

The main purpose of this article is to describe all effective algebraic cycles with minimal cohomology class in the Jacobian of a complex curve. More precisely, let  $(A, \theta)$  be a complex principally polarized abelian variety of dimension  $g$ . For  $0 \leq d \leq g$ , the cohomology class  $\theta_d = \theta^d/d!$  is *minimal*, i.e. non-divisible, in  $H^{2d}(A, \mathbf{Z})$ . When  $(A, \theta)$  is isomorphic to the Jacobian  $(JC, \theta)$  of a curve  $C$  of genus  $g$ , the image of the symmetric product  $C^{(g-d)}$  by any Abel-Jacobi map is a subvariety  $W_{g-d}(C)$  of  $JC$  with class  $\theta_d$  ([ACGH], p. 25).

Our main result (theorem 5.1) implies that *any effective algebraic cycle in  $JC$  with class  $\theta_d$  is a translate of either  $W_{g-d}(C)$  or  $-W_{g-d}(C)$ .*

For  $1 < d < g$ , I know of only one other family of principally polarized abelian varieties with an effective algebraic cycle with minimal class: in the 5-dimensional intermediate Jacobian  $JT$  of a cubic threefold  $T$  in  $\mathbf{P}^4$ , the image by any Abel-Jacobi map of the Fano surface of lines contained in  $T$  is a surface with class  $\theta_3$  ([CG], [B]).

I suspect that these should be the only examples of effective algebraic cycles with class  $\theta_d$  on abelian varieties of dimension  $g$ , when  $1 < d < g$ . This holds for any  $g$  and  $d = g - 1$  by Matsusaka's criterion ([M]), and for  $g = 4$  and  $d = 2$  by a result of Ran ([R1]).

In a second part, the main theorem is used to prove a weak version of this conjecture: *for  $1 < d < g$ , the Jacobian locus (resp. the locus of intermediate Jacobians of cubic threefolds) is an irreducible component of the set of principally polarized abelian varieties of dimension  $g$  for which  $\theta_d$  (resp.  $\theta_3$ ) is the class of an effective algebraic cycle.* This result was first proved by Barton and Clemens ([BC]) for  $g = 4$  and  $d = 2$ ; a slightly weaker form of it appeared later in [R1] (corollary III.1), but the proof is incomplete.

Throughout this article, we will be working over the field of complex numbers. A variety will be a reduced projective (complex) scheme of finite type.

The author would like to thank the M.S.R.I., where this research was done, for its hospitality and support.

## I. SUBVARIETIES WITH MINIMAL CLASSES IN JACOBIANS

Let  $(JC, \theta)$  be the Jacobian of a smooth curve  $C$  of genus  $g$ . The aim of this part is to prove that any effective algebraic cycle in  $JC$  with class  $\theta_d$  is a translate of either  $W_{g-d}(C)$  or  $-W_{g-d}(C)$ .

---

(\*) Partially supported by N.S.F. Grant DMS 92-03919 and the European Science Project "Geometry of Algebraic Varieties", Contract no. SCI-0398-C (A).

## 1. Non-degenerate subvarieties

This notion was introduced by Ran ([R1], [R2]). Let  $A$  be an abelian variety of dimension  $g$  and let  $W$  be a subvariety of  $A$  of pure dimension  $d$ . Let  $W_{\text{reg}}$  be the smooth part of  $W$ . We say that  $W$  is *non-degenerate* if the restriction map  $H^0(A, \Omega_A^d) \rightarrow H^0(W_{\text{reg}}, \Omega_{W_{\text{reg}}}^d)$  is injective. By [R1], lemma II.1, this is equivalent to each one of the following properties:

- the cup-product map  $\cdot[W] : H^{d,0}(A) \rightarrow H^{g, g-d}(A)$  is injective, where  $[W] \in H^{g-d, g-d}(A)$  is the cohomology class of  $W$ ,
- the contraction map  $\cdot\{W\} : H^{g,d}(A) \rightarrow H^{g-d,0}(A)$  is injective, where  $\{W\} \in H_{d,d}(A)$  is the homology class of  $W$ .

This allows the extension of the definition to effective algebraic cycles. Note that on a principally polarized abelian variety, an effective algebraic cycle with class a multiple of a minimal class is non-degenerate.

## 2. The property $(\mathcal{P})$

Let  $V$  and  $W$  be two irreducible subvarieties of an abelian variety  $A$  and let  $f : V \times W \rightarrow A$  be the addition map. We will say that  $V$  *has property  $(\mathcal{P})$  with respect to  $W$*  if, for  $v$  generic in  $V$ , the only irreducible subvariety of  $f^{-1}(v + W)$  which dominates both  $v + W$  via  $f$ , and  $W$  via the second projection, is  $\{v\} \times W$ .

(2.1) Note that this implies that  $\{v\} \times W$  is a component of  $f^{-1}(v + W)$ , hence that the latter has the same dimension as  $W$  at some point, hence that  $f$  is generically finite onto its image; in particular,  $\dim(V) + \dim(W) \leq \dim(A)$ . Note also that if  $V$  has property  $(\mathcal{P})$  with respect to  $W$ , then any translate of  $V$  has property  $(\mathcal{P})$  with respect to any translate of  $W$ .

EXAMPLE 2.2. If  $C$  is a curve of genus  $g$ , it is not difficult to check that  $W_d(C)$  has property  $(\mathcal{P})$  with respect to  $W_e(C)$  whenever  $d + e \leq g$ .

LEMMA 2.3. – *Let  $V$  and  $W$  be two irreducible subvarieties of an abelian variety  $A$  and let  $g : W \times W \rightarrow A$  be the subtraction map. Then,  $V$  has property  $(\mathcal{P})$  with respect to  $W$  if and only if, for  $v$  generic in  $V$ , the only irreducible subvariety of  $g^{-1}(V - v)$  which dominates both factors  $W$ , is the diagonal.*

**Proof.** Let  $f : V \times W \rightarrow A$  be the addition map. Let  $v$  be generic in  $V$  and consider the automorphism  $h$  of  $A \times A$  defined by  $h(x, y) = (v + x - y, y)$ . One checks that  $h(g^{-1}(V - v)) = f^{-1}(v + W)$ . The proposition then follows from the fact that, if  $Z$  is a variety contained in  $g^{-1}(V - v)$ , then  $f(h(Z)) = v + p_1(Z)$  and  $p_2(h(Z)) = p_2(Z)$ . ■

PROPOSITION 2.4. – *Let  $V$  and  $W$  be two irreducible subvarieties of an abelian variety  $A$ . Assume that  $V$  has property  $(\mathcal{P})$  with respect to  $W$ . Then:*

- *the variety  $(-V)$  has property  $(\mathcal{P})$  with respect to  $W$ ,*

of  $V$  which meets  $\Omega$ , then  $U$  has property  $(\mathcal{P})$  with respect to  $W$ .

**Proof.** The first point follows from lemma 2.3. Let  $g : W \times W \rightarrow A$  be the difference map. Let  $\Omega$  be the open set of points  $v$  in  $V$  such that the only irreducible subvariety of  $g^{-1}(V - v)$  which dominates both factors is the diagonal. By the lemma,  $\Omega$  is dense in  $V$ . Take  $u$  in  $\Omega \cap U$ ; since  $U - u \subset V - u$ , any irreducible subvariety of  $g^{-1}(U - u)$  which dominates both factors is also a subvariety of  $g^{-1}(V - u)$ , hence is equal to the diagonal. By lemma 2.3, this proves the second point. ■

Although the following result can be checked directly, it is an easy consequence of lemma 2.3.

**PROPOSITION 2.5.** – *Let  $C$  be a curve of genus  $g$  and let  $V$  be an irreducible subvariety of  $JC$ . Assume that  $V$  has property  $(\mathcal{P})$  with respect to some  $W_d(C)$ . Then  $V$  also has property  $(\mathcal{P})$  with respect to any  $W_e(C)$  for  $0 \leq e \leq d$ .*

**Proof.** Let  $g : W_d(C) \times W_d(C) \rightarrow JC$  and  $h : W_e(C) \times W_e(C) \rightarrow JC$  be the difference maps. Let  $v$  be generic in  $V$ , and let  $Z$  be an irreducible subvariety of  $h^{-1}(v - V)$  which dominates both factors. Then  $\{z + (w, w) \mid z \in Z, w \in W_{d-e}(C)\}$  is an irreducible subvariety of  $g^{-1}(v - V)$  which dominates both factors. By lemma 2.3, it is the diagonal of  $W_d(C) \times W_d(C)$ , hence  $Z$  is the diagonal of  $W_e(C) \times W_e(C)$ . By lemma 2.3 again, this proves that  $V$  has property  $(\mathcal{P})$  with respect to  $W_e(C)$ . ■

### 3. Ran's theorem

Let  $V$  and  $W$  be two subvarieties of  $A$ , of respective pure codimensions  $d$  and  $g - d$ . Assume that  $s = V \cdot W > 0$ . The addition map  $V \times W \rightarrow A$  is then surjective, hence generically étale. It follows that for  $x$  generic in  $A$ , the varieties  $V$  and  $x - W$  meet transversally at distinct smooth points  $v_1, \dots, v_s$ . Let  $P_i : T_0A \rightarrow T_0A$  be the projector with image  $T_{v_i}V$  and kernel  $T_{x-v_i}W$ . Let  $c(V, W)$  be the endomorphism  $\sum_{i=1}^s \bigwedge^{g-d} P_i$  of  $\bigwedge^{g-d} T_0A$ ; Ran proves ([R1], theorem 2) that the transposed endomorphism  ${}^t c(V, W)$  of  $\bigwedge^{g-d} T_0^*A \simeq H^{g-d,0}(A)$  is equal to the composition:

$$H^{g-d,0}(A) \xrightarrow{\cdot[V]} H^{g,d}(A) \xrightarrow{\cdot\{W\}} H^{g-d,0}(A) .$$

In particular, when  $V$  and  $W$  are non-degenerate,  $c(V, W)$  is an automorphism.

The following result of Ran on non-degenerate subvarieties with minimal intersection number motivates the somewhat abstruse definition of “property  $(\mathcal{P})$ ”.

**THEOREM 3.1.** (Ran) – *If  $V$  and  $W$  are non-degenerate subvarieties of a  $g$ -dimensional principally polarized abelian variety  $(A, \theta)$ , of respective pure codimensions  $d$  and  $g - d$ , then  $V \cdot W \geq \binom{g}{d}$ . If moreover  $W$  is irreducible and  $V \cdot W = \binom{g}{d}$ , then  $V$  is irreducible and has property  $(\mathcal{P})$  with respect to  $W$ .*

$$\binom{g}{d} = \text{rank}(c(V, W)) \leq V \cdot W .$$

If there is equality, it follows from (7.4) in [R2] that  $W$  meets at most one component of  $V$ . But by [R1], corollary II.6,  $W$  meets each component of  $V$ . This implies that  $V$  is irreducible. To prove property  $(\mathcal{P})$ , we keep the same notation as above and proceed as in the proof of [R1], theorem 5. If  $\alpha_i$  spans the line  $\bigwedge^{g-d} T_{v_i} V$  in  $\bigwedge^{g-d} T_0 A$ , then  $\{\alpha_1, \dots, \alpha_s\}$  (with  $s = \binom{g}{d}$ ) is a basis for  $\bigwedge^{g-d} T_0 A$ . The choice of an identification  $\bigwedge^n T_0 A \simeq \mathbf{C}$  induces an isomorphism  $\bigwedge^d T_0 A \simeq \bigwedge^{g-d} T_0^* A$ . Let  $\beta_i$  be an element of  $\bigwedge^d T_{x-v_i} W$  such that  $\beta_i(\alpha_i) = 1$ . Then  $c(V, W) = \sum_{i=1}^s \alpha_i \otimes \beta_i$ . In particular, for  $i \neq j$ :

$$\beta_i(c(V, W)^{-1}(\alpha_j)) = 0 .$$

Fix  $v = v_1$  in  $V$  and let  $x$  vary in  $v + W$ , so that  $v \in V \cap (x - W)$ . We get:

$$\bigwedge^d T_{x-v_i} W \wedge c(V, W)^{-1} \left( \bigwedge^{g-d} T_v V \right) = 0$$

for  $i > 1$ . Since  $W$  is non-degenerate, the points  $x - v_1, \dots, x - v_s$  must therefore describe a proper subvariety of  $W$ . This proves the last part of the theorem, since  $p_2 f^{-1}(x) = W \cap (x - V) = \{x - v, x - v_1, \dots, x - v_s\}$ . ■

For example, if  $C$  is a curve of genus  $g$ , the theorem implies that  $W_d(C)$  has property  $(\mathcal{P})$  with respect to  $W_{g-d}(C)$  for  $d \leq g$ . Together with proposition 2.5, this proves the claim of example 2.2.

#### 4. An auxiliary result

Let  $C$  be a smooth curve of genus  $g$  and let  $n > 1$ . We define a subset  $\mathcal{T}_n$  of  $C^{(n)}$  as follows: consider all surjective morphisms  $C \rightarrow C'$  of degree  $r > 1$ , where  $C'$  is a smooth irrational curve. Such a morphism induces a map  $\psi : C' \rightarrow C^{(r)}$ ; let  $\mathcal{T}_n$  be the union of all  $\psi(C') + C^{(n-r)}$ , for  $r \leq n$ , obtained in this way, of the inverse image in  $C^{(n)}$  of  $W_n^1(C)$  and of the diagonal  $2C + C^{(n-2)}$ . For  $n \leq g$ , it is a proper closed subvariety of  $C^{(n)}$ .

**PROPOSITION 4.1.** – *Let  $(JC, \theta)$  be the Jacobian of a smooth curve  $C$  of genus  $g \geq 2$  and let  $Z$  be a subvariety of  $C^{(n)}$  of pure codimension  $m$ . Then, for any  $E$  in  $C^{(n-m)}$ , the variety  $Z$  meets  $E + C^{(m)}$ . Moreover, if there exists  $E$  in  $C^{(n-m)}$  such that  $Z \cdot (E + C^{(m)}) = 1$ , then either  $Z$  is contained in  $\mathcal{T}_n$ , or there exist points  $c_1, \dots, c_m$  of  $C$  such that  $Z = c_1 + \dots + c_m + C^{(n-m)}$ .*

**Proof.** The first part follows from the fact that the cohomology class of  $E + C^{(m)}$  is the  $(n - m)$ -fold self-intersection of the cohomology class of the ample divisor  $C^{(n-1)}$  in  $C^{(n)}$  ([ACGH], pp. 309, 310).

Assume now that  $Z \cdot (E + C^{(m)}) = 1$ . We first do the case when  $Z$  is a curve. We may assume that  $Z$  is not contained in  $x + C^{(n-1)}$  for any  $x$  in  $C$ . Since  $Z \cdot (x + C^{(n-1)}) = 1$

that  $Z = \{x + \tau(x) \mid x \in C\}$ . Let  $\Gamma$  be the curve  $\{(x, \tau(x)) \mid x \in C\}$  in  $C \times C^{(n-1)}$ . If the induced morphism  $\phi : \Gamma \rightarrow Z$  is not birational,  $Z$  is contained in  $\mathcal{T}_n$ .

Since  $Z$  is smooth, we are left with the case where  $\phi$  is an isomorphism. If  $\tau$  is constant, the proof is over. Assume therefore that  $\tau(C)$  is a curve. If  $n = 2$ , we get  $Z = \{x + \tau(x) \mid x \in C\}$ , where  $\tau$  is an involution of  $C$ . In particular,  $Z$  is contained in  $\mathcal{T}_2$ . We assume  $n \geq 3$  and proceed by induction. Let us show that the curve  $\tau(C)$  satisfies the same property as  $Z$ . Let  $x$  be a point of  $C$  and assume that  $\tau(y) = x + D$  and  $\tau(y') = x + D'$  are both in  $x + C^{(n-1)}$  for some  $y$  and  $y'$  on  $C$ . Then,  $y + x + D$  and  $y' + x + D'$  are on  $Z$  hence the hypothesis on  $Z$  implies  $y + D = y' + D'$ . Therefore, either  $y = y'$  and  $D = D'$ , or there exists an element  $E$  of  $C^{(n-3)}$  such that  $D = y' + E$  and  $D' = y + E$ . Then, both  $(y, \tau(y))$  and  $(y', \tau(y'))$  are sent by addition to  $x + y + y' + E$  hence, since  $\phi$  is an isomorphism, we get again  $y = y'$  and  $D = D'$ . It follows that  $\tau(C)$  and  $x + C^{(n-2)}$  have a single common point and one checks that the intersection is transverse. We can therefore apply the induction hypothesis to  $\tau(C)$ : since we have assumed that  $Z$  is not contained in any  $x + C^{(n-1)}$ , the curve  $\tau(C)$  is in  $\mathcal{T}_{n-1}$ . It follows that  $Z$  is contained in  $C + \mathcal{T}_{n-1}$ , hence in  $\mathcal{T}_n$ . This finishes the proof of the lemma when  $Z$  is a curve.

We now do the general case, by induction on the dimension of  $Z$ . Let  $Z$  of dimension  $n - m > 1$  satisfy the hypothesis. For  $x$  generic in  $C$ , the inverse image of  $Z$  by the map  $C^{(n-1)} \rightarrow C^{(n)}$  which sends  $D$  to  $x + D$  satisfies the same property. Therefore, either  $Z$  is contained in  $C + \mathcal{T}_{n-1}$ , hence in  $\mathcal{T}_n$ , or there exists a morphism  $\tau : C \rightarrow C^{(m)}$  such that:

$$Z = \{x + \tau(x) + C^{(n-m-1)} \mid x \in C\}.$$

But  $\tau$  must then be constant. This finishes the proof of the proposition. ■

## 5. The main theorem

We now prove our main result.

**THEOREM 5.1.** – *Let  $(JC, \theta)$  be the Jacobian of a smooth curve  $C$  of genus  $g$  and let  $V$  be an effective non-degenerate algebraic  $(g - d)$ -cycle in  $JC$  such that  $V \cdot W_d(C) = \binom{g}{d}$ . Then  $V$  is a translate of either  $W_{g-d}(C)$  or  $-W_{g-d}(C)$ .*

We refer to §1 for the definition of a non-degenerate cycle. Recall that any effective algebraic cycle with class  $\theta_d$  satisfies the hypotheses of the theorem.

**Proof.** By replacing any multiple component of  $V$  by a sum of translates, we may assume that  $V$  is a subvariety. Ran's theorem 3.1 implies that  $V$  is irreducible and has property  $(\mathcal{P})$  with respect to  $W_d(C)$ . We first prove by induction on  $g - d$  that any  $(g - d)$ -dimensional irreducible subvariety  $V$  of  $JC$  which has property  $(\mathcal{P})$  with respect to  $W_d(C)$  is a translate of some  $W_r(C) - W_{g-d-r}(C)$ , with  $0 \leq r \leq g - d$ . This is obvious for  $g - d = 0$ , hence we assume  $g - d > 0$ .

For any positive integer  $e$ , we define the addition map  $f_e^+ : V \times W_e(C) \rightarrow JC$  and the subtraction map  $f_e^- : V \times W_e(C) \rightarrow JC$ . When  $e \leq d$ , proposition 2.5 implies that  $V$

from (2.1) that  $f_e^+$  and  $f_e^-$  are finite onto their images.

PROPOSITION 5.2. – *Let  $V$  be an irreducible subvariety of  $J\mathcal{C}$  of codimension  $d < g$  having property  $(\mathcal{P})$  with respect to  $W_d(\mathcal{C})$ . Then there exists an irreducible subvariety  $U$  of  $J\mathcal{C}$  such that either  $V = U + \mathcal{C}$  or  $V = U - \mathcal{C}$ .*

**Proof.** The maps  $f_d^\pm$  are not birational ([R1], lemma II.15). Let  $n$  be the smallest integer such that either  $f_n^+$  or  $f_n^-$  is not birational onto its image. Since both the hypotheses and conclusion of the lemma hold for  $V$  if and only if they do for  $-V$  (proposition 2.4), we may assume that  $f_n^+$  is not birational onto its image. For  $v$  generic in  $V$ , there exists an irreducible component  $\Gamma$  of  $(f_n^+)^{-1}(v + W_n(\mathcal{C}))$ , distinct from  $\{v\} \times W_n(\mathcal{C})$ , which dominates  $v + W_n(\mathcal{C})$ . Since  $V$  has property  $(\mathcal{P})$  with respect to  $W_n(\mathcal{C})$  (proposition 2.5), the projection  $p_2(\Gamma)$  is a subscheme  $G$  of  $W_n(\mathcal{C})$  of dimension  $m < n$ . For  $D$  generic in  $G$ , there exists a subvariety  $S_D$  of  $V$  of dimension  $n - m$  such that:

$$S_D + D \subset v + W_n(\mathcal{C}) .$$

Let  $E$  be generic in  $W_m(\mathcal{C})$ . By proposition 4.1, the subvariety  $E + W_{n-m}(\mathcal{C})$  of  $W_n(\mathcal{C})$  meets  $G$  at some point  $D = E + E'$ . For the same reason, the subvariety  $S_D + D - v$  of  $W_n(\mathcal{C})$  meets  $E' + W_m(\mathcal{C})$  at  $r$  points  $E' + E_1, \dots, E' + E_r$  with  $r > 0$ , i.e. there exist points  $v_1, \dots, v_r$  of  $S_D \subset V$  such that:

$$v_i + (E + E') - v = E' + E_i , \quad \text{i.e.} \quad v_i - E_i = v - E .$$

Since  $m < n$ , the map  $f_m^- : V \times W_m(\mathcal{C}) \rightarrow J\mathcal{C}$  is birational onto its image. Therefore, since  $v$  and  $E$  are generic, we must have  $E_i = E$  for all  $i$  and  $v \in S_D$ . In other words:

$$(S_D + D - v) \cap (E' + W_m(\mathcal{C})) = \{D\}$$

as sets. This is actually a scheme-theoretic equality; if  $\epsilon$  is a tangent vector such that:

$$\begin{aligned} \epsilon &\in T_D(S_D + D - v) = T_v S_D \subset T_v V \\ \epsilon &\in T_D(E' + W_m(\mathcal{C})) = T_E W_m(\mathcal{C}) , \end{aligned}$$

then  $(\epsilon, \epsilon)$  is in the kernel of the differential of  $f_m^-$  at  $(v, E)$ , hence is 0.

By proposition 4.1,  $E' + W_m(\mathcal{C})$  meets each component of  $S_D + D - v$ , which must therefore be irreducible. Recall that since  $f_n^+(\Gamma) = v + W_n(\mathcal{C})$ , the subvarieties  $S_D + D - v$  cover  $W_n(\mathcal{C})$  as  $D$  varies in  $G$ . In particular, we may assume that  $S_D + D - v$  is not contained in the image of  $\mathcal{T}_n$  (defined in §4) in  $W_n(\mathcal{C})$ . It then follows from proposition 4.1 applied to the strict transform of  $S_D + D - v$  in  $\mathcal{C}^{(n)}$  that there exists an effective divisor  $E_D$  of degree  $m$  such that:

$$S_D + D - v = W_{n-m}(\mathcal{C}) + E_D .$$

and if  $x$  is any point of  $C$ , the point  $v - c + x$  is in  $S_D$  hence in  $V$ . Since  $v$  is generic, this implies that some irreducible component  $T$  of the scheme:

$$\{ (v, c) \in V \times C \mid v - c + C \subset V \}$$

dominates  $V$ . Note that  $U = f_1^-(T)$  satisfies  $U + C \subset V$ . In particular,  $\dim U < \dim V \leq \dim T$ . In general, if  $f_1^-$  contracts a subvariety  $F$  of  $V \times C$  to a point, the projection  $F \rightarrow C$  is an isomorphism. This implies that  $U = f_1^-(T)$  has dimension  $\dim T - 1$ , and that  $V = p_1(T) \subset U + C$ , hence  $V = U + C$ . ■

Replacing  $V$  with  $-V$  if necessary, we may assume that  $V = U + C$ . The following proposition shows that  $U$  satisfies the induction hypothesis.

**PROPOSITION 5.3.** – *Let  $V$  be an irreducible subvariety of  $J\mathcal{C}$  of codimension  $d < g$  having property  $(\mathcal{P})$  with respect to  $W_d(C)$ . Assume that  $V = U + C$ . Then  $U$  has property  $(\mathcal{P})$  with respect to  $W_{d+1}(C)$ .*

**Proof.** By proposition 2.4,  $U$  has property  $(\mathcal{P})$  with respect to  $W_d(C)$ , hence also with respect to  $C$  (proposition 2.5).

Let  $u$  be generic in  $U$  and let  $\phi_{d+1}^+ : U \times W_{d+1}(C) \rightarrow J\mathcal{C}$  be the addition map. Let  $\Gamma_0$  be an irreducible subvariety of  $U \times W_{d+1}(C)$  such that  $\phi_{d+1}^+(\Gamma_0) = u + W_{d+1}(C)$  and  $p_2(\Gamma_0) = W_{d+1}(C)$ . Let  $\Gamma$  be an irreducible subvariety of  $U \times C^{(d+1)}$  which maps onto  $\Gamma_0$  by the natural map:

$$\Pi_U = \text{Id}_U \times \pi_{d+1} : U \times C^{(d+1)} \rightarrow U \times W_{d+1}(C) .$$

We have  $\phi_{d+1}^+ \Pi_U(\Gamma) = u + W_{d+1}(C)$  and  $p_2(\Gamma) = C^{(d+1)}$ . Moreover, since  $\phi_{d+1}^+ \Pi_U$  is generically finite onto its image, the dimension of  $\Gamma$  is  $d + 1$ . We need to show that  $\Gamma = \{u\} \times C^{(d+1)}$ .

For any  $c$  in  $C$ , each component of  $\Gamma \cap (\phi_{d+1}^+ \Pi_U)^{-1}(u + c + W_d(C))$  that dominates  $u + c + W_d(C)$  has dimension  $d$ . Since  $p_2(\Gamma) = C^{(d+1)}$ , at least one of these components, say  $\Gamma_c$ , projects onto a divisor  $Z_c$  in  $C^{(d+1)}$ . We use the elementary:

**LEMMA 5.4.** – *Let  $Z$  be an irreducible divisor in  $C^{(d+1)}$  and let  $g : C \times C^{(d)} \rightarrow C^{(d+1)}$  be the natural map. Then, there exists an irreducible divisor  $Z'$  in  $C \times C^{(d)}$  such that  $g(Z') = Z$  and  $p_2(Z') = C^{(d)}$ .*

**Proof.** By proposition 4.1,  $g^*Z \cdot (C \times \{D\}) = Z \cdot (C + D)$  is non-zero for any point  $D$  in  $C^{(d)}$ , hence  $p_2(g^{-1}(Z)) = C^{(d)}$ . Some component  $Z'$  of  $g^{-1}(Z)$  must satisfy the same property. Its dimension is then  $\geq d$ , hence it must map onto  $Z$  by the finite map  $g$ . ■

Pick such a divisor  $Z'_c$  for  $Z_c$  and let  $\Gamma'_c$  be a component of  $Z'_c \times_{Z_c} \Gamma_c$  that projects

$$\begin{array}{ccccc}
\Gamma'_c & \subset & \mathbb{U} \times \mathbb{C} \times \mathbb{C}^{(d)} & \xrightarrow{\text{Id}_{\mathbb{U}} \times g} & \mathbb{U} \times \mathbb{C}^{(d+1)} & \supset & \Gamma_c \\
\downarrow & & \downarrow p_{23} & & \downarrow p_2 & & \downarrow \\
Z'_c & \subset & \mathbb{C} \times \mathbb{C}^{(d)} & \xrightarrow{g} & \mathbb{C}^{(d+1)} & \supset & Z_c \\
\downarrow & & \downarrow p_2 & & & & \\
\mathbb{C}^{(d)} & = & \mathbb{C}^{(d)} & & & & 
\end{array}$$

Since  $\text{Id}_{\mathbb{U}} \times g$  is finite, it maps  $\Gamma'_c$  onto  $\Gamma_c$ . Set  $\phi_{1,d}^+ = \phi_1^+ \times \text{Id}_{\mathbb{C}^{(d)}}$  and consider:

$$\mathbb{U} \times \mathbb{C} \times \mathbb{C}^{(d)} \xrightarrow{\phi_{1,d}^+} \mathbb{V} \times \mathbb{C}^{(d)} \xrightarrow{\Pi_{\mathbb{V}}} \mathbb{V} \times \mathbb{W}_d(\mathbb{C}) \xrightarrow{\phi_d^+} \mathbb{J}\mathbb{C} .$$

Then:

$$\begin{aligned}
\phi_d^+ \Pi_{\mathbb{V}} \phi_{1,d}^+(\Gamma'_c) &= \phi_{d+1}^+ \Pi_{\mathbb{U}}(\Gamma_c) = u + c + \mathbb{W}_d(\mathbb{C}) \\
p_2 \Pi_{\mathbb{V}} \phi_d^+(\Gamma'_c) &= \pi_d p_3(\Gamma'_c) = \mathbb{W}_d(\mathbb{C}) .
\end{aligned}$$

Since  $\mathbb{V}$  has property  $(\mathcal{P})$  with respect to  $\mathbb{W}_d(\mathbb{C})$ , this implies:

$$\Pi_{\mathbb{V}} \phi_{1,d}^+(\Gamma'_c) = \{u + c\} \times \mathbb{W}_d(\mathbb{C})$$

for  $c$  generic. Therefore:

$$\Gamma_c = \{u'\} \times (c' + \mathbb{C}^{(d)}) \quad \text{and} \quad Z_c = c' + \mathbb{C}^{(d)} ,$$

for some points  $u'$  of  $\mathbb{U}$  and  $c'$  of  $\mathbb{C}$  such that  $u' + c' = u + c$ . The union as  $c$  varies in  $\mathbb{C}$  of all  $Z_c$ 's must be  $\mathbb{C}^{(d+1)}$ . It follows that  $c'$  must describe the whole of  $\mathbb{C}$ . Recall that  $\mathbb{U}$  has property  $\mathcal{P}$  with respect to  $\mathbb{C}$ , hence that the only component of  $(\phi_1^+)^{-1}(u + \mathbb{C})$  which dominates both  $u + \mathbb{C}$  via  $\phi_1^+$  and  $\mathbb{C}$  via  $p_2$  is  $\{u\} \times \mathbb{C}$ . This implies  $\Gamma_c = \{u\} \times (c + \mathbb{C}^{(d)})$  for  $c$  generic, hence the proposition. ■

It then follows from our induction hypothesis that  $\mathbb{U}$  is a translate of some  $\mathbb{W}_{r-1}(\mathbb{C}) - \mathbb{W}_{g-d-r}(\mathbb{C})$ , hence that  $\mathbb{V}$  is a translate of  $\mathbb{W}_r(\mathbb{C}) - \mathbb{W}_{g-d-r}(\mathbb{C})$ .

When  $\mathbb{C}$  is hyperelliptic,  $-\mathbb{C}$  is a translate of  $\mathbb{C}$ , hence  $-\mathbb{W}_{g-d-r}(\mathbb{C})$  is a translate of  $\mathbb{W}_{g-d-r}(\mathbb{C})$ , and  $\mathbb{V}$  is a translate of  $\mathbb{W}_{g-d}(\mathbb{C})$ . To conclude the proof of the theorem, we may therefore assume that  $\mathbb{C}$  is non-hyperelliptic.

hyperelliptic. Then:

$$(W_r(C) - W_{g-d-r}(C)) \cdot W_d(C) = \binom{g-d}{r} \binom{g}{d}.$$

**Proof.** The subtraction map  $W_r(C) \times W_{g-d-r}(C) \rightarrow \mathbf{JC}$  is birational onto its image. In fact, if  $(D, E)$  and  $(D', E')$  are two distinct elements of  $W_r(C) \times W_{g-d-r}(C)$  such that  $D - E \equiv D' - E'$ , then:

- either  $h^0(C, D + E') = 1$ , in which case  $(D, E)$  belongs to the divisor  $\{(D, E) \mid \text{Supp}(D) \cap \text{Supp}(E) \neq \emptyset\}$ ,
- or  $h^0(C, D + E') > 1$ , in which case  $(D, E)$  belongs to:

$$\{ (D, E) \mid \exists L \in W_{g-d}^1(C) \quad D, E \leq |L| \}.$$

This locus has dimension  $\max_{a>0}(\dim W_{g-d}^a(C) + 2a)$ , which is less than  $g - d$  if  $C$  is non-hyperelliptic (Martens' theorem, [ACGH], p. 191).

If  $*$  denotes the Pontryagin product on  $H^\bullet(\mathbf{JC}, \mathbf{Q})$ , it follows that the cohomology class of  $(W_r(C) - W_{g-d-r}(C))$  is:

$$\theta_{g-r} * \theta_{d+r} = \binom{g-d}{r} \theta_d.$$

Its intersection number with  $W_d(C)$  is therefore:

$$\binom{g-d}{r} \theta_d \cdot \theta_{g-d} = \binom{g-d}{r} \binom{g}{d}.$$

This proves the lemma. ■

But this intersection number is equal to  $V \cdot W_d(C)$ , hence to  $\binom{g}{d}$ . It follows that either  $r = g - d$ , or  $r = 0$ , hence either  $V$  is a translate of  $W_{g-d}(C)$ , or it is a translate of  $-W_{g-d}(C)$ . ■

## 6. Preliminaries

For any positive integer  $g$ , let  $\mathcal{A}_g$  be the moduli space of complex principally polarized abelian varieties of dimension  $g$ , let  $\mathcal{J}_g$  be the closure in  $\mathcal{A}_g$  of the subvariety which corresponds to Jacobians of smooth curves of genus  $g$ , and let  $\mathcal{CT}_5$  be the closure in  $\mathcal{A}_5$  of the subvariety which corresponds to intermediate Jacobians of smooth cubic threefolds. For  $0 < d \leq g$ , the subset  $\mathcal{C}_{g,d}$  of  $\mathcal{A}_g$  which corresponds to principally polarized abelian varieties  $(A, \theta)$  for which  $\theta_d$  is the class of an effective algebraic cycle (so that  $\mathcal{C}_{g,1} = \mathcal{C}_{g,g} = \mathcal{A}_g$ ), is *closed* in  $\mathcal{A}_g$ . Indeed, let  $\mathcal{X} \rightarrow S$  be a versal family of abelian varieties of dimension  $g$  with a relatively ample line bundle  $\mathcal{L}$  on  $\mathcal{X}$  which induces principal polarizations on the fibers, such that the classification morphism  $S \rightarrow \mathcal{A}_g$  is a finite cover. Fix an embedding of  $\mathcal{X}$  in some projective space  $\mathbf{P}_S^N$ , using for example the sections of  $\mathcal{L}^{\otimes 3}$ . Chow coordinates show that the family of effective cycles in  $\mathcal{X}$  whose fibers over  $S$  have codimension  $d$  and degree  $\theta_d(3\theta)^{g-d}$ , is projective over  $S$ . In particular, so is its closed subset which parametrizes effective cycles with class  $\theta_d$  in the fibers. Consequently, its image in  $S$  is also closed, hence so is its image  $\mathcal{C}_{g,d}$  in  $\mathcal{A}_g$ .

## 7. Degenerations of abelian varieties

Let  $\Delta = \{t \in \mathbf{C} \mid |t| < 1\}$ . A degeneration of principally polarized abelian varieties of dimension  $g+1$  will be a proper family  $\mathcal{X} \rightarrow \Delta$ , with  $\mathcal{X}$  smooth, whose fibers over  $\Delta^*$  are principally polarized abelian varieties of dimension  $g+1$  and whose central fiber  $X$  is a projective variety whose normalization is a  $\mathbf{P}^1$ -bundle  $\mathbf{P}$  over a principally polarized abelian variety  $(A, \theta)$  of dimension  $g$ . More precisely, there exists an element  $a$  of  $A$  such that  $\mathbf{P} = \mathbf{P}(\mathcal{O}_A \oplus \mathcal{O}_A(\widehat{a}))$ , where  $\widehat{a}$  is the image of  $a$  by the canonical isomorphism  $\phi_\theta : A \rightarrow \text{Pic}^0(A)$ . The bundle  $p : \mathbf{P} \rightarrow A$  has two disjoint sections  $\mathbf{P}_0$  and  $\mathbf{P}_\infty$ , and  $X$  is obtained from  $\mathbf{P}$  by identifying any point  $x$  on  $\mathbf{P}_0$  with the point  $x - a$  on  $\mathbf{P}_\infty$ . The singular locus  $X_s$  of  $X$  will always be identified with  $A$  via the isomorphism  $X_s \rightarrow \mathbf{P}_0 \rightarrow A$ . By [N], theorem 16.1, we may also assume the existence of a relatively ample line bundle  $\mathcal{L}$  on  $\mathcal{X}$  which induces the principal polarization on the smooth fibers and which restricts on  $X$  to an ample line bundle whose pull-back to  $\mathbf{P}$  is  $\mathcal{O}_{\mathbf{P}}(p^*\Theta + \mathbf{P}_0) \simeq \mathcal{O}_{\mathbf{P}}(p^*\Theta_a + \mathbf{P}_\infty)$ , for a suitable representative  $\Theta$  of the polarization  $\theta$ . Another (more accessible) reference for the existence of  $\mathcal{L}$  is [HW], proposition 4.1.3 (the proof is given in the surface case, but works in general).

The projective varieties  $X$  with their polarizations correspond to the boundary points of  $\mathcal{A}_{g+1}$  in a suitable partial compactification  $\bar{\mathcal{A}}_{g+1}$ , where they form a divisor  $\partial\mathcal{A}_{g+1}$ . There is a surjective map  $q : \partial\mathcal{A}_{g+1} \rightarrow \mathcal{A}_g$  which, in the above notation, sends  $X$  with its polarization to  $(A, \theta)$ , and whose generic fiber is isomorphic to  $A/\pm 1$ . For any subvariety  $\mathcal{B}$  of  $\mathcal{A}_{g+1}$ , we will denote by  $\partial\mathcal{B}$  the intersection of  $\partial\mathcal{A}_{g+1}$  with the closure of  $\mathcal{B}$  in  $\bar{\mathcal{A}}_{g+1}$ .

**PROPOSITION 7.1.** – *For  $1 < d < g+1$ , the image by  $q$  of the boundary  $\partial\mathcal{C}_{g+1,d}$  is*

**Proof.** Keeping the same notation, we assume moreover that the smooth fibers of  $\mathcal{X} \rightarrow \Delta$  correspond to elements of  $\mathcal{C}_{g+1,d}$ . Then, there exists a subvariety  $\mathcal{Z}$  of  $\mathcal{X}$  such that the cycles associated with the fibers of  $\mathcal{Z} \rightarrow \Delta$  over  $\Delta^*$  are sums of effective cycles, each with class  $\theta_d$ . Over the open interval  $(0,1)$ , the scheme  $\mathcal{Z}$  contains a subscheme whose fibers have class  $\theta_d$ . Its topological closure meets the central fiber  $X$  in a union  $Z = Z_1 \cup \dots \cup Z_s$  of irreducible algebraic subvarieties of codimension  $d$ . For  $i = 1, \dots, s$ , let  $Y_i$  be the closure in  $\mathbf{P}$  of the inverse image of  $Z_i$  by the bijection  $\mathbf{P} - \mathbf{P}_\infty \rightarrow X$ . Then:

$$(7.2) \quad \sum_{i=1}^s m_i [Y_i] = (p^*\theta + [\mathbf{P}_0])^d / d! = p^*\theta_d + [\mathbf{P}_0] \cdot p^*\theta_{d-1},$$

for some positive integers  $m_1, \dots, m_s$ . It follows that the image of  $\sum_{i=1}^s m_i Y_i$  in  $A$  has class  $\theta_{d-1}$ , and that its intersection with  $\mathbf{P}_\infty$  has class  $\theta_d$ . This proves the proposition. ■

## 8. Proof of the theorem

**THEOREM 8.1.** – *For  $1 < d < g$ , the locus  $\mathcal{J}_g$  is a component of  $\mathcal{C}_{g,d}$ . Moreover,  $\mathcal{CT}_5$  is a component of  $\mathcal{C}_{5,3}$  and is not contained in  $\mathcal{C}_{5,2}$ .*

**Proof.** The proof of the first part is by induction on  $g$ . Since this property is empty for  $g = 2$ , we assume that it holds for some  $g \geq 2$  and show that it then holds in dimension  $g + 1$ . Let  $1 < d < g + 1$  and let  $\mathcal{F}$  be a component of  $\mathcal{C}_{g+1,d}$  which contains either  $\mathcal{J}_{g+1}$  or  $\mathcal{CT}_5$  (if  $g = 4$ ). It is known that  $q(\partial\mathcal{J}_{g+1}) = \mathcal{J}_g$  and  $q(\partial\mathcal{CT}_5) = \mathcal{J}_4$ . It follows from proposition 7.1 and our induction hypothesis that  $\mathcal{J}_g$  is a component of  $q(\partial\mathcal{F})$ . With the notation of the proof of proposition 7.1, we may therefore assume that  $(A, \theta)$  is the Jacobian  $(\text{JC}, \theta)$  of a generic smooth curve  $C$  of genus  $g$ . Then,  $\partial\mathcal{J}_g \cap q^{-1}\{(\text{JC}, \theta)\}$  corresponds to the image of the surface  $C - C$  in  $\text{JC}/\pm 1$ . For  $g = 4$ , the curve  $C$  has two pencils  $g_3^1$  and  $h_3^1$  of degree 3, and  $\partial\mathcal{CT}_5 \cap q^{-1}\{(\text{JC}, \theta)\}$  corresponds to the point  $\pm(g_3^1 - h_3^1)$  of  $\text{JC}/\pm 1$  ([C]).

We will show that  $\partial\mathcal{C}_{g+1,d} \cap q^{-1}\{(\text{JC}, \theta)\}$  is equal to  $(C - C) \cup \{\pm(g_3^1 - h_3^1)\}$  for  $g = 4$  and  $d = 3$ , and to  $C - C$  otherwise.

Keeping the notation of the proof of proposition 7.1, the image in  $\text{JC}$  of the effective cycle  $Y = \sum_{i=1}^s m_i Y_i$  has class  $\theta_{d-1}$ . By theorem 5.1, it is a translate of  $\pm W_{g-d+1}(C)$  which, possibly after translation and action of  $(-1)_X$ , we may assume to be  $W_{g-d+1}(C)$ . Consequently, after reindexing if necessary, we have  $m_1 = 1$ , the morphism  $Y_1 \rightarrow p(Y_1)$  is birational, and  $Y_i = p^{-1}(p(Y_i))$  for  $i > 1$ . Identity (7.2) and theorem 5.1 then imply that:

**Case 1:** either  $Y = Y_1$  maps birationally onto  $W_{g-d+1}(C)$  and meets  $\mathbf{P}_0$  and  $\mathbf{P}_\infty$  along subvarieties which are translates of  $\pm W_{g-d}(C)$ ,

**Case 2:** or  $Y = Y_1 + Y_2$ , where  $Y_1$  has class  $[\mathbf{P}_0] \cdot p^*\theta_{d-1}$  and maps isomorphically onto  $W_{g-d+1}(C)$ , and  $Y_2$  is the pull-back by  $p$  of a translate  $V$  of  $\pm W_{g-d}(C)$ . In this case,

lar locus  $X_s$  of  $X$ , which may well happen, contrary to what is asserted in [BC] on p. 64), or it meets neither  $\mathbf{P}_0$  nor  $\mathbf{P}_\infty$ . In the latter case, the line bundle  $\mathcal{O}_{\mathbf{P}}(\mathbf{P}_0 - \mathbf{P}_\infty) = p^* \mathcal{O}_{\text{JC}}(\hat{a})$  is trivial on  $Y_1$ . Since the restriction  $\text{Pic}^0(\text{JC}) \rightarrow \text{Pic}^0(W_{g-d+1}(\mathbb{C}))$  is injective, this implies that  $a$  is 0 hence is in  $\mathbb{C} - \mathbb{C}$ . We may therefore assume that  $Y_1$  is contained in  $\mathbf{P}_0$ .

The following result makes the structure of the limit cycle more precise.

LEMMA 8.2. – *Let  $z$  be a point of  $Z \cap X_s$ . Then either  $z$  belongs to a component of  $Z$  contained in  $X_s$ , or each of the two branches of  $X$  contains a local component of  $Z$  at  $z$ .*

**Proof.** Let  $\varepsilon : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  be the blow-up of the smooth subscheme  $X_s$ . The strict transform  $\tilde{X}$  of  $X$  can be identified with  $\mathbf{P}$ . Let  $\tilde{Z}$  be the strict transform of  $Z$ . If no component of  $Z$  through  $z$  is contained in  $X_s$ , the scheme  $Z \cap X_s = \tilde{Z} \cap X_s$  has pure codimension 2 in  $\tilde{Z}$  at  $z$ . It follows that its ideal in  $\tilde{Z}$  is not invertible at  $z$ , hence that the rational curve  $\varepsilon^{-1}(z)$  is contained in  $\tilde{Z}$ . This curve meets  $\tilde{X}$  at two points (one on  $\mathbf{P}_0$ , the other on  $\mathbf{P}_\infty$ ) which correspond to the two branches of  $X$  at  $z$ . It follows that the intersection of  $\tilde{Z}$  with the Cartier divisor  $\tilde{X}$  is non-empty, hence has dimension  $\dim \tilde{Z} - 1 = g + 1 - d$ , at each of these two points. Consequently, the intersection of  $Z$  with each of the two branches of  $X$  has dimension  $g + 1 - d$  at  $z$ , and this proves the lemma. ■

In case 1, this means that  $V = Y \cap \mathbf{P}_0$  and  $Y \cap \mathbf{P}_\infty$  must be identified through the glueing process, hence that  $Y \cap \mathbf{P}_\infty = V - a$ . In particular, both  $V$  and  $V - a$  are contained in  $W_{g-d+1}(\mathbb{C})$ .

In case 2, both  $V$  and  $V - a$  must be again contained in  $W_{g-d+1}(\mathbb{C})$ .

Recall that in both cases,  $V$  is a translate of  $\pm W_{g-d}(\mathbb{C})$ .

LEMMA 8.3. – *Let  $C$  be a smooth curve of genus  $g$  and let  $a$  be an element of  $\text{JC}$ . Assume that some translate of  $W_{m+1}(\mathbb{C})$  contains  $\varepsilon W_m(\mathbb{C})$  and  $\varepsilon W_m(\mathbb{C}) - a$ , for some  $\varepsilon = \pm 1$  and  $0 \leq m \leq g - 2$ . Then either  $a \in \mathbb{C} - \mathbb{C}$ , or  $\varepsilon = -1$ ,  $g = 4$  and  $a$  is the difference of two  $g_3^1$ 's.*

**Proof.** Let  $L_a$  be the line bundle of degree 0 on  $C$  associated with  $a$ . If  $\varepsilon = 1$ , there exists a line bundle  $L$  on  $C$  of degree 1 such that:

$$h^0(C, L(c_1 + \cdots + c_m)) > 0 \quad \text{and} \quad h^0(C, L \otimes L_a^{-1}(c_1 + \cdots + c_m)) > 0,$$

for any points  $c_1, \dots, c_m$  of  $C$ . Riemann-Roch implies that both  $L$  and  $L \otimes L_a^{-1}$  have a non-zero-section, hence that  $a \in \mathbb{C} - \mathbb{C}$ .

If  $\varepsilon = -1$ , we may assume that  $C$  is non-hyperelliptic and  $0 < m \leq g - 3$  (otherwise,  $-W_{m+1}(\mathbb{C})$  is a translate of  $W_{m+1}(\mathbb{C})$ ). There exists a line bundle  $L$  on  $C$  of degree  $2m + 1$  such that:

$$h^0(C, L(-c_1 - \cdots - c_m)) > 0 \quad \text{and} \quad h^0(C, L \otimes L_a^{-1}(-c_1 - \cdots - c_m)) > 0,$$

$g_{2m+1}^m$ 's. By [ACGH], exercise B-7, p. 138, this implies under our assumptions that  $m = g - 3$ , and that the residual series of both  $L$  and  $L \otimes L_a^{-1}$  are  $g_3^1$ 's. If  $g > 4$ , there is at most one  $g_3^1$  and  $a = 0$ . If  $g = 4$ , there are at most two  $g_3^1$ 's and  $a$  is either 0 or their difference. ■

This proves that  $\partial\mathcal{C}_{g+1,d}$  coincides over  $\mathcal{J}_g$  with  $\partial(\mathcal{J}_{g+1} \cup \mathcal{CT}_5)$  for  $g = 4$  and  $d = 3$ , and with  $\partial\mathcal{J}_{g+1}$  otherwise. In particular:

$$\begin{aligned} \dim \partial\mathcal{F} &\leq 2 + \dim \mathcal{J}_g = 3g - 1 && \text{if } \mathcal{F} \supset \mathcal{J}_{g+1}, \\ \dim \partial\mathcal{F} &\leq \dim \mathcal{J}_4 = \dim \mathcal{CT}_5 - 1 && \text{if } \mathcal{F} \supset \mathcal{CT}_5. \end{aligned}$$

Since  $\partial\mathcal{F}$  is the intersection of  $\mathcal{F}$  with the Cartier divisor  $\partial\mathcal{A}_{g+1}$ , we have  $\dim \mathcal{F} \leq \dim \partial\mathcal{F} + 1$ . It follows that either  $\mathcal{F} = \mathcal{J}_{g+1}$  or  $\mathcal{F} = \mathcal{CT}_5$ , which finishes the proof of the induction step.

To conclude, note that if  $C$  is non-hyperelliptic of genus 4, then  $g_3^1 - h_3^1$  is not in  $C - C$  if it is non-zero. Therefore,  $\mathcal{CT}_5$  is not contained in  $\mathcal{C}_{5,2}$ . ■

Note that we only needed for  $C - C$  to be a component of  $\partial\mathcal{C}_{g+1,d} \cap q^{-1}\{(\text{JC}, \theta)\}$  for the proof. Our more precise result, combined with the fact that  $\mathcal{J}_4 = \mathcal{C}_{4,3} = \mathcal{C}_{4,2}$ , proves:

$$\partial\mathcal{C}_{5,2} = \partial\mathcal{J}_5 \quad \text{and} \quad \partial\mathcal{C}_{5,3} = \partial(\mathcal{J}_5 \cup \mathcal{CT}_5).$$

This makes the following conjecture plausible, at least in dimension 5:

CONJECTURE – For  $1 < d < g$  and  $(g, d) \neq (5, 3)$ , then  $\mathcal{C}_{g,d} = \mathcal{J}_g$ ; furthermore,  $\mathcal{C}_{5,3} = \mathcal{J}_5 \cup \mathcal{CT}_5$ .

A more tractable version would be the weaker:

CONJECTURE' – For  $0 < d < g$ , then  $\mathcal{C}_{g,d} \cap \mathcal{C}_{g,g-d} = \mathcal{J}_g$ .

## REFERENCES

- [ACGH] Arbarello, E., Cornalba, M., Griffiths, P., Harris, J., *Geometry of algebraic curves. I. Grundlehren 267*, Springer-Verlag, New York, 1985.
- [BC] Barton, C., Clemens, C.H., *A result on the integral Chow ring of a generic principally polarized complex Abelian variety of dimension four*, *Comp. Math.* **34** (1977), 49–67.
- [B] Beauville, A., *Sous-variétés spéciales des variétés de Prym*, *Comp. Math.* **45** (1982), 357–383.
- [CG] Clemens, C.H., Griffiths, P.A., *The Intermediate Jacobian of the Cubic Threefold*, *Ann. of Math.* **95** (1972), 281–356.
- [C] Collino, A., *A cheap proof of the irrationality of most cubic threefolds*, *Boll. Un. Mat. Ital.* **16** (1979), 451–465.
- [HW] Hulek, K., Weintraub, S., *The Principal Degenerations of Abelian Surfaces and Their Polarizations*, *Math. Ann.* **286** (1990), 281–307.
- [M] Matsusaka, T., *On a characterization of a Jacobian Variety*, *Mem. Coll. Sc. Kyoto, Ser. A*, **23** (1959), 1–19.
- [N] Namikawa, Y., *A New Compactification of the Siegel Space and Degeneration of Abelian Varieties, I, II*, *Math. Ann.* **221** (1976), 97–141, 201–241.
- [R1] Ran, Z., *On subvarieties of abelian varieties*, *Invent. Math.* **62** (1981), 459–479.
- [R2] Ran, Z., *A Characterization of Five-Dimensional Jacobian Varieties*, *Invent. Math.* **67** (1982), 395–422.

Olivier DEBARRE

URA D 0752 du CNRS  
 Mathématique – Bâtiment 425  
 91405 Orsay Cedex – France  
 debarre@matups.matups.fr

Department of Mathematics  
 The University of Iowa  
 Iowa City IA 52242 – U.S.A.  
 debarre@math.uiowa.edu