POINTS OF LOW DEGREE ON SMOOTH PLANE CURVES

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1. Introduction

The purpose of this note is to provide some applications of a theorem of Faltings ([Fa1]) to smooth plane curves, using ideas from [A] and [AH].

Let C be a smooth projective plane curve defined by an equation of degree d with rational coefficients. We show:

Theorem 1.– *If $d \geq 7$, the curve C has only finitely many points whose field of definition has degree $\leq d-2$ over $\mathbf{Q}$.*

The result still holds for $d < 7$, provided that the complex curve C has no morphisms of degree $\leq d-2$ onto an elliptic curve, an assumption which we show automatically holds for $d \geq 7$. This result is sharp in the sense that if C has a rational point, there exist infinitely many points on C with field of definition of degree $\leq d-1$. These points come from the intersection of C with a rational line through a rational point. We show further:

Theorem 2.– *If $d \geq 8$, all but finitely many points of C whose field of definition has degree $\leq d-1$ over $\mathbf{Q}$ arise as the intersection of C with a rational line through a rational point of C.*

In particular, if C has no rational points, there are only finitely many points whose field of definition has degree $\leq d-1$ over $\mathbf{Q}$.

Again, the result still holds for $d = 6$ or 7, provided that C has no morphisms of degree $\leq d-1$ onto an elliptic curve, and for $d = 5$, provided that C has no morphisms onto an elliptic curve.

Both results remain valid if $\mathbf{Q}$ is replaced by any number field.

These theorems apply in particular to the Fermat curves F_d with equation $X^d + Y^d = Z^d$, which is the case we had in mind when we started this investigation. Moreover, we can extend the results to all $d \geq 3$ in this case, with the one exception $d \neq 6$ (see § 6).

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2. Notation

For a projective curve C, we denote by $\mathrm{C}^{(d)}$ the symmetric product varieties of C. We also denote by $\mathrm{J(C)}$ the Jacobian variety of C and by $\mathrm{W}_d(\mathrm{C})$ the image of $\mathrm{C}^{(d)}$ under the Abel-Jacobi map to $\mathrm{J(C)}$ defined with respect to a chosen base point on C. It corresponds to isomorphism classes of line bundles on C of degree d which have a non-zero section.

3. Faltings' theorem

We first remark that theorem 1 follows from the statement that the set of $\mathbf{Q}$–rational points on the symmetric product $\mathrm{C}^{(n)}$ is finite for any $n \leq d-2$. This is obtained by simply observing that any point on $\mathrm{C(K)}$ with $[\mathrm{K}:\mathbf{Q}] = n$, together with its conjugates, forms a divisor of degree n invariant by $\mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q})$, and hence defines a $\mathbf{Q}$–rational point on $\mathrm{C}^{(n)}$. Furthermore, since C is a smooth plane curve of degree d, it has no pencils of degree $\leq d-2$ ([ACGH] p. 56, exercise 18.(i)), hence $\mathrm{C}^{(n)}$ maps isomorphically onto $\mathrm{W}_n(\mathrm{C})$. Thus the proof of theorem 1 is reduced to showing that $\mathrm{W}_n(\mathrm{C})(\mathbf{Q})$ is finite for all $n \leq d-2$. We use the following beautiful result of Faltings:

Theorem (Faltings, [Fa1]) – *Let A be an abelian variety defined over a number field K. If X is a subvariety of A which does not contain any translate of a positive-dimensional abelian subvariety of A, then X contains only finitely many K–rational points.*

It is therefore enough to show that $\mathrm{W}_{d-2}(\mathrm{C})$ does not contain any non-zero abelian variety.

The situation in theorem 2 is a bit more complicated since the morphism:

$$\psi : \mathrm{C}^{(d-1)} \longrightarrow \mathrm{W}_{d-1}(\mathrm{C})$$

is no longer an isomorphism: each pencil of degree $d-1$ on C corresponds to a rational curve in $\mathrm{C}^{(d-1)}$ which is contracted by ψ. By [ACGH] p. 56, exercise 18.(ii), all such pencils are given by the lines through a fixed point x of C. Let R_x be the corresponding rational curve in $\mathrm{C}^{(d-1)}$. Since ψ induces an isomorphism outside of the union of all R_x, any rational point of $\mathrm{C}^{(d-1)}$ corresponds either to a rational point of $\mathrm{W}_{d-1}(\mathrm{C})$, or to a rational point of some R_x. Now let x be a point of C such that $\mathrm{R}_x(\mathbf{Q})$ is non-empty and let D be the divisor on C that corresponds to a point of $\mathrm{R}_x(\mathbf{Q})$. The points of D are then on a unique line l (which passes through x) and, since D is invariant under the action of $\mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q})$, so is l, which is therefore rational. It follows that the divisor $l \cdot \mathrm{C}$ is rational, hence so is $x = l \cdot \mathrm{C} - \mathrm{D}$. This reduces the proof of theorem 2 to showing that $\mathrm{W}_{d-1}(\mathrm{C})(\mathbf{Q})$ is finite. As above, by Faltings' theorem, it is enough to show that $\mathrm{W}_{d-1}(\mathrm{C})$ does not contain any positive-dimensional abelian variety.

4. Linear systems on smooth plane curves

Before proceeding to the proof of the theorems, we gather here some elementary facts about linear systems on smooth plane curves, which we will deduce from the following result of Coppens and Kato. Let H be a hyperplane section on a smooth plane curve C of degree d and let D be an effective divisor on C which belongs to a base-point-free pencil. Then we have:

THEOREM (Coppens-Kato, [CK]) – *If* $\deg(\mathrm{D}) < k(d-k)$ *for some integer* k, *the linear system* $|(k-1)\mathrm{H}-\mathrm{D}|$ *is non-empty.*

We assume now that $d \geq 5$. Here are the consequences that we need:

(4.1) *If* $\deg(\mathrm{D}) \leq 2d-5$, *then either* $\mathrm{D} \equiv \mathrm{H}$ *or* $\mathrm{D} \equiv \mathrm{H} - x$ *for some point* x *on* C.

(4.2) *If* $\deg(\mathrm{D}) = 2d-4$, *then* $\mathrm{D} \equiv 2\mathrm{H} - x_1 - x_2 - x_3 - x_4$ *for some points* x_1, x_2, x_3 *and* x_4 *on* C, *no three of them collinear.*

This follows from the theorem with $k=3$, except for $d=5$. In the latter case, Riemmann-Roch says that the 6 points of D are on a conic, which is what we need.

(4.3) *If* $\deg(\mathrm{D}) = 2d-3$ *and* $d \geq 7$, *then* $\mathrm{D} \equiv 2\mathrm{H} - x_1 - x_2 - x_3$ *for some points* x_1, x_2 *and* x_3 *on* C, *not collinear.*

(4.4) *If* $\deg(\mathrm{D}) = 2d-2$ *and* $d \geq 8$, *then* $\mathrm{D} \equiv 2\mathrm{H} - x_1 - x_2$ *for some points* x_1 *and* x_2 *on* C.

(4.5) *If* $\deg(\mathrm{D}) = 2d-2$, $\dim|\mathrm{D}| \geq 2$ *and* $d \geq 6$, *then* $\mathrm{D} \equiv 2\mathrm{H} - x_1 - x_2$ *for some points* x_1 *and* x_2 *on* C.

For $d \geq 7$, (4.3) yields $|2\mathrm{H} - (\mathrm{D} - x)| \neq \emptyset$ for all $x \in \mathrm{C}$, hence $|2\mathrm{H} - \mathrm{D}| \neq \emptyset$. For $d = 6$, Riemmann-Roch gives $\dim|3\mathrm{H} - \mathrm{D}| \geq 1$. If $|3\mathrm{H} - \mathrm{D}|$ is a base-point-free pencil, then $|\mathrm{D} - \mathrm{H}| \neq \emptyset$ by (4.2). Since $|\mathrm{D}|$ is base-point-free, $\dim|\mathrm{D}| \geq 3$ and $\dim|3\mathrm{H} - \mathrm{D}| \geq 2$ by Riemmann-Roch, contradiction. Hence $|3\mathrm{H} - \mathrm{D}|$ contains a pencil of degree one less, and $|3\mathrm{H} - \mathrm{D} - (\mathrm{H} - x)| \neq \emptyset$ for some x by (4.1). The conclusion follows since x is fixed in $|2\mathrm{H} + x|$.

5. Proof of the theorems

We only need to prove that, under the hypotheses of theorem 1 and theorem 2, $\mathrm{W}_{d-2}(\mathrm{C})$ and $\mathrm{W}_{d-1}(\mathrm{C})$ respectively do not contain any non-zero abelian varieties. This will follow from the following proposition.

PROPOSITION 1. – *Let* C *be a smooth plane curve of degree* $d \geq 5$. *Then:*

(i) *The variety* $\mathrm{W}_{d-1}(\mathrm{C})$ *does not contain any abelian variety of dimension* ≥ 2.

(ii) *If the variety* $\mathrm{W}_{d-2}(\mathrm{C})$ *contains an elliptic curve* E, *the inclusion is induced by a morphism* $\mathrm{C} \to \mathrm{E}$ *of degree* $d-2$ *and* $d \leq 6$.

(iii) *If* $d \geq 6$, *and if the variety* $W_{d-1}(C)$ *contains an elliptic curve* E, *then* $d \leq 7$ *and the inclusion is induced by a morphism* $C \to E$ *of degree* $d-1$ *or* $d-2$.

Proof. Let $1 \leq e \leq d-1$ and assume that $W_e(C)$ contains an abelian variety A of dimension $h > 0$. Let A_2 be the image of the abelian variety $A \times A$ under the addition map $W_e(C) \times W_e(C) \to W_{2e}(C)$ and let r be the dimension of the linear system on C which corresponds to a generic point of A_2. It follows from [A], lemma 8 that $r \geq h$. We may assume that A is not contained in $x + W_{e-1}(C)$ for any point x in C. In this case, the linear system on C that corresponds to a generic point of A_2 is base-point-free.

Assume first $h \geq 2$. Since $r \geq h$, we get a family of base-point-free linear systems of degree $\leq 2d-2$ and dimension ≥ 2 parametrized by the abelian variety A. By (4.5) and (4.1), we have $d = 5$, $e = d-1 = 4$ and $h = r = 2$. By [A] lemmas 13 and 14, the morphisms $C \to \mathbf{P}^2$ which correspond to points of A_2 factor through a fixed morphism $p : C \to B$ of degree $n > 1$ onto a curve B of genus $\geq h = 2$. Moreover, the induced morphims $B \to \mathbf{P}^2$ are birational onto their images, which have therefore degree $8/n$. This implies $n = 2$ and $g(B) = 2$. Let σ be the involution associated with the double cover p, and let H be a hyperplane section of C. Since the embedding of C as a smooth plane curve is unique (this follows for example from (4.1)), one has $\sigma^*(H) \equiv H$ hence σ is induced by a projective automorphism τ of $\mathbf{P}^2$. By Riemann-Hurwitz, σ has exactly 6 fixed points, hence τ is the symmetry with respect to a line. But then, the fixed points of σ are the intersection of this line with C, hence there cannot be 6 of them since C has degree 5. Therefore, this case does not occur.

This takes care of (i) and we now assume $h = 1$.

If $e \leq d-2$, fact (4.1) yields $e = d-2$ and $r = 1$, and fact (4.2) yields a morphism from A_2 to $W_4(C)$. Since $r = h = 1$, the embedding of the elliptic curve A in $W_{d-2}(C)$ is then induced by a morphism $C \to A$ of degree $d-2$ ([A], lemma 13). Moreover, by lemma 8 of [A], the elliptic curve $(A_2)_2$ parametrizes pencils on C of degree 8, hence (4.1) implies $8 > 2d-5$, i.e. $d \leq 6$. This proves (ii).

If $e = d-1$ and $d \geq 8$, fact (4.4) yields a morphism from A_2 to $W_2(C)$. By lemma 8 of [A], the elliptic curve $(A_2)_2$ parametrizes pencils on C of degree 4, which contradicts (4.1) since $4 \leq 2d-5$. One gets the same contradiction if $d \geq 6$ and $r \geq 2$ using (4.5). It follows that $d \leq 7$ and $r = h = 1$, and we conclude as above with [A], lemma 13. This proves (iii). ∎

6. Fermat curves

Both theorems apply in particular to the Fermat curves F_d defined by the equation $X^d + Y^d = Z^d$, at least for $d \geq 8$. For small d, the situation is the following:

- for $d = 3$, 4, 5 or 7, it is known that $J(F_d)(\mathbf{Q})$, hence also its subvarieties

$W_e(F_d)(\mathbf{Q})$ for all e, are finite ([F1], [F2]). This of course implies both theorems. For $d=4$, Faddeev also shows that in addition to its four rational points, F_4 has exactly twelve points defined over quadratic fields, and that the lines through each of the four rational points of F_4 account for *all* points of F_4 in all cubic fields.

- for $d=6$, there is a morphism of degree 4 from F_6 onto the elliptic curve F_3. In particular, $W_4(F_6)$ does contain an elliptic curve and our whole method of proof collapses. However, since $J(F_3)(\mathbf{Q})$ is finite, this does not say anything about the finiteness of $W_4(F_6)(\mathbf{Q})$. On the other hand, $J(F_6)(\mathbf{Q})$ is known to be infinite ([F2]). One may use here a stronger theorem of Faltings ([Fa2]), which says that *if* X *is a subvariety of an abelian variety* A *defined over a number field* K, *then the set* $X(K)$ *lies inside a finite union of* K*-rationally defined translates of abelian subvarieties of* A *contained in* X. Consequently, if any morphism of degree 4 from F_6 onto an elliptic curve has image F_3, theorem 1 will hold for F_6. If, in addition, there are no morphisms of degree 5 from F_6 onto an elliptic curve, theorem 2 will hold for F_6.

We mentioned in the introduction that our two theorems remained valid over any number field K. This applies in particular to Fermat curves for $d \geq 7$ (for theorem 1) and $d \geq 8$ (for theorem 2). Both theorems remain valid for F_5: the absolutely simple factors of its Jacobian are surfaces ([KR] theorem 2), and $W_4(F_5)$ cannot contain an abelian surface by proposition 1.(i). However, theorem 1 fails trivially for F_3, for F_4 (this curve has a morphism of degree 2 onto the elliptic curve E with equation $U^2W^2+V^4=W^4$, and as soon as $E(K)$ becomes infinite, so will $F_4^{(2)}(K)$) and for F_6 (for the same reason, since there is a morphism of degree 4 from F_6 onto the elliptic curve F_3). As far as F_7 is concerned, theorem 1 holds, and theorem 2 holds if and only if there are no morphisms of degree 6 from F_7 onto an elliptic curve.

We cannot resist giving a different proof of theorem 1 for Fermat curves when d is an odd *prime* number p which satisfies $p \equiv 2 \pmod 3$ as a nice application of the following result of [DF] (proposition 3.3): *for a complex projective curve* C *of genus* g, *the variety* $W_d(C)$ *cannot contain an abelian variety of dimension* $> d/2$ *for* $d<g$. In fact, it is known in this case that the absolutely simple factors of $J(F_p)$ are all of dimension $\frac{p-1}{2}$ ([KR], theorem 2), so that $W_{p-2}(F_p)$ cannot contain any non-zero abelian variety.

It would be very interesting to know specifically which points constitute the finite sets of algebraic points in the theorems for the Fermat curves. Of course, this extends the already difficult question, posed by Fermat, of showing that there are only three rational points if d is odd and four if d is even.

Assume again that d is an odd prime number p. The easiest way to produce algebraic points of degree $\leq p-3$ is to take the other points of intersection of F_p with the line through the three known rational points. In the affine patch where $Z=1$, this is just the line $y=1-x$, and F_p is defined by $x^p+y^p=1$. The x-coordinates of these other points of intersection are then just the roots of

$$\frac{x^p+(1-x)^p-1}{x(x-1)}=0\ .$$

One sees, by considering the equation $x^p+(1-x)^p-1=0$ and its derivative, that the factor x^2-x+1 always occurs with multiplicity one or two depending on whether p is 5 or 1 (mod 6) respectively, so we obtain x-coordinates η, and η^{-1}, where η is a primitive sixth root of unity. The other factor, of degree $p-5$ or $p-7$, is irreducible over $\mathbf{Q}$ for $p\leq 101$ (checked using MAPLE), but the authors do not know if this is always the case. Also, the authors do not know of any other points on F_p of degree $\leq p-2$, i.e. which do not lie on the line $y=1-x$. Gross and Rohrlich show that this line accounts for all the points of degree $\leq (p-1)/2$ on F_p for the primes $p=3$, 5, 7 and 11 (see [GR]).

It is interesting to note that the linear equivalence class described above produces the only known points of infinite order on the Mordell-Weil group $\mathrm{J}(\mathrm{F}_p)(\mathbf{Q})$. More specifically, Gross and Rohrlich take the conjugate quadratic points $\mathrm{P}=(\eta,\eta^{-1},1)$, and $\overline{\mathrm{P}}=(\eta^{-1},\eta,1)$, and form the divisor $\mathrm{P}+\overline{\mathrm{P}}-2\infty$ on $\mathrm{Div}^0(\mathrm{F}_p)$. Then, they show that for $p>7$, the linear equivalence class in $\mathrm{J}(\mathrm{F}_p)$ of the divisor $\mathrm{P}+\overline{\mathrm{P}}-2\infty$ represents a point of infinite order.

Finally, one would like to have, if not a complete description, at least an upper bound on the cardinality of the finite sets of algebraic points in theorem 1. It is natural to begin by trying to bound the number of $\mathbf{Q}$–rational points on F_p. The greatest success in this regard is Kummer's proof of Fermat's Last Theorem for p a regular prime (see [W]). For general p, all bounds depend on the rank of $\mathrm{J}(\mathrm{F}_p)(\mathbf{Q})$. One approach is to come up with bounds which are exponential in the Mordell-Weil rank as in Bombieri's version of Faltings' Theorem ([B]). Another approach is to use Coleman's "Effective Chabauty," applying his theory of p-adic abelian integrals ([C1], [C2]). In this case, one needs to know that the rank of the Mordell-Weil group $\mathrm{J}(\mathrm{F}_p)(\mathbf{Q})$ is less than its dimension (i.e. the genus of the curve). This is known to hold in the case when p is regular ([F3]). However, McCallum has shown that this would hold for all p if one has a certain bound on the ideal class group Cl of $\mathbf{Q}(\zeta)$, where ζ is a primitive p^{th} root of unity. In particular, if $\mathrm{Cl}[p]$ denotes the subgroup of Cl of elements killed by p, then he shows that

$$\mathrm{rank}_{\mathbf{Z}}\,\mathrm{J}_s(\mathbf{Q})\leq\frac{p-7}{4}+2\,\mathrm{rank}_{\mathbf{Z}/p\mathbf{Z}}(\mathrm{Cl}[p]).$$

He then goes on to show that if $\mathrm{rank}_{\mathbf{Z}/p\mathbf{Z}}(\mathrm{Cl}[p])<\frac{p+5}{8}$, then the number of $\mathbf{Q}$–rational points on F_p is $\leq 2p-3$ ([Mc]). The second author has begun to apply Coleman's theory to the symmetric products of curves in his Ph.D. thesis (to appear).

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