbf{P}^r$ that corresponds to a generic point in A_2 is a curve of degree a divisor d' of $2d$, with normalization N . Moreover, one has $r \geq n$ ([AH], lemma 1). Castelnuovo's bound ([ACGH], p.116 and [B] when $\text{char}(k) > 0$) then gives:

$$g(N) \leq m(d' - 1) - \frac{m(m + 1)(r - 1)}{2} ,$$

where $m = \lfloor \frac{d'-1}{r-1} \rfloor$. Hence:

$$\begin{aligned} g(\mathbb{N}) &\leq m \left(d' - 1 - \frac{(m+1)(r-1)}{2} \right) \\ &< \frac{d'-1}{r-1} \left(d' - 1 - \frac{d'-1}{2} \right) \\ &\leq \frac{(d'-1)^2}{2(n-1)} \leq \frac{(2d-1)^2}{2(n-1)}. \end{aligned}$$

This finishes the proof of the theorem. ■

In particular, in a principally polarized abelian variety (X, λ) of dimension n , any *smooth* curve numerically equivalent to $c\theta_{\min}^{n-1}$ has genus $< \frac{(2cn-1)^2}{2(n-1)}$. For curves in *generic* principally polarized abelian varieties of dimension n , I conjecture the stronger inequality $g(C) \leq cn + (c-1)^2$.

The theorem also gives a lower bound on the degree of any curve in a *generic* complex polarized abelian variety of dimension n , whose only merit is to go to infinity with n .

COROLLARY 5.2.— *Let C be a curve in a generic complex polarized abelian variety (X, λ) of dimension n and let c be the integer such that C is numerically equivalent to $c\lambda_{\min}^{n-1}$. Then:*

$$c > \sqrt{\frac{n}{8}} - \frac{1}{4}.$$

Proof. We may assume that λ is a principal polarization and that $n > 12$. Let \mathbb{N} be the normalization of C . Corollary 5.5 in [AP] yields $g(\mathbb{N}) > 1 + n(n+1)/4$, which, combined with the proposition, gives what we want. ■

We can get better bounds on the genus when d/n is small.

PROPOSITION 5.3.— *Let C be an irreducible curve that generates a complex polarized abelian variety (X, λ) of dimension n . Let \mathbb{N} be the normalization of C and let $d = C \cdot \lambda$. Then:*

- (i) *If $d < 2n$, then $g(\mathbb{N}) \leq d$.*
- (ii) *If $d = 2n$, then $g(\mathbb{N}) < \frac{3d}{2} = 3n$.*
- (iii) *If $d \leq 3n$, then $g(\mathbb{N}) \leq 4d$.*
- (iv) *If $d \leq 4n$, then $g(\mathbb{N}) \leq 6d$.*

Proof. We keep the notation of the proof of theorem 5.1. In particular, $W_d(\mathbb{N})$ contains an abelian variety A of dimension n . If $2n > d$, it follows from proposition 3.3 of [DF] that $g(\mathbb{N}) \leq d$. Recall that we proved earlier that the morphisms that correspond to generic points in A_2 are *birational* onto their image. It follows from corollary 3.6 of *loc.cit.* that $g(\mathbb{N}) < 3d/2$ when $d = 2n$. This proves (ii). We will do (iv) only, (iii) being analogous. First, we may assume that the embedding of A in $W_d(\mathbb{N})$ satisfies the

minimality assumptions made in [A1]. Let A_k be the image of $A \times \cdots \times A$ in $W_{kd}(\mathbb{N})$ under the addition map and let r_k be the maximum integer such that A_k is contained in $W_{kd}^{r_k}(\mathbb{N})$. If $g(\mathbb{N}) > 6d$, we get, as in the proof of proposition 3.8 of [DF], the inequalities $r_6 \geq 8n + 2$ and $n \leq 6d - 3r_6$. It follows that $d \geq (n + 3r_6)/6 \geq (25n + 6)/6 > 4n$. This proves (iv). ■

The inequality (ii) should be compared with the inequality $g(C) \leq 2n + 1$ proved by Welters in [W] when $\text{char}(k) = 0$ for any irreducible curve C numerically equivalent to $2\theta_{\min}^{n-1}$ on a principally polarized abelian variety (X, θ) of dimension n (so that $C \cdot \theta = 2n$). Equality is obtained only with the Prym construction.

6. Curves of low degrees

Let C be an irreducible curve that generates a principally polarized abelian variety (X, θ) of dimension n . We keep the same notation: N is the normalization of C and $q : X \rightarrow JN$ is the induced morphism. From (2.2), we get that the square of the monic polynomial $Q_C(T) = (T\theta - q^*\theta_N)^n/n!$ has integral coefficients (and is the characteristic polynomial of $\alpha(C, \theta)$). It follows that Q_C itself has integral coefficients, and we get from theorem 2.1 and (3.4):

- (i) The roots of Q_C are all real and positive.
- (ii) The sum of the roots of Q_C is the degree $d = C \cdot \theta$.
- (iii) The product of the roots of Q_C is the degree of the polarization $q^*\theta_N$.

Smyth obtained in [S] a lower bound on the trace of a totally real algebraic integer in terms of its degree. His results can be partially summarized as follows.

THEOREM 6.1.(Smyth)— *Let σ be a totally positive algebraic integer of degree m . Then $\text{Tr}(\sigma) > 1.7719m$, unless σ belongs to an explicit finite set, in which case $\text{Tr}(\sigma) = 2m - 1$ and $\text{Nm}(\sigma) = 1$.*

It is tempting to conjecture:

(6.2) **CONJECTURE C_m** — *Let σ be a totally positive algebraic integer of degree m . Then $\text{Tr}(\sigma) \geq 2m - 1$. If there is equality, then $\text{Nm}(\sigma) = 1$.*

(6.3) The inequality in the conjecture follows from Smyth's theorem for $m \leq 8$ (and holds also for $m = 9$ according to further calculations). Smyth also worked out a list of all totally positive algebraic integers σ for which $\text{Tr}(\sigma) - \text{deg}(\sigma) \leq 6$. It follows from this list that *the full conjecture holds for $m \leq 7$.*

There are infinitely many examples for which the conjectural bound is obtained: if M is an odd prime, the algebraic integer $4\cos^2(\pi/2M)$ is totally positive, has degree $(M - 1)/2$, trace $M - 2$ and norm 1.

PROPOSITION 6.4.— *Let C be an irreducible curve that generates a principally polarized abelian variety (X, θ) of dimension n and let Q_C be the polynomial defined above. Then, if $|Q_C(0)| = 1$, the curve C is smooth, X is isomorphic to its Jacobian and C is canonically embedded.*

Proof. By fact (iii) above, the polarization $q^*\theta_N$ is principal. The proposition then follows from the following lemma. ■

LEMMA 6.5.— *Let (JN, θ_N) be the Jacobian of a smooth curve, let X be a non-zero abelian variety and let $q : X \rightarrow JN$ be a morphism. Assume that $q^*\theta_N$ is a principal polarization. Then q is an isomorphism.*

Proof. Since $q^*\theta_N$ is a principal polarization, q is a closed immersion. By Mumford's proof of Poincaré's complete reducibility theorem ([Mu], p.173), there exist another abelian subvariety Y of JN and an isogeny $f : X \times Y \rightarrow JN$ such that $f^*\theta_N$ is the product of the induced polarizations on each factor. As in *loc.cit.*, for any k -scheme S , the set $(X \cap Y)(S)$ is contained in $K(q^*\theta_N)(S)$, which is trivial. Hence f is an isomorphism of polarized varieties. But a Jacobian with its canonical principal polarization cannot be a product, hence Y is 0 and q is an isomorphism. ■

We now give a result on curves on *simple* abelian varieties. The part that depends on the validity of conjecture 6.2 holds in particular for $n \leq 7$.

THEOREM 6.6.— *Let C be an irreducible curve in a simple principally polarized abelian variety (X, θ) of dimension n . Assume that either $C \cdot \theta \leq 1.7719n$, or that conjecture C_m holds for all divisors m of n and $C \cdot \theta < 2n$. Then, the curve C is smooth, X is isomorphic to its Jacobian and C is canonically embedded.*

Proof. Since X is simple, the polynomial $P_{\alpha(C, \theta)}$, hence also its “square root” Q_C , are powers of an irreducible polynomial R of degree some divisor m of n . If the degree of C , which is equal to the sum of the roots of Q_C , is $\leq 1.7719n$, the sum of the roots of R is also $\leq 1.7719m$. It follows from theorem 6.1 that $|R(0)| = 1$. On the other hand, if $C \cdot \theta < 2n$, the sum of the roots of R is also $< 2m$, hence, since C_m is supposed to hold, we also have $|R(0)| = 1$. The theorem then follows in both cases from proposition 6.4. ■

It follows from the proof of the theorem that C has degree $2n - m$ for some divisor m of n . In particular, for n prime, either C has degree n and θ is the canonical principal polarization, or it has degree $2n - 1$.

If one wants curves of degree between n and $2n$ in a simple abelian variety X , and if one believes in conjecture 6.2, X needs to be a *Jacobian with real multiplications* (in the sense that the ring $\text{End}(X) \otimes \mathbf{Q}$ contains a totally real number field different from \mathbf{Q}). Examples have been constructed in [Me] (see also [TTV]). More precisely, for any integer $M \geq 4$, Mestre constructs an explicit 2-dimensional family of complex hyperelliptic

Jacobians JC of dimension $[M/2]$ whose endomorphism rings contain a subring isomorphic to $\mathbf{Z}[T]/G_M(T)$, where:

$$G_M(T) = \prod_{0 < k \leq [M/2]} \left(T - 4 \cos^2 \frac{k\pi}{M} \right),$$

whose elements are invariant under the Rosati involution. By (2.2), they correspond to polarizations on JC . Take M odd and set $n = \dim(\text{JC}) = (M-1)/2$. Then, the endomorphism of X that corresponds to T gives rise to a *principal* polarization on JC , with respect to which the degree of C , canonically embedded, is $2n-1$. Therefore, for any $n \geq 2$, we have examples of *complex principally polarized abelian varieties of dimension n that contain curves of degree $2n-1$* . They are simple if $2n+1$ is prime. For $n=2$, these examples are Humbert surfaces, which contain curves of degree 3 ([vG], p.221).

If the assumption X simple is dropped, much less can be said. If Q is a monic polynomial with integral coefficients whose roots are all real, we will say that a curve C has *real multiplications by Q* if there is an endomorphism of JC whose characteristic polynomial (see § 2) is Q^2 . If $k = \mathbf{C}$, this is the same as asking that the characteristic polynomial of the endomorphism acting on the space of first-order differentials of C be Q .

PROPOSITION 6.7.— *Let C be an irreducible curve that generates a principally polarized abelian variety (X, θ) of dimension n . Then, if $C \cdot \theta = n+1$, the curve C is smooth, X is isomorphic to its Jacobian and C is canonically embedded. Moreover, the curve C has real multiplications by $(T-1)^{n-2}(T^2-3T+1)$.*

Proof. By theorem 6.1 and Smyth's list in [S], the polynomial Q_C can only be $(T-1)^{n-1}(T-2)$ or $(T-1)^{n-2}(T^2-3T+1)$. By proposition 6.4, we need only exclude the first case. By theorem 2.1, there exist a polarized elliptic curve (X', λ') and a morphism $f' : X \rightarrow X'$ such that $f'^* \lambda' = q^* \theta_N - \theta$. Similarly, there exist an $(n-1)$ -dimensional polarized abelian variety (X'', λ'') and a morphism $f'' : X \rightarrow X''$ such that $f''^* \lambda'' = 2\theta - q^* \theta_N$. The isogeny $(f', f'') : (X, \theta) \rightarrow (X', \lambda') \times (X'', \lambda'')$ is a morphism of polarized abelian varieties. Since θ is principal, it is an isomorphism and λ' and λ'' are both principal polarizations. Then, $(X, q^* \theta_N)$ is isomorphic to $(X', 2\lambda') \times (X'', \lambda'')$. In particular, the pull-back of θ_N by $X'' \rightarrow \text{JN}$ is a principal polarization. By lemma 6.5, this cannot occur. ■

In the next case where $\deg(C) = n+2$, the same techniques give partial results.

PROPOSITION 6.8.— *Let C be an irreducible curve that generates a principally polarized abelian variety (X, θ) of dimension $n > 2$. Assume that $\text{char}(k) \neq 2, 3$. Then, if $C \cdot \theta = n+2$, one of the following possibilities occurs:*

- (i) *the curve C is smooth of genus n , X is isomorphic to its Jacobian and C is canonically embedded. Moreover, the curve C has real multiplications by $(T-1)^{n-2}(T^2-4T+1)^2$, $(T-1)^{n-3}(T^3-5T^2+6T-1)$ or $(T-1)^{n-4}(T^2-3T+1)^2$.*

- (ii) *the curve C is smooth of genus n and bielliptic, i.e. there exists a morphism of degree 2 from C onto an elliptic curve E . The abelian variety X is the quotient of JC by an element of order 3 that comes from E .*
- (iii) *The normalization N of C has genus n and real multiplications by $(T-1)^{n-2}(T^2-4T+2)$ or $(T-1)^{n-3}(T-2)(T^2-3T+1)$. There is an isogeny $JN \rightarrow X$ of degree 2, and either C is smooth, or it has one node and N is hyperelliptic.*
- (iv) *the curve C is smooth and bielliptic of genus $n+1$, and has real multiplications by $T(T-1)^{n-2}(T^2-4T+2)$ or $T(T-1)^{n-3}(T-2)(T^2-3T+1)$. The abelian variety X is the ‘‘Prym variety’’ associated with the bielliptic structure.*

REMARKS 6.9.1) Mestre’s construction for $N = 7$ gives examples of curves of degree 5 in principally polarized abelian varieties of dimension 3, which fit into case (i) of the proposition. Example 6.11 below show that case (ii) does occur. These are the only examples I know of.

2) In general, if a curve C has real multiplications by a polynomial $(T-a)^m Q(T)$, where a is an integer and $Q(a) \neq 0$, then there is a morphism from JC onto an abelian variety of dimension m (this follows for example from [MK], theorem 2, p. 98).

Proof. By theorem 6.1 and Smyth’s list in [S], the polynomial Q_C can only be $(T-1)^{n-2}(T-2)^2$, $(T-1)^{n-2}(T^2-4T+1)^2$, $(T-1)^{n-3}(T^3-5T^2+6T-1)$, $(T-1)^{n-4}(T^2-3T+1)^2$, $(T-1)^{n-1}(T-3)$, $(T-1)^{n-2}(T^2-4T+2)$ or $(T-1)^{n-3}(T-2)(T^2-3T+1)$. The first polynomial is excluded as in proposition 6.7 (use $n > 2$). If the constant term is ± 1 , the same proof as above yields that we are in case (i).

If $Q_C(T) = (T-1)^{n-1}(T-3)$, as in the proof of proposition 6.7, there exist a polarized elliptic curve (X', λ') and a morphism $f' : X \rightarrow X'$ with connected kernel X'' such that $f'^* \lambda' = q^* \theta_N - \theta$ or equivalently $q^* \theta_N = \theta + (\deg \lambda')[X'']$. The identity $\frac{1}{n!} (T\theta - q^* \theta_N)^n = (T-1)^{n-1}(T-3)$ yields $(\deg \lambda')(\deg \theta|_{X''}) = 2$. If $\deg \lambda' = 2$, one gets a contradiction as in the proof of proposition 6.7. If λ' is principal, one has $\deg((q^* \theta_N)|_{X''}) = 2$. We use the following result.

LEMMA 6.10.– *Let (JN, θ_N) be the Jacobian of a smooth curve, let X be a non-zero abelian variety and let $r : X \rightarrow JN$ be a morphism with finite kernel. Assume that $\deg(r^* \theta_N)$ is $\leq \dim(X)$ and prime to $\text{char}(k)$. Then $g(N) < \dim(X) + \deg(r^* \theta_N)$.*

Proof. Let K be the kernel of r and let $\iota : X/K \rightarrow JN$ be the induced embedding. By Poincaré’s complete reducibility theorem ([Mu], p.173), there exist an abelian subvariety X' of JN and an isogeny $f : X/K \times X' \rightarrow JN$ such that the pull-back $f^* \theta_N$ is the product of the induced polarizations. Note that $\deg(\iota^* \theta_N)$ divides $\deg(r^* \theta_N)$. In particular, under our

assumptions, the polarization $\iota^*\theta_N$ is *separable* and has a non-empty base locus F , of dimension $\geq \dim(X) - \deg(r^*\theta_N)$. If Θ is a theta divisor for JN , it follows from the equation of $f^*\Theta$ given in [D], proposition 9.1, that $f(F \times X')$ is contained in Θ . Lemma 5.1 from [DF] (which is valid in any characteristic) then yields $\dim(F \times X') + \dim(X') \leq g(N) - 1$, from which the lemma follows. ■

Since $\text{char}(k) \neq 2$, it follows from the lemma applied to the inclusion $X'' \rightarrow JN$ that $g(N) = n$ hence that the morphism $q : X \rightarrow JN$ is an isogeny of degree 3. It is not difficult to see (using for example [D] § 9) that since $\text{char}(k) \neq 2, 3$, there is a commutative diagram of separable isogenies:

$$\begin{array}{ccccc}
X'' \times X' & \xrightarrow{3:1} & X'' \times E & \xrightarrow{3:1} & X'' \times X' \\
\downarrow 4:1 & & \downarrow 4:1 & & \downarrow 4:1 \\
X & \xrightarrow{q} & JN & \xrightarrow{p} & X
\end{array}$$

where E is the quotient of X' by a subgroup of order 3. The middle vertical arrow induces an injection of E into JN whose image has degree 2 with respect to θ_N . By duality, one gets a morphism $f : N \rightarrow E$ of degree 2. In this situation, one checks that since $n > 2$, for any two points x and y of N , one cannot have $x - y \equiv f^*e$, for $e \neq 0$ in E . Thus C , image of N in X by p , is smooth.

If $Q_C(T) = (T - 1)^{n-2}(T^2 - 4T + 2)$ or $(T - 1)^{n-3}(T - 2)(T^2 - 3T + 1)$, the polarization $q^*\theta_N$ has degree 2. It follows from lemma 6.10 that:

- either $g(N) = n$ and C is the image of N by an isogeny $p : JN \rightarrow X$ of degree 2. In particular, either C is smooth or N is hyperelliptic and C is obtained by identifying two Weierstrass points of N (so that, in a sense, C is bielliptic).

- or $g(N) = n + 1$ and q is a closed immersion. The proof of lemma 6.10 yields an elliptic curve X' in JN and an isogeny $f : X \times X' \rightarrow JN$ of degree 4. Moreover, $\deg(\theta_N)|_{X'} = 2$, hence the morphism $N \rightarrow X'$ obtained by duality has degree 2. One checks as above that C is smooth. The abelian variety X is the Prym variety associated with the bielliptic structure, i.e. is isomorphic to the quotient JN/X' . It remains to prove the statement about real multiplications. With the notation of (2.2), we calculate the characteristic polynomial of the endomorphism $\beta_{\theta_N}(p^*\theta)$ of JN . If t is any integer, one has:

$$\begin{aligned}
\deg(t \text{Id}_{JN} - \beta_{\theta_N}(p^*\theta)) &= \deg(t \theta_N - p^*\theta)^2 \\
&= \left(\frac{1}{4} \deg(t f^*\theta_N - f^*p^*\theta) \right)^2 \\
&= \left(\frac{1}{4} \deg(t (\theta_N)|_{X'}) \deg(t q^*\theta_N - q^*p^*\theta) \right)^2 \\
&= \frac{t^2}{4} \deg(t \phi_{q^*\theta_N} - \phi_{q^*p^*\theta}) .
\end{aligned}$$

Set $\alpha = \alpha(C, \theta)$. Using (3.1) and (3.3), we get:

$$\begin{aligned} \deg(t \text{Id}_{\mathbf{JN}} - \beta_{\theta_N}(p^*\theta)) &= \frac{t^2}{4} \deg(t \phi_\theta \circ \alpha - \phi_{\alpha^*\theta}) \\ &= \frac{t^2}{4} \deg(t \text{Id}_{\text{Pic}^0(X)} - \text{Pic}^0(\alpha)) \deg(\phi_\theta \circ \alpha) \\ &= P_{\text{Pic}^0(\alpha)}(t) t^2 = Q_C(t)^2 t^2 . \end{aligned}$$

It follows that \mathbf{N} has real multiplications by $\mathbf{T} \mathbf{Q}_C(\mathbf{T})$. This finishes the proof of the proposition. ■

EXAMPLE 6.11 Case (ii) of the proposition does occur as a particular case of the following construction. Let C be a smooth curve of genus n with a morphism of degree r onto an elliptic curve E . Assume that r is prime to $\text{char}(k)$ and that the induced morphism $E \rightarrow \mathbf{J}C$ is a closed immersion. Let s be an integer prime to $\text{char}(k)$ and congruent to 1 modulo r , and let $q : \mathbf{J}C \rightarrow X$ be the quotient by a cyclic subgroup of order s of E . There exist an abelian variety Y of dimension $n - 1$ with a polarization λ_Y of type $(1 | \cdots | 1 | r)$ and an isogeny $f : E \times Y \rightarrow \mathbf{J}C$ with kernel isomorphic to $(\mathbf{Z}/r\mathbf{Z})^2$, such that $f^*\theta_C = pr_1^*(r\lambda_E) \otimes pr_2^*\lambda_Y$, where λ_E is the polarization on E defined by a point. Because $s \equiv 1 \pmod{r}$, one checks that there exists a principal polarization θ on X such that $f^*q^*\theta = pr_1^*(rs\lambda_E) \otimes pr_2^*\lambda_Y$. We claim that *the degree of the curve $q(C)$ on X with respect to the principal polarization θ is $n + s - 1$* . In fact, one has:

$$f^*\theta_C^{n-1}/(n-1)! \sim r\lambda_E (pr_2^*\lambda_Y)^{n-2}/(n-2)! + (pr_2^*\lambda_Y)^{n-1}/(n-1)!$$

hence

$$\begin{aligned} f^*\theta_C^{n-1}/(n-1)! \cdot f^*q^*\theta &= rs \deg \lambda_Y + r(n-1) \deg \lambda_Y \\ &= r^2(s+n-1) . \end{aligned}$$

It follows that $C \cdot q^*\theta = n + s - 1$, which proves our claim.

When $\text{char}(k) = 0$, this construction yields examples of curves of degree $n + t$ in principally polarized abelian varieties of dimension n , for any $n \geq 2$ and $t \geq 2$.

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O. DEBARRE, Université Paris-Sud, Mathématique, Bâtiment 425, 91405 Orsay Cedex, France, and Department of Mathematics, The University of Iowa, Iowa City, IA 52242, U.S.A.

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