

The diagonal property for abelian varieties

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Dedicated to Roy Smith on his 65th birthday.

ABSTRACT. We study complex abelian varieties of dimension g that have a vector bundle of rank g with a section that vanishes at a single point, with multiplicity 1

The authors of [PSP] study which varieties X (defined, say, over an algebraically closed field) satisfy what they call “the diagonal property”: *there exists a vector bundle of rank $\dim(X)$ on $X \times X$ with a section whose zero-scheme is the diagonal.* This property is known to hold for all flag varieties SL_n/P ([F1]) and, using a variant of Serre’s construction, they show that it holds for projective surfaces which have a cohomologically trivial line bundle. In particular, it holds for abelian surfaces.

The diagonal property implies the weaker “point property”: *for each point x of X , there exists a vector bundle of rank $\dim(X)$ on X with a section whose zero-scheme is x .* When X is a group variety, these two properties are equivalent (it is even enough to have the point property for a single point x).

In this note, we study these (equivalent) properties when X is a complex abelian variety. We show in §1 that a general non-principally polarized abelian variety of a sufficiently high dimension does not have these properties. Using Picard bundles, we show in §2 that these properties hold when X is the Jacobian of a smooth curve (or a product of such). However, Lange pointed out that in any dimension, principally polarized abelian varieties that have the point property are dense in their moduli space (Remark 2.4). I do not know any principally polarized abelian variety that does not have the point property, although I would like to think that Jacobians of curves are the only principally polarized abelian varieties *with Picard number 1* that have this property, thereby giving us another (partial) solution to the Schottky problem. In §3 and 4, we prove a necessary condition for the point property to hold, and use it to get restrictions on possible vector bundles.

We work over the complex numbers, although the results of §2 and 3 are valid over an algebraically closed field of arbitrary characteristic.

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1. Non-principally polarized abelian varieties

Let X be an abelian variety of dimension g . If \mathcal{E} is a vector bundle of rank g on X with a section whose zero-scheme is the origin o , we have $c_g(\mathcal{E}) = [o]$ in the Chow group of X . We use simple-minded numerics coming from the fact that the number

$$\chi(X, \mathcal{E}) = \int_X \text{ch}_g(\mathcal{E})$$

(this is the Hirzebruch-Riemann-Roch formula; [F2], Corollary 15.2.1) is an integer.

If (X, ℓ) is a very general polarized abelian variety of type $(\delta_1 | \cdots | \delta_g)$ ([BL], §3.1), we know by Mattuck's theorem ([BL], Theorem 17.4.1) that $c_i(\mathcal{E})$ is, in cohomology, a rational multiple of ℓ^i . Since $\frac{\ell^i}{\delta_1 \cdots \delta_i i!}$ is a nondivisible integral class in $H^{2i}(X, \mathbf{Z})$, we may write

$$(1.1) \quad c_i(\mathcal{E}) = a_i \frac{\ell^i}{\delta_1 \cdots \delta_i i!} \quad \text{with } a_i \in \mathbf{Z}.$$

We obtain ([Mc], p. 20)

$$(1.2) \quad \chi(X, \mathcal{E}) = \delta_1 \cdots \delta_g D\left(\frac{a_1}{\delta_1}, \dots, \frac{a_g}{\delta_1 \cdots \delta_g}\right)$$

where

$$(1.3) \quad D(b_1, \dots, b_g) := \begin{vmatrix} b_1 & 1 & 0 & \cdots & \cdots & 0 \\ b_2 & b_1 & 1 & \ddots & & \vdots \\ b_3/2! & b_2/2! & b_1 & \ddots & \ddots & \vdots \\ b_4/3! & b_3/3! & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ b_g/(g-1)! & b_{g-1}/(g-1)! & \cdots & b_3/3! & b_2/2! & b_1 \end{vmatrix}$$

PROPOSITION 1.1. *A very general nonprincipally polarized abelian variety of type $(\delta_1 | \cdots | \delta_g)$ whose dimension g is greater than some prime factor of δ_g/δ_1 does not have the point property.*

PROOF. We may assume $\delta_1 = 1$. Expanding the determinant (1.3), we see that the integer $\chi(X, \mathcal{E})$ can be written as $(-1)^{g-1} \frac{a_g}{(g-1)!} + \delta_g \frac{a}{(g-1)!}$ for some integer a . It follows that a_g is divisible by $\gcd(\delta_g, (g-1)!)$, hence the proposition. \square

2. Jacobians of curves

Let C be a smooth connected projective curve of genus $g \geq 2$ and let JC be its Jacobian, endowed with its canonical principal polarization θ . Any element ξ of JC defines a numerically trivial line bundle P_ξ on JC .

Fixing a point c of C , we view the curve C as embedded in JC by sending a point x of C to the isomorphism class of $\mathcal{O}_C(x - c)$ and we define W_i as the i -fold sum $C + \cdots + C$, with the convention $W_0 = \{o\}$.

Let \mathcal{P} be the Poincaré line bundle on $C \times JC$, uniquely defined by the properties

$$\mathcal{P}|_{\{c\} \times JC} \simeq \mathcal{O}_{JC} \quad \text{and} \quad \mathcal{P}|_{C \times \{\xi\}} \simeq P_\xi|_C \quad \text{for all } \xi \in JC.$$

Following [S], §2, Definition (see also [Mk], and [Mu], Definition 4.1), we define the *Picard bundle* by¹

$$\mathcal{F} = R^1 q_* (\mathcal{P} \otimes p^* \mathcal{O}_C(-c))$$

where $p : C \times JC \rightarrow C$ and $q : C \times JC \rightarrow JC$ are the projections. By [S], *the sheaf \mathcal{F} is locally free of rank g on JC* . Moreover, if ι is the involution $\xi \mapsto K_C - (2g-2)c - \xi$ of JC , the morphism² $\pi : \mathbf{P}(\iota^* \mathcal{F}) \rightarrow JC$ is isomorphic to the Abel-Jacobi map

$$\begin{aligned} \alpha : \quad C^{(2g-1)} &\longrightarrow JC \\ x_1 + \cdots + x_{2g-1} &\longmapsto \mathcal{O}_C(x_1 + \cdots + x_{2g-1} - (2g-1)c) \end{aligned}$$

and the divisor $C^{(2g-2)} + c$ in $C^{(2g-1)}$ represents the ample line bundle $\mathcal{O}_{\mathbf{P}(\iota^* \mathcal{F})}(1)$ ([S], Theorem 2). The Chern classes of \mathcal{F} were computed by Mattuck in [Mk], §6, Corollary (see also [S], §4, and [G], Corollary 3 to Theorem 4); he obtains:

$$(2.1) \quad c_{g-i}(\mathcal{F}) = [W_i] \quad \text{for all } i \in \{0, \dots, g\},$$

in the Chow group of JC .

THEOREM 2.1. *For all $i \in \{0, \dots, g\}$, we have*

$$h^i(JC, \mathcal{F} \otimes P_\xi) = \begin{cases} \binom{g-1}{i} & \text{if } -\xi \in C; \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the (scheme-theoretic) zero-locus of any nonzero section of \mathcal{F} is $W_0 = \{o\}$.

The first statement of the theorem was also obtained in [Mu], Proposition 4.3, using the Fourier transform (see also [G], Corollary 2 to Theorem 4).

PROOF. We have

$$\begin{aligned} H^i(JC, \mathcal{F} \otimes P_\xi) &\simeq H^i(JC, \iota^* \mathcal{F} \otimes \iota^* P_\xi) \\ &\simeq H^i(\mathbf{P}(\iota^* \mathcal{F}), \mathcal{O}_{\mathbf{P}(\iota^* \mathcal{F})}(1) \otimes \pi^* P_{-\xi}) \\ &\simeq H^i(C^{(2g-1)}, \mathcal{O}_{C^{(2g-1)}}(C^{(2g-2)} + c) \otimes \alpha^* P_{-\xi}) \\ &\simeq \wedge^i H^1(C, c - \xi) \otimes \text{Sym}^{2g-1-i} H^0(C, c - \xi) \end{aligned}$$

by [I], §3.1. This space vanishes if $\xi \notin -C$, and if ξ represents $\mathcal{O}_C(c - c')$, it is isomorphic to $\wedge^i H^0(C, K_C - c')^\vee$.

In particular, the zero-locus of a nonzero section of $\iota^* \mathcal{F}$ is the set of points ξ' of JC such that $\alpha^{-1}(\xi') \subset C^{(2g-2)} + c$, i.e., c is a base-point for the linear system $|\mathcal{O}_C((2g-1)c) \otimes P_{\xi'}|$. This is $\{\iota(o)\}$. By (2.1), the order of vanishing at this point must be 1, and this proves the last statement of the theorem. \square

COROLLARY 2.2. *The Jacobian of any smooth projective curve satisfies the point property.*

Of course, any (finite) product of varieties that satisfy the point property also satisfies this property.

The vector bundle \mathcal{F} is known to be stable ([K]), and its Euler characteristic is 0 (by Theorem 2.1, or [Mk], §8).

¹This is the sheaf denoted by \mathcal{F}_{-1} (or F_{-1}) in [S] and [Mu], and by P_{2g-1} in [K] and [EL]. In the terminology of [Mu], the sheaf $\mathcal{O}_C(-c)$, viewed as a (torsion) sheaf on JC , is IT_1 and its Fourier-Mukai transform is \mathcal{F} .

²As in [S], we use Grothendieck's convention for projectivization

We may ask to what extent \mathcal{F} is unique. In dimension 2, Mukai proved the following ([Mu], Theorem 5.4): *on a principally polarized abelian surface (X, θ) , any stable vector bundle of rank 2 with first Chern class θ and second Chern class 1 is a translate of \mathcal{F} tensored by a numerically trivial line bundle.* However, the vector bundle \mathcal{E} constructed in [PSP] to prove the point property for an abelian surface X is different: it appears as an extension

$$(2.2) \quad 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow P \otimes \mathcal{I}_o \longrightarrow 0$$

where $P = \det(\mathcal{E})$ can be any nontrivial numerically trivial line bundle on X (\mathcal{E} is of course not stable, only semistable).

PROPOSITION 2.3. *The Picard bundle does not deform outside the closure of the Jacobian locus in the moduli space \mathcal{A}_g of principally polarized abelian varieties of dimension g .*

PROOF. In the terminology of [Mu], any small deformation \mathcal{G} of \mathcal{F} remains WIT_{g-1} ([Mu], Proposition 4.3), and $\widehat{\mathcal{G}}$ is WIT_1 ; since it is nonzero, its support must have dimension at least 1. Since $\widehat{\mathcal{F}}$ has support $-C$, the support of $\widehat{\mathcal{G}}$ has dimension at most 1 and must be a deformation of $-C$ (and, possibly, finitely many points). This can only happen if the deformed abelian variety is still a Jacobian (or a product of such) by Matsusaka's criterion ([Ma]). \square

REMARK 2.4 (Lange). For any $g > 0$, the set of principally polarized abelian varieties that satisfy the point property is *dense* in \mathcal{A}_g . This can be seen as follows: any abelian variety isogeneous to E^g , where E is an elliptic curve with complex multiplication, is *isomorphic* to the product of g elliptic curves ([BL], Exercise 5.6.(10)), hence has the point property. Moreover, the corresponding subset of \mathcal{A}_g is dense ([L]). These varieties all have Picard number g^2 . An explicit example of such a principally polarized abelian fourfold which is not a Jacobian can be found in [D], §5.

3. A necessary condition

Following ideas of [PSP], we get a necessary condition, on any smooth projective variety, for the point property to be satisfied.

PROPOSITION 3.1. *Let X be a smooth projective variety of dimension g and let \mathcal{E} be a vector bundle of rank g on X with a section s whose zero-scheme is a single point x . All sections of $\det(\mathcal{E}) \otimes \omega_X$ then vanish at x .*

PROOF. Let \mathcal{L} be the invertible sheaf $\det(\mathcal{E}) = \wedge^g \mathcal{E}$. The long exact sequence (Koszul complex)

$$0 \rightarrow \wedge^g \mathcal{E}^\vee \rightarrow \dots \rightarrow \wedge^2 \mathcal{E}^\vee \rightarrow \mathcal{E}^\vee \xrightarrow{s^\vee} \mathcal{I}_x \rightarrow 0$$

determines an element $[s]$ of $\text{Ext}_X^{g-1}(\mathcal{I}_x, \mathcal{L}^\vee)$. The short exact sequence $0 \rightarrow \mathcal{I}_x \rightarrow \mathcal{O}_X \rightarrow \mathbf{C}_x \rightarrow 0$ induces another exact sequence

$$\text{Ext}_X^{g-1}(\mathcal{I}_x, \mathcal{L}^\vee) \xrightarrow{\beta} \text{Ext}_X^g(\mathbf{C}_x, \mathcal{L}^\vee) \rightarrow H^g(X, \mathcal{L}^\vee) \xrightarrow{\gamma} \text{Ext}_X^g(\mathcal{I}_x, \mathcal{L}^\vee) \rightarrow 0$$

The image of $\beta([s])$ by the morphism

$$(3.1) \quad \text{Ext}_X^g(\mathbf{C}_x, \mathcal{L}^\vee) \rightarrow H^0(X, \mathcal{E}xt_{\mathcal{O}_X}^g(\mathbf{C}_x, \mathcal{L}^\vee)) \simeq \mathbf{C}$$

is nonzero because \mathcal{E} is locally free (compare with the proof of **[PSP]**, Proposition 1). Since, by Serre duality, the vector space on the left-hand-side of (3.1) is also 1-dimensional, β is surjective, hence γ is bijective and its Serre-dual is

$$H^0(X, \mathcal{L} \otimes \omega_X \otimes \mathcal{I}_x) \xrightarrow{\sim} H^0(X, \mathcal{L} \otimes \omega_X)$$

In other words, all sections of $\mathcal{L} \otimes \omega_X$ vanish at x . \square

Of course, $\det(\mathcal{E}) \otimes \omega_X$ might have no nonzero sections, in which case the proposition says nothing at all.

4. Principally polarized abelian varieties

Let now (X, θ) be a very general principally polarized abelian variety of dimension g and let \mathcal{E} be a vector bundle of rank g on X . We may, as in (1.1), write in cohomology $c_i(\mathcal{E}) = a_i \theta^i / i!$, with $a_i \in \mathbf{Z}$, and $\chi(X, \mathcal{E}) = D(a_1, \dots, a_g)$ (see (1.2)). This number is an integer, and this implies various congruences, one example of which is the following.

PROPOSITION 4.1. *If $p := g - 1$ is prime, we have*

$$a_1 a_{g-1} \equiv a_g \pmod{p}$$

In particular, if $a_g = 1$, a_1 is prime to p .

PROOF. Expanding the determinant (1.3) along its last row, we get

$$\chi(X, \mathcal{E}) = (-1)^{g-1} \frac{1}{(g-1)!} (a_g - a_1 a_{g-1}) + a$$

where a is a rational number with a denominator whose prime factors are all $< p$. Since $\chi(X, \mathcal{E})$ is an integer, this proves the proposition. \square

Combining this result with Proposition 3.1, we obtain the following.

COROLLARY 4.2. *Let (X, θ) be a very general principally polarized abelian variety of dimension g and let \mathcal{E} be a vector bundle of rank g on X with a section whose zero-scheme is the origin.*

If \mathcal{E} is stable, or if \mathcal{E} is semistable and $g - 1$ is prime, $c_1(\mathcal{E}) = \theta$.

In particular, if $g = 2$ and \mathcal{E} is stable, it follows from the result of Mukai mentioned in §2 that \mathcal{E} is a translate of the vector bundle \mathcal{F} constructed there, tensored by a numerically trivial line bundle.

PROOF. Write $c_1(\mathcal{E}) = a_1 \theta$. If \mathcal{E} is stable (resp. semistable), we have $a_1 \geq 1$ (resp. $a_1 \geq 0$). On the other hand, by Proposition 3.1, the linear system $|\det(\mathcal{E})|$ has a base-point, hence $a_1 \leq 1$ by the Lefschetz Theorem. Finally, by Proposition 4.1, if $g - 1$ is prime, $a_1 \neq 0$. This proves the corollary. \square

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