SINGULARITIES OF DIVISORS OF LOW DEGREE ON ABELIAN VARIETIES

OLIVIER DEBARRE AND CHRISTOPHER D. HACON

Abstract. Building on previous work of Kollár, Ein, Lazarsfeld, and Hacon, we show that ample divisors of low degree on an abelian variety have mild singularities in case the abelian variety is simple or the degree of the polarization is two.

1. Introduction

Since Kollár used in [Ko1] the Kawamata–Viehweg vanishing theorem to settle classical conjectures about singularities of theta divisors in complex principally polarized abelian varieties, the subject has known spectacular developments. Ein and Lazarsfeld, using generic vanishing theorems of Green and Lazarsfeld, proved in [EL] that irreducible theta divisors are normal and gave an optimal bound on the dimension of the locus of points of given multiplicity of a multitheta divisor. Hacon determined in [H1] exactly when this bound is attained and obtained in [H2] results for ample divisors of degree 2.

In this article, we investigate more generally abelian varieties with an indecomposable polarization of degree smaller than the dimension. When the degree increases, many special cases begin to appear, often due to the presence of reducible divisors that represent the polarization. In order to avoid overly technical statements, we restrict ourselves to two cases: the case where the ambient abelian variety is *simple,* and the case of polarizations of degree 2 (thereby completing Hacon's above-mentioned results). Although we obtained almost complete results for

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polarizations of degree 3, we chose not to inflict their very technical proofs on the unsuspecting reader.

We refer to Theorems 7, 8, 9, and 10 for more precise formulations and quote only the following results. Let (A,ℓ) be a polarized abelian variety of degree d and dimension g, and let D be any effective divisor that represents $m\ell$, with $m>0$.

Assume A is *simple* and $g>(d+1)^2/4$. *If $m=1$, the divisor D is normal and has rational singularities; if $k\geq 2$, the set of points of multiplicity at least mk on D has codimension $>k$ in A.*

Assume instead (A,ℓ) only *indecomposable,* $d=2$, and $g>2$. *If $m=1$, the divisor D, if prime, is normal and has rational singularities; if $k\geq 1$, the set of points of multiplicity at least mk on D has codimension $\leq k$ in A if and only if (A,ℓ) is a double étale cover of a product of at least k nonzero principally polarized abelian varieties.*

The proofs of these results systematically use generic vanishing theorems and precise descriptions of cohomological loci attached to various situations (see §5). We work over the complex numbers.

2. Singularities of pairs

We just need a quick review of the basic terminology relative to the singularities of a pair (A,D) consisting of an effective $\mathbf{Q}$-divisor D in a smooth projective variety A.

A *log resolution* of the pair (A,D) is a proper birational morphism $\mu : A' \to A$ such that the union of $\mu^{-1}(D)$ and the exceptional locus of μ is a divisor with simple normal crossing support. Write

$$\mu^*D - K_{A'/A} = \sum a_i D_i$$

where the D_i are distinct prime divisors on A'. The pair (A,D) is

- *log canonical* if $a_i \leq 1$ for all i;
- *log terminal* if $a_i < 1$ for all i;
- *canonical* if $a_i \leq 0$ for all i such that D_i is μ-exceptional;

for all log resolutions μ. The *multiplier ideal sheaf* associated to the pair (A,D) is

$$\mathscr{I}(A,D) = \mu_*\big(\mathscr{O}_{A'}(-\sum \lfloor a_i \rfloor D_i)\big)$$

One sees that

$$\begin{aligned}(A,D) \text{ log canonical} &\iff \mathscr{I}(A,tD)=\mathscr{O}_A \text{ for all } t\in\mathbf{Q}\cap(0,1)\\ (A,D) \text{ log terminal} &\iff \mathscr{I}(A,D)=\mathscr{O}_A\end{aligned}$$

Assume now that D is a prime divisor in A. The *adjoint ideal sheaf* $\mathscr{J}(A,D) \subset \mathscr{O}_A$ is defined in [EL], Proposition 3.1. For any desingularization $f : X \to D$, it fits into an exact sequence

$$0 \to \omega_A \to \omega_A(D) \otimes \mathscr{J}(A,D) \to f_*\omega_X \to 0 \tag{1}$$

of sheaves on A and ([EL], Proposition 3.1; [Ko2], Corollary 7.9.2, Theorem 7.9, and Theorem 11.1.1)

$$\begin{aligned} (A,D) \text{ canonical} &\iff \mathscr{J}(A,D) = \mathscr{O}_A \\ &\iff D \text{ is normal and has rational singularities} \end{aligned}$$

Furthermore, for any positive integers $m \geq 1$ and $k \geq 2$, we have

$$\begin{aligned} (A, \tfrac{1}{m}D) \text{ log canonical} &\implies \left\lfloor \tfrac{1}{m+1}D \right\rfloor = 0 \text{ and} \\ &\qquad \operatorname{codim}_A(\operatorname{Sing}_{mk} D) \geq k \\ (A, \tfrac{1}{m}D) \text{ log terminal} &\implies \left\lfloor \tfrac{1}{m}D \right\rfloor = 0 \text{ and} \\ &\qquad \operatorname{codim}_A(\operatorname{Sing}_{mk} D) > k \\ (A,D) \text{ canonical} &\implies \operatorname{codim}_A(\operatorname{Sing}_k D) > k \end{aligned} \tag{2}$$

where, for any positive integer ℓ, we denote by $\operatorname{Sing}_\ell D$ the set of points of multiplicity $\geq \ell$ on D.

3. Polarized abelian varieties

Let A be an abelian variety of dimension g. Any line bundle L on A induces a morphism $\varphi_L : A \to \operatorname{Pic}^0(A)$ defined by $\varphi_L(a) = \tau_a^* L \otimes L^{-1}$, where $\tau_a : A \to A$ is the translation $x \mapsto x - a$. This morphism only depends on the numerical equivalence class $[L]$ of L and will also be denoted by $\varphi_{[L]}$. We denote its kernel by $K(L)$ or $K([L])$. If D is a divisor on A, we write $[D]$ for $[\mathscr{O}_A(D)]$ and $K(D)$ for $K([D])$. The line bundle L is ample if and only if $K(L)$ is finite, in which case this group has order d^2, where

$$d = h^0(A,L) = \frac{1}{g!} c_1(L)^g$$

is the *degree* of L. A polarization on A is a numerical equivalence class of ample line bundles on A. A polarization of degree 1 is called *principal* and a divisor representing it is called a theta divisor. A polarization ℓ is of type (d) if $K(\ell) \simeq (\mathbf{Z}/d\mathbf{Z})^2$. If d is *prime*, any polarization of degree d is of type (d).

A polarized abelian variety (A, ℓ) is *indecomposable* if it is not the product of nonzero polarized abelian varieties. If $g \geq 2$, a general element of ℓ is prime.

4. Reducible divisors in indecomposable polarizations

We gather in this section elementary results about divisors in polarized abelian varieties that are either nonprime or have components with high multiplicities.

Lemma 1. *Let (A,ℓ) be an indecomposable polarized abelian variety. If the restriction of ℓ to an abelian subvariety A_1 of A is principal, either $A_1 = 0$ or $A_1 = A$.*

Proof. Let A_2 be the neutral component of the kernel of

$$f : A \xrightarrow{\varphi_\ell} \mathrm{Pic}^0(A) \longrightarrow \mathrm{Pic}^0(A_1)$$

The sum map $p : A_1 \times A_2 \to A$ is an isogeny of polarized abelian varieties whose kernel is isomorphic to $A_1 \cap A_2$. Since ℓ induces a principal polarization on A_1, the restriction of f to A_1 is injective, i.e., $A_1 \cap A_2 = \{0\}$ and p is injective. □

We now study indecomposable polarizations that contain nonprime divisors. The situation is manageable when A is simple or the degree is 2. For degrees at least 3, more and more exceptional cases arise.

Proposition 2. *Let (A,ℓ) be an indecomposable polarized abelian variety of degree d and dimension $g \geq d$ such that ℓ contains a nonprime divisor E. Then,*

a) *the abelian variety A is* not *simple;*

b) *if $d = 2$, there exist a* decomposable *principally polarized abelian variety $(B,[\Theta])$ and an isogeny $p : A \to B$ of degree d such that $E = p^*\Theta$.*

Proof. Write $E = E_1 + E_2$, with E_1 and E_2 effective and nonzero, and let, for $j \in \{1,2\}$, $A_j = K(E_j)^0$, $B_j = A/A_j$, and $g_j = \dim B_j > 0$. Since $A_1 \cap A_2$ is contained in $K(\ell)$, it is finite, hence $g_1 + g_2 \geq g$. There is an ample divisor D_j on B_j which pulls back to E_j, and

$$d = \frac{1}{g!}(E_1+E_2)^g = \frac{1}{g_1!(g-g_1)!}E_1^{g_1}E_2^{g-g_1} + \cdots + \frac{1}{(g-g_2)!g_2!}E_1^{g-g_2}E_2^{g_2}$$

The first term of this sum is

$$\begin{aligned}\frac{1}{(g-g_1)!}\deg(D_1)[A_1]\cdot E_2^{g-g_1} &= \deg(D_1)\deg(E_2|_{A_1})\\ &= \deg(D_1)\deg(\ell|_{A_1}) > 0\end{aligned}$$

and similarly for the last term, hence, by the Teissier–Hovanski inequalities

$$E_1^iE_2^{g-i} \geq \left(E_1^{g_1}E_2^{g-g_1}\right)^{\frac{i-g+g_2}{g_1+g_2-g}} \cdot \left(E_1^{g-g_2}E_2^{g_2}\right)^{\frac{g_1-i}{g_1+g_2-g}}$$

for $g_1 \geq i \geq g - g_2$, all terms in the sum are positive integers.

When A is simple, we have $g_1 = g_2 = g$, hence $d > g$. This proves a).

We now assume $d = 2$ and prove b). By Lemma 1, ℓ does not restrict to a principal polarization on A_j unless $A_j = 0$, i.e., $g_j = g$. The only possibility is $g_1 + g_2 = g$, the polarization $[D_j]$ on B_j is principal, the map $p : A \to B_1 \times B_2$ is an isogeny, and $E = p^*(D_1 \times B_2) + p^*(B_1 \times D_2)$. □

We use these results to bound the multiplicities of the components of elements of $m\ell$.

Corollary 3. *Let (A, ℓ) be an indecomposable polarized abelian variety of degree d and dimension $g \geq d$ and let $D \in m\ell$, with $m > 0$.*

a) *If A is simple, we have $\left\lfloor \frac{1}{m+1} D \right\rfloor = 0$. Moreover, $\left\lfloor \frac{1}{m} D \right\rfloor = 0$ unless $D = mE$, with $E \in \ell$.*

b) *If $d = 2$, we have $\left\lfloor \frac{1}{m+1} D \right\rfloor = 0$. Moreover, $\left\lfloor \frac{1}{m} D \right\rfloor = 0$ unless $D = mE$, with $E \in \ell$ prime, or there are nonzero principally polarized abelian varieties $(B_1, [\Theta_1])$ and $(B_2, [\Theta_2])$, and an isogeny $p : A \to B_1 \times B_2$ of degree 2, such that*

$$D = mp^*(\Theta_1 \times B_2) + p^*(B_1 \times D_2)$$

with $D_2 \in |m\Theta_2|$.

Proof. Assume $D = rD_1 + D_2$, with D_1 and D_2 effective nonzero and $r \geq m$. We follow [H1], Lemmas 2.2 and 2.3: the class $\ell - D_1 = \frac{r-m}{r}\ell + \frac{1}{r} D_2$ is nef and nonzero, hence effective. It follows that ℓ contains a nonprime divisor of the form $D_1 + E_2$, with E_2 effective nonzero, and

- either $r > m$ and E_2 is ample, contradicting Proposition 2;
- or $r = m$ and D_2 is numerically equivalent to mE_2, in which case Proposition 2.b) applies.

This proves the corollary. □

Remark 4. Polarized abelian varieties that satisfy the condition in Proposition 2.b) are all obtained as follows. Let d be any positive integer and, for $j \in \{1, 2\}$, let $(B_j, [\Theta_j])$ be a nonzero principally polarized abelian variety of dimension g_j. Endow $B = B_1 \times B_2$ with the product polarization $[\Theta]$. Choose a point β_j in B_j of order d and consider the cyclic isogeny $p : A \to B$ of degree d associated with the point $\varphi_\Theta(\beta_1, \beta_2)$ of $\mathrm{Pic}^0(B)$. The divisor $p^*\Theta$ is reducible and defines a polarization ℓ of type (d) on A.

Let $p_j : A_j \to B_j$ be the cyclic isogeny of degree d associated with $\varphi_{\Theta_j}(\beta_j)$ and let ℓ_j be the polarization $[p_j^*\Theta_j]$ (of type (d)) induced on A_j. There is a factorization

$$p_1 \times p_2 : A_1 \times A_2 \xrightarrow{\pi} A \xrightarrow{p} B$$

Another way to construct (A, ℓ) is to start from nonzero polarized abelian varieties (A_1, ℓ_1) and (A_2, ℓ_2) of type (d), choose elements $\alpha_1 \in K(\ell_1)$ and $\alpha_2 \in K(\ell_2)$ or order d, and take the quotient of $A_1 \times A_2$ by the subgroup generated by (α_1, α_2). For more details, see [D1], Proposition 9.1.

Assume d is *prime.* A polarized abelian variety of degree d is decomposable if and only if it has a nonzero principally polarized abelian factor. It follows that the polarized abelian variety (A, ℓ) obtained by the above construction is indecomposable if and only if both polarized abelian varieties (A_1, ℓ_1) and (A_2, ℓ_2) are indecomposable.

5. Cohomological loci in $\mathrm{Pic}^0(A)$

Let A be an abelian variety. For any coherent sheaf $\mathscr{F}$ on A and integer i, we define reduced subvarieties of $\mathrm{Pic}^0(A)$ by setting

$$\begin{aligned} V_i(\mathscr{F}) &= \{P \in \mathrm{Pic}^0(A) \mid H^i(A, \mathscr{F} \otimes P) \neq 0\} \\ V_{>0}(\mathscr{F}) &= \bigcup_{i>0} V_i(\mathscr{F}) \end{aligned}$$

We investigate the geometry of these loci when $\mathscr{F}$ is the tensor product of an ample line bundle with an ideal sheaf.

Lemma 5. *Let L be an ample line bundle of degree d on an abelian variety A of dimension g, with base locus $\mathrm{Bs}\,|L|$, and let Z be a subscheme of A, with ideal sheaf $\mathscr{I}$. Set $V_i = V_i(L \otimes \mathscr{I})$ and*

$$h = h^0(A, L \otimes \mathscr{I} \otimes P)$$

for P general in $\mathrm{Pic}^0(A)$. The following properties hold.

a) *For $i > \dim Z + 1$, the set V_i is empty.*
b) *If $h = 0$, the set $V_{>0}$ is nonempty.*
c) *We have $h \leq d$, and $h = d$ if and only if Z is empty.*
d) *If $h = d - 1 > 0$ and the polarized abelian variety $(A, [L])$ is indecomposable,*
   - *either Z has finite length and $V_1 = \mathrm{Pic}^0(A)$;*
   - *or Z is a single reduced point z and $V_1 = \varphi_L(\mathrm{Bs}\,|L| - z)$, so that $\dim V_1 \geq g - d$.*
e) *If A is simple and $0 < h < d$, we have $\dim Z \leq d - 1 - h$ and $\dim V_{>0} \geq g - (d+1)^2/4$.*

Note that $h = h^0(A, L \otimes \mathscr{I} \otimes P) = \chi(A, L \otimes \mathscr{I})$ for $P \notin V_{>0}$.

Proof. For $i > \dim Z+1$, we have $H^i(A, L\otimes\mathscr{I}\otimes P) \simeq H^i(A, L\otimes P) = 0$ for all $P \in \operatorname{Pic}^0(A)$. This proves a).

If $V_{>0}$ is empty,

$$h = h^0(A, L \otimes \mathscr{I} \otimes P) = \chi(A, L \otimes \mathscr{I})$$

for all $P \in \operatorname{Pic}^0(A)$. If $h = 0$, we have $H^i(A, L \otimes \mathscr{I} \otimes P) = 0$ for all integers i and all $P \in \operatorname{Pic}^0(A)$. This is impossible by [M], Corollary 2.4, and b) is proved.

We have $h = d$ if and only if all sections of L vanish on general translates of Z; this happens if and only if Z is empty. This proves c).

Let us now prove d). Set

$$J = \{(s, a) \in \mathbf{P}H^0(A, L) \times A \mid s|_{Z+a} \equiv 0\}$$

The fiber of a point a of A for the second projection $q : J \to A$ is isomorphic to $\mathbf{P}H^0(A, L \otimes P_{\varphi_L(a)} \otimes \mathscr{I})$. If $h > 0$, a unique irreducible component I of J dominates A, and $\dim I = g + h - 1$.

Let $p : I \to \mathbf{P}H^0(A, L)$ be the first projection. Since any nonempty $F_s = q\big(p^{-1}(s)\big)$ satisfies $Z + F_s \subset \operatorname{div}(s)$, we have

$$g-1 \geq \dim F_s \geq \dim I - \dim p(I) \geq g+h-1-\dim p(I) \geq g-(d-h)$$

Assume $h = d - 1 > 0$. Then p is surjective and F_s has dimension $g - 1$. The divisor of a general section s being prime, the inclusion $z + F_s \subset \operatorname{div}(s)$ is an equality for all z in Z. This implies that Z is finite. If $V_1 \neq \operatorname{Pic}^0(A)$, the length of Z is $d - h = 1$, so that $Z = \{z\}$, and an element $P = P_{\varphi_L(a)}$ of $\operatorname{Pic}^0(A)$ satisfies $H^1(A, L \otimes P \otimes \mathscr{I}) \neq 0$ if and only if the restriction

$$H^0(A, L \otimes P) \to H^0(Z, L \otimes \mathscr{O}_Z \otimes P) \simeq \mathbf{C}_z$$

is not surjective; in other words, if all sections of $L \otimes P$ vanish at z, i.e., $z \in \operatorname{Bs}|L \otimes P| = \operatorname{Bs}|L| - a$. This proves d).

Assume A is *simple* and $0 < h < d$. Then Z is nonempty and the inclusion $Z + F_s \subset \operatorname{div}(s)$ implies ([D2], Corollaire 2.7)

$$\dim Z \leq g - 1 - \dim F_s \leq d - 1 - h$$

For a general in A, the subvariety $p(q^{-1}(a))$ of $\mathbf{P}H^0(A, L)$ is a linear subspace of dimension $h-1$. It must vary with a, because a nonzero s does not vanish on all translates of Z. It follows that the linear span of $p(I)$ has dimension at least h. For $s_1, \ldots, s_{h+1}$ general elements in $p(I)$, one has ([D2], Corollaire 2.4)

$$\dim(F_{s_1} \cap \cdots \cap F_{s_{h+1}}) \geq g - (h + 1)(d - h) \geq g - (d + 1)^2/4$$

For $a \in F_{s_1} \cap \cdots \cap F_{s_{h+1}}$, the sections $s_1, \ldots, s_{h+1}$ all vanish on $Z + a$, hence $h^0(A, L \otimes \mathscr{I} \otimes P_{\varphi_L(a)}) \geq h + 1$. This implies $P_{\varphi_L(a)} \in V_{>0}$ and proves e). □

Assume now that there is a smooth projective variety X with a morphism $f : X \to A$ such that the sheaf $\mathscr{F}$ on A is a direct summand of $f_*\omega_X$. Let B be an abelian variety with a morphism $\pi : A \to B$. For all integers i and j, and any torsion point P_0 in $\operatorname{Pic}^0(A)$, *every irreducible component of* $V_i(R^j\pi_*(\mathscr{F} \otimes P_0))$ *is an abelian subvariety of* $\operatorname{Pic}^0(B)$ *of codimension at least* i *translated by a torsion point* (this applies in particular to the loci $V_i(\mathscr{F})$ in $\operatorname{Pic}^0(A)$).

When $P_0 = 0$, this follows from [HP]: write $f_*\omega_X = \mathscr{F} \oplus \mathscr{G}$, let $P \in \operatorname{Pic}^0(B)$ be a general point of an irreducible component V of $V_i(R^j\pi_*(f_*\omega_X))$, and set $m = h^i(B, R^j\pi_*(\mathscr{F}) \otimes P)$ and $n = h^i(B, R^j\pi_*(\mathscr{G}) \otimes P)$. Then V is an irreducible component of the locus

$$\{Q \in \operatorname{Pic}^0(B) \mid h^i(B, R^j\pi_*(f_*\omega_X) \otimes Q) \geq m + n\}$$

and the statement follows from [HP], Theorem 2.2.a) and b).

For the general case, associate to the torsion element f^*P_0 in $\operatorname{Pic}^0(X)$ a cyclic étale cover $p : X' \to X$. Then $\omega_X \otimes f^*P_0$ is a direct summand of $p_*\omega_{X'}$, hence $\mathscr{F} \otimes P_0$ is a direct summand of $f_*p_*\omega_{X'}$.

Lemma 6. *Under the hypotheses and notation of Lemma 5, assume further that there is an exact sequence*

$$0 \to \mathscr{O}_A^{\oplus \varepsilon} \to L \otimes \mathscr{I} \to \mathscr{F} \to 0$$

for some $\varepsilon \in \mathbf{N}$*, where* $\mathscr{F}$ *is a direct summand of a pushforward of a dualizing sheaf. The following properties hold.*

a) *Every irreducible component of* $V_i(L \otimes \mathscr{I})$ *is an abelian subvariety of* $\operatorname{Pic}^0(A)$ *of codimension at least* i *translated by a torsion point.*
b) *If the support of* $\mathscr{F}$ *is not contained in any nonample divisor of* A*, we have, for any* $i > 0$ *such that* $V_i(L \otimes \mathscr{I})$ *is nonempty,* $\dim Z \geq i - 1 + \dim V_i(L \otimes \mathscr{I})$.

Proof. Since $V_i(L \otimes \mathscr{I}) - \{0\} = V_i(\mathscr{F}) - \{0\}$, item a) holds. Let us prove b). Since b) follows from Lemma 5.a) when $V_i(L \otimes \mathscr{I})$ is finite, we may pick a common irreducible component V of $V_i(\mathscr{F})$ and of $V_i(L \otimes \mathscr{I})$ of maximal *positive* dimension. Let $B = \operatorname{Pic}^0(V)$ and let $\pi : A \to B$ be the induced morphism. Let $P_0 \in \operatorname{Pic}^0(A)$ be a torsion point such that $V = P_0 + \pi^* \operatorname{Pic}^0(B)$.

We know that $V_k(R^j\pi_*(\mathscr{F} \otimes P_0))$ has codimension at least k in $\operatorname{Pic}^0(B)$. It follows that for general $P \in \operatorname{Pic}^0(B)$, and all $k > 0$ and

$j \geq 0$,

$$H^k(B, R^j\pi_*(\mathscr{F} \otimes P_0) \otimes P) = 0$$

hence

$$H^0(B, R^i\pi_*(\mathscr{F} \otimes P_0) \otimes P) \simeq H^i(A, \mathscr{F} \otimes P_0 \otimes \pi^* P) \neq 0 \tag{3}$$

because $P_0 \otimes \pi^* P$ is in $V \subset V_i(\mathscr{F})$. We have an exact sequence

$$R^i\pi_*(P_0)^{\oplus\varepsilon} \longrightarrow R^i\pi_*(L \otimes \mathscr{I} \otimes P_0) \longrightarrow R^i\pi_*(\mathscr{F} \otimes P_0) \xrightarrow{\delta} R^{i+1}\pi_*(P_0)^{\oplus\varepsilon}$$

The sheaves $R^j\pi_*(P_0)$ are direct sums of numerically trivial line bundles on B (this follows from the proof of [K], Theorem 1). By a result of Kollár ([Ko3], Theorem 3.4; [HP], Theorem 2.1), the sheaf $R^i\pi_*(\mathscr{F} \otimes P_0)$ is torsion-free on $\pi(\operatorname{Supp} \mathscr{F})$, which is B by hypothesis. It follows that the support of the sheaf $R^i\pi_*(L \otimes \mathscr{I} \otimes P_0)$ is B: if it is not, the map δ is generically injective, hence injective; but a twist of $R^{i+1}\pi_*(P_0)^{\oplus\varepsilon}$ by a general $P \in \operatorname{Pic}^0(B)$ has no nonzero section, contradicting (3).

Since $R^i\pi_*(L \otimes P_0) = 0$, the short exact sequence

$$0 \to L \otimes \mathscr{I} \otimes P_0 \to L \otimes P_0 \to L \otimes \mathscr{O}_Z \otimes P_0 \to 0$$

yields a surjection

$$R^{i-1}\pi_*(L \otimes \mathscr{O}_Z \otimes P_0) \twoheadrightarrow R^i\pi_*(L \otimes \mathscr{I} \otimes P_0)$$

hence the support of $R^{i-1}\pi_*(L \otimes \mathscr{O}_Z \otimes P_0)$ is also B. In particular, all fibers of $\pi|_Z : Z \to B$ have dimension at least $i-1$, and b) follows. □

6. Singularities of ample divisors in abelian varieties

This section contains the central results of this article. Let (A, ℓ) be a polarized abelian variety. We study the singularities of a divisor in $m\ell$, with $m > 0$, when the dimension is large enough with respect to the degree.

Theorem 7. *Let (A, ℓ) be a* simple *polarized abelian variety of degree d and dimension $g > (d+1)^2/4$.*

a) *Every divisor in ℓ is prime, normal, and has rational singularities.*
b) *If $m \geq 2$ and D is a divisor in $m\ell$, the pair $(A, \frac{1}{m}D)$ is log terminal unless $D = mE$, with $E \in \ell$.*

In particular, in case b), the pair $(A, \frac{1}{m}D)$ is log canonical. In view of further investigations, it is natural to conjecture that the conclusion of the theorem hold under the weaker assumption $g > d$. We can prove the conjecture for $d \leq 3$. For $g > 4 = d$, we can prove that the pair $(A, \frac{1}{m}D)$ is log terminal when A is general. For $5 \geq g \geq d$, we can prove that the pair $(A, \frac{1}{m}D)$ is log canonical when A is simple. However, because of the technical nature of our arguments, we do not pursue this here.

Note that, in any dimension ≥ 2, and for any $d \geq 3$, there are examples (obtained by the construction of Remark 4) of indecomposable (but not simple!) polarized abelian varieties (A, ℓ) of degree d and, for any $m \geq d-1$, of pairs $(A, \frac{1}{m}D)$ that are *not* log canonical because D has a component of multiplicity $\left\lfloor \frac{md}{d-1} \right\rfloor > m$ (see (2)).

However, for polarizations of degree 2, the results of Theorem 7 can be extended to the case where the polarized abelian variety is only indecomposable.

Theorem 8. *Let* (A, ℓ) *be an* indecomposable *polarized abelian variety of degree* $d \leq 2$ *and dimension* $g > d$*.*

a) *Every prime divisor in* ℓ *is normal and has rational singularities.*
b) *If* $m \geq 2$ *and* D *is a divisor in* $m\ell$ *such that* $\left\lfloor \frac{1}{m}D \right\rfloor = 0$*, the pair* $(A, \frac{1}{m}D)$ *is log terminal.*

Note that ℓ may very well contain reducible elements (see §4). As to b), we explained in Corollary 3 exactly when the assumption $\left\lfloor \frac{1}{m}D \right\rfloor = 0$ fails to hold (recall that in any event, the pair $(A, \frac{1}{m}D)$ is always log canonical when $g \geq d$, as proved in [H2], Theorem 4.1).

Proof of Theorems 7 and 8. We set up the notation in order to give a uniform presentation for all cases. Note first that by Proposition 2.a), under the hypotheses of Theorem 7, any divisor that represents ℓ is prime.

Let L be an ample line bundle on A that represents ℓ, let E be a prime divisor in $|L|$, and let D be a divisor in $m\ell$. We let $\mathscr{I}_0 = \mathscr{I}_{Z_0}$ be the adjoint ideal $\mathscr{J}(A, E)$ and we let $\mathscr{I}_1 = \mathscr{I}_{Z_1}$ be the multiplier ideal $\mathscr{I}(A, \frac{1}{m}D)$. We have:

$$Z_0 = \varnothing \iff \text{the pair } (A, E) \text{ is canonical}$$
$$Z_1 = \varnothing \iff \text{the pair } (A, \tfrac{1}{m}D) \text{ is log terminal}$$

so that we must prove (under suitable assumptions) that Z_t is empty for $t \in \{0, 1\}$. We set as above, for P general in $\mathrm{Pic}^0(A)$,

$$h_t = h^0(A, L \otimes \mathscr{I}_t \otimes P) \in \{0, \dots, d\}$$

The point is to prove $h_t = d$ (Lemma 5.c)).

We set $V_i = V_i(L \otimes \mathscr{I}_t)$ and $V_{>0} = \bigcup_{i>0} V_i$ (see §5).

Case $t = 0$**.** The exact sequence (1) shows that Lemma 6 applies with $\varepsilon = 1$ and $\mathscr{F} = f_*\omega_X$, where $f : X \to E$ is a desingularization. In particular, $V_{>0} \neq \operatorname{Pic}^0(A)$, hence

$$h_0 = h^0(A, L \otimes \mathscr{I}_0 \otimes P) = \chi(A, L \otimes \mathscr{I}_0 \otimes P) = \chi(X, \omega_X)$$

for P general in $\operatorname{Pic}^0(A)$. Since E is not fibered by (nonzero) abelian varieties, we obtain $h_0 > 0$ by [EL], Theorem 3.

Case $t = 1$. In this case, $L \otimes \mathscr{I}_1$ is a direct summand of the pushforward of a dualizing sheaf. This can be seen as follows. Let $\mu : A' \to A$ be a log resolution of the pair (A, D). Set $L' = \mu^* L \otimes \mathscr{O}_{A'}(-\lfloor \frac{1}{m}\mu^* D \rfloor)$. The divisor

$$\mu^* D - m\Big\lfloor \frac{1}{m}\mu^* D\Big\rfloor \in |mL'|$$

defines a cyclic cover $g : X \to A'$ of degree m, and X is normal with rational singularities. Let $\nu : X' \to X$ be a desingularization; then $\omega_{A'} \otimes L'$ is a direct summand of $g_*\omega_X = g_*\nu_*\omega_{X'}$ ([EV], p. 33).

It follows that $\mu_* g_* \nu_* \omega_{X'}$ splits as a direct sum of m torsion free sheaves, one of these being

$$\begin{aligned}
\mu_*(\omega_{A'} \otimes L') &= \mu_*\Big(\omega_{A'/A} \otimes \mu^* L \otimes \mathscr{O}_{A'}\Big(- \Big\lfloor \frac{1}{m}\mu^* D \Big\rfloor\Big)\Big) \\
&= L \otimes \mu_*\Big(\omega_{A'/A} \otimes \mathscr{O}_{A'}\Big(- \Big\lfloor \frac{1}{m}\mu^* D \Big\rfloor\Big)\Big) \\
&= L \otimes \mathscr{I}_1
\end{aligned}$$

Lemma 6 therefore applies again, with $\varepsilon = 0$. Moreover, since $g > d$ and $\lfloor \frac{1}{m} D \rfloor = 0$, following the proof of [H1], Theorem 1, one obtains $h_1 > 0$.

If A is simple, $V_{>0}$ is finite, and Lemma 5.e) implies Theorem 7.

We now prove Theorem 8. Since $h_t > 0$, we need only consider the case $d = 2$.

If $h_t = 1$, since V_1 has codimension at least 1, the scheme Z_t is a single point and V_1 has dimension $\geq g - 2$ (Lemma 5.d)). This contradicts the fact that V_1 has dimension 0 (Lemma 6.b)).

Hence $h_t = 2$ and Theorem 8 is proved. □

We now interpret our results in terms of dimensions of loci of singularities.

Theorem 9. *Let (A, ℓ) be a* simple *polarized abelian variety of degree d and dimension $g > (d+1)^2/4$. Let m and k be positive integers. For all $D \in m\ell$, we have*

$$\dim \operatorname{Sing}_{mk} D < g - k$$

unless $k = 1$ and $D = mE$, with $E \in \ell$.

Proof. According to Theorem 7, the hypotheses imply that the pair (A, D) is canonical for $m = 1$ and the pair $(A, \frac{1}{m}D)$ is log terminal for $m \geq 2$, unless $D = mE$, with $E \in \ell$. Since the pair (A, E) is then also canonical, the theorem follows from (2). □

In the case of a polarization of degree 2, we get a more precise result, analogous to [EL], Corollary 2, and [H1], Corollary 2.

Theorem 10. *Let (A, ℓ) be an* indecomposable *polarized abelian variety of degree* 2 *and dimension $g > 2$ and let m and k be positive integers. The following properties are equivalent:*

- (i) *for some D in $m\ell$, the locus $\operatorname{Sing}_{mk} D$ contains an irreducible component of codimension k in A;*
- (ii) *the polarized abelian variety (A, ℓ) is a double étale cover of a product of k nonzero principally polarized abelian varieties.*

Proof. If (ii) holds, the polarization is represented by an étale cover of the theta divisor of a product of k nonzero principally polarized abelian varieties, hence (i) holds.

Assume (i). By Theorem 8.b) and (2), D must have a component of multiplicity at least m. If $D = mE$, with $E \in \ell$ prime, the pair (A, E) is canonical (Theorem 8.a)) and by (2), this contradicts (i). Therefore, by Corollary 3.b), there are nonzero principally polarized abelian varieties $(B_1, [\Theta_1])$ and $(B_2, [\Theta_2])$ and an isogeny $p : A \to B_1 \times B_2$ such that

$$D = mp^*(\Theta_1 \times B_2) + p^*(B_1 \times D_2)$$

where $D_2 \in |m\Theta_2|$.

If S is a component of $\operatorname{Sing}_{mk} D$ of maximal dimension, there is an integer $l \leq k$ such that

$$S \subset \operatorname{Sing}_l \Theta_1 \times \operatorname{Sing}_{m(k-l)} D_2$$

From [Ko1], Theorem 17.1, we get

$$\operatorname{codim}(\operatorname{Sing}_l \Theta_1) \geq l$$

From [EL], Proposition 3.5, we get

$$\operatorname{codim}(\operatorname{Sing}_{m(k-l)} D_2) \geq k - l$$

Since S has codimension k, both inequalities must be equalities. By [EL], Corollary 2, $(B_1, [\Theta_1])$ splits as a product of l nonzero principally polarized abelian varieties, and by [H1], Corollary 2, $(B_2, [\Theta_2])$ splits as the product of at least $k - l$ principally polarized abelian varieties, so that (ii) holds. □

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Mathématique – IRMA – UMR 7501, Université Louis Pasteur, 7, rue René Descartes, 67084 Strasbourg Cédex, France

E-mail address: debarre@math.u-strasbg.fr

URL: http://www-irma.u-strasbg.fr/∼debarre

Department of Mathematics, University of Utah, 155 South 1400 East, Room 233, Salt Lake City, UT 84112, USA

E-mail address: hacon@math.utah.edu

URL: http://www.math.utah.edu/∼hacon