FULTON-HANSEN AND BARTH-LEFSCHETZ THEOREMS FOR SUBVARIETIES OF ABELIAN VARIETIES

Olivier DEBARRE(*)

Sommese showed that a large part of the geometry of a *smooth* subvariety of a complex abelian variety depends on "how ample" its normal bundle is (see § 1 for more details). Unfortunately, the only known way of measuring this ampleness uses rather strong properties of the ambiant abelian variety.

We show that a notion of non-degeneracy due to Ran is a good substitute for ampleness of the normal bundle. It can be defined as follows: *an irreducible subvariety* V *of an abelian variety* X *is geometrically non-degenerate if for any abelian variety* Y *quotient of* X, *the image of* V *in* Y *either is* Y *or has same dimension as* V. This property does not require V to be smooth; for smooth subvarieties, it is (strictly) weaker than ampleness of the normal bundle.

Our main result is a Fulton-Hansen type theorem for an irreducible subvariety V of an abelian variety: the dimension of the "secant variety" of V along a subvariety S (defined as $\mathrm{V}-\mathrm{S}$), and that of its "tangential variety" along S (defined in the smooth case as the union of the projectivized tangent spaces to V at points of S, translated at the origin) differ by 1. Corollaries include a new proof of the finiteness of the Gauss map and an estimate on the ampleness of the normal bundle of a smooth geometrically non-degenerate subvariety.

We also complement Sommese's work with a new Barth-Lefschetz theorem for subvarieties of abelian varieties whose proof is based on an idea of Schneider and Zintl. Let C be a smooth curve in an abelian variety X; we apply this result to give an estimate on the dimension of the singular locus of $\mathrm{C}+\cdots+\mathrm{C}$ in X.

We work over the field of complex numbers.

1. Geometrically non-degenerate subvarieties

Recall ([S1]) that a line bundle L on an irreducible projective variety V is k–ample if, for some $m>0$, the line bundle L^m is generated by its global sections and the fibers of the associated map $\phi_{\mathrm{L}^m}:\mathrm{V}\to\mathbf{P}^{\mathrm{N}}$ are all of dimension $\le k$. A vector bundle E on V is k–ample if the line bundle $\mathcal{O}_{\mathbf{P}\mathrm{E}^*}(1)$ is k–ample. Ordinary ampleness coincide with 0–ampleness.

(*) Partially supported by N.S.F. Grant DMS 94-00636 and the European HCM project "Algebraic Geometry in Europe" (AGE), Contract CHRXCT-940557.

$f : \mathrm{V} \to \mathrm{X}$. Let V^0 be the open set of smooth points of V at which f is unramified. Define the normal bundle to f as the vector bundle on V^0 quotient of $f^*(\mathrm{TX})_{|\mathrm{V}^0}$ by TV^0 .

For any $x \in \mathrm{X}$, let τ_x be the translation by x . For any $v \in \mathrm{V}$, the differential of the map $\tau_{-f(v)}f$ at v is a linear map $\mathrm{T}_v\mathrm{V} \to \mathrm{T}_0\mathrm{X}$, which we will simply denote by f_* .

Proposition 1.1.– *Under the above assumptions, let* S *be a complete irreducible subvariety of* V^0 . *The following properties are equivalent*

(i) *the restriction to* S *of the normal bundle to* f *is* k*–ample;*

(ii) *for any hyperplane* H *in* $\mathrm{T}_0\mathrm{X}$, *the set* $\{ s \in \mathrm{S} \mid f_*(\mathrm{T}_s\mathrm{V}) \subset \mathrm{H} \}$ *has dimension* $\leq k$.

Proof. Let N be the restriction to S of the normal bundle to f and let $\iota : \mathbf{P}\mathrm{N}^* \to \mathbf{P}f^*(\mathrm{T}^*\mathrm{X})_{|\mathrm{S}}$ be the canonical injection. The morphism

$$\phi : \mathbf{P}\mathrm{N}^* \xrightarrow{\iota} \mathbf{P}f^*(\mathrm{T}^*\mathrm{X})_{|\mathrm{S}} \simeq \mathbf{P}\mathrm{T}_0^*\mathrm{X} \times \mathrm{S} \xrightarrow{pr_1} \mathbf{P}\mathrm{T}_0^*\mathrm{X}$$

satisfies

$$\phi^*\mathcal{O}_{\mathbf{P}\mathrm{T}_0^*\mathrm{X}}(1) = \iota^*\mathcal{O}_{\mathbf{P}f^*(\mathrm{T}^*\mathrm{X})_{|\mathrm{S}}}(1) = \mathcal{O}_{\mathbf{P}\mathrm{N}^*}(1) \ .$$

It follows that N is k–ample if and only if the fibers of ϕ have dimension $\leq k$ ([S1], prop. 1.7). The proposition follows, since the restriction of the projection $\mathbf{P}\mathrm{N}^* \to \mathrm{S}$ to any fiber of ϕ is injective. ■

When X is simple, the normal bundle to any smooth subvariety of X is ample ([H]). More generally, the normal bundle to any smooth subvariety of X is k–ample, where k is the maximum dimension of a proper abelian subvariety of X ([S1], prop. 1.20).

Following Ran, we will say that a d-dimensional irreducible subvariety V of X is *geometrically non-degenerate* if the kernel of the restriction $\mathrm{H}^0(\mathrm{X}, \Omega^d_{\mathrm{X}}) \to \mathrm{H}^0(\mathrm{V}_{\mathrm{reg}}, \Omega^d_{\mathrm{V}_{\mathrm{reg}}})$ contains no non-zero decomposable forms. This property holds if and only if for any abelian variety Y quotient of X , the image of V in Y either is Y or has same dimension as V ([R1], lemma II.12).

Examples. 1) A divisor is geometrically non-degenerate if and only if it is ample; a curve is geometrically non-degenerate if and only if it generates X . Any geometrically non-degenerate subvariety of positive dimension generates X , but the converse is false in general. However, any irreducible subvariety of a *simple* abelian variety is geometrically non-degenerate.

2) If ℓ is a polarization on X and V is an irreducible subvariety of X with class a rational multiple of ℓ^c , it follows from [R1], cor. II.2 and II.3 that V is non-degenerate in the sense of [R1], II, hence geometrically non-degenerate. In particular, the subvarieties $\mathrm{W}_d(\mathrm{C})$ of the Jacobian of a curve C are geometrically non-degenerate; it can be checked that their normal bundle is ample when they are smooth (use prop. 1.1).

Definition 1.2.– *An irreducible subvariety* V *of an abelian variety* X *is* k*–geometrically non-degenerate if and only if for any abelian variety* Y *quotient of* X*, the image of* V *in* Y *either is* Y *or has dimension* $\geq \dim(\mathrm{V}) - k$*.*

Proposition 1.3.– *In an abelian variety, any smooth irreducible subvariety with* k*–ample normal bundle is* k*–geometrically non-degenerate.*

Proof. Let $\pi : \mathrm{X} \to \mathrm{Y}$ be a quotient of X such that $\pi(\mathrm{V}) \neq \mathrm{Y}$. The tangent spaces to V along a smooth fiber of $\pi_{|\mathrm{V}}$ are all contained in a fixed hyperplane, hence general fibers of $\pi_{|\mathrm{V}}$ have dimension $\leq k$ by prop. 1.1. ■

The converse is not true, as the construction sketched below shows, but a partial converse will be obtained in 2.3. Roughly speaking, if Y is a quotient of X, and if the image W of V in Y is not Y, k–geometrical nondegeneracy requires that the general fibers of $\mathrm{V} \to \mathrm{W}$ be of dimension $\leq k$, whereas k–ampleness of the normal bundle requires that every fiber of $\mathrm{V} \to \mathrm{W}$ be of dimension $\leq k$.

Let $\mathrm{L_E}$ be an ample line bundle on an elliptic curve E, with linearly independent sections s_1, s_2 defining a morphism $\mathrm{E} \to \mathbf{P}^1$ with ramification points $(e_1, 1), \ldots, (e_4, 1) \in \mathbf{P}^1$. Let $\mathrm{L_Y}$ be an ample line bundle on a simple abelian variety Y of dimension ≥ 3, with linearly independent sections t_1, t_2, t_3 such that $\operatorname{div}(t_3)$, $\mathrm{F} = \operatorname{div}(t_1) \cap \operatorname{div}(t_2) \cap \operatorname{div}(t_3)$ and $\operatorname{div}(e_i t_1 + t_2) \cap \operatorname{div}(t_3)$ are smooth for $i = 1, \ldots, 4$ (such a configuration can be constructed using results from [D2]). Set $\mathrm{X} = \mathrm{E} \times \mathrm{Y}$ and define a subvariety V of X by the equations $s_1 t_1 + s_2 t_2 = t_3 = 0$; then V is smooth of codimension 2, geometrically non-degenerate, but its normal bundle is not ample (for all $e \in \mathrm{E}$ and $f \in \mathrm{F}$, one has $\mathrm{T}_{(e,f)}\mathrm{V} \subset \mathrm{T}_f\big(\operatorname{div}(t_3)\big)$), only 1–ample (cor. 2.3).

Proposition 1.4.– *Let* X *be an abelian variety and let* V *and* W *be irreducible subvarieties of* X*. Define a morphism* $\phi : \mathrm{V}^r \times \mathrm{W} \to \mathrm{X}^r$ *by* $\phi(v_1, \ldots, v_r, w) = (v_1 - w, \ldots, v_r - w)$*. If* V *is* k*–geometrically non-degenerate,*

$$\dim \phi(\mathrm{V}^r \times \mathrm{W}) \geq \min\big(r \dim(\mathrm{X}), r \dim(\mathrm{V}) + \dim(\mathrm{W}) - k\big) .$$

Proof. Assume first $r \dim(\mathrm{V}) + \dim(\mathrm{W}) - k \geq r \dim(\mathrm{X})$. Let $\pi : \mathrm{X} \to \mathrm{X}/\mathrm{K}$ be a quotient of X. I claim that $r \dim \pi(\mathrm{V}) + \dim \pi(\mathrm{W}) \geq r \dim(\mathrm{X}/\mathrm{K})$. If $\pi(\mathrm{V}) = \mathrm{X}/\mathrm{K}$, this is obvious; otherwise, we have $\dim \pi(\mathrm{V}) \geq \dim(\mathrm{V}) - k_0$, where $k_0 = \min\big(k, \dim(\mathrm{K})\big)$, hence

$$\begin{aligned} r \dim \pi(\mathrm{V}) + \dim \pi(\mathrm{W}) &\geq r(\dim(\mathrm{V}) - k_0) + \dim(\mathrm{W}) - \dim(\mathrm{K}) \\ &\geq r \dim(\mathrm{X}) + k - r k_0 - \dim(\mathrm{K}) \geq r \dim(\mathrm{X}/\mathrm{K}) . \end{aligned}$$

It follows that $(\mathrm{V}, \ldots, \mathrm{V}, \mathrm{W})$ (where V is repeated r times) fills up X in the sense of [D1], (1.10); th. 2.1 of *loc.cit.* then implies that ϕ is onto.

in X that generates X. Let W' be the sum of W and s copies of C; then $r\dim(\mathrm{V}) + \dim(\mathrm{W}') - k = r\dim(\mathrm{X})$ and the first case shows that the sum of the image of ϕ and s curves is X^r. The proposition follows. ∎

We obtain a nice characterization of k–geometrically non-degenerate varieties.

Corollary 1.5.– *An irreducible subvariety* V *of an abelian variety* X *is* k*–geometrically non-degenerate if and only if it meets any subvariety of* X *of dimension* $\geq \mathrm{codim}(\mathrm{V}) + k$.

Proof. Assume that V meets any subvariety of X of dimension $\geq \mathrm{codim}(\mathrm{V}) + k$ and let $\pi : \mathrm{X} \to \mathrm{Y}$ be a quotient of X. If $\pi(\mathrm{V}) \neq \mathrm{Y}$, there exists a subvariety W of Y of dimension $\dim(\mathrm{Y}) - \dim\pi(\mathrm{V}) - 1$ that does not meet $\pi(\mathrm{V})$. Since V does not meet $\pi^{-1}(\mathrm{W})$,

$$\mathrm{codim}(\mathrm{V}) + k > \dim\pi^{-1}(\mathrm{W}) = \dim(\mathrm{X}) - \dim\pi(\mathrm{V}) - 1$$

hence $\dim\pi(\mathrm{V}) \geq \dim(\mathrm{V}) - k$ and V is k–geometrically non-degenerate. Conversely, assume V is k–geometrically non-degenerate; let W be an irreducible subvariety of X of dimension $\geq \mathrm{codim}(\mathrm{V}) + k$. Proposition 1.4 shows that $\mathrm{V} - \mathrm{W} = \mathrm{X}$, hence V meets W. ∎

2. A Fulton-Hansen-type result

Fulton and Hansen proved in [FH] (*cf.* also [FL1], [Z1], [Z2]) a beautiful result that relates the dimension of the tangent variety and that of the secant variety of a subvariety of a projective space. We prove an analogous result for a subvariety of an abelian variety.

Let X be an abelian variety and let V be a variety with a morphism $f : \mathrm{V} \to \mathrm{X}$. Recall that f is unramified along a subvariety S of V if $\Delta_{\mathrm{S}} = \{(v, s) \in \mathrm{V} \times \mathrm{S} \mid v = s\}$ is an open subscheme of $\mathrm{V} \times_{\mathrm{X}} \mathrm{S}$. Following [FL1], we will say that f is *weakly unramified* along S if Δ_{S} is a connected component of $\mathrm{V} \times_{\mathrm{X}} \mathrm{S}$, ignoring scheme structures. In that case, if $p : \mathrm{V} \times \mathrm{S} \to \mathrm{X}$ is the morphism defined by $p(v, s) = f(v) - f(s)$ and $\epsilon : \tilde{\mathrm{X}} \to \mathrm{X}$ is the blow-up of the origin, there exists a commutative diagram

$$\begin{tikzcd}
\tilde{\mathrm{Y}} \arrow[r, "\tilde{p}"] \arrow[d, "\alpha"'] & \tilde{\mathrm{X}} \arrow[d, "\epsilon"'] \\
\mathrm{V} \times \mathrm{S} \arrow[r, "p"] & \mathrm{X}
\end{tikzcd}$$

where α is the blow-up of $\mathrm{V} \times_{\mathrm{X}} \mathrm{S}$. Let E be the exceptional divisor above $\Delta_{\mathrm{S}} \subset \mathrm{V} \times \mathrm{S}$ and set $\mathrm{T}(\mathrm{V}, \mathrm{S}) = \tilde{p}(\mathrm{E})$. It is a subscheme of $\mathbf{P}\mathrm{T}_0\mathrm{X}$ contained in $\bigcup_{s \in \mathrm{S}} \mathbf{P}\mathrm{T}_s\mathrm{V}$ and equal

$\mathrm{T(V,S)}$ is the set of limits in $\tilde{\mathrm{X}}$ of $\big(f(v)-f(s)\big)$, as $v\in\mathrm{V}$ and $s\in\mathrm{S}$ converge to the same point. Obviously, $\dim\mathrm{T(V,S)}<\dim\big(f(\mathrm{V})-f(\mathrm{S})\big)$.

Theorem 2.1.– *Let* X *be an abelian variety and let* V *be an irreducible projective variety with a morphism* $f:\mathrm{V}\to\mathrm{X}$. *Let* S *be a complete irreducible subvariety of* V *along which* f *is weakly unramified. Then* $\dim\mathrm{T(V,S)}=\dim\big(f(\mathrm{V})-f(\mathrm{S})\big)-1$.

We begin with a lemma.

Lemma 2.2.– *Let* C *be an irreducible projective curve with a morphism* $g:\mathrm{C}\to\mathrm{X}$ *such that* $g(\mathrm{C})$ *is a smooth curve through the origin. Assume that* g *is unramified at some point* $c_0\in\mathrm{C}$ *with* $g(c_0)=0$ *and that* $\mathbf{P}\mathrm{T}_0g(\mathrm{C})\notin\mathrm{T(V,S)}$. *The morphism* $h:\mathrm{V}\times\mathrm{C}\to\mathrm{X}$ *defined by* $h(v,c)=f(v)-g(c)$ *is weakly unramified along* $\mathrm{S}\times\{c_0\}$ *and* $\mathrm{T}(\mathrm{V}\times\mathrm{C},\mathrm{S}\times\{c_0\})$ *is contained in the cone over* $\mathrm{T(V,S)}$ *with vertex* $\mathbf{P}\mathrm{T}_0g(\mathrm{C})$.

One can prove that $\mathrm{T}(\mathrm{V}\times\mathrm{C},\mathrm{S}\times\{c_0\})$ is actually equal to the cone.

Proof. Let Γ be a smooth irreducible curve, let γ_0 be a point on Γ and let $q=(q_1,q_2,q_3):\Gamma\to(\mathrm{V}\times\mathrm{C})\times_{\mathrm{X}}\mathrm{S}$ be a morphism with $q(\gamma_0)=(s_0,c_0,s_0)$. We need to prove that $q(\Gamma)\subset\Delta'_{\mathrm{S}}$, where $\Delta'_{\mathrm{S}}=\{(s,c_0,s)\mid s\in\mathrm{S}\}$; since Δ_{S} is a connected component of $\mathrm{V}\times_{\mathrm{X}}\mathrm{S}$, it suffices to show that q_2 is constant. Suppose the contrary; then (q_1,q_3) lifts to a morphism $\tilde{q}_{13}:\Gamma\to\tilde{\mathrm{Y}}$ and g to a morphism $\tilde{g}:\mathrm{C}\to\tilde{\mathrm{X}}$. Since $p(q_1,q_3)=gq_2$, one has $\tilde{p}\tilde{q}_{13}=\tilde{g}q_2$ hence $\tilde{g}(c_0)=\tilde{p}\big(\tilde{q}_{13}(\gamma_0)\big)\in\mathrm{T(V,S)}$. This contradicts the hypothesis since $\tilde{g}(c_0)$ is the point $\mathbf{P}\mathrm{T}_0g(\mathrm{C})$ of $\mathbf{P}\mathrm{T}_0\mathrm{X}$. This proves the first part of the lemma.

The second part is similar: let $\tilde{\mathrm{Z}}\to(\mathrm{V}\times\mathrm{C})\times\mathrm{S}$ be the blow-up of $(\mathrm{V}\times\mathrm{C})\times_{\mathrm{X}}\mathrm{S}$, let Γ be a smooth irreducible curve with a point $\gamma_0\in\Gamma$ and let $\tilde{q}:\Gamma\to\tilde{\mathrm{Z}}$ be a morphism such that $\tilde{q}(\gamma_0)$ is in the exceptional divisor above Δ'_{S}. Write $q=\alpha\tilde{q}=(q_1,q_2,q_3)$ and keep the same notation as above. Then $pq(\Gamma)$ is contained in the surface $pq_{13}(\Gamma)-g(\mathrm{C})$ hence $\tilde{p}\tilde{q}(\gamma_0)$ belongs to the line in $\mathbf{P}\mathrm{T}_0\mathrm{X}$ through $\tilde{p}\big(\tilde{q}_{13}(\gamma_0)\big)$ and $\tilde{g}(c_0)=\mathbf{P}\mathrm{T}_0g(\mathrm{C})$. This proves the lemma. ∎

Proof of the theorem. We proceed by induction on the codimension of $f(\mathrm{V})-f(\mathrm{S})$. Assume $f(\mathrm{V})-f(\mathrm{S})=\mathrm{X}$; if $\mathrm{T(V,S)}\neq\mathbf{P}\mathrm{T}_0\mathrm{X}$, pick a point $u\notin\mathrm{T(V,S)}$ and a smooth projective curve C' in X tangent to u at 0, and such that the restriction induces an injection $\mathrm{Pic}^0(\mathrm{X})\to\mathrm{Pic}^0(\mathrm{C}')$. Let C be a smooth curve with a connected ramified double cover $g:\mathrm{C}\to\mathrm{C}'$ unramified at a point c_0 above 0; the map $\mathrm{Pic}^0(\mathrm{C}')\to\mathrm{Pic}^0(\mathrm{C})$ induced by g is injective.

Since p is surjective, C' generates X and C is smooth, th. 3.6 of [D1] implies that $(\mathrm{V}\times\mathrm{S})\times_{\mathrm{X}}\mathrm{C}$ is connected. If $h:\mathrm{V}\times\mathrm{C}\to\mathrm{X}$ is defined by $h(v,c)=f(v)-g(c)$, it follows that $(\mathrm{V}\times\mathrm{C})\times_{\mathrm{X}}\mathrm{S}$ is also connected. On the other hand, the lemma implies that the set $\{(s,c_0,s)\mid s\in\mathrm{S}\}$ is a connected component of, hence is equal to, $(\mathrm{V}\times\mathrm{C})\times_{\mathrm{X}}\mathrm{S}$. It follows that $h^{-1}\big(f(\mathrm{S})\big)=\mathrm{S}\times\{c_0\}$. Since $g^{-1}(0)$ consists of 2 distinct points, this is absurd, hence $\mathrm{T(V,S)}=\mathbf{P}\mathrm{T}_0\mathrm{X}$ and the theorem holds in this case.

morphism $f' : \mathrm{V} \times \mathrm{C}' \to \mathrm{X}$ defined by $f'(v, c') = f(v) + c'$ is weakly unramified along $\mathrm{S} \times \{0\}$, and $\dim \mathrm{T}(\mathrm{V} \times \mathrm{C}', \mathrm{S} \times \{0\}) \leq \dim \mathrm{T}(\mathrm{V}, \mathrm{S}) + 1$. It follows from the induction hypothesis that

$$\dim \mathrm{T}(\mathrm{V}, \mathrm{S}) \geq \dim\big(f(\mathrm{V}) + \mathrm{C}' - f(\mathrm{S})\big) - 2 = \dim\big(f(\mathrm{V}) - f(\mathrm{S})\big) - 1 ,$$

which proves the theorem. ■

The following corollary provides a partial converse to prop. 1.3.

COROLLARY 2.3.– *Let* X *be an abelian variety of dimension* n *and let* V *be an irreducible projective variety of dimension* d *with a morphism* $f : \mathrm{V} \to \mathrm{X}$ *such that* $f(\mathrm{V})$ *is* k*–geometrically non-degenerate. Let* V^0 *be the open set of smooth points of* V *at which* f *is unramified. The restriction of the normal bundle to* f *to any complete irreducible subvariety* S *of* V^0 *is* $(n - d - 1 + k)$*–ample.*

Proof. By prop. 1.1, we must show that for any hyperplane H in $\mathrm{T}_0\mathrm{X}$, any irreducible component S_H of $\{s \in \mathrm{S} \mid f_*(\mathrm{T}_v\mathrm{V}) \subset \mathrm{H}\}$ has dimension $\leq n - d - 1 + k$. But $\mathrm{T}(\mathrm{V}, \mathrm{S}_\mathrm{H})$ is contained in H and the theorem gives $f(\mathrm{V}) - f(\mathrm{S}_\mathrm{H}) \neq \mathrm{X}$. Since f is unramified along S_H and $f(\mathrm{V})$ is k–geometrically non-degenerate, prop. 1.4 implies that $f(\mathrm{V}) - f(\mathrm{S}_\mathrm{H})$ has dimension $\geq d + \dim(\mathrm{S}_\mathrm{H}) - k$; this proves the corollary. ■

It should be noted that the corollary also follows from the main result of [Z3] (cor. 1), whose proof is unfortunately so sketchy (to say the least) that I could not understand it.

COROLLARY 2.4.– *Let* X *be an abelian variety and let* V *be an irreducible projective variety with a morphism* $f : \mathrm{V} \to \mathrm{X}$. *Let* L *be a linear subspace of* $\mathrm{T}_0\mathrm{X}$ *and let* S *be a complete irreducible subvariety of* V *along which* f *is unramified. Assume that* $\dim\big(f_*(\mathrm{T}_s\mathrm{V}) \cap \mathrm{L}\big) < m$ *for all* $s \in \mathrm{S}$, *and let* $\Delta_{f(\mathrm{S})}$ *be the small diagonal in* $f(\mathrm{S})^m$. *Then*

$$\dim\big(f(\mathrm{V})^m - \Delta_{f(\mathrm{S})}\big) < m \dim(\mathrm{X}) - \dim(\mathrm{L}) + m .$$

In particular, if $m \leq \dim(\mathrm{L})$ *and* $f(\mathrm{V})$ *is* k*–geometrically non-degenerate,*

$$m \dim(\mathrm{V}) + \dim(\mathrm{S}) < m \dim(\mathrm{X}) - \dim(\mathrm{L}) + m + k .$$

Proof. Let $r = \dim(\mathrm{L})$; the variety $\mathrm{N} = \{\, [t_1, \ldots, t_m] \in \mathbf{P}(\mathrm{L}^m) \mid t_1 \wedge \cdots \wedge t_m = 0\}$ has codimension $r - m + 1$ in $\mathbf{P}(\mathrm{L}^m)$. Consider the morphism $f^m : \mathrm{V}^m \to \mathrm{X}^m$ and the subvariety Δ_S of V^m. The hypothesis imply that in $\mathbf{P}(\mathrm{T}_0\mathrm{X}^m)$, the intersection of $\mathrm{T}(\mathrm{V}^m, \Delta_\mathrm{S})$ and $\mathbf{P}(\mathrm{L}^m)$ is contained in N. It follows that

$$\begin{aligned}\dim \mathrm{T}(\mathrm{V}^m, \Delta_\mathrm{S}) &\leq \dim(\mathrm{N}) + \dim \mathbf{P}(\mathrm{T}_0\mathrm{X}^m) - \dim \mathbf{P}(\mathrm{L}^m) \\ &= \dim \mathbf{P}(\mathrm{T}_0\mathrm{X}^m) - (r - m + 1) .\end{aligned}$$

The first inequality of the corollary follows from th. 2.1, and the second from prop. 1.4. ■

We keep the same setting: X is an abelian variety and V an irreducible projective variety of dimension d with a morphism $f : \mathrm{V} \to \mathrm{X}$. Let V^0 be the open set of smooth points of V at which f is unramified; define the *Gauss map* $\gamma : \mathrm{V}^0 \to \mathrm{G}(d, \mathrm{T}_0\mathrm{X})$ by $\gamma(v) = f_*(\mathrm{T}_v\mathrm{V})$. The following result was first proved by Ran ([R2]), and by Abramovich ([A]) in all characteristics.

Proposition 3.1.– *Let X be an abelian variety and let V be an irreducible projective variety with a morphism $f : \mathrm{V} \to \mathrm{X}$. If S is a complete irreducible variety contained in a fiber of the Gauss map, $f(\mathrm{V})$ is stable by translation by the abelian variety generated by $f(\mathrm{S})$. In particular, the Gauss map of a smooth projective subvariety of X invariant by translation by no non-zero abelian subvariety of X is finite.*

Proof. Under the hypothesis of the proposition, $\mathrm{T}(\mathrm{V}, \mathrm{S})$ has dimension $\dim(\mathrm{V}) - 1$; th. 2.1 implies $f(\mathrm{V}) - f(\mathrm{S}) = f(\mathrm{V})$, hence the proposition. ■

For any linear subspace L of $\mathrm{T}_0\mathrm{X}$ and any integer $m \leq \dim(\mathrm{L})$, let $\Sigma_{\mathrm{L},m}$ be the Schubert variety $\{\mathrm{M} \in \mathrm{G}(d, \mathrm{T}_0\mathrm{X}) \mid \dim(\mathrm{L} \cap \mathrm{M}) \geq m\}$; its codimension in $\mathrm{G}(d, \mathrm{T}_0\mathrm{X})$ is $m(\operatorname{codim}(\mathrm{L}) - d + m)$.

Proposition 3.2.– *Let X be an abelian variety and let V be an irreducible projective variety of dimension d with a morphism $f : \mathrm{V} \to \mathrm{X}$ such that $f(\mathrm{V})$ is k–geometrically non-degenerate. Let $\gamma : \mathrm{V}^0 \to \mathrm{G}(d, \mathrm{T}_0\mathrm{X})$ be the Gauss map, let L be a linear subspace of $\mathrm{T}_0\mathrm{X}$ and let m be an integer $\leq \dim(\mathrm{L})$. Any complete subvariety S of V^0 of dimension $\geq \operatorname{codim} \Sigma_{\mathrm{L},m} + (m-1)\big(\dim(\mathrm{L}) - m\big) + k$ meets $\gamma^{-1}(\Sigma_{\mathrm{L},m})$.*

Proof. Apply cor. 2.4. ■

The hypothesis could probably be weakened to $\dim(\mathrm{S}) \geq \operatorname{codim} \Sigma_{\mathrm{L},m} + k$ (see next proposition); the proposition gives that for $m = 1$ or $\dim(\mathrm{L})$. The corresponding Schubert varieties are $\Sigma_{\mathrm{L},1} = \{\mathrm{M} \in \mathrm{G}(d, \mathrm{T}_0\mathrm{X}) \mid \mathrm{L} \cap \mathrm{M} \neq 0\}$ and $\Sigma_{\mathrm{L},\dim(\mathrm{L})} = \{\mathrm{M} \in \mathrm{G}(d, \mathrm{T}_0\mathrm{X}) \mid \mathrm{L} \subset \mathrm{M}\}$.

More generally, a result of Fulton and Lazarsfeld imposes strong restrictions on the image of the Gauss map of smooth subvarieties with ample normal bundle which I believe should also hold for geometrically non-degenerate subvarieties.

Proposition 3.3.– *Let X be an abelian variety and let V be a smooth irreducible projective variety of dimension d with an unramified morphism $f : \mathrm{V} \to \mathrm{X}$ and Gauss map $\gamma : \mathrm{V} \to \mathrm{G}(d, \mathrm{T}_0\mathrm{X})$. Assume that the normal bundle to f is ample; any subvariety S of $\gamma(\mathrm{V})$ meets any subvariety of $\mathrm{G}(d, \mathrm{T}_0\mathrm{X})$ of codimension $\leq \dim(\mathrm{S})$.*

Proof. If Q is the universal quotient bundle on $\mathrm{G}(d, \mathrm{T}_0\mathrm{X})$, the pull-back $\gamma^*(\mathrm{Q})$ is isomorphic to the normal bundle $f^*\mathrm{TX}/\mathrm{TV}$, hence is ample. It follows from [FL2] that for each Schubert variety Σ_λ of codimension m in $\mathrm{G}(d, \mathrm{T}_0\mathrm{X})$ and each irreducible subvariety S of V of dimension m, one has $\int_{\mathrm{S}} \gamma^*[\Sigma_\lambda] > 0$. Now the class of any irreducible subvariety (not all zero) of the Schubert classes; this implies $\int_{\mathrm{S}} \gamma^*[\mathrm{Z}] > 0$, hence $\mathrm{S} \cap \gamma^{-1}(\mathrm{Z}) \neq \emptyset$. ■

Regarding the Gauss map of a smooth subvariety of an abelian variety, Sommese and Van de Ven also proved in [SV] a strong result for higher relative homotopy groups of pull-backs of *smooth* subvarieties of the Grassmannian.

4. A Barth-Lefschetz-type result

Sommese has obtained very complete results on the homotopy groups of *smooth* subvarieties of an abelian variety. For example, he proved in [S2] that if V is a smooth subvariety of dimension d of an abelian variety X, with k–ample normal bundle, $\pi_q(\mathrm{X},\mathrm{V}) = 0$ for $q \leq 2d - n - k + 1$. For arbitrary subvarieties, we have the following:

Theorem 4.1.– *Let* X *be an abelian variety and let* V *be a* k*–geometrically non-degenerate normal subvariety of* X *of dimension* $> \frac{1}{2}\big(\dim(\mathrm{X}) + k\big)$. *Then* $\pi_1^{\mathrm{alg}}(\mathrm{V}) \simeq \pi_1^{\mathrm{alg}}(\mathrm{X})$.

Proof. The case $k = 0$ is cor. 4.2 of [D1]. The general case is similar, since the hypothesis implies that the pair (V,V) satisfies condition $(*)$ of [D1]. ■

Going back to smooth subvarieties, I will give an elementary proof of (a slight improvement of) the cohomological version of Sommese's theorem, based on the following vanishing theorem ([LP]) and the ideas of [SZ].

Vanishing Theorem 4.2 (Le Potier, Sommese).– *Let* E *be a* k*–ample rank* r *vector bundle on a smooth irreducible projective variety* V *of dimension* d. *Then*

$$\mathrm{H}^q(\mathrm{V}, \mathrm{E}^* \otimes \Omega^p_{\mathrm{V}}) = 0 \qquad \textit{for} \qquad p + q \leq d - r - k\ .$$

Recall also the following elementary lemma from [SZ]:

Lemma 4.3.– *Let* $0 \to \mathrm{F} \to \mathrm{E}_0 \to \mathrm{E}_1 \to \ldots \to \mathrm{E}_k \to 0$ *be an exact sequence of sheaves on a scheme* V. *Assume* $\mathrm{H}^s(\mathrm{V},\mathrm{E}_i) = 0$ *for* $0 \leq i < k$ *and* $s \leq q$ *; then* $\mathrm{H}^q(\mathrm{V},\mathrm{F}) \simeq \mathrm{H}^{q-k}(\mathrm{V},\mathrm{E}_k)$.

Theorem 4.4.– *Let* V *be a smooth irreducible subvariety of dimension* d *of an abelian* n*–fold* X *and let* $\mathcal{L}$ *be a nef line bundle on* V. *Assume that the normal bundle* N *of* V *in* X *is a direct sum* $\oplus \mathrm{N}_i$, *where* N_i *is* k_i*–ample of rank* r_i. *For* $j > 0$,

$$\mathrm{H}^q(\mathrm{V}, \mathrm{S}^j\mathrm{N}^* \otimes \mathcal{L}^{-1}) = 0 \qquad \textit{for} \qquad q \leq d - \max(r_i + k_i)\ .$$

Proof. Since $\mathrm{S}^j\mathrm{N}^*$ is a direct summand of $\mathrm{S}^{j-1}\mathrm{N}^* \otimes \mathrm{N}^*$, it is enough to show, by induction on j, that $\mathrm{H}^q(\mathrm{V}, \mathrm{S}^j\mathrm{N}^* \otimes \mathrm{N}_i^* \otimes \mathcal{L}^{-1})$ vanishes for $j \geq 0$ and $q \leq d - r_i - k_i$. Since $\mathrm{N}_i \otimes \mathcal{L}$ is k_i–ample, the case $j = 0$ follows from Le Potier's theorem. For $j \geq 1$, tensor the exact sequence

$$0 \to \mathrm{S}^j\mathrm{N}^* \to \mathrm{S}^{j-1}\mathrm{N}^* \otimes \Omega^1_{\mathrm{X}|\mathrm{V}} \to \ldots \to \Omega^j_{\mathrm{X}|\mathrm{V}} \to \Omega^j_{\mathrm{V}} \to 0$$

by $\mathrm{N}_i^* \otimes \mathcal{L}^{-1}$. Since Ω^1_{X} is trivial, the induction hypothesis and the lemma give $\mathrm{H}^q(\mathrm{V}, \mathrm{S}^j\mathrm{N}^* \otimes \mathrm{N}_i^* \otimes \mathcal{L}^{-1}) \simeq \mathrm{H}^{q-j}(\mathrm{V}, \Omega^j_{\mathrm{V}} \otimes \mathrm{N}_i^* \otimes \mathcal{L}^{-1})$, and this group vanishes for $q \leq d - r_i - k_i$ by Le Potier's theorem. ■

THEOREM 4.5.– *Let* V *be a smooth irreducible subvariety of dimension* d *of an abelian* n*–fold* X. *Assume that its normal bundle a direct sum* $\oplus \mathrm{N}_i$, *where* N_i *is* k_i*–ample of rank* r_i. *Then*

a) $\mathrm{H}^q(\mathrm{X},\mathrm{V};\mathbf{C}) = 0$ *for* $q \leq d - \max(r_i + k_i) + 1$;

b) *for all nonzero elements* P *of* $\mathrm{Pic}^0(\mathrm{V})$, *the cohomology groups* $\mathrm{H}^q(\mathrm{V},\mathrm{P})$ *vanish for* $q \leq d - \max(r_i + k_i)$.

Remarks 4.6. 1) It is likely that a) should hold for cohomology with integral coefficients.

2) If the normal bundle is k–ample, we get $\mathrm{H}^q(\mathrm{X},\mathrm{V};\mathbf{C}) = 0$ for $q \leq 2d - n - k + 1$. If the normal bundle is a sum of ample line bundles, $\mathrm{H}^q(\mathrm{X},\mathrm{V};\mathbf{C}) = 0$ for $q \leq d$; in particular, the restriction $\mathrm{H}^0(\mathrm{X},\Omega^d_{\mathrm{X}}) \to \mathrm{H}^0(\mathrm{V},\Omega^d_{\mathrm{V}})$ is injective and V is non-degenerate in the sense of [R1], II, hence also geometrically non-degenerate.

3) By [GL], $\mathrm{H}^q(\mathrm{V},\mathrm{P}) = 0$ for P outside of a subset of codimension $\geq d - q$ of $\mathrm{Pic}^0(\mathrm{V})$. By [S], this subset is a union of translates of abelian subvarieties of X by torsion points.

Proof of the theorem. For a), it is enough by Hodge theory to study the maps

$$\mathrm{H}^i(\mathrm{X},\Omega^j_{\mathrm{X}}) \longrightarrow \mathrm{H}^i(\mathrm{V},\Omega^j_{\mathrm{X}|\mathrm{V}}) \xrightarrow{\psi} \mathrm{H}^i(\mathrm{V},\Omega^j_{\mathrm{V}}) .$$

Since Ω^j_{X} is trivial, we only need look at $\phi\colon \mathrm{H}^i(\mathrm{X},\mathcal{O}_{\mathrm{X}}) \to \mathrm{H}^i(\mathrm{V},\mathcal{O}_{\mathrm{V}})$ and ψ. We begin with ψ. We may assume $j > 0$. Let M_j be the kernel of the surjection $\Omega^j_{\mathrm{X}|\mathrm{V}} \to \Omega^j_{\mathrm{V}} \to 0$. The long exact sequence of th. 4.4 gives

$$0 \to \mathrm{S}^j\mathrm{N}^* \to \mathrm{S}^{j-1}\mathrm{N}^* \otimes \Omega^1_{\mathrm{X}|\mathrm{V}} \to \ldots \to \mathrm{N}^* \otimes \Omega^{j-1}_{\mathrm{X}|\mathrm{V}} \to \mathrm{M}_j \to 0 .$$

The lemma and the theorem then yield

$$\mathrm{H}^i(\mathrm{V},\mathrm{M}_j) \simeq \mathrm{H}^{i+j-1}(\mathrm{V},\mathrm{S}^j\mathrm{N}^*) = 0$$

for $i + j - 1 \leq d - \max(k_i + r_i)$, since $j > 0$. This implies that ψ has the required properties.

For $i = 0$, the map ψ is $\mathrm{H}^0(\mathrm{X},\Omega^j_{\mathrm{X}}) \to \mathrm{H}^0(\mathrm{V},\Omega^j_{\mathrm{V}})$. By Hodge symmetry, this proves that ϕ also has the required properties, hence the first point.

For b), we may assume $d - \max(k_i + r_i) \geq 1$, in which case the first point implies $\mathrm{Pic}^0(\mathrm{X}) \simeq \mathrm{Pic}^0(\mathrm{V})$. Let $\mathrm{P} \in \mathrm{Pic}^0(\mathrm{X})$ be nonzero; the same proof as above yields $\mathrm{H}^0(\mathrm{V},\Omega^q_{\mathrm{V}} \otimes \mathrm{P}_{|\mathrm{V}}) = 0$ for $q \leq d - \max(k_i + r_i)$. The theorem follows from the existence of an anti-linear isomorphism $\mathrm{H}^0(\mathrm{V},\Omega^q_{\mathrm{V}} \otimes \mathrm{P}_{|\mathrm{V}}) \simeq \mathrm{H}^q(\mathrm{V},\mathrm{P}^*_{|\mathrm{V}})$ ([GL]). ∎

abelian variety X, write C_d for the subvariety $\mathrm{C}+\cdots+\mathrm{C}$ (d times) of X. Recall that if C is general of genus n and $d<n$, the singular locus of $\mathrm{C}_d=\mathrm{W}_d(\mathrm{C})$ in the Jacobian JC has dimension $2d-n-2$.

Proposition 4.7.– *Let* X *be an abelian variety of dimension* n *and let* C *be a smooth irreducible curve in* X. *Assume that* C *generates* X *and that its Gauss map is birational onto its image. Then, for* $d<n$, *the singular locus of* C_d *has dimension* $\geq 2d-n-1$ *unless* X *is isomorphic to the Jacobian of* C *and* C *is canonically embedded in* X.

Proof. Let $\gamma:\mathrm{C}_{\mathrm{reg}}\to\mathbf{P}\mathrm{T}_0\mathrm{X}$ be the Gauss map and let $\pi:\mathrm{C}^{(d)}\to\mathrm{C}_d$ be the sum map. The image of the differential of π at the point $(c_1\cdot\ldots\cdot c_d)$ is the linear subspace of $\mathrm{T}_0\mathrm{X}$ generated by $\gamma(c_1),\ldots,\gamma(c_d)$. Since C generates X, the curve $\gamma(\mathrm{C})$ is non-degenerate; it follows that for $c_1,\ldots,c_d$ general, the points $\gamma(c_1),\ldots,\gamma(c_d)$ span a $(d-1)$–plane whose intersection with the curve $\gamma(\mathrm{C})$ consists only of these points. Thus π is birational. Moreover, if $x=c_1+\cdots+c_d$ is smooth on C_d, then $\gamma(c_i)\in\tau_x^*(\mathrm{T}_x\mathrm{C}_d)\cap\gamma(\mathrm{C})$ hence $\pi^{-1}(x)$ is finite. By Zariski's Main Theorem, π induces an isomorphism between $\pi^{-1}\big((\mathrm{C}_d)_{\mathrm{reg}}\big)$ and $(\mathrm{C}_d)_{\mathrm{reg}}$.

Let s be the dimension of the singular locus of C_d and assume $-1\leq s\leq 2d-n-2$. Let L be a very ample line bundle on X; the intersection W of C_d with $(s+1)$ general elements of $|\mathrm{L}|$ is smooth of dimension ≥ 2 and contained in $(\mathrm{C}_d)_{\mathrm{reg}}$. If H is a hyperplane in $\mathrm{T}_0\mathrm{X}$ and $x=c_1+\cdots+c_d\in\mathrm{W}$, the inclusion $\mathrm{T}_x\mathrm{C}_d\subset\mathrm{H}$ implies $\gamma(c_i)\in\mathbf{P}\mathrm{H}\cap\gamma(\mathrm{C})$; the restriction of $\mathrm{N}_{\mathrm{C}_d/\mathrm{X}}$ to W is ample by prop. 1.1. Since $\mathrm{N}_{\mathrm{W}/\mathrm{X}}$ is the direct sum of this restriction and of $(s+1)$ copies of L, the restriction $\mathrm{H}^1(\mathrm{X},\mathcal{O}_{\mathrm{X}})\to\mathrm{H}^1(\mathrm{W},\mathcal{O}_{\mathrm{W}})$ is bijective by th. 4.5. On the other hand, the line bundle $\pi^*\mathrm{L}$ is nef and big on $\mathrm{C}^{(d)}$, hence the Kawamata-Viehweg vanishing theorem ([K], [V]) implies

$$\mathrm{H}^1(\mathrm{C}^{(d)},\mathcal{O}_{\mathrm{C}^{(d)}})\subset\mathrm{H}^1(\pi^{-1}(\mathrm{W}),\mathcal{O}_{\pi^{-1}(\mathrm{W})})\simeq\mathrm{H}^1(\mathrm{W},\mathcal{O}_{\mathrm{W}})\ .$$

Since $\mathrm{H}^1(\mathrm{C},\mathcal{O}_{\mathrm{C}})\simeq\mathrm{H}^1(\mathrm{C}^{(d)},\mathcal{O}_{\mathrm{C}^{(d)}})$ ([M]), we get $h^1(\mathrm{C},\mathcal{O}_{\mathrm{C}})\leq h^1(\mathrm{X},\mathcal{O}_{\mathrm{X}})$ and there must be equality because C generates X. Thus, the inclusion $\mathrm{C}\subset\mathrm{X}$ factors through an isogeny $\phi:\mathrm{JC}\to\mathrm{X}$. Since π is birational, the inverse image $\phi^{-1}(\mathrm{C}_d)$ is the union of $\deg(\phi)$ translates of $\mathrm{W}_d(\mathrm{C})$. But any two translates of $\mathrm{W}_d(\mathrm{C})$ meet along a locus of dimension $\geq 2d-n>s$, hence ϕ is an isomorphism. ■

References

[A] Abramovich, D., *Subvarieties of semiabelian varieties*, Comp. Math. **90** (1994), 37–52.

[D1] Debarre, O., *Théorèmes de connexité et variétés abéliennes*, Am. J. of Math. **117** (1995), 1–19.

[D2] Debarre, O., *Sur les variétés abéliennes dont le diviseur thêta est singulier en codimension* 3, Duke Math. J. **56** (1988), 221–273.

[FH] Fulton, W., Hansen, J., *A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings*, Ann. of Math. **110** (1979), 159–166.

in *Algebraic Geometry*, Proceedings of the Midwest Algebraic Geometry Conference, Chicago 1980, Springer Lecture Notes **862**.

[FL2] Fulton, W., Lazarsfeld, R., *Positive polynomials for ample vector bundles*, Ann. of Math. **118** (1983), 35–60.

[GL] Green, M., Lazarsfeld, R., *Deformation Theory, Generic Vanishing Theorems and Some Conjectures of Enriques, Catanese and Beauville*, Invent. Math. **90** (1987), 389–407.

[H] Hartshorne, R., *Ample Vector Bundles on Curves*, Nagoya Math. J. **43** (1971), 73–89.

[K] Kawamata, Y., *A Generalisation of Kodaira-Ramanujan's Vanishing Theorem*, Math. Ann. **261** (1982), 43–46.

[LP] Le Potier, J., *Annulation de la cohomologie à valeurs dans un fibré vectoriel holomorphe positif de rang quelconque*, Math. Ann. **218** (1975), 35–53.

[M] Macdonald, I., *Symmetric Products of an Algebraic Curve*, Topology **1** (1962), 319–343.

[R1] Ran, Z., *On subvarieties of abelian varieties*, Invent. Math. **62** (1981), 459–479.

[R2] Ran, Z., *The structure of Gauss-like maps*, Comp. Math. **52** (1984), 171–177.

[SZ] Schneider, M., Zintl, J., *The theorem of Barth-Lefschetz as a consequence of Le Potier's vanishing theorem*, Manuscr. Math. **80** (1993), 259–263.

[S] Simpson, C., *Subspaces of Moduli Spaces of Rank One Local Systems*, Ann. Sc. Ecole Norm. Sup. **26** (1993), 361–401.

[S1] Sommese, A., *Submanifolds of Abelian Varieties*, Math. Ann. **233** (1978), 229–256.

[S2] Sommese, A., *Complex Subspaces of Homogeneous Complex Manifolds II. Homotopy Results*, Nagoya Math. J. **86** (1982), 101–129.

[SV] Sommese, A., Van de Ven, A., *Homotopy groups of pullbacks of varieties*, Nagoya Math. J. **102** (1986), 79–90.

[V] Viehweg, E., *Vanishing Theorems*, J. reine Angew. Math. **335** (1982), 1–8.

[Z1] Zak, F., *Structure of Gauss Maps*, Funct. Anal. Appl. **21** (1987), 32–41.

[Z2] Zak, F., *Tangents and Secants of Algebraic Varieties*, Transl. Math. Mono. **127**, A.M.S., Providence, 1993.

[Z3] Zak, F., *Theorems on Tangents to Subvarieties of Complex Tori*, Math. Notes Acad. Sc. USSR **43** (1988), 219–225.

Olivier Debarre
IRMA – Mathématique – CNRS URA 001
Université Louis Pasteur
7, rue René Descartes
67084 Strasbourg Cedex – France
e-mail: debarre@math.u-strasbg.fr

FULTON-HANSEN AND BARTH-LEFSCHETZ THEOREMS FOR SUBVARIETIES OF ABELIAN VARIETIES

Olivier Debarre

Abstract: We prove the following Fulton-Hansen type result for an irreducible subvariety V of an abelian variety X: the dimension of the "secant variety" of V along a subvariety S (defined as $V - S$), and that of its "tangential variety" along S (defined in the smooth case as the union of the projectivized tangent spaces to V at points of S, translated at the origin) differ by 1. Corollaries include a new proof of the finiteness of the Gauss map and an estimate on the ampleness of the normal bundle, for certain smooth subvarieties of X. We also prove, using ideas of Schneider and Zintl, a new Barth-Lefschetz theorem for smooth subvarieties of X. Let C be a smooth curve in X; we apply this result to give an estimate on the dimension of the singular locus of $C + \cdots + C$ in X.