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The aim of this article is to apply the Yau-Zaslow-Beauville method ([YZ], [B1]) to compute the Euler characteristic of the generalized Kummer varieties attached to a complex abelian surface (a calculation also done in [GS] by different methods). The argument is very geometric : given an ample line bundle L with $h^0(L) = n$ on an abelian surface A , such that each curve in $|L|$ is integral, we construct a projective symplectic $(2n - 2)$ -dimensional variety $J^d(A)$ with a Lagrangian fibration $J^d(A) \rightarrow |L|$ whose fiber over a point corresponding to a smooth curve C is the kernel of the Albanese map $J^d C \rightarrow A$. The Yau-Zaslow-Beauville method shows that the Euler characteristic of $J^d(A)$ is n times the number of genus 2 curves in $|L|$, to wit $n^2 \sigma(n)$ (where $\sigma(n) = \sum_{m|n} m$). The latter computation was also done in [G], where a general conjecture (proved in [BL1] for K3 surfaces and in [BL2] for abelian surfaces) is stated : if N_r^n is the number of genus $r + 2$ curves in $|L|$ passing through r general points of A , one should have

$$\sum_{n \in \mathbb{N}} N_r^n q^n = \left(\sum_{n \in \mathbb{N}} n \sigma(n) q^n \right)^r \left(\sum_{n \in \mathbb{N}} n^2 \sigma(n) q^n \right)$$

for L sufficiently ample (for example a sufficiently high power of an ample line bundle). It is remarkable that this identity should hold when L has type $(1, n)$ for any n ; it does not in general give the right number of genus 2 curves for other types (see remark 2.3).

Unlike the case of K3 surfaces, none of these varieties $J^d(A)$ seem to be birationally isomorphic to the generalized Kummer variety $K_{n-1}(A)$, a symplectic desingularization of a fiber of the sum morphism $A^{(n)} \rightarrow A$ introduced by Beauville in [B2]. However, using the Mukai-Fourier transform for sheaves on an abelian surface, a degeneration to the case when A is a product of elliptic curves and a result of Huybrechts, we prove that $K_{n-1}(A)$ and $J^{n-2}(A)$ are diffeomorphic, hence have the same Euler characteristic $n^3 \sigma(n)$. I would like to thank Huybrechts very much for his help with theorem 3.4.

1. The symplectic variety $J^d(A)$

Let A be a complex abelian surface with a polarization ℓ of type $(1, n)$. Assume that each curve with class ℓ is integral (this holds for generic (A, ℓ)). Let \hat{A} be the dual abelian surface. Let $\phi_\ell : A \rightarrow \hat{A}$ be the morphism associated with the polarization ℓ ; there exists a factorization $n \text{Id}_A : A \xrightarrow{\phi_\ell} \hat{A} \xrightarrow{\phi_{\hat{\ell}}} A$, where $\hat{\ell}$ is a polarization on \hat{A} of type $(1, n)$.

We denote by $\text{Pic}^\ell(A)$ the component of the Picard group of A corresponding to line bundles with class ℓ , by $\{\ell\}$ the component of the Hilbert scheme that parametrizes curves in A with class ℓ , by $\mathcal{C} \rightarrow \{\ell\}$ the universal family, and by $\bar{\mathcal{C}} \rightarrow \{\ell\}$ the compactified Picard scheme of this family ([AK]).

The variety $\bar{\mathcal{C}}$ splits as a disjoint union $\coprod_{d \in \mathbb{Z}} \bar{\mathcal{C}}^d$, where $\bar{\mathcal{C}}^d$ is a projective variety of dimension $2n + 2$, which parameterizes pairs (C, \mathcal{L}) where C is a curve on A with class ℓ and \mathcal{L} is a torsion free, rank 1 coherent sheaf on C of degree d (i.e. with

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$\chi(\mathcal{L}) = d + 1 - g(C) = d - n$). According to Mukai ([M1], ex. 0.5), $\bar{\mathcal{J}}^d\mathcal{C}$ can be viewed as a connected component of the moduli space of simple sheaves \mathcal{L} on A , and therefore is smooth, and admits a (holomorphic) symplectic structure. There is a natural morphism

$$\begin{array}{ccccc} \alpha : \bar{\mathcal{J}}^d\mathcal{C} & \longrightarrow & \{\ell\} & \longrightarrow & \text{Pic}^\ell(A) \\ (C, \mathcal{L}) & \longmapsto & C & \longmapsto & [\mathcal{O}_A(C)] \end{array}$$

also defined by $\alpha(\mathcal{L}) = \det \mathcal{L}$. For each smooth curve C in $\{\ell\}$, the inclusion $C \subset A$ induces an Abel-Jacobi map $J^d C \rightarrow A$; this defines a rational map

$$\beta : \bar{\mathcal{J}}^d\mathcal{C} \dashrightarrow A \times \{\ell\} \longrightarrow A$$

which is regular since A is an abelian variety and $\bar{\mathcal{J}}^d\mathcal{C}$ is normal. Let $J^d(A)$ be a fiber of the map $(\alpha, \beta) : \bar{\mathcal{J}}^d\mathcal{C} \rightarrow \text{Pic}^\ell(A) \times A$ (they are all isomorphic). Note that $J^d(A)$, $J^{d+2n}(A)$ and $J^{-d}(A)$ are isomorphic.

Proposition 1.1.— *The symplectic structure on $\bar{\mathcal{J}}^d\mathcal{C}$ induces a symplectic structure on the $(2n - 2)$ -dimensional variety $J^d(A)$.*

Proof. Recall that there is a canonical isomorphism $T_{\mathcal{L}}\bar{\mathcal{J}}^d\mathcal{C} \simeq \text{Ext}^1(\mathcal{L}, \mathcal{L})$, and that the symplectic form ω is the pairing

$$\text{Ext}^1(\mathcal{L}, \mathcal{L}) \otimes \text{Ext}^1(\mathcal{L}, \mathcal{L}) \rightarrow \text{Ext}^2(\mathcal{L}, \mathcal{L}) \xrightarrow{\text{Tr}} H^2(A, \mathcal{O}_A) \simeq \mathbf{C}$$

The map $T_{\mathcal{L}}\alpha$ is the trace map $T : \text{Ext}^1(\mathcal{L}, \mathcal{L}) \rightarrow H^1(A, \mathcal{O}_A)$, whereas the tangent map at the origin to the map $\iota : \text{Pic}^0(A) \rightarrow \bar{\mathcal{J}}^d\mathcal{C}$ defined by $\iota(P) = P \otimes \mathcal{L}$ is the dual $T^* : H^1(A, \mathcal{O}_A) \rightarrow \text{Ext}^1(\mathcal{L}, \mathcal{L})$. Since $\alpha\iota$ is constant, $T \circ T^* = 0$; in particular

$$\text{Ker } T \supset \text{Im } T^* = (\text{Ker } T)^\perp .$$

Note also that $\beta\iota\phi_\ell = n\text{Id}_A$ (use the Morikawa-Matsusaka endomorphism), hence $T_{\mathcal{L}}\beta \circ T^* = T\phi_\ell$ and $\text{Ker } T_{\mathcal{L}}\beta \cap \text{Im } T^* = \{0\}$. Since both $\text{Ker } T$ and $\text{Ker } T_{\mathcal{L}}\beta$ have codimension 2, this implies

$$\text{Ker } T = (\text{Ker } T \cap \text{Ker } T_{\mathcal{L}}\beta) \oplus (\text{Ker } T)^\perp ,$$

and the restriction of ω to $\text{Ker } T \cap \text{Ker } T_{\mathcal{L}}\beta = T_{\mathcal{L}}J^d(A)$ is non-degenerate. ■

The map α restricts to a morphism $\alpha : J^d(A) \rightarrow |\mathbf{L}|$ whose fiber $K^d(C)$ over the point corresponding to a smooth curve C is the (connected) kernel of the Abel-Jacobi map $\beta : J^d C \rightarrow A$; it is a Lagrangian fibration.

2. The Euler characteristic of $J^d(A)$

We calculate the Euler characteristic of $J^d(A)$ by using the Lagrangian fibration $\alpha : J^d(A) \rightarrow |\mathbf{L}|$, as in [B1].

Proposition 2.1.— *Let C be an integral element of $|\mathbf{L}|$. The Euler characteristic of $K^d(C)$ is n if the normalization of C has genus 2, and 0 otherwise.*

Proof. Let $\eta : \tilde{C} \rightarrow C$ be the normalization. There is a commutative diagram (as in §2 of [B1], we may restrict ourselves to the case $d = 0$ and drop the superscript d)

$$\begin{array}{ccccccc}
K(C) & \longrightarrow & \bar{J}C & \xrightarrow{\beta} & A & & \\
& & \cup & & \uparrow \pi & & \\
& & JC & \xrightarrow{\eta^*} & J\tilde{C} & \longrightarrow & 0 \\
& & \cup & & \uparrow & & \\
(\eta^*)^{-1}(\text{Ker } \pi) & \longrightarrow & & & \text{Ker } \pi & \longrightarrow & 0
\end{array}$$

By lemma 2.1 of *loc.cit.*, the group JC acts freely on $\bar{J}C$. Note also that for M in JC and \mathcal{L} in $\bar{J}C$,

$$\beta(M \otimes \mathcal{L}) = \beta(M) + \beta(\mathcal{L})$$

because this is true when \mathcal{L} is invertible, and JC is dense in $\bar{J}C$. It follows that $(\eta^*)^{-1}(\text{Ker } \pi)$ acts (freely) on $K(C)$. As in prop. 2.2 of *loc.cit.*, it follows that $e(K(C)) = 0$ if $\text{Ker } \pi$ is infinite, that is if $g(\tilde{C}) > 2$.

Assume now that \tilde{C} has genus 2. The situation here is much simpler than in *loc.cit.*, because the normalization η of C is *unramified* : it is the restriction to \tilde{C} of the isogeny $\pi : J\tilde{C} \rightarrow A$. If $\check{C} \rightarrow C$ is the minimal unibranch partial normalization (*cf. loc.cit.*), it follows that $\tilde{C} \rightarrow \check{C}$ is an unramified homeomorphism, hence an isomorphism ([Gr], 18.12.6).

There is a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Ker } \pi & \longrightarrow & J\tilde{C} & \xrightarrow{\pi} & A \\
& & \cap & & \cap \eta_* & & \parallel \\
0 & \longrightarrow & K(C) & \longrightarrow & \bar{J}C & \xrightarrow{\beta} & A
\end{array}$$

and an exact sequence

$$1 \rightarrow \mathcal{O}_{\tilde{C}}^*/\mathcal{O}_C^* \rightarrow JC \rightarrow J\tilde{C} \rightarrow 0 .$$

If one chooses a line bundle M on C corresponding to a point of $\mathcal{O}_{\tilde{C}}^*/\mathcal{O}_C^*$ as in the proof of prop. 3.3 of *loc.cit.*, it acts on $\bar{J}C$, hence on $K(C)$. Beauville's reasoning proves that M acts *freely* on the complement of $\eta_*J\tilde{C}$ in $\bar{J}C$, hence also on the complement of $\text{Ker } \pi$ in $K(C)$. It follows that $e(K(C)) = e(\text{Ker } \pi) = n$. ■

As a corollary, we get, assuming that each curve with class ℓ is integral,

$$e(J^d(A)) = n \cdot \#\{ C \in |L| \mid g(\tilde{C}) = 2 \} .$$

It remains to count the number of (integral) genus 2 curves in $|L|$ or, which amounts to the same, the number of morphisms $f : \tilde{C} \rightarrow A$ such that $f_*\tilde{C} \in |L|$, where \tilde{C} is a smooth curve of genus 2, modulo automorphisms of \tilde{C} . Let us do this calculation for any polarization ℓ of degree n on a simple abelian surface A . To f one associates an isogeny $\pi : J\tilde{C} \rightarrow A$ such that $\pi_*\theta = \ell$ (where θ is the principal polarization on $J\tilde{C}$), hence $\pi^*\ell = r\theta$, where r

The number of isomorphism classes of isogenies $\pi : \tilde{A} \rightarrow A$ as above is also the number $N(\ell)$ of isomorphism classes of isogenies $\hat{\pi} : \hat{A} \rightarrow \tilde{A}$, where $\hat{\pi}^* \theta = \hat{\ell}$, hence also the number of subgroups G of $\text{Ker } \phi_{\hat{\ell}}$ that are maximal totally isotropic for the Weil form. This kernel is (non-canonically) isomorphic to $H \times H^*$, where H is an abelian group of cardinal n and the Weil form is given by $\epsilon((x, x^*), (y, y^*)) = y^*(x) \cdot x^*(y)^{-1}$. If $K = p_1(G) \subset H$ and $K' = G \cap (\{0\} \times H^*)$, there exists a group homomorphism $u : K \rightarrow H^*/K'$ such that

$$G = \{ (x, x^*) \in K \times H^* \mid u(x) = \overline{x^*} \}.$$

The fact that G is totally isotropic of cardinal n yields $K' = K^\perp$ and $u : K \rightarrow K^*$ *symmetric*. Note that the latter condition is empty if K is cyclic.

When ℓ has type $(1, n)$, the group H is cyclic and

$$N(\ell) = \sum_{K < \mathbf{Z}/n\mathbf{Z}} \# \text{Hom}(K, K^*) = \sum_{m|n} m = \sigma(n).$$

Corollary 2.2.— *Assume each curve with class ℓ is integral; then $e(J^d(A)) = n^3 \sigma(n)$.*

Remark 2.3. Suppose ℓ is p times a principal polarization (p prime), so that $H \simeq (\mathbf{Z}/p\mathbf{Z})^2$ and $\ell^2/2 = p^2$. The subgroups K are $(\mathbf{Z}/p\mathbf{Z})$ -vector subspaces of H , hence

$$N(\ell) = \sum_{K < H} \# \text{Hom}^{\text{sym}}(K, K^*) = \sum_{e=0}^2 \sum_{K, \dim K=e} p^{\epsilon(e+1)/2} = 1 + p(p+1) + p^3 \neq \sigma(p^2).$$

In this case, the formula given in the introduction does *not* give the right number of genus 2 curves in $|\mathbf{L}|$ (note that these curves are all integral and are interchanged by monodromy). In [V], Vainsencher defines a scheme Σ whose length, when finite, should count the number of nodes of curves in $|\mathbf{L}|$ with $p^2 - 1$ nodes (hence of genus 2); here however, Σ is not finite because $|\mathbf{L}|$ contains non-reduced curves.

3. A degeneration of $J^{n-2}(A)$

Our aim is to relate the symplectic variety $J^d(A)$ constructed above with the generalized Kummer variety $K_{n-1}(A)$. Contrary to the case of K3 surfaces, these varieties do not seem to be birational for general A (except when $n = 2$). Using the Mukai-Fourier transform, we will relate $J^d(A)$ with a moduli space of rank $(n - d)$ -sheaves on the dual surface \hat{A} , then degenerate the situation to the case where A is a product of elliptic curves and $d = n - 2$, where one can prove that this moduli space is birational to $K_{n-1}(A)$.

For any sheaf F on A , we denote by $\mathcal{F}^\bullet F$ the cohomology sheaves of the Mukai-Fourier transform of F (see [M2]). If only $\mathcal{F}^j F$ is non-zero, we say that F has weak index j , and we write $\hat{F} = \mathcal{F}^j F$; in that case, \hat{F} has weak index $2 - j$, and $\hat{\mathcal{F}}\hat{F} \simeq (-1)^* F$ (*loc.cit.*, cor. 2.4). If $H^i(A, F \otimes P_{\hat{x}}) = 0$ for all $\hat{x} \in \hat{A}$ and all $i \neq j$, we say that F has index j ; it implies that F has weak index j .

For any \hat{x} in \hat{A} , we denote by $P_{\hat{x}}$ the corresponding line bundle on A ; we identify the dual of \hat{A} with A , so that, for any x in A , P_x is a line bundle on \hat{A} .

Let C be a generic curve in $\{\ell\}$; the translate of the surface $\text{Pic}^0(A)$ by a generic point \mathcal{L} in $J^d C$ does not meet the d -dimensional subvariety $W_d(C)$ of $J^d C$, as soon as $g(C) > 2 + d$, i.e. $d < n - 1$. In that case, one has $H^0(A, \mathcal{L} \otimes P_{\hat{x}}) = 0$ for all \hat{x} in \hat{A} , so that \mathcal{L} has index 1, and $\hat{\mathcal{L}}$ is a locally free simple sheaf on \hat{A} of rank $n - d$, first Chern class $\hat{\ell}$ and Euler characteristic 0.

Proposition 3.1.— *Let (A, ℓ) be a polarized abelian surface of type $(1, n)$ whose Néron-Severi group is generated by ℓ . For $d < n - 1$ and \mathcal{L} generic in $\bar{\mathcal{J}}^d \mathcal{C}$, the vector bundle $\hat{\mathcal{L}}$ on \hat{A} is $\hat{\ell}$ -stable.*

Proof. We follow [FL] : assume $\hat{\mathcal{L}}$ is not stable, and look at torsion-free non-zero quotients of $\hat{\mathcal{L}}$ of smallest degree, and among those, pick one, Q , of smallest rank. Because $\text{NS}(\hat{A}) = \mathbf{Z}\hat{\ell}$, the degree of Q is non-positive. The proofs of lemmes 2 and 3 of [FL] apply without change : Q has index 1 and if K be the kernel of $\hat{\mathcal{L}} \rightarrow Q$, the sheaf $\mathcal{F}^2 K$ has finite support. Consider the exact sequence

$$0 \rightarrow \mathcal{F}^1 K \rightarrow (-1)^* \mathcal{L} \rightarrow \hat{Q} \rightarrow \mathcal{F}^2 K \rightarrow 0 ;$$

since $c_1(\hat{Q}) \cdot \ell = c_1(Q) \cdot \hat{\ell} \leq 0$ ([FL], lemme 1), the torsion sheaf \hat{Q} has finite support, hence index 0. But this index is also $2 - \text{ind } Q = 1$; this contradiction proves the proposition. ■

For each $d < n - 1$, we have constructed a birational rational map between $\bar{\mathcal{J}}^d \mathcal{C}$ and an irreducible component $\mathcal{M}_{\hat{A}}^0(n - d, \hat{\ell}, 0)$ of the moduli space $\mathcal{M}_{\hat{A}}(n - d, \hat{\ell}, 0)$ of $\hat{\ell}$ -semi-stable sheaves on \hat{A} of rank $n - d$, first Chern class $\hat{\ell}$ and Euler characteristic 0. This map is a morphism if $d < 0$. Let us interpret the maps α and β in this context. Let (C, \mathcal{L}) be a pair corresponding to a point of $\bar{\mathcal{J}}^d \mathcal{C}$; it follows from the exact sequence $0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_C(x) \rightarrow \mathbf{C}_x \rightarrow 0$ that $\det \widehat{\mathcal{O}_C(x)} \simeq \det \widehat{\mathcal{O}_C} \otimes \mathbf{P}_{-x}$, hence

$$\det \hat{\mathcal{L}} \simeq \det \widehat{\mathcal{O}_C} \otimes \mathbf{P}_{-\beta(\mathcal{L})} \simeq \det \widehat{\mathcal{O}_A(-C)} \otimes \mathbf{P}_{-\beta(\mathcal{L})} .$$

Hence, the fibers of (α, β) are also the fibers of the map $\bar{\mathcal{J}}^d \mathcal{C} \rightarrow \text{Pic}^\ell(A) \times \text{Pic}^{-\ell}(\hat{A})$ which sends \mathcal{L} to $(\det \mathcal{L}, \det \mathcal{F}^\bullet \mathcal{L})$. Let $\Omega : \mathcal{M}_{\hat{A}}^0(n - d, \hat{\ell}, 0) \rightarrow \text{Pic}^{-\ell}(A) \times \text{Pic}^{\hat{\ell}}(\hat{A})$ be the map $E \mapsto (\det \mathcal{F}^\bullet E, \det E)$, and let $M_{n-d}(\hat{A})$ be a fiber. We have proved the following.

Proposition 3.2.— *Let (A, ℓ) be a polarized abelian surface of type $(1, n)$ whose Néron-Severi group is generated by ℓ . For $d < n - 1$, the Fourier-Mukai transform induces a birational isomorphism between $\bar{\mathcal{J}}^d \mathcal{C}$ and an irreducible component of $\mathcal{M}_{\hat{A}}(n - d, \hat{\ell}, 0)$ which sends $J^d(A)$ onto $M_{n-d}(\hat{A})$.*

We will now study the case where $n - d = 2$ and A is the product of two general elliptic curves F and G , with ℓ of bidegree $(1, n)$. One has $\hat{A} = \hat{F} \times \hat{G}$, and $\hat{\ell}$ has bidegree $(n, 1)$. To avoid non-stable semi-stable sheaves, we will study the moduli space $\mathcal{M}'_{\hat{A}}$ of rank 2 sheaves on \hat{A} with first Chern class $\hat{\ell}$ and Euler characteristic 0 which are semi-stable for the polarization $\hat{\ell}'$ of bidegree $(n + 1, 1)$, and call $M'(\hat{A})$ a fiber of the map $\Omega : \mathcal{M}'_{\hat{A}} \rightarrow \text{Pic}^{-\ell}(A) \times \text{Pic}^{\hat{\ell}}(\hat{A})$ defined above.

Proposition 3.3.— *In the above situation where $A = F \times G$, the moduli space $\mathcal{M}'_{\hat{A}}$ is smooth and birational to $\hat{A}^{(n)} \times A$, and the variety $M'(\hat{A})$ is smooth and birational to $K_{n-1}(\hat{A})$.*

Proof. Let E be an $\hat{\ell}'$ -semi-stable rank 2 torsion free sheaf on \hat{A} with first Chern class $\hat{\ell}$ and Euler characteristic 0. Let $x \in A$; by semi-stability of E^* , one has $H^2(\hat{A}, E \otimes \mathbf{P}_x^{-1}) = 0$, hence $h^0(\hat{A}, E \otimes \mathbf{P}_x^{-1}) = h^1(\hat{A}, E \otimes \mathbf{P}_x^{-1})$. Since $\hat{\mathcal{A}}E$ is non-zero, for at least one x , these numbers are non-zero and there is an inclusion $\mathbf{P}_x \hookrightarrow E$; let K be the kernel of $E \rightarrow E/\mathbf{P}_x \rightarrow (E/\mathbf{P}_x)/(E/\mathbf{P}_x)_{\text{tors}}$. There is an exact sequence

$$(*) \quad 0 \rightarrow K \rightarrow E \rightarrow \mathcal{I}_Z \otimes K' \rightarrow 0 ,$$

where K' is a line bundle. The line bundle K has bidegree (a, b) , with a and b non-negative and $b(n+1) + a \leq (2n+1)/2$ (by $\hat{\ell}'$ -semi-stability); hence $b = 0$ and Z is a subscheme of \hat{A} of length $n - a$.

Set $M = K' \otimes K^{-1}$. By Serre duality, $\text{Ext}_{\hat{A}}^1(\mathcal{I}_Z \otimes K', K)$ and $H^1(\hat{A}, \mathcal{I}_Z \otimes M)^*$ are isomorphic. Assume $H^0(\hat{A}, \mathcal{I}_Z \otimes M) = 0$; one has

$$h^1(\hat{A}, \mathcal{I}_Z \otimes M) = \text{length}(Z) - \chi(\hat{A}, M) = a$$

and $a > 0$ (otherwise $\mathcal{I}_Z \otimes K'$ would be a subsheaf of E with $\hat{\ell}'$ -slope $2n+1$), and E depends on at most $2n+3-a$ parameters (2 for K , 2 for K' , $2(n-a)$ for Z and $a-1$ for the extension). Since each component of $\mathcal{M}'_{\hat{A}}$ has dimension $2n+2$, this forces $a = 1$ for E generic. Let \mathcal{M}^0 be the subset of $\mathcal{M}'_{\hat{A}}$ parametrized in this fashion.

Assume now $H^0(\hat{A}, \mathcal{I}_Z \otimes M) \neq 0$; one checks (by projecting onto $|M|$), that the set of pairs (Z, D) with $D \in |M|$ and $Z \subset D$, has dimension $\leq n - 2a - 1 + n - a$. Hence E depends on at most $2n - 3a - 1 - \chi(\hat{A}, \mathcal{I}_Z \otimes M) + 4 = 2n - 2a + 3$ parameters. For E generic, this forces $a = 0$, Z reduced and $h^0(\hat{A}, \mathcal{I}_Z \otimes M) = 1$. This yields a component of $\mathcal{M}'_{\hat{A}}$ which can be parametrized as follows. Let $Z = (\hat{f}_1, \hat{g}_1) + \dots + (\hat{f}_n, \hat{g}_n)$ be generic in $\hat{A}^{(n)}$, set $L = \mathcal{O}_{\hat{F}}(\hat{f}_1 + \dots + \hat{f}_n)$, and let $f \in F$ and $\hat{g} \in \hat{G}$. The vector space $\text{Ext}_{\hat{A}}^1(\mathcal{I}_Z \otimes p_{\hat{F}}^* L \otimes p_{\hat{G}}^* \mathcal{O}_{\hat{G}}(\hat{g}), \mathcal{O}_{\hat{A}})$ has dimension 1, hence there is a unique extension

$$0 \rightarrow p_{\hat{F}}^* P_f \rightarrow E \rightarrow \mathcal{I}_Z \otimes p_{\hat{F}}^*(L \otimes P_f) \otimes p_{\hat{G}}^* \mathcal{O}_{\hat{G}}(\hat{g}) \rightarrow 0,$$

where E is locally free (it satisfies the Cayley-Bacharach condition; see for example th. 5.1.1 of [HL]) and stable (the only thing to check is $H^0(\hat{A}, E \otimes p_{\hat{F}}^* P_{-f} \otimes p_{\hat{G}}^* \mathcal{O}_{\hat{G}}(-\hat{g})) = 0$, and this is true because the extension is non-trivial). This yields a rational map

$$\phi : \hat{A}^{(n)} \times F \times \hat{G} \dashrightarrow \mathcal{M}'_{\hat{A}}$$

which is birational onto its image: given a locally free E as above, one recovers f and the $(\hat{g}_i - \hat{g})$'s by noting that the set $C_E = \{x \in A \mid H^0(\hat{A}, E \otimes P_x) \neq 0\}$ is

$$(\{-f\} \times G) \cup \bigcup_{i=1}^n (F \times \{[\mathcal{O}_{\hat{G}}(\hat{g}_i - \hat{g})]\}),$$

the \hat{f}_i 's because $\text{Ext}_{\hat{A}}^1(\mathcal{I}_Z \otimes p_{\hat{F}}^* L \otimes p_{\hat{G}}^* \mathcal{O}_{\hat{G}}(\hat{g}), \mathcal{O}_{\hat{A}})$ must be non-zero, and \hat{g} by noting that $\det E \simeq p_{\hat{F}}^*(L \otimes P_{2f}) \otimes p_{\hat{G}}^* \mathcal{O}_{\hat{G}}(\hat{g})$. Because $H^0(\hat{A}, E \otimes p_{\hat{F}}^*(P_{-f} \otimes \mathcal{O}_{\hat{F}}(-f_1)) \otimes p_{\hat{G}}^* \mathcal{O}_{\hat{G}}(\hat{g}_1 - \hat{g}))$ is non-zero, there exists an exact sequence $(*)$ with K of bidegree $(1, 0)$. This proves that the set \mathcal{M}^0 defined above is contained in the image of ϕ , which must therefore be $\mathcal{M}'_{\hat{A}}$.

Finally, E has weak index 1, $\mathcal{F}^1 E$ has support on C_E , and fixing $\det \mathcal{F}^1 E$ amounts to fixing $[\mathcal{O}_A(C_E)]$. It follows that taking a fiber of Ω amounts to fixing f , $\sum(\hat{g}_i - \hat{g})$, $\sum \hat{f}_i$ and \hat{g} ; hence $M'(\hat{A})$ is birational to $K_{n-1}(\hat{A})$. ■

The following proof is due to D. Huybrechts, and uses ideas from prop. 2.2 of [GH].

Theorem 3.4.— *Let (A, ℓ) be a polarized abelian surface of type $(1, n)$ whose Néron-Severi group is generated by ℓ . The symplectic varieties $J^{n-2}(A)$, $M_2(\hat{A})$ and $K_{n-1}(\hat{A})$ are deformation equivalent. In particular, they are all irreducible symplectic.*

Proof. Let $f : \hat{\mathcal{A}} \rightarrow \mathcal{S}$ be a family of polarized abelian surfaces, where \mathcal{S} is smooth connected quasi-projective, with a relative polarization $\hat{\mathcal{L}}$ of type $(1, n)$, such that the fiber over a point $0 \in \mathcal{S}$ is $\hat{\mathbb{F}} \times \hat{\mathbb{G}}$ with a polarization of bidegree $(n, 1)$, and such that the Néron-Severi group of a very general fiber of f has rank 1. Let $g : \mathcal{M} \rightarrow \mathcal{S}$ be the (projective) relative moduli space of $\hat{\mathcal{L}}$ -semi-stable sheaves of rank 2 with first Chern class $\hat{\ell}$ and Euler characteristic 0 on the fibers of f (cf. [HL], th. 4.3.7, p. 92).

Lemma 3.5.— *Under the hypothesis of the theorem, any rank 2 torsion free sheaf on $\hat{\mathcal{A}}$ with first Chern class $\hat{\ell}$ which is either simple or semi-stable is stable.*

Proof. Assume that a rank 2 torsion free sheaf E on $\hat{\mathcal{A}}$ with first Chern class $\hat{\ell}$ is not stable. There exists an exact sequence

$$0 \rightarrow K \rightarrow E \rightarrow \mathcal{I}_Z \otimes K' \rightarrow 0 ,$$

where K and K' are line bundles on $\hat{\mathcal{A}}$ with $c_1(K) = k\hat{\ell}$, $c_1(K') = (1 - k)\hat{\ell}$ and $k > 0$. This proves that E is not semi-stable; moreover, $K \otimes K'^{-1}$ is ample, hence there exists a non-zero morphism $u : K' \rightarrow K$, which induces an endomorphism $E \rightarrow \mathcal{I}_Z \otimes K' \xrightarrow{u} K \rightarrow E$ which is not a homothety, and E is not simple. ■

By the lemma, the (closed) locus of non-stable points in \mathcal{M} does not project onto \mathcal{S} . By replacing \mathcal{S} with an open subset, we may assume that there are no such points. Let now $\mathcal{S} \rightarrow \mathcal{S}$ be the (smooth) relative moduli space of simple sheaves on the fibers of f (see [AK]). There are embeddings $\mathcal{M} \subset \mathcal{S}$ and $\mathcal{M}'_{\hat{\mathbb{F}} \times \hat{\mathbb{G}}} \subset \mathcal{S}_0$ as closed and open subsets. Let $\mathcal{S}' = \mathcal{S} - (\mathcal{S}_0 - \mathcal{M}'_{\hat{\mathbb{F}} \times \hat{\mathbb{G}}})$; it is open in \mathcal{S} , hence smooth over \mathcal{S} . Let \mathcal{M}' be the closure of $g^{-1}(\mathcal{S} - \{0\})$ in \mathcal{S}' ; the fibers of $g' : \mathcal{M}' \rightarrow \mathcal{S}$ are projective away from 0, and contained in $\mathcal{M}'_{\hat{\mathbb{F}} \times \hat{\mathbb{G}}}$ over 0. Norton's criterion ([N]) shows that points in \mathcal{M}'_0 are separated in the moduli space of simple sheaves on $\hat{\mathcal{A}}$ (because they are stable), hence in \mathcal{S} ; therefore, \mathcal{M}' is separated. By semi-continuity, \mathcal{M}'_0 is a closed subset of $\mathcal{M}'_{\hat{\mathbb{F}} \times \hat{\mathbb{G}}}$ of the same dimension, hence they are equal. Using the lemma, we get, after shrinking \mathcal{S} again, a proper family $g' : \mathcal{M}' \rightarrow \mathcal{S}$ with projective irreducible smooth fibers which coincide with $g : \mathcal{M} \rightarrow \mathcal{S}$ away from 0.

For A general, $J^{n-2}(A)$ is birationally isomorphic to $M_2(\hat{A})$ by prop. 3.2, and we just saw that the latter deforms to $M'(\hat{\mathbb{F}} \times \hat{\mathbb{G}})$, itself birationally isomorphic to $K_{n-1}(\hat{\mathbb{F}} \times \hat{\mathbb{G}})$ by prop. 3.3; in particular, the symplectic varieties are all irreducible symplectic. Since birationally isomorphic smooth projective irreducible symplectic varieties are deformation equivalent ([H], th. 10.12), the theorem is proved. ■

Corollary 3.6.— *Let (A, ℓ) be a general polarized abelian surface of type $(1, n)$. The moduli space $\mathcal{M}_A(2, \ell, 0)$ is smooth irreducible.*

Corollary ([GS] 3.7.— *Let A be an abelian surface. The Euler characteristic of $K_{n-1}(A)$ is $n^3 \sigma(n)$.*

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