ON COVERINGS OF SIMPLE ABELIAN VARIETIES

by Olivier Debarre

Abstract. — To any finite covering $f : Y \to X$ of degree d between smooth complex projective manifolds, one associates a vector bundle E_f of rank $d-1$ on X whose total space contains Y. It is known that E_f is ample when X is a projective space ([L1]), a Grassmannian ([M]), or a lagrangian Grassmannian ([KM]). We show an analogous result when X is a simple abelian variety and f does not factor through any nontrivial isogeny $X' \to X$. This result is obtained by showing that E_f is M-regular in the sense of Pareschi–Popa, and that any M-regular sheaf is ample.

Résumé (*Sur les revêtements des variétés abéliennes simples*). — À tout revêtement fini $f : Y \to X$ de degré d entre variétés projectives lisses complexes, on associe un fibré vectoriel E_f de rang $d-1$ sur X dont l'espace total contient Y. On sait que E_f est ample lorsque X est un espace projectif ([L1]), une grassmannienne ([M]) ou une grassmannienne lagrangienne ([KM]). Nous montrons un résultat analogue lorsque X est une variété abélienne simple et que f ne se factorise par aucune isogénie non triviale $X' \to X$. Ce résultat est obtenu en montrant que E_f est M-régulier au sens de Pareschi–Popa, puis que tout faisceau M-régulier est ample.

1. Introduction

We work over the complex numbers. Let $f : Y \to X$ be a finite surjective morphism of degree d between smooth projective varieties of the same dimension n. The morphism f is flat, hence the sheaf $f_*\mathscr{O}_Y$ is locally free. We may

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define a locally free sheaf E_f of rank $d-1$ on X as the dual of the kernel of the trace map $\mathrm{Tr}_{Y/X} : f_*\mathscr{O}_Y \to \mathscr{O}_X$, so that

$$f_*\mathscr{O}_Y = \mathscr{O}_X \oplus E_f^*$$

By duality for a finite flat morphism, we have

$$f_*\omega_{Y/X} = \mathscr{O}_X \oplus E_f$$

Our aim is to prove the following statement conjectured in [D1].

THEOREM 1.1. — *Let X be a* simple *abelian variety, let Y be a smooth* connected *projective variety, and let $f : Y \to X$ be a finite cover. If f does not factor through any nontrivial isogeny $X' \to X$, the vector bundle E_f is ample.*

For a more general statement, see Theorem 4.1. See also the remarks at the end of this article for more comments. Even if X is not simple, the vector bundle E_f is known to be nef ([PS], Theorem 1.17; [L2], Example 6.3.59) and its restriction to a general complete intersection curve in X to be ample ([HKP], Lemma 2.7).

The ampleness of E_f has a number of consequences, as explained in [L2], Example 6.3.56. In our case, one new statement beyond the Fulton–Hansen-type results already obtained in [D1] is the following: under the hypotheses of the theorem, the induced morphism

$$H^i(f, \mathbf{C}) : H^i(X, \mathbf{C}) \to H^i(Y, \mathbf{C})$$

is bijective for $i \le n-d+1$ ([L2], Theorem 7.1.16).

When moreover $d \le n$, the morphism $\pi_1(f) : \pi_1(Y) \to \pi_1(X)$ is bijective.(1) In particular, the group $H_1(Y, \mathbf{Z})$ is isomorphic to $H_1(X, \mathbf{Z})$, hence is torsion-free, and so is $H^2(Y, \mathbf{Z})$ by the universal coefficient theorem.

When $d \le n-1$, the morphism $H^2(f, \mathbf{Z}) : H^2(X, \mathbf{Z}) \to H^2(Y, \mathbf{Z})$ is injective with finite cokernel, hence so is $\mathrm{Pic}(f) : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$. It seems likely that those two maps are bijective.

The proof is a simple application of the results of [PP] about global generation of sheaves on an abelian variety. More precisely, it is based on the remark that any M-regular sheaf (§ 3) on an abelian variety is ample (Corollary 3.2).

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(1) For algebraic fundamental groups, this is [D1], Corollaire 6.2; for topological fundamental groups, this is [D2], Exercice VIII.5, where the hypothesis $d \le n$ is unfortunately missing.

2. Ample sheaves

To any coherent sheaf $\mathscr{F}$ on a scheme X of finite type over $\mathbf{C}$, one associates the X-scheme

$$\mathbf{P}(\mathscr{F}) = \operatorname{Proj}\Big(\bigoplus_{m\geq 0}\mathbf{Sym}^m\mathscr{F}\Big)$$

and an invertible sheaf $\mathscr{O}_{\mathbf{P}(\mathscr{F})}(1)$ on $\mathbf{P}(\mathscr{F})$. The sheaf $\mathscr{F}$ is said to be ample if $\mathscr{O}_{\mathbf{P}(\mathscr{F})}(1)$ is.

Well-known properties of ampleness for locally free sheaves (see for example [L2], Chapter 6) still hold in this general setting:

a) the sheaf $\mathscr{F}$ is ample if and only if, for any coherent sheaf $\mathscr{G}$ on X, the sheaf $\mathscr{G}\otimes\mathbf{Sym}^m\mathscr{F}$ is globally generated for all $m\gg 0$ ([Ku], Theorem 1);

b) any quotient of an ample sheaf is ample ([Ku], Proposition 1);

c) if $\pi : Y\to X$ is a finite morphism, $\mathscr{F}$ is ample if and only if $\pi^*\mathscr{F}$ is (this is because $\mathbf{P}(\pi^*\mathscr{F}) = \mathbf{P}(\mathscr{F})\times_X Y$ and $\mathscr{O}_{\mathbf{P}(\mathscr{F})}(1)$ pulls back, by a finite morphism, to $\mathscr{O}_{\mathbf{P}(\pi^*\mathscr{F})}(1)$);

d) if X is proper and $\mathscr{F}$ is globally generated, $\mathscr{F}$ is ample if and only if, for any curve C in X, the restriction $\mathscr{F}\otimes\mathscr{O}_C$ has no trivial quotient (Gieseker's Lemma).

3. Continuously generated sheaves

Following [PP], Definition 2.10, we say that a coherent sheaf $\mathscr{F}$ on an irreducible projective variety X is *continuously globally generated* if, for any nonempty subset U of $\operatorname{Pic}^0(X)$, the sum of the twisted evaluation maps

$$\bigoplus_{\xi\in U} H^0(X,\mathscr{F}\otimes P_\xi)\otimes P_\xi^\vee \to \mathscr{F}$$

is surjective, where, for any element ξ of $\operatorname{Pic}^0(X)$, we denote by P_ξ the corresponding numerically trivial line bundle on X. This property is equivalent to the existence of a positive integer N such that for $(\xi_1,\dots,\xi_N)$ general in $\operatorname{Pic}^0(X)^N$, the analogous map

$$\bigoplus_{i=1}^{N} H^0(X,\mathscr{F}\otimes P_{\xi_i})\otimes P_{\xi_i}^\vee \to \mathscr{F} \tag{1}$$

is surjective. Being a quotient of a direct sum of numerically trivial line bundles, a continuously globally generated sheaf is nef. Our aim is to show that under certain circumstances, it is ample.

Proposition 3.1. — *A coherent sheaf $\mathscr{F}$ on an irreducible projective variety X is continuously globally generated if and only if there exists a connected abelian Galois étale cover $\pi : Y\to X$ such that $\pi^*(\mathscr{F}\otimes P_\xi)$ is globally generated for all $\xi\in\operatorname{Pic}^0(X)$.*

Proof. — Assume $\mathscr{F}$ is continuously globally generated and let $\xi_0 \in \operatorname{Pic}^0(X)$. Since torsion points are dense in $\operatorname{Pic}^0(X)^N$, the open subset of $\operatorname{Pic}^0(X)^N$ of points for which the map (1) is surjective and all $h^0(X, \mathscr{F} \otimes P_{\xi_i})$ are minimal contains a point of the type

$$(\xi_0 + \eta_1(\xi_0), \dots, \xi_0 + \eta_N(\xi_0))$$

where $(\eta_1(\xi_0), \dots, \eta_N(\xi_0))$ is torsion, hence contains also $U_{\xi_0} + (\eta_1(\xi_0), \dots, \eta_N(\xi_0))$, where U_{ξ_0} is a neighborhood of ξ_0 in $\operatorname{Pic}^0(X)$. Since $\operatorname{Pic}^0(X)$ is quasi-compact, it is covered by finitely many such neighborhoods, say $U_{\xi_1}, \dots, U_{\xi_M}$.

Let $\pi : Y \to X$ be a connected abelian Galois étale cover such that the kernel of $\operatorname{Pic}^0(\pi) : \operatorname{Pic}^0(X) \to \operatorname{Pic}^0(Y)$ contains all $\eta_i(\xi_j)$, for $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, M\}$. Fix $j \in \{1, \dots, M\}$; the map

$$\bigoplus_{i=1}^N H^0(X, \mathscr{F} \otimes P_\xi \otimes P_{\eta_i(\xi_j)}) \otimes \pi^* P_\xi^\vee \otimes \pi^* P_{\eta_i(\xi_j)}^\vee \longrightarrow \pi^* \mathscr{F}$$

is surjective for all $\xi \in U_{\xi_j}$. But this map is

$$\bigoplus_{i=1}^N H^0(X, \mathscr{F} \otimes P_\xi \otimes P_{\eta_i(\xi_j)}) \otimes \pi^* P_\xi^\vee \longrightarrow \pi^* \mathscr{F}$$

and since each $H^0(X, \mathscr{F} \otimes P_\xi \otimes P_{\eta_i(\xi_j)})$ is a vector subspace of $H^0(Y, \pi^*(\mathscr{F} \otimes P_\xi))$, the sheaf $\pi^*(\mathscr{F} \otimes P_\xi)$ is globally generated for all $\xi \in U_{\xi_j}$, hence for all $\xi \in \operatorname{Pic}^0(X)$.

For the converse, assume that there exists a connected abelian Galois étale cover $\pi : Y \to X$ such that the evaluation map

$$H^0(Y, \pi^*(\mathscr{F} \otimes P_\xi)) \otimes \mathscr{O}_Y \to \pi^*(\mathscr{F} \otimes P_\xi)$$

is surjective for all $\xi \in \operatorname{Pic}^0(X)$. Since π is finite, the map

$$H^0(X, \mathscr{F} \otimes P_\xi \otimes \pi_* \mathscr{O}_Y) \otimes \pi_* \mathscr{O}_Y \to \mathscr{F} \otimes P_\xi \otimes \pi_* \mathscr{O}_Y$$

is also surjective. If we let $\operatorname{Ker}(\operatorname{Pic}^0(\pi)) = \{\eta_1, \dots, \eta_N\}$, we have $\pi_* \mathscr{O}_Y = \bigoplus_{i=1}^N P_{\eta_i}$, the map

$$\begin{gathered} \Big(\textstyle\bigoplus_{i=1}^N H^0(X, \mathscr{F} \otimes P_\xi \otimes P_{\eta_i})\Big) \otimes \Big(\textstyle\bigoplus_{i=1}^N P_{\eta_i}\Big) \\ \downarrow \\ \mathscr{F} \otimes P_\xi \otimes \Big(\textstyle\bigoplus_{i=1}^N P_{\eta_i}\Big) \end{gathered}$$

is surjective, and so is

$$\bigoplus_{i=1}^N H^0(X, \mathscr{F} \otimes P_\xi \otimes P_{\eta_i}) \otimes P_{\eta_i}^\vee \to \mathscr{F} \otimes P_\xi$$

In other words, the map (1) is surjective for $(\xi_1, \dots, \xi_N) = (\xi + \eta_1, \dots, \xi + \eta_N)$, for all $\xi \in \operatorname{Pic}^0(X)$. Choosing ξ_0 such that $h^0(X, \mathscr{F} \otimes P_{\xi_0 + \eta_i})$ takes the general (minimal) value for each i in $\{1,\dots,N\}$, we obtain that the map (1) is still surjective for $(\xi_1,\dots,\xi_N)$ in a neighborhood of $(\xi_0+\eta_1,\dots,\xi_0+\eta_N)$. This proves that $\mathscr{F}$ is continuously globally generated. □

Corollary 3.2. — *Let X an irreducible projective variety with a finite map to an abelian variety. Any continuously globally generated coherent sheaf on X is ample.*

The converse is in general false: if L is an ample line bundle on an abelian variety A of dimension g, a general map $(L^{-d})^{\oplus g} \to (L^{-1})^{\oplus 2g}$ is injective for $d \gg 0$ and its cokernel is an ample vector bundle E ([L2], Theorem 6.3.65). If $g \geq 2$, we have $H^0(A, E\otimes P_\xi)=0$ for all $\xi\in \operatorname{Pic}^0(A)$, hence E cannot be continuously globally generated.

Proof. — Let $\mathscr{F}$ be a continuously globally generated coherent sheaf on X. By Proposition 3.1, there exists a connected abelian Galois étale cover $\pi: Y\to X$ such that $\pi^*(\mathscr{F}\otimes P_\xi)$ is globally generated for all $\xi\in\operatorname{Pic}^0(X)$.

Let C be a curve in Y. If there is a trivial quotient $\pi^*\mathscr{F}|_C \twoheadrightarrow \mathscr{O}_C$, we have also surjections $\pi^*(\mathscr{F}\otimes P_\xi)|_C \twoheadrightarrow \pi^*P_\xi|_C$ for each $\xi\in\operatorname{Pic}^0(X)$. Since $\pi^*(\mathscr{F}\otimes P_\xi)$ is globally generated, so is $\pi^*P_\xi|_C$. This implies that the composition $\operatorname{Pic}^0(X)\to\operatorname{Pic}^0(Y)\to\operatorname{Pic}^0(C)$ is zero, hence that $\pi(C)$ is contracted by any map from X to an abelian variety. This contradicts our hypothesis, hence $\pi^*\mathscr{F}|_C$ has no trivial quotient.

By Gieseker's Lemma, $\pi^*\mathscr{F}$ is ample, and so is $\mathscr{F}$ (§ 2). □

4. The main theorem

Following [PP], Definition 2.1, we say that a coherent sheaf $\mathscr{F}$ on an abelian variety A is *M-regular* if

$$\operatorname{codim}_{\operatorname{Pic}^0(A)} \operatorname{Supp}\big(R^i\hat{\mathscr{S}}(\mathscr{F})\big) > i$$

for all $i>0$ ($R^i\hat{\mathscr{S}}$ is the ith Fourier–Mukai functor). This is the case if

$$\operatorname{codim}_{\operatorname{Pic}^0(A)}\{\xi\in\operatorname{Pic}^0(A)\mid H^i(A,\mathscr{F}\otimes P_\xi)\neq 0\} > i$$

for all $i>0$. We refer to [Mu] and [PP] for more details. For our purposes, the main result of [PP] (Proposition 2.13) is that *an M-regular coherent sheaf on an abelian variety is continuously globally generated.*

Theorem 4.1. — *Let X be a smooth connected projective variety with a finite map to a* simple *abelian variety, let Y be a smooth* connected *projective variety with a finite surjective map $f: Y\to X$. If f factors through no nontrivial connected abelian Galois étale covering of X, the vector bundle $E_f\otimes\omega_X$ is ample.*

Proof. — Let n be the common dimension of X and Y, and let $\alpha : X \to A$ be a finite map to a simple abelian variety such that $\mathrm{Pic}^0(\alpha) : \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(X)$ is injective. Set $g = \alpha \circ f$. By [GL1], Theorem 1, [GL2], Theorem 0.1, and [EL], Remark 1.6 (see also [EL], Theorem 1.2), every irreducible component of the set

$$V_i = \{\xi \in \mathrm{Pic}^0(A) \mid H^{n-i}(Y, g^* P_\xi^\vee) \neq 0\}$$

is a translated abelian subvariety of $\mathrm{Pic}^0(A)$ of codimension at least i. In particular, since A is simple, V_i is finite for $i > 0$.

Since Y is connected, we have

$$\begin{aligned} V_n &= \{\xi \in \mathrm{Pic}^0(A) \mid H^0(Y, g^* P_\xi^\vee) \neq 0\} \\ &= \{\xi \in \mathrm{Pic}^0(A) \mid g^* P_\xi^\vee \simeq \mathscr{O}_Y\} \\ &= \mathrm{Ker}(\mathrm{Pic}^0(g) : \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(Y)\} \end{aligned}$$

hence $V_n = \{0\}$ since both $\mathrm{Pic}^0(\alpha)$ and $\mathrm{Pic}^0(f)$ are injective (f factors through no nontrivial abelian étale covering of X). Consider now

$$\begin{aligned} W_i &= \{\xi \in \mathrm{Pic}^0(A) \mid H^i(X, E_f \otimes \omega_X \otimes \alpha^* P_\xi) \neq 0\} \\ &= \{\xi \in \mathrm{Pic}^0(A) \mid H^i(A, \alpha_*(E_f \otimes \omega_X) \otimes P_\xi) \neq 0\} \end{aligned}$$

By Serre duality on Y,

$$\begin{aligned} V_i &= \{\xi \in \mathrm{Pic}^0(A) \mid H^i(Y, \omega_Y \otimes g^* P_\xi) \neq 0\} \\ &= \{\xi \in \mathrm{Pic}^0(A) \mid H^i(X, f_*\omega_Y \otimes \alpha^* P_\xi) \neq 0\} \end{aligned}$$

Since $f_*\omega_Y = f_*\omega_{Y/X} \otimes \omega_X = \omega_X \oplus (E_f \otimes \omega_X)$, we have $W_i \subset V_i$ and $W_n = \varnothing$. It follows that W_i is finite, hence $\mathrm{codim}(W_i) > i$ for each $i > 0$, so that the sheaf $\alpha_*(E_f \otimes \omega_X)$ on A is M-regular, hence continuously globally generated. It is therefore ample by Corollary 3.2, and, since α is finite, so are $\alpha^*\big(\alpha_*(E_f \otimes \omega_X)\big)$ and its quotient $E_f \otimes \omega_X$ (§ 2). □

In the following remarks, we keep the hypotheses and notation of the theorem and its proof.

Remark 4.2. — The proof of the theorem shows that the sheaf $\alpha_*(E_f \otimes \omega_X)$ is continuously globally generated. In particular, if f is not an isomorphism, $E_f \otimes \omega_X$ has nonzero sections, hence $p_g(Y) > p_g(X)$.

Remark 4.3. — The simplicity of the abelian variety in the theorem is essential: if B is an abelian variety and $g = (f, \mathrm{Id}_B) : Y \times B \to X \times B$, we have $E_g = p^* E_f$, where $p : X \times B \to X$ is the first projection, hence $E_g \otimes \omega_{X \times B} = p^*(E_f \otimes \omega_X)$ is not ample if B is nonzero. The locus W_i for g contains $\mathrm{Pic}^0(A) \times \{0\}$ for $i \leq \dim(B)$; in particular, for $i = \dim(B)$, it is an abelian subvariety of codimension i of $\mathrm{Pic}^0(A \times B)$.

REMARK 4.4. — If X is not an abelian variety, ω_X is already ample (see, e.g., [D1], Théorème 6.9) and one can show that the hypothesis that f does not factor through a nontrivial connected abelian Galois étale covering of X is unnecessary. If X is a (simple) abelian variety, any finite cover $Y \to X$ factorizes as $Y \xrightarrow{f} X' \xrightarrow{\rho} X$ where ρ is an isogeny and f satisfies the hypotheses of the theorem.

REMARK 4.5. — Assume $X = A$ and let d be the degree of f. For all $i \geq d-1$, the set W_i is empty, i.e.,

$$H^i(A, E_f \otimes P_\xi) = 0 \qquad \text{for all } \xi \in \operatorname{Pic}^0(A),$$

by Le Potier's vanishing theorem ([L2], Theorem 7.3.5). This does not hold in general for $0 \leq i < d-1$, as shown by the following example. Take an elliptic curve C, with origin o_C. Let L be a very ample line bundle on A and let $Y \subset C \times A$ be a general (smooth) element of $|\mathscr{O}_C((n+1)o_C) \boxtimes L|$. Following the proof of [L2], Lemma 6.3.43, one sees that the second projection $f : Y \to A$ is finite (of degree $d = n+1$). By the Lefschetz theorem, the induced morphism

$$H^{n-i}(C \times A, \mathscr{O}_{C\times A}) \to H^{n-i}(Y, \mathscr{O}_Y)$$

is bijective for $i > 0$ and injective for $i = 0$. In particular, $H^{n-i}(f, \mathscr{O})$ is not surjective for $0 \leq i < n$, hence $0 \in W_i$, i.e.,

$$H^i(A, E_f) \neq 0 \qquad \text{for all } 0 \leq i < d-1 = n.$$

In particular, $H^{n-1}(A, E_f) \neq 0$, and it follows from [Mu], Proposition 2.7, that the M-regular vector bundle E_f does not satisfy Mukai's condition WIT_0 when $n > 1$ (sheaves that satisfy condition WIT_0 are M-regular).

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