SESHADRI CONSTANTS OF ABELIAN VARIETIES*

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February 27, 2003

This note grew out of discussions with Lazarsfeld on the possibility of characterizing Jacobians among all principally polarized abelian varieties by means of their Seshadri constant.

Buser and Sarnak define in [BuS] a metric invariant associated with a principally polarized abelian variety and prove that it takes much smaller values on Jacobians than it does on general principally polarized abelian varieties. In [L], Lazarsfeld relates this invariant with the Seshadri constant and deduces that the latter is again smaller for Jacobians than for *general* principally polarized abelian varieties.

Unfortunately, the picture is not as pretty as one could have hoped. There are in each dimension at least 4 indecomposable principally polarized abelian varieties that are not Jacobians but have Seshadri constant 2 (Example 3(3)), whereas general Jacobians of dimension n have Seshadri constants that grow at least like $\log n$.

However, as Lazarsfeld pointed out, it follows from a celebrated conjecture of van Geemen and van der Geer (see Remark 2) that indecomposable principally polarized abelian varieties with Seshadri constant < 2 should be hyperelliptic Jacobians.

We were unable to settle this question in general. Nevertheless, in dimension 4, using the fact that the above mentioned conjecture is known ([I]), we

*This is not the text of the talk given at the Fano Conference, which was entitled "Fano varieties and polytopes" and can be found at `http://irmasrv1.u-strasbg.fr/~debarre/turin.ps`, or, in an expanded form, as [D1].

give an almost complete picture of the situation (the dimension 2 case was done in [S] and the appendix of [B2]; the dimension 3 case in [BS]).

Another question that we leave unsettled is the determination of the rate of growth of the maximal value of Seshadri constants of Jacobians (see Section 3).

1 Definitions and bounds

Let X be a smooth projective variety of dimension $n \geq 2$ and let D be a nef divisor on X. For any curve Γ on X and any point x on Γ, we set, following [De],

$$\begin{aligned} \varepsilon(X, D, \Gamma, x) &= \frac{D \cdot \Gamma}{\operatorname{mult}_x \Gamma} \\ \varepsilon(X, D, x) &= \inf_{x \in \Gamma} \varepsilon(X, D, \Gamma, x) \end{aligned}$$

and we define the *Seshadri constant* of the pair (X, D) by

$$\varepsilon(X, D) = \inf_{x \in X} \varepsilon(X, D, x)$$

Of course, this number only depends on the numerical equivalence class d of D, so we may also use the notation $\varepsilon(X, d)$.

The Seshadri criterion states

$$d \text{ ample } \Leftrightarrow \varepsilon(X, d) > 0$$

One has $\varepsilon(X, md, x) = m\varepsilon(X, d, x)$. If $\alpha_x : \widetilde{X} \to X$ is the blow up of x, with e the class of the exceptional divisor E, we have

$$\varepsilon(X, d, x) = \max\{\varepsilon \geq 0 \mid \alpha_x^* d - \varepsilon e \text{ nef}\}$$

If Y is a smooth subvariety of X passing through x, we have

$$\varepsilon(X, d, x) \leq \varepsilon(Y, d|_Y, x) \tag{1}$$

Let Y be an irreducible subvariety of X of dimension $r > 0$ and let $\widetilde{Y}$ be its strict transform on $\widetilde{X}$. Since the $\mathbf{R}$-divisor class $\alpha_x^* d - \varepsilon(X, d, x)e$ is nef, one has

$$0 \leq (\alpha_x^* d - \varepsilon(X, d, x)e)^r \cdot \widetilde{Y} = (\alpha_x^* d^r + \varepsilon(X, d, x)^r (-e)^r) \cdot \widetilde{Y}$$

hence

$$\varepsilon(X,d,x) \leq \sqrt[r]{\frac{d^r \cdot Y}{\operatorname{mult}_x Y}} \tag{2}$$

We define $\varepsilon(X,d,Y,x)$ to be the right-hand side of this inequality. Taking $Y = X$, we get

$$\varepsilon(X,d,x) \leq \sqrt[n]{d^n} \tag{3}$$

The Nakai-Moishezon criterion for $\mathbf{R}$-divisors proved in [CP] implies that there exists an irreducible subvariety Y of X for which there is equality in (2). We say that Y *computes the Seshadri constant of* (X,d,x). If moreover Y is a subvariety of X of smallest dimension with this property, we say that Y *minimally computes the Seshadri constant of* (X,d,x).

For any positive integers m and k, set

$$\{md\}_{x,k} = \{mD\}_{x,k} = \{M \sim mD \mid \operatorname{mult}_x M \geq k\} \tag{4}$$

and define $|mD|_{x,k}$ similarly, replacing numerical equivalence $\sim$ with linear equivalence $\equiv$.

Lemma 1 *Let Y be an irreducible subvariety of X of dimension r. If*

- *either $r = 1$ and $\varepsilon(X,d,Y,x) < \frac{k}{m}$,*
- *or $r \geq 2$, the subvariety Y minimally computes the Seshadri constant of (X,d,x), and $\varepsilon(X,d,x) \leq \frac{k}{m}$,*

the subvariety Y is contained in the base locus of $\{md\}_{x,k}$.

PROOF. We keep the same notation as above. Let M be an element of $\{md\}_{x,k}$. The divisor $\widetilde{M} = \alpha_x^* M - kE$ is effective.

If $r = 1$ and $\varepsilon(X,d,Y,x) < \frac{k}{m}$, we have

$$\widetilde{M} \cdot \widetilde{Y} = mD \cdot Y - k \operatorname{mult}_x Y < 0$$

hence Y is contained in M.

Assume $r \geq 2$ and set $\varepsilon = \varepsilon(X,d,x)$. We have

$$\begin{aligned}
(\alpha_x^* D - \varepsilon E)^{r-1} \cdot \widetilde{M} \cdot \widetilde{Y} &= (\alpha_x^* D - \varepsilon E)^{r-1} \cdot (\alpha_x^* M - kE) \cdot \widetilde{Y} \\
&= \frac{k}{\varepsilon}(\alpha_x^* D - \varepsilon E)^{r-1} \cdot (\frac{m\varepsilon}{k}\alpha_x^* D - \varepsilon E) \cdot \widetilde{Y} \\
&= \frac{k}{\varepsilon}(\alpha_x^* D - \varepsilon E)^{r} \cdot \widetilde{Y} \\
&\quad + \frac{k}{\varepsilon}(\frac{m\varepsilon}{k} - 1)(\alpha_x^* D - \varepsilon E)^{r-1} \cdot \alpha_x^* D \cdot \widetilde{Y}
\end{aligned}$$

If there is equality in (2) and $\varepsilon \leq \frac{k}{m}$, this is non-positive because the first term of the sum vanishes and $\alpha_x^* D - \varepsilon E$ is nef. If r is minimal, the intersection of $\widetilde{M}$ with $\widetilde{Y}$ cannot be proper, hence Y is contained in M. □

2 Seshadri constants of abelian varieties

For an abelian variety X, the real number $\varepsilon(X, d, x)$ is independent of x, hence is equal to $\varepsilon(X, d)$. We set $\varepsilon(X, d, Y) = \varepsilon(X, d, Y, 0)$ for any subvariety Y of X.

One has

$$\varepsilon((X, d) \times (X', d')) = \min(\varepsilon(X, d), \varepsilon(X', d'))$$

and if $f : X' \to X$ is an isogeny,

$$\varepsilon(X, d) \leq \varepsilon(X', f^* d) \leq \deg(f)\, \varepsilon(X, d)$$

If D is an effective ample divisor on X with class d, the base locus B of $\{D\}_{0,1}$ is finite, because it satisfies $B + D \subset D$; therefore, $\varepsilon(X, d) \geq 1$ by Lemma 1.

Remark 2 (Lazarsfeld) It follows from Lemma 1 that

$$\{0\} \text{ is isolated in the base locus of } \{md\}_k \quad \Rightarrow \quad \varepsilon(X, d) \geq \frac{k}{m} \tag{5}$$

Following work of Welters ([W]), van Geemen and van der Geer conjectured in [vGvG] that for an indecomposable principally polarized abelian variety (X, θ) and symmetric theta divisor Θ,

$$(X, \theta) \text{ is } \textit{not} \text{ a Jacobian} \quad \Leftrightarrow \quad \text{the base locus of } |2\Theta|_{0,4} \text{ is } \{0\} \tag{6}$$

(By [W], the base locus of $|2\Theta|_{0,4}$ on a Jacobian JC consists of the surface $C - C$ and, possibly, isolated points; the base locus of $\{2\theta\}_{0,4}$ does not seem to be known, although 0 is certainly not an isolated point when C is hyperelliptic by virtue of (5) and Theorem 7.)

Together with (5), this conjecture implies

$$\varepsilon(X, \theta) < 2 \quad \Rightarrow \quad (X, \theta) \text{ is decomposable or a Jacobian} \tag{7}$$

Conjecture (6) is known in dimension 4 by [I], Theorem 4(i). For (X, θ) general of dimension at least 4, it is known by [BD] that the base locus of $|2\Theta|_{0,4}$ is finite (see Example 3(2) below), hence $\varepsilon(X, \theta) \geq 2$.

Examples 3 (1) By [BDDG], the base locus of the linear system $|2\Theta|_{0,4}$ on the (5-dimensional) intermediate Jacobian (X,θ) of a smooth cubic threefold is $\{0\}$, hence $\varepsilon(X,\theta) \geq 2$.

(2) In each dimension $n \geq 4$, a principally polarized abelian variety (X,θ) of dimension n for which the base locus of $|2\Theta|_{0,4}$ is $\{0\}$ is constructed in [BD], théorème 2. It is isogeneous to the product of n elliptic curves on which the polarization has degree 2, hence $\varepsilon(X,\theta) = 2$.

(3) More generally, if the principally polarized abelian variety (X,θ) contains an abelian subvariety Y of dimension r, we get from (2) the bound

$$\varepsilon(X,\theta) \leq \sqrt[r]{\theta^r \cdot Y} = \sqrt[r]{\deg(\theta|_Y)r!}$$

In particular, if Y is an elliptic curve or abelian surface on which θ has degree 2 (this situation is studied in great details in [D4], Appendice), we have $\varepsilon(X,\theta) \leq 2$.

For any variety Z, we denote by $\mathrm{Sing}_k(Z)$ the set of points with multiplicity at least k on Z. For an irreducible subset S of an abelian variety X, we denote by $\langle S \rangle$ the abelian subvariety generated by S, i.e., the intersection of all abelian subvarieties of X that contain $S - S$.

Proposition 4 *Let (X,d) be a polarized abelian variety. Let Γ be an irreducible curve on X, let Y be an irreducible subvariety of X of dimension at least 2, and let k be a positive integer.*

(a) *If $\varepsilon(X,d,\Gamma) < k$, we have $\Gamma + \mathrm{Sing}_k(D) \subset D$ for any effective divisor D with class d.*

(b) *If $\varepsilon(X,d,\Gamma) = k$, we have $\Gamma + S \subset D$ for any effective divisor D with class d and each irreducible component S of $\mathrm{Sing}_k(D)$ such that $\dim(\langle S \rangle \cap \langle \Gamma \rangle) > 0$.*

(c) *If Y minimally computes the Seshadri constant of (X,d) and $\varepsilon(X,d) \leq k$, we have $Y + \mathrm{Sing}_k(D) \subset D$ for any effective divisor D with class d.*

PROOF. For any point x of $\mathrm{Sing}_k(D)$, the divisor $D - x$ is in $\{D\}_k$, hence (a) and (c) follow from Lemma 1.

Let us prove (b). We assume $\varepsilon(X, d, \Gamma) = k$. Let y be a non-zero point of Γ. If x is in $\mathrm{Sing}_k(D) \cap (D - y)$, the point $x + y$ is also in $D \cap (\Gamma + x)$. If $\Gamma + x$ is not contained in D, we get, by [F], Corollary 12.4,

$$D \cdot (\Gamma + x) \geq (\mathrm{mult}_x D)(\mathrm{mult}_0 \Gamma) + (\mathrm{mult}_{x+y} D)(\mathrm{mult}_y \Gamma) > k \,\mathrm{mult}_0 \Gamma$$

This contradicts our assumption, hence

$$\Gamma + (\mathrm{Sing}_k(D) \cap (D - y)) \subset D$$

Let S be an irreducible component of $\mathrm{Sing}_k(D)$. If $\bigcup_{y \in \Gamma - \{0\}} (S \cap (D - y))$ is not dense in S, the $S \cap (D - y)$ are constant divisors in S. In particular, the composition

$$\Gamma \longrightarrow X \xrightarrow{\varphi_D} \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0(S)$$

is constant. This implies that $\langle S \rangle \cap \langle \Gamma \rangle$ is finite and proves (b). $\square$

Examples 5 (1) Let (X, d) be a polarized abelian variety. If $\varepsilon(X, d) = 1$, it follows from the proposition that (X, d) is a product of an elliptic curve with another polarized abelian variety. This is a result of Nakamaye ([N]).

(2) Let (X, θ) be the 5-dimensional intermediate Jacobian of a smooth cubic threefold. By (3), $\varepsilon(X, \theta) \leq \sqrt[5]{5!} \approx 2.605 < 3$. Since a suitable theta divisor Θ has a triple point at the origin ([Be]), it follows from the proposition that any curve Γ with $\varepsilon(X, \theta, \Gamma) < 3$ is contained in Θ.

3 Seshadri constants of Jacobians

We first introduce some notation: let $C^{(2)}$ be the double symmetric product of a smooth projective curve C of genus n, with quotient map $\pi : C \times C \to C^{(2)}$, and let Δ be the diagonal in $C^{(2)}$. We denote by c the numerical equivalence class in $C^{(2)}$ of $C + x$, for any point x of C, and by δ the class of $\frac{1}{2}\Delta$. We have the relations

$$c^2 = c \cdot \delta = 1 \qquad \delta^2 = -(n-1)$$

Given a g^1_d on C, we set

$$\Gamma(g^1_d) = \{x + y \in C^{(2)} \mid H^0(C, g^1_d(-x - y)) \neq 0\} \tag{8}$$

The numerical equivalence class of this curve is $dc - \delta$.

As shown in [L], Jacobians have small Seshadri constants among all principally polarized abelian varieties.

Proposition 6 (Lazarsfeld, [L]) *For a Jacobian (JC,θ) of dimension n,*

$$\varepsilon(JC,\theta) \leq \sqrt{n}$$

If there exists a surjective morphism $C \to \mathbf{P}^1$ of degree d,

$$\varepsilon(JC,\theta) \leq \frac{dn}{n+d-1} < d$$

The proposition is proved by taking for Y, in (2), the surface $C - C$ and the curve $\Gamma(g^1_d)$, respectively.

The Buser–Sarnak invariant. Lazarsfeld found a very interesting connection between the Seshadri constant of a principally polarized abelian variety (X,θ) and a metric invariant defined in [BuS] as follows.

Write (X,θ) as a quotient of a complex vector space V by a lattice Λ. The principal polarization θ defines a positive definite Hermitian form H on V and we set

$$m(X,\theta) = \min_{x\in\Lambda-\{0\}} H(x,x)$$

The main result of [L] is (see also [B1] for non-principal polarizations)

$$\varepsilon(X,\theta) \geq \frac{\pi}{4} m(X,\theta) \tag{9}$$

The lower bound for the maximal value for $m(X,\theta)$ given in [BuS], (1.12), implies, together with (3),

$$\frac{n}{4e} \approx \frac{1}{4}\sqrt[n]{2n!} \leq \max_{(X,\theta)\in\mathscr{A}_n} \varepsilon(X,\theta) \leq \sqrt[n]{n!} \approx \frac{n}{e}$$

(the maximal value is attained for a very general principally polarized abelian variety). For Jacobians, the maximal value for $m(JC,\theta)$ is much smaller: it grows like a multiple of $\log n$ ([BuS]). It follows from (9) and Proposition 6 that there is a positive constant c such that

$$c\log n \leq \max_{C\in\mathscr{M}_n} \varepsilon(JC,\theta) \leq \sqrt{n}$$

It would be interesting to determine the actual rate of growth of this maximum. To this end, one may try to construct many elements of $\{m\theta\}_{0,k}$ with

k/m large. However, it seems, as Pauly remarked, that all known constructions of such divisors (including constructions via vector bundles) lead to a base locus that contains the difference variety $C^{(t)} - C^{(t)}$ with $t \sim \sqrt{n}$, and this is too large for our purposes.

We now prove the main result of this section.

Theorem 7 *For a Jacobian (JC, θ) of dimension $n \geq 2$, we have the following.*

- $\varepsilon(JC, \theta) = \frac{2n}{n+1}$ *if C is hyperelliptic.*[1]
- $\varepsilon(JC, \theta) = \frac{12}{7}$ *if C is not hyperelliptic and $n = 3$.*
- $\varepsilon(JC, \theta) = 2$ *if C is not hyperelliptic and $n = 4$.*
- $\varepsilon(JC, \theta) = 2$ *if C is bielliptic and $n \geq 5$.*
- $\varepsilon(JC, \theta) > 2$ *if C is not hyperelliptic, not bielliptic, and $n \geq 5$.*

When $n = 3$, we recover in a simpler way the results of [BS].

Proof. We use the notation (4). By [W], the base locus of $|2\Theta|_{0,4}$ consists of the surface $C - C$ and, possibly, isolated points. It follows from Lemma 1 and Proposition 4 that

- either $\varepsilon(JC, \theta) > 2$;
- or $\varepsilon(JC, \theta)$ is minimally computed by the surface $C - C$, hence is equal to $\sqrt{n}$;
- or $\varepsilon(JC, \theta)$ is minimally computed by an irreducible curve Γ and
  - either $\varepsilon(JC, \theta) < 2$ and Γ is contained in $C - C$;
  - or $\varepsilon(JC, \theta) = 2$ and $\Gamma + S \subset \Theta$ for each irreducible component S of $\mathrm{Sing}(\Theta)$ such that $\dim(\langle S \rangle \cap \langle \Gamma \rangle) > 0$.

For $n \geq 5$, we know from [T] and [W] that

[1] This was first shown by Lazarsfeld.

- either each irreducible component of $\operatorname{Sing}(\Theta)$ generates JC and

$$\{x \in JC \mid x + \operatorname{Sing}(\Theta) \subset \Theta\} = C - C$$

- or C is bielliptic, i.e., is a double cover of an elliptic curve E, in which case the curve E embeds in $C - C$ and satisfies $\varepsilon(JC, \theta, E) = 2$.

It follows that either $\varepsilon(JC, \theta)$ is > 2, or it equals $\sqrt{n}$, or it is minimally computed by an irreducible curve in $C - C$. Let $\partial : C \times C \to JC$ be the difference map. We have

$$\partial^*\theta = \pi^*((n-1)c + \delta)$$

hence

$$\inf_{\Gamma \subset C - C} \frac{\theta \cdot \Gamma}{\operatorname{mult}_0 \Gamma} = \inf_{\Gamma \subset C \times C} \frac{\partial^*\theta \cdot \Gamma}{\Delta \cdot \Gamma} = \inf_{\Gamma \subset C^{(2)}} \frac{((n-1)c + \delta) \cdot \Gamma}{\delta \cdot \Gamma} = 1 + \frac{n-1}{\mu_C}$$

where $\mu_C = \min\{\mu \geq 0 \mid \mu c - \delta \text{ nef on } C^{(2)}\}$. Since $(\mu_C c - \delta)^2 \geq 0$ implies $\mu_C \geq \sqrt{n} + 1$, i.e., $1 + \dfrac{n-1}{\mu_C} \leq \sqrt{n}$, we get

$$\varepsilon(JC, \theta) \leq 1 + \frac{n-1}{\mu_C}$$

and there is equality if the right-hand side is ≤ 2. The theorem therefore follows from the next proposition. □

Proposition 8 *Let C be a smooth projective curve of genus $n \geq 2$. Set*

$$\mu_C = \min\{\mu \geq 0 \mid \mu c - \delta \text{ nef on } C^{(2)}\}$$

If C is hyperelliptic, $\mu_C = n + 1$.
If C is not hyperelliptic,

- $\mu_C = \frac{14}{5}$ *when $n = 3$;*
- $\mu_C = 3$ *when $n = 4$;*
- $\mu_C \leq n - 1$ *when $n \geq 5$, and there is equality if and only if C is bielliptic.*

The real number μ_C was studied in [K] and [CK], where it is proved that for a *general* curve C of genus n, one has

$$\sqrt{n}+1 \leq \mu_C \leq \frac{n}{[\sqrt{n}]-1}+1$$

and there is equality on the left when n is a perfect square. This yields a description of the ample cone of $C^{(2)}$ in this case. Further results on the ample cones of other symmetric products of general curves of even genus are obtained in [P].

Kouvidakis also shows that if the curve C has genus $n \geq (d-1)^2$ and has a g^1_d such that the curve $\Gamma(g^1_d)$ is irreducible (this is the case for example if the ramification of the associated morphism is simple), $\mu_C = \frac{n}{d-1}+1$. The number μ_C measures in some sense how special the curve C is with respect to Brill–Noether theory.

PROOF OF THE PROPOSITION. Let $\sigma : C^{(2)} \to JC$ be the sum morphism. It is finite if and only if C is not hyperelliptic, and

$$\sigma^*\theta \sim (n+1)c - \delta$$

hence $(n+1)c-\delta$ is nef on $C^{(2)}$, ample if and only if C is not hyperelliptic. This proves $\mu_C = n+1$ when C is hyperelliptic. We assume from now on that C is not hyperelliptic.

Case $n=3$**.** There is an involution τ on $C^{(2)}$ that sends $x+y$ to the unique element of $|K_C - x - y|$, and $\tau^*\delta = 8c - 3\delta$. Write the class of an irreducible curve on $C^{(2)}$ as $rc - s\delta + \gamma$, with $\gamma \in \langle c, \delta\rangle^\perp$. If the curve is neither the diagonal Δ nor $\tau(\Delta)$, it must satisfy

$$(rc - s\delta + \gamma)\cdot\delta \geq 0 \qquad \text{and} \qquad (rc - s\delta + \gamma)\cdot(8c - 3\delta) \geq 0$$

This is equivalent to

$$r \geq -2s \qquad \text{and} \qquad r \geq \frac{14s}{5}$$

Since the convex cone spanned by δ and $\tau^*\delta$ contains the convex cone in $\mathbf{R}c \oplus \mathbf{R}\delta$ defined by these two inequalities, $\mu c - \delta$ is nef if and only if

$$(\mu c - \delta)\cdot\delta \geq 0 \qquad \text{and} \qquad (\mu c - \delta)\cdot(8c - 3\delta) \geq 0$$

This is equivalent to $\mu \geq \frac{14}{5}$, hence the proposition is proved in this case.

Case $n = 4$. The curve C has a base-point-free g^1_3 and the corresponding curve $\Gamma(g^1_3)$, defined in (8), is irreducible with class $3c-\delta$ and self-intersection 0, hence $\mu_C \leq 3$. Since in any case $\mu_C \geq \sqrt{n}+1$, the proposition follows in this case.

Case $n \geq 5$: the class $(n-1)c-\delta$ is nef. If not, there exists an irreducible curve Γ on $C^{(2)}$ on which it has negative degree. For any g^1_{n-1}, the curve $\Gamma(g^1_{n-1})$ has class $(n-1)c-\delta$ hence contains Γ. In particular, for any other h^1_{n-1}, the curves $\Gamma(g^1_{n-1})$ and $\Gamma(h^1_{n-1})$ have a common component. If g^1_{n-1} and h^1_{n-1} are base-point-free, this means, by [C], Theorem (1.8), that the corresponding morphisms $C \to \mathbf{P}^1$ both factor through a morphism $f : C \to C'$ whose degree d divides $n-1$ and $2 \leq d \leq (n-1)/2$.

By [ACGH], Ex. VIII.F, p. 372, there is a component of $W^1_{n-1}(C)$ whose general point corresponds to a base-point-free g^1_{n-1}. If C' is rational, we have

$$\dim\bigl(W^1_d(C)\bigr) \geq n-4 \geq 2d-3$$

and this contradicts Martens' theorem ([ACGH], Theorem IV(5.1), p. 191).

It follows that C' is irrational, does not depend on the g^1_{n-1}, and

$$\dim(JC') \geq \dim\bigl(W^1_{\frac{n-1}{d}}(C')\bigr) \geq n-4 \geq 2\frac{n-1}{d}-3$$

Using Martens' theorem again, we see that this is impossible.

Curves of genus $n \geq 5$ for which the class $(n-1)c-\delta$ is not ample. When C is a double cover of an elliptic curve E, the curve E embeds in the surface $C^{(2)}$ and $E \cdot c = 1$, $E \cdot \delta = n-1$. It follows that $(n-1)c-\delta$ is not ample.

Conversely, assume that $(n-1)c-\delta$ is not ample. We want to prove that C is bielliptic. Since $(n-1)c-\delta$ is nef and its self-intersection is positive, there exists an irreducible curve Γ in $C^{(2)}$ such that $((n-1)c-\delta)\cdot\Gamma = 0$. For any g^1_{n-1}, the curve $\Gamma(g^1_{n-1})$ contains Γ if it meets it. Since, for any one-dimensional family of g^1_{n-1}, these curves cover $C^{(2)}$, there is an $(n-5)$-dimensional family of g^1_{n-1} with this property. A general point of this family induces a morphism $C \to \mathbf{P}^1$ of degree $e \leq n-1$ and there is an $(e-4)$-dimensional family of such base-point-free g^1_e.

Subcase $e \geq 5$. As above, any two such morphisms factor through a morphism $f : C \to C'$ whose degree d divides e and $2 \leq d \leq e/2$. If C' is rational, we have

$$\dim\bigl(W^1_d(C)\bigr) \geq e - 4 \geq 2d - 4$$

and this contradicts Martens' theorem.

It follows that C' is irrational, does not depend on the g^1_e in the family, and

$$\dim(JC') \geq \dim\bigl(W^1_{\frac{e}{d}}(C')\bigr) \geq e - 4 \geq 2\frac{e}{d} - 4$$

Using Martens' theorem again, we see that the only possible case is $g(C') = 2$, $d = 2$ and $e = 6$. The curve C then has a g^1_4 and Γ is contained in $\Gamma(g^1_4)$ because, since $n > e = 6$,

$$\Gamma(g^1_4) \cdot \Gamma = (4c - \delta) \cdot \Gamma < ((n-1)c - \delta) \cdot \Gamma = 0$$

We are reduced to the subcase below.

Subcase $e = 4$**.** We have a g^1_4 on C such that the curve $\Gamma(g^1_4)$ contains Γ. It must therefore be reducible, since $((n-1)c - \delta) \cdot \Gamma(g^1_4) \neq 0$. The Galois group G of the corresponding morphism $f : C \to \mathbf{P}^1$ is a subgroup of $\mathscr{S}_4$ that operates transitively, and irreducible components of $\Gamma(g^1_4)$ are in one-to-one correspondence with the G-orbits of pairs of elements of $\{1, \ldots, 4\}$. According to the analysis of [D2], p. 10, either G is isomorphic to D_4 or $\mathbf{Z}/4\mathbf{Z}$ and there are two such orbits, or G is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^2$ and there are three orbits; in each case, there is an involution s on C such that f factors through the quotient $C \to C/s$ and either Γ is the curve $\Gamma_s = \{x + s(x) \mid x \in C\}$, or $\Gamma = \Gamma(g^1_4) - \Gamma_s$ and s has fixed points. Since $\Gamma_s \cdot c = 1$ and

$$\Gamma_s \cdot \delta = \frac{1}{2}\#\{\text{fixed points of } s\} = n + 1 - 2g(C/s)$$

we have

$$((n-1)c - \delta) \cdot \Gamma_s = 2g(C/s) - 2$$

and

$$((n-1)c - \delta) \cdot (\Gamma(g^1_4) - \Gamma_s) = 2n - 4 - 2g(C/s)$$

The case $\Gamma = \Gamma(g^1_4) - \Gamma_s$ does not happen since it implies $g(C/s) = 3$, which is absurd since s has fixed points. It follows that $\Gamma = \Gamma_s$ and C/s is elliptic.

This finishes the proof of the proposition. $\square$

Remark 9 Assume $n \geq 9$ and that the curve C is neither hyperelliptic nor trigonal, but has a g^1_4 such that $\Gamma(g^1_4)$ is reducible. It follows from the proof of the proposition that the corresponding morphism $C \to \mathbf{P}^1$ factors through the quotient of C by an involution s. When there are two orbits, s has fixed points, μ_C is the smallest number such that

$$(\mu_C c - \delta) \cdot \Gamma_s \geq 0 \qquad (\mu_C c - \delta) \cdot (\Gamma(g^1_4) - \Gamma_s) \geq 0$$

hence

$$\mu_C = \max\{n + 1 - 2g(C/s), g(C/s) + 1\}$$

When there are three orbits, i.e., three involutions s_1, s_2, and s_3, we have

$$\mu_C = \max_{1 \leq j \leq 3}\{n + 1 - 2g(C/s_j)\}$$

(note that $\sum_{j=1}^{3} g(C/s_j) = n$).

4 Seshadri constants of principally polarized abelian varieties in low dimensions

Let (X, θ) be a principally polarized abelian variety. Theorem 7 implies the following.

In dimension 2 (this was first proved in [S], Proposition 2),

$$\begin{cases} \varepsilon(X,\theta) = 1 & \text{if } (X,\theta) \text{ is decomposable,} \\ \varepsilon(X,\theta) = \frac{4}{3} & \text{otherwise.} \end{cases}$$

In dimension 3 (this was first proved in [BS], Theorem 1),

$$\begin{cases} \varepsilon(X,\theta) = 1 & \text{if } (X,\theta) \text{ is decomposable,} \\ \varepsilon(X,\theta) = \frac{3}{2} & \text{if } (X,\theta) \text{ is a hyperelliptic Jacobian,} \\ \varepsilon(X,\theta) = \frac{12}{7} & \text{if } (X,\theta) \text{ is a non-hyperelliptic Jacobian.} \end{cases}$$

In dimension 4,

$$\begin{cases} \varepsilon(X,\theta) = 1 & \text{if } (X,\theta) \text{ is decomposable with a factor an elliptic curve,} \\ \varepsilon(X,\theta) = \frac{4}{3} & \text{if } (X,\theta) \text{ is a product of indecomposable abelian surfaces,} \\ \varepsilon(X,\theta) = \frac{8}{5} & \text{if } (X,\theta) \text{ is a hyperelliptic Jacobian,} \\ \varepsilon(X,\theta) = 2 & \text{if } (X,\theta) \text{ is a non-hyperelliptic Jacobian,} \\ 2 \leq \varepsilon(X,\theta) \leq \sqrt[4]{4!} \approx 2.213 & \text{if } (X,\theta) \text{ is indecomposable and not a Jacobian.} \end{cases}$$

A principally polarized abelian variety (X, θ) of dimension 4 that contains an abelian surface X_1 on which θ has degree 2 is never a Jacobian. It is isogeneous to a product $X_1 \times X_2$ and is indecomposable if and only if each X_i, with the induced polarization, is. In this case, we get, using Example 3(3),

$$\varepsilon(X, \theta) = \varepsilon(X_1, \theta|_{X_1}) = \varepsilon(X_2, \theta|_{X_2}) = 2$$

The last equality strengthen a result of [B2], Theorem A.1(c); it can also be checked directly as a consequence of Proposition 4. By the same token, conjecture (7) implies that the Seshadri constant of *any* indecomposable polarized abelian variety of degree 2 of dimension at least 2 should be at least 2.

In dimension ≥ 5, the Seshadri constant is > 2 for non-hyperelliptic and non-bielliptic Jacobians, whereas it is 2 for some non-Jacobians (Examples 3(2) and 3(3)). One cannot therefore expect to distinguish Jacobians among all principally polarized abelian varieties by means of their Seshadri constant, with the possible exception of hyperelliptic Jacobians.

It is interesting to compare the values above with the values for the Buser–Sarnak invariant $m(X, \theta)$ given in Appendix 1 of [BuS].

For $n = 1$, the maximum for $m(X, \theta)$ is $\frac{2}{\sqrt{3}}$ and is attained for the elliptic curve with j invariant 0.

For $n = 2$, the maximum is $\sqrt{2}$ and is attained for the Jacobian of the curve $y^2 = x^5 - x$. The automorphism group of this curve is $\mathrm{GL}_2(\mathbf{F}_3)$ and has order 48.

For $n = 3$, the maximum is unknown, but is less than $(64/3)^{1/6} \approx 1.665$ and more than its value $\frac{4}{\sqrt{7}} \approx 1.512$ for the Jacobian of the Klein curve with equation $xy^3 + yz^3 + zx^3 = 0$ in $\mathbf{P}^2$. The automorphism group of this curve is $\mathrm{PSL}_2(\mathbf{F}_7)$ and has order 168. Its Jacobian is isomorphic (without the polarization) to the threefold product of the elliptic curve associated with the lattice $\mathbf{Z} \oplus \mathbf{Z}(1 + \sqrt{-7})/2$.

For $n = 4$, the maximum among all principally polarized abelian varieties (the maximum among all Jacobians is unknown) is 2 and is attained for a remarkable abelian variety studied in [V], [D3] and [BuS] whose automorphism group has order 46080 and which is isomorphic (without the polarization) to the fourfold product of the elliptic curve with j-invariant 1728. This elliptic curve has degree 2 with respect to the polarization, hence the Seshadri constant is 2 (this is a particular case of Example 3(2)).

In all these cases, Lazarsfeld's inequality (9) is not sharp.

5 Principally polarized abelian varieties with small Seshadri constant

Let (X,θ) be a principally polarized abelian variety of dimension n and let Γ be an irreducible curve in X such that

$$\varepsilon = \varepsilon(X,\theta,\Gamma) < 2$$

We expect (X,θ) to be isomorphic to the Jacobian of a hyperelliptic curve C and, at least for $n \geq 5$, the curve Γ to be isomorphic to the only curve that computes the Seshadri constant of (JC,θ), to wit $2C - g_2^1$ (this follows from the proof of Proposition 8).

We discuss now a partial result in this direction. Let $f : X \to X$ be the multiplication by 2. Pick an irreducible component Γ_0 of $f^{-1}(\Gamma)$ and set

$$G = \{x \in X[2] \mid \Gamma_0 + x = \Gamma_0\}$$

This group is independent of the choice of Γ_0. Choose a decomposition

$$X[2] = G \oplus H$$

The irreducible components of $f^{-1}(\Gamma)$ are the $\Gamma_h = \Gamma_0 + h$, for $h \in H$. If $m = \operatorname{mult}_0 \Gamma$, we have for each $x \in X[2]$,

$$m = \operatorname{mult}_x f^{-1}(\Gamma) = \sum_{h\in H} \operatorname{mult}_x \Gamma_h \tag{10}$$

and

$$\theta \cdot \Gamma_0 = \frac{\theta \cdot f^{-1}(\Gamma)}{\operatorname{Card}(H)} = \frac{f^*\theta \cdot f^{-1}(\Gamma)}{4\operatorname{Card}(H)} = \frac{(\theta\cdot\Gamma)\operatorname{Card}(X[2])}{4\operatorname{Card}(H)} = \frac{\varepsilon m \operatorname{Card}(G)}{4}$$

Pick a symmetric theta divisor Θ. The set

$$\Lambda = \{x \in X[2] \mid \Gamma_0 \subset \Theta_x\}$$

is invariant by translation by G. One has, using (10),

$$\begin{aligned}
\sum_{k\in(X[2]-\Lambda)/G} \Theta\cdot\Gamma_k &\geq \sum_{\substack{k\in(X[2]-\Lambda)/G\\ x\in\Theta[2]}} \operatorname{mult}_x \Gamma_k \qquad \text{because } \Gamma_k \not\subset \Theta\\
&\geq \sum_{k\in H,\ x\in\Theta[2]} \operatorname{mult}_x \Gamma_k - \sum_{k\in\Lambda/G,\ x\in\Theta[2]} \operatorname{mult}_x \Gamma_k\\
&\geq m\operatorname{Card}(\Theta[2]) - \sum_{k\in\Lambda/G,\ x\in X[2]} \operatorname{mult}_x \Gamma_k
\end{aligned}$$

We have, using (10) again,

$$\begin{aligned}\sum_{k\in\Lambda/G,\ x\in X[2]} \operatorname{mult}_x \Gamma_k &= \sum_{k\in\Lambda/G,\ g\in G,\ h\in H} \operatorname{mult}_{g+h} \Gamma_k \\ &= \sum_{k\in\Lambda/G,\ g\in G,\ h\in H} \operatorname{mult}_{g+k} \Gamma_h \\ &= \sum_{k\in\Lambda/G,\ g\in G} m = m \operatorname{Card}(\Lambda)\end{aligned}$$

Putting everything together, we get

$$\begin{aligned} m \operatorname{Card}(\Theta[2] - \Lambda) &\leq \sum_{k\in(X[2]-\Lambda)/G} \theta\cdot\Gamma_k \\ &= (\theta\cdot\Gamma_0) \operatorname{Card}((X[2]-\Lambda)/G) \\ &\leq \frac{\varepsilon m \operatorname{Card}(G)}{4} \operatorname{Card}((X[2]-\Lambda)/G)\end{aligned}$$

It follows that

$$\left(1-\frac{\varepsilon}{4}\right) \operatorname{Card}(\Lambda) \geq \operatorname{Card}(\Theta[2]) - \frac{\varepsilon}{4} \operatorname{Card}(X[2]) \geq 2^{g-1}(2^g-1) - \frac{\varepsilon}{4} 2^{2g}$$

hence

$$\operatorname{Card}(\Lambda) \geq \left(\frac{2^{2g}}{4-\varepsilon}\right)(2-\varepsilon-2^{1-g})$$

If $\varepsilon < 2 - 2^{1-g}$, the curve Γ_0 is contained in "many" symmetric translates of Θ. Unfortunately, we were unable to make further progress in this direction.

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