UNEXPECTED ISOMORPHISMS BETWEEN HYPERKÄHLER FOURFOLDS

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Abstract. We study smooth projective hyperkähler fourfolds which are deformations of Hilbert squares of K3 surfaces and are equipped with a polarization of fixed degree and divisibility. When their Picard number is 2, we describe their nef and movable cones and their biregular and birational automorphism groups. We then apply these results to determine which of these varieties are isomorphic to Hilbert squares of K3 surfaces, varieties of lines on cubic fourfolds, or double EPW sextics.

1. Introduction

We consider smooth projective hyperkähler fourfolds F which are deformations of Hilbert squares of K3 surfaces (one says that F is of $\mathrm{K3}^{[2]}$-type), equipped with a polarization h, considered as a primitive element of the lattice $(H^2(F,\mathbf{Z}), q_{BBJ})$, where q_{BBJ} is the Beauville–Bogomolov–Fujiki quadratic form. The *square* (or *degree*) of h is the positive even integer $q_{BBJ}(h)$ and its *divisibility* is the integer $\gamma \in \{1,2\}$ such that $h \cdot H^2(F,\mathbf{Z}) = \gamma\mathbf{Z}$.

Smooth polarized hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type of degree $2n$ and divisibility γ admit an irreducible quasi-projective coarse moduli space $\mathscr{M}_{2n}^{(\gamma)}$ of dimension 20 ([GHS3, Remark 3.17]; the case $\gamma = 2$ only occurs when $n \equiv -1 \pmod 4$). When $n = 1$ and $\gamma = 1$, the corresponding hyperkähler fourfolds are the so-called double EPW sextics (see Example 2.9); when $n = 3$ and $\gamma = 2$, they are the varieties of lines contained in cubic hypersurfaces in $\mathbf{P}^5$ (see Example 2.5); when $n = 11$ and $\gamma = 2$, these fourfolds were geometrically described in [DV] (see Example 2.6); when $n = 19$ and $\gamma = 2$, these fourfolds are "varieties of sums of powers" and were studied in [IR1] (see Example 2.7).

A fourfold corresponding to a very general point of $\mathscr{M}_{2n}^{(\gamma)}$ has Picard number 1 and, except when $n = 1$, trivial birational automorphism group (Proposition 2.11). The set of points (called *special*) of $\mathscr{M}_{2n}^{(\gamma)}$ corresponding to fourfolds with Picard number at least 2 is called the *Noether–Lefschetz locus*. It is a countable union of hypersurfaces in $\mathscr{M}_{2n}^{(\gamma)}$, which we label as $\mathscr{C}_{2n,2e}^{(\gamma)}$, where the positive even integer $2e$ is called the *discriminant* (and can take infinitely many values). The irreducible components of these hypersurfaces were shown in [BLMM, Theorem 1.5] to generate (over $\mathbf{Q}$) the Picard group of $\mathscr{M}_{2n}^{(\gamma)}$.

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When n is a prime number, we describe these irreducible components (see Proposition 2.3 when $\gamma = 2$ and Proposition 2.8 when $\gamma = 1$). This extends the work of Hassett in the case $n = 3$ and $\gamma = 2$ ([H1]). The case $n = \gamma = 1$ was worked out in [DIM].

Our first main result is, when e is moreover divisible by n, the determination of the nef and movable cones of the fourfolds F corresponding to very general points of $\mathscr{C}^{(2)}_{2n,2e}$ (Proposition 4.2) and the computation of the biregular and birational automorphism groups of F (Proposition 4.7). These groups are either trivial, infinite cyclic, or infinite dihedral, depending on the solvability or non-solvability of Pell-type equations related to n and e. The proofs use results from [BHT] and the Torelli theorem for hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type ([V]; Theorems 2.1 and 2.2). Similar ideas were used in [AV] to produce interesting automorphisms on suitable deformations of any hyperkähler manifold with second Betti number at least 5.

Hilbert squares of K3 surfaces are obviously hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type and their Picard number is at least 2 but they carry no preferred polarization. Given a polarized K3 surface (S, f_S) with Picard group $\mathbf{Z}f_S$, the ample cone of the Hilbert square $S^{[2]}$ was determined in [BM2] and it only depends on the degree $e := f_S^2/2$ (Theorem 3.4). Fixing an element h of this cone defines a point of $\mathscr{M}^{(\gamma)}_{2n}$, where $2n := q_{BBJ}(h)$ and $\gamma := \mathrm{div}(h)$, which is special. This implies that the fourfolds corresponding to general points of some components of the Noether–Lefschetz loci in $\mathscr{M}^{(\gamma)}_{2n}$ are isomorphic to Hilbert squares of K3 surfaces.

Our second main result is the precise determination of the components of the Noether–Lefschetz loci that one obtains in this fashion, i.e., we determine, given n and γ, for which positive integers e the fourfolds corresponding to general points of some irreducible component of $\mathscr{C}^{(\gamma)}_{2n,2e}$ are isomorphic to Hilbert squares of K3 surfaces (Proposition 5.1). We show that there are always infinitely many such components and that their union is dense for the euclidean topology (Propositions 5.5 and 5.18). This proves in particular that $\mathscr{M}^{(1)}_{2n}$ is non-empty for all positive integers n, that $\mathscr{M}^{(2)}_{2n}$ is non-empty for all positive integers $n \equiv -1 \pmod 4$, and that in both cases, the set of points corresponding to Hilbert squares of K3 surfaces is dense.

In the case $n = 3$ and $\gamma = 2$, the fourfolds involved are the varieties of lines contained in cubic hypersurfaces in $\mathbf{P}^5$ and our result extends prior results of Hassett's ([H1]). We also identify in Proposition 5.10 the components of the Noether–Lefschetz locus in $\mathscr{M}^{(1)}_{2}$ that one obtains in the same fashion, thereby determining which double EPW sextics with Picard number 2 are isomorphic to Hilbert squares of K3 surfaces: by Corollary 5.11, they correspond exactly to those Hilbert squares of polarized K3 surfaces with Picard number 1 that have a non-trivial involution (they were determined in [BCNS]).

There are other "unexpected isomorphisms" between hyperkähler fourfolds F of $\mathrm{K3}^{[2]}$-type. If the Picard number of F is 1, the polarization is uniquely determined as the ample generator of the Picard group and nothing interesting happens. But if the Picard number of F is 2 (so we are considering very general points of a component of the Noether–Lefschetz locus in some moduli space $\mathscr{M}^{(\gamma)}_{2n}$), one can choose a different polarization on F, thereby defining an isomorphism between F and the fourfold corresponding to a very general point of some component of the Noether–Lefschetz locus in some another moduli space.

We investigate these unexpected isomorphisms in two instances. We determine when the fourfolds corresponding to general points of some component of the Noether–Lefschetz locus in $\mathscr{M}_{2n}^{(\gamma)}$ are isomorphic to varieties of lines contained in a cubic fourfold (Proposition 5.14). We show that there are always infinitely many such components and that their union is dense for the euclidean topology (Proposition 5.18). Finally, when n divides e, we determine which varieties in $\mathscr{C}_{2n,2e}^{(2)}$ are double EPW sextics; when this is the case, F has infinitely many involutions $(\iota_m)_{m\in\mathbf{Z}}$ and all the quotients F/ι_m are EPW sextics (Proposition 5.22).

The article is organized as follows.

In Section 2, we recall a few facts about the moduli spaces $\mathscr{M}_{2n}^{(\gamma)}$ and their period maps and state the Torelli theorem (Section 2.1). We define the Noether–Lefschetz locus in $\mathscr{M}_{2n}^{(\gamma)}$ and determine its irreducible components when n is a prime number (Section 2.2). Finally, we determine the biregular and birational automorphism groups of a very general element of $\mathscr{M}_{2n}^{(\gamma)}$ (Section 2.3): they are both trivial except in the case of double EPW sextics ($n=\gamma=1$), which have a non-trivial biregular involution.

We study in Section 3 the geometry of Hilbert squares of K3 surfaces. We begin with a quick survey of elementary results on Pell-type equations which will be needed throughout the article (Section 3.1). We then explain how to deduce from the results of [BM2] descriptions of the nef and movable cones of Hilbert squares of K3 surfaces with Picard number 1 (Section 3.2), and of the contractions of their two extremal rays (Section 3.3). We then determine their birational automorphism group (Section 3.4; the biregular automorphism group was determined in [BCNS]) and completely characterize Hilbert squares that are *ambiguous* in the sense of [H1] (i.e., when there is an isomorphism $S^{[2]}\simeq \bar{S}^{[2]}$ for some other K3 surface $\bar{S}$ which is not induced by an isomorphism $S\simeq\bar{S}$; see Section 3.5).

In Section 4, we perform a similar study of hyperkähler fourfolds of K3$^{[2]}$-type with a polarization of degree $2n$, divisibility 2, and Picard number 2 (that includes the case $n=3$ of varieties of lines contained in cubic hypersurfaces in $\mathbf{P}^5$): when n divides e, we determine their nef and movable cones (Section 4.2) and their biregular and birational automorphism groups (Section 4.3).

In Section 5, we study unexpected isomorphisms between hyperkähler fourfolds of K3$^{[2]}$-type with Picard number 2. We first determine for which e a general element of some component $\mathscr{C}_{2n,2e}^{(\gamma)}$ of the Noether–Lefschetz locus in $\mathscr{M}_{2n}^{(\gamma)}$ is the Hilbert square of a K3 surface (Section 5.1). The cases of varieties of lines contained in cubic hypersurfaces ($n=3$, $\gamma=2$), Debarre–Voisin fourfolds ($n=11$, $\gamma=2$), varieties of sums of powers ($n=19$, $\gamma=2$), IKKR fourfolds ($n=2$, $\gamma=1$), and double EPW sextics ($n=1$, $\gamma=1$) are explained in Sections 5.2, 5.3, 5.4, 5.5, and 5.6. In Section 5.7, we answer an analogous question: for which e is a general element of $\mathscr{C}_{2n,2e}^{(\gamma)}$ the variety of lines contained in a cubic fourfold? We also introduce and study ambiguous cubic fourfolds (Section 5.11). Finally, in Section 5.12, when n divides e, we determine when a general element in an irreducible component of $\mathscr{C}_{2n,2e}^{(2)}$ is a double EPW sextic.

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2. Hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type: moduli spaces, period maps, Noether–Lefschetz loci, and automorphisms

In this section, we review a few known facts about the irreducible quasi-projective 20-dimensional coarse moduli spaces $\mathscr{M}_{2n}^{(\gamma)}$ of hyperkähler fourfolds F of $\mathrm{K3}^{[2]}$-type with a polarization of fixed (positive) square $2n$ and divisibility $\gamma \in \{1, 2\}$ in the lattice $H^2(F, \mathbf{Z})$ (see [GHS3, Remark 3.17] for their construction). We introduce the period map, now known to be an open embedding thanks to work of Verbitsky and Markman (Theorems 2.1 and 2.2). We study, in each space $\mathscr{M}_{2n}^{(\gamma)}$, the (Noether–Lefschetz) locus of hyperkähler fourfolds with Picard number at least 2 (also called *special*); it is a countable union of hypersurfaces which we describe in Propositions 2.3 and 2.8. Finally, we show in Proposition 2.11 that a very general point in each moduli space $\mathscr{M}_{2n}^{(\gamma)}$ corresponds to a fourfold with trivial birational automorphism group (with the one exception $n = \gamma = 1$).

2.1. **The period maps and Torelli theorems for polarized hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type.** Given a lattice Λ, we denote by $\Lambda(t)$ the lattice obtained by multiplying the quadratic form by t. We let $\Lambda^\vee$ be the dual lattice $\mathrm{Hom}(\Lambda, \mathbf{Z})$ and define the *discriminant group* $D(\Lambda) := \Lambda^\vee/\Lambda$. Let x be a non-zero element of Λ; we define its *divisibility* $\mathrm{div}(x)$ as the positive generator of the subgroup $x \cdot \Lambda$ of $\mathbf{Z}$ and set $x_* := x/\mathrm{div}(x) \in D(\Lambda)$. When Λ is an even lattice, we endow the group $D(\Lambda)$ with the $\mathbf{Q}/2\mathbf{Z}$-valued quadratic form $\mathbf{q}$ defined by $\mathbf{q}(x) = x^2 \pmod{2\mathbf{Z}}$. The *stable orthogonal group* $\widetilde{O}(\Lambda)$ is the subgroup of the orthogonal group $O(\Lambda)$ that consists of automorphisms of Λ that act trivially on $D(\Lambda)$.

Let F be a hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type. The lattice $(H^2(F, \mathbf{Z}), q_{BBJ})$ is isomorphic to the even lattice

$$L_{\mathrm{K3}^{[2]}} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus I_1(-2)$$

with signature $(3, 20)$, where U is the lattice $\mathbf{Z}^2$ with intersection matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, E_8 the unique positive definite even unimodular lattice of rank 8, and I_1 the lattice $\mathbf{Z}$ with intersection matrix (1). The discriminant group of $L_{\mathrm{K3}^{[2]}}$ is $\mathbf{Z}/2\mathbf{Z}$.

By Eichler's criterion ([GHS2, Lemma 3.5]), primitive elements of $L_{\mathrm{K3}^{[2]}}$ with fixed positive square $2n$ and fixed divisibility γ form a single $O(L_{\mathrm{K3}^{[2]}})$-orbit. We fix one such element h_0. If $\gamma = 1$, we have

$$h_0^\perp \simeq U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus I_1(-2) \oplus I_1(-2n) =: L^{(1)}_{\mathrm{K3}^{[2]}, 2n}, \tag{1}$$

a lattice with discriminant group $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2n\mathbf{Z}$. If $\gamma = 2$, we have $n \equiv -1 \pmod 4$ and

$$h_0^\perp \simeq U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus \begin{pmatrix} -2 & -1 \\ -1 & -\frac{n+1}{2} \end{pmatrix} =: L^{(2)}_{\mathrm{K3}^{[2]}, 2n}, \tag{2}$$

a lattice with discriminant group $\mathbf{Z}/n\mathbf{Z}$ ([GHS3, Example 7.12]).

We now describe the period mapping. The space

$$\Omega_{h_0} := \{x \in \mathbf{P}(L_{\mathrm{K3}^{[2]}} \otimes \mathbf{C}) \mid x \cdot h_0 = 0,\ x \cdot x = 0,\ x \cdot \bar{x} > 0\}$$

has two connected components, interchanged by complex conjugation, which are Hermitian symmetric domains of type IV. It is acted on by the group

$$O(L_{\mathrm{K3}^{[2]}}, h_0) := \{g \in O(L_{\mathrm{K3}^{[2]}}) \mid g(h_0) = h_0\}.$$

By results of Baily–Borel and Griffiths, the quotient is an irreducible quasi-projective variety and the *period map*

$$\wp_{2n}^{(\gamma)} \colon \mathscr{M}_{2n}^{(\gamma)} \longrightarrow O(L_{\mathrm{K3}^{[2]}}, h_0) \backslash \Omega_{h_0} \tag{3}$$

is algebraic. Alternatively, one has

$$\Omega_{h_0} := \{x \in \mathbf{P}(L^{(\gamma)}_{\mathrm{K3}^{[2]},2n} \otimes \mathbf{C}) \mid x \cdot x = 0,\ x \cdot \bar{x} > 0\}$$

and the group $O(L_{\mathrm{K3}^{[2]}}, h_0)$ can be identified with the stable orthogonal group $\widetilde{O}(L^{(\gamma)}_{\mathrm{K3}^{[2]},2n})$ ([GHS2, Proposition 3.12 and Corollary 3.13]).

Verbitsky's Torelli theorem takes the following form.[1]

Theorem 2.1 (Verbitsky)**.** *For each positive degree $2n$ and each divisibility γ, the period map*

$$\wp_{2n}^{(\gamma)} \colon \mathscr{M}_{2n}^{(\gamma)} \longrightarrow O(L_{\mathrm{K3}^{[2]}}, h_0) \backslash \Omega_{h_0}$$

is an open embedding.

We will also use the following form of this theorem (see [Dr, Theorem 1.1.26]).

Theorem 2.2 (Verbitsky, Markman)**.** *Let F_1 and F_2 be hyperkähler fourfolds of* K3$^{[2]}$*-type, with respective polarizations h_1 and h_2. Let $\Phi \colon H^2(F_1, \mathbf{Z}) \xrightarrow{\sim} H^2(F_2, \mathbf{Z})$ be an isometry of Hodge structures such that $\Phi(h_1) = h_2$. There exists an isomorphism $\varphi \colon F_2 \xrightarrow{\sim} F_1$ such that $\Phi = \varphi^*$.*

2.2. **Special polarized hyperkähler fourfolds.** A hyperkähler fourfold corresponding to a very general point of $\mathscr{M}_{2n}^{(\gamma)}$ has Picard number 1. The *Noether–Lefschetz locus* (or special locus) is the subset of $\mathscr{M}_{2n}^{(\gamma)}$ corresponding to hyperkähler fourfolds with Picard number at least 2. It can be described as follows.

Let K be a primitive, rank-2, signature-$(1,1)$ sublattice of $L_{\mathrm{K3}^{[2]}}$ containing h_0. The codimension-2 subspace $\mathbf{P}(K^\perp \otimes \mathbf{C})$ in $\mathbf{P}(L_{\mathrm{K3}^{[2]}} \otimes \mathbf{C})$ cuts out a hypersurface in Ω_{h_0} whose image in $O(L_{\mathrm{K3}^{[2]}}, h_0) \backslash \Omega_{h_0}$ will be denoted by $\mathscr{D}^{(\gamma)}_{2n,K}$. We also denote by $\mathscr{C}^{(\gamma)}_{2n,K} \subset \mathscr{M}^{(\gamma)}_{2n}$ the inverse image of $\mathscr{D}^{(\gamma)}_{2n,K}$ by $\wp^{(\gamma)}_{2n}$. Since the divisors $\mathscr{D}^{(\gamma)}_{2n,K}$ are $\mathbf{Q}$-Cartier, each $\mathscr{C}^{(\gamma)}_{2n,K}$ is either empty or everywhere of codimension 1. The Noether–Lefschetz locus is then equal to the countable union of hypersurfaces $\bigcup_K \mathscr{C}^{(\gamma)}_{2n,K}$.

For each positive integer d, the unions

$$\mathscr{D}^{(\gamma)}_{2n,d} := \bigcup_{K,\ \mathrm{disc}(K^\perp) = -d} \mathscr{D}^{(\gamma)}_{2n,K} \qquad \text{and} \qquad \mathscr{C}^{(\gamma)}_{2n,d} := (\wp^{(\gamma)}_{2n})^{-1}(\mathscr{D}^{(\gamma)}_{2n,d}) = \bigcup_{K,\ \mathrm{disc}(K^\perp) = -d} \mathscr{C}^{(\gamma)}_{2n,K} \subset \mathscr{M}^{(\gamma)}_{2n}$$

[1]This is a particular case of [GHS3, Theorem 3.14] since, in our case, $\widehat{O}^+(L^{(\gamma)}_{\mathrm{K3}^{[2]},2n}) = \widetilde{O}^+(L^{(\gamma)}_{\mathrm{K3}^{[2]},2n})$ ([GHS3, Remark 3.15]). The derivation of this result from Verbitsky's Torelli theorem ([V]) is explained in [M2] (see also [Dr, Proposition 1.1.27]).

are finite[2] and the hyperkähler fourfolds corresponding to points of $\mathscr{C}^{(\gamma)}_{2n,d}$ are called *special of discriminant* d (the lattice $K^\perp$ has signature $(2,19)$, hence d is positive).

We now work out the structure of the loci $\mathscr{D}^{(\gamma)}_{2n,d}$ and $\mathscr{C}^{(\gamma)}_{2n,d}$. We begin with the case when the divisibility γ is 2.

Proposition 2.3. *Let n be a positive number such that $n \equiv -1 \pmod 4$ and let d be a positive integer.*

a) *The locus $\mathscr{D}^{(2)}_{2n,d}$ is non-empty if and only if d is even and $d/2$ is a square modulo n. If this is the case, $\mathscr{C}^{(2)}_{2n,d}$ either has pure codimension 1 in $\mathscr{M}^{(2)}_{2n}$ or is empty, the latter occurring only for finitely many values of d.*

b) *If n is* prime *and d satisfies the conditions in* a)*, $\mathscr{D}^{(2)}_{2n,d}$ is an irreducible hypersurface and $\mathscr{C}^{(2)}_{2n,d}$ is either an irreducible hypersurface or is empty.*

Proof. Let (u,v) be a standard basis for a hyperbolic plane U contained in $L_{\mathrm{K3}^{[2]}}$ and let ℓ be a basis for the $I_1(-2)$ factor (so that $\ell^2=-2$). We may take

$$h_0 := 2\Big(u+\frac{n+1}{4}v\Big)+\ell$$

(it has the correct square and divisibility). The even lattice $h_0^\perp$ is spanned by $\mathbf{x} := v+\ell$, $\mathbf{y} := -u+\frac{n+1}{4}v$, and $M := \{u,v,\ell\}^\perp = U^{\oplus 2}\oplus E_8(-1)^{\oplus 2}$; the matrix of the intersection form on $\mathbf{Z}\mathbf{x}\oplus\mathbf{Z}\mathbf{y}$ is the matrix $\begin{pmatrix} -2 & -1 \\ -1 & -\frac{n+1}{2}\end{pmatrix}$ in (2) and the discriminant group $D(h_0^\perp)\simeq \mathbf{Z}/n\mathbf{Z}$ is generated by $(\mathbf{x}-2\mathbf{y})_* = (\mathbf{x}-2\mathbf{y})/n$, with $\mathbf{q}((\mathbf{x}-2\mathbf{y})_*) = 2/n$.

Let (h_0,κ') be a basis for K, so that $\mathrm{disc}(K) = 2n\kappa'^2-(h_0\cdot\kappa')^2$. Since $\mathrm{div}(h_0)=\gamma=2$, the integer $h_0\cdot\kappa'$ is even and since κ'^2 is also even (because $L_{\mathrm{K3}^{[2]}}$ is an even lattice), we have $4 \mid \mathrm{disc}(K)$ and $-\mathrm{disc}(K)/4$ is a square modulo n. Since the discriminant of $L_{\mathrm{K3}^{[2]}}$ is 2, the integer $d = |\,\mathrm{disc}(K^\perp)|$ is either $2\,|\,\mathrm{disc}(K)|$ or $\frac12\,|\,\mathrm{disc}(K)|$, hence it is even and $d/2$ is a square modulo n, as desired.

Conversely, we need to construct, for any integer d with these properties, an actual lattice K such that $d = -\,\mathrm{disc}(K^\perp)$. Given any integer a, we set

$$\begin{aligned}\kappa &:= a(\mathbf{x}-2\mathbf{y})+nw\\ &= a(v+\ell)-2a(-u+\tfrac{n+1}{4}v)+nw\\ &= ah_0+n(-av+w),\end{aligned}$$

[2]Write $K=\mathbf{Z}h_0\oplus\mathbf{Z}\kappa'$. A generator of $K\cap h_0^\perp$ is $\kappa := ((h_0\cdot\kappa')h_0-2n\kappa')/\gcd(h_0\cdot\kappa',2n)$. Its divisibility divides the discriminant of $h_0^\perp$ hence can take only finitely many values. The formula from [GHS3, Lemma 7.5] gives $\kappa^2 = \mathrm{disc}(\kappa^\perp)\,\mathrm{div}(\kappa)^2/\,\mathrm{disc}(h_0^\perp) = d\,\mathrm{div}(\kappa)^2/\,\mathrm{disc}(h_0^\perp)$; hence κ^2 may only take finitely many values. The number of $O(L^{(\gamma)}_{\mathrm{K3}^{[2]},2n})$-orbits of such κ is therefore finite, and so is the number of $\widetilde{O}(L^{(\gamma)}_{\mathrm{K3}^{[2]},2n})$-orbits for K.

where w is a primitive vector in M. The divisibility of κ in $h_0^\perp$ is n and $\kappa_* = \kappa/n = a(\mathbf{x}-2\mathbf{y})_*$ in $D(h_0^\perp)$. Since $\kappa^2 = 2a^2n + n^2w^2$, the formula from [GHS3, Lemma 7.5] reads

$$d = \left|\operatorname{disc}(K^\perp)\right| = \left|\frac{\kappa^2 \operatorname{disc}(h_0^\perp)}{\delta^2}\right| = \left|\frac{(2a^2n + n^2w^2)n}{n^2}\right| = 2a^2 + nw^2. \tag{4}$$

Since w^2 can be any even number, this proves that $d/2$ can take any positive value which is a square modulo n. Finally, since the map $\wp_{2n}^{(2)}$ is an open embedding, its image can only avoid finitely many of the hypersurfaces $\mathscr{D}_{2n,d}^{(2)}$. This proves a).

We now assume that n is prime and prove b). Given a lattice K containing h_0 with $\operatorname{disc}(K^\perp) = -d$, we let κ be a generator of $K \cap h_0^\perp$. With the notation above, we have $\kappa = ((h_0 \cdot \kappa')h_0 - 2n\kappa')/m$, where $m := \gcd(h_0 \cdot \kappa', 2n)$ is even and $\kappa^2 = \frac{2n}{m^2}\operatorname{disc}(K)$. Formula (4) then gives

$$d = \left|\operatorname{disc}(K^\perp)\right| = \left|\frac{\kappa^2 \operatorname{disc}(h_0^\perp)}{\operatorname{div}(\kappa)^2}\right| = \left|\frac{2n^2 \operatorname{disc}(K)}{m^2 \operatorname{div}(\kappa)^2}\right|.$$

Since n is odd, m is even, and, as we saw above, $|\operatorname{disc}(K)| \in \{d/2, 2d\}$, the only possibility is $|\operatorname{disc}(K)| = 2d$ and $m \operatorname{div}(\kappa) = 2n$. Since n is prime, either $(m, \operatorname{div}(\kappa), \kappa^2) = (2, n, nd)$ and $n \nmid d$ (because $n \nmid m$ and $d = -\frac{1}{2}\operatorname{disc}(K) \equiv \frac{1}{2}(h_0 \cdot \kappa')^2 \pmod{n}$), or $n \mid d$ and $(m, \operatorname{div}(\kappa), \kappa^2) = (2n, 1, d/n)$.

Given d, the integer κ^2 is therefore uniquely defined by d: if we write $d = 2a^2 + 2nm$, we have

- either $n \mid a$, $\kappa^2 = d/n$, and $\kappa_* = 0$;
- or $n \nmid a$, $\kappa^2 = nd$, $\kappa_* = \kappa/n$, and $\mathbf{q}(\kappa_*) = 2a^2/n \pmod{2\mathbf{Z}}$.

In the second case, $\kappa_* = \pm a(\mathbf{x}-2\mathbf{y})_*$; it follows that in all cases, κ_* is also uniquely defined, up to sign, by d.

By Eichler's criterion ([GHS2, Lemma 3.5]), the group $\widetilde{O}(h_0^\perp)$ acts transitively on the set of primitive vectors $\kappa \in h_0^\perp$ of given square and fixed $\kappa_* \in D(h_0^\perp)$. Since κ and $-\kappa$ give rise to the same lattice K (obtained as the saturation of $\mathbf{Z}h_0 \oplus \mathbf{Z}\kappa$), we have shown that the action of the group $O(L_{\mathrm{K3}^{[2]}}, h_0)$ on the set of lattices $K \ni h_0$ such that $\operatorname{disc}(K^\perp) = -d$ is also transitive. The corresponding locus $\mathscr{D}_{2n,d}^{(2)}$ is therefore irreducible when non-empty. □

Remark 2.4. If n is a positive number such that $n \equiv -1 \pmod 4$, we will prove in Proposition 4.2 that the locus $\mathscr{C}_{2n,2n}^{(2)}$ is empty. In the other direction, we will see in Remark 5.2 that the locus $\mathscr{C}_{2n,2}^{(2)}$ is non-empty.

Example 2.5 (Degree 6, divisibility 2)**.** By Proposition 2.3, the locus $\mathscr{D}_{6,d}^{(2)}$ is non-empty if and only if d is positive and $d \equiv 0$ or $2 \pmod 6$; it is then an irreducible hypersurface.

If $W \subset \mathbf{P}^5$ is a smooth cubic fourfold, the variety $F(W)$ of lines contained in W is a hyperkähler fourfold, with its Plücker polarization g of square 6 and divisibility 2 ([BD], [H1, Proposition 2.1.2], [GHS3, Example 4.2]). These fourfolds fill out a dense open subset $\mathscr{U}_6^{(2)}$ in $\mathscr{M}_6^{(2)}$.

Special cubic fourfolds were originally defined by Hassett in [H1] as follows: let $h \in H^2(W,\mathbf{Z})$ be the hyperplane class; W is said to be special (of discriminant d) if the lattice $H^4(W,\mathbf{Z})\cap H^{2,2}(W)$ contains a primitive, rank-2, discriminant-d lattice K which contains the class h^2 (since the lattice $H^4(W,\mathbf{Z})$ is unimodular, this is the same as asking $\operatorname{disc}(K^\perp)=-d$). Because of the existence of an isomorphism

$$\alpha_{BD}\colon H^4(W,\mathbf{Z})(-1)_{h^2} \xrightarrow{\sim} H^2(F(W),\mathbf{Z})_g \tag{5}$$

of polarized Hodge structures ([BD]), this is equivalent to $F(W)$ being special of discriminant d in our sense. Hassett proved that smooth cubic fourfolds of discriminant d exists if and only if $d\equiv 0,2 \pmod 6$ and $d\geq 8$; the locus $\mathscr{C}^{(2)}_{6,d}$ is then an irreducible hypersurface in $\mathscr{M}^{(2)}_6$ which meets $\mathscr{U}^{(2)}_6$ ([H1, Theorems 1.0.1, 3.2.3, and 4.3.1]).

By [Dr, Theorem 2.1.7], the locus $\mathscr{C}^{(2)}_{6,2}$ is also an irreducible hypersurface in $\mathscr{M}^{(2)}_6$, disjoint from $\mathscr{U}^{(2)}_6$ (in the notation of Section 3.2, it consists of pairs $(S^{[2]},2f-\delta)$, where S is a polarized K3 surface of degree 2; see Proposition 5.5). By [Dr, Theorem 2.2.13] or Remark 2.4, the locus $\mathscr{C}^{(2)}_{6,6}$ is empty. In fact, the image of the period map $\wp^{(2)}_6$ is exactly the complement of $\mathscr{D}^{(2)}_{6,6}$ in the period domain ([La, Theorem 1.1]).

Example 2.6 (Degree 22, divisibility 2)**.** General elements of $\mathscr{M}^{(2)}_{22}$ were described in [DV]: in the Grassmannian $\operatorname{Gr}(6,10)$, smooth zero loci of sections of the third exterior power of the canonical rank-6 subbundle form a dense open subset $\mathscr{U}^{(2)}_{22}$ of $\mathscr{M}^{(2)}_{22}$.

By Proposition 2.3, the locus $\mathscr{D}^{(2)}_{22,d}$ is non-empty if and only if d is positive, even, and $d\equiv 0,2,6,8,10,\text{ or }18 \pmod{22}$; it is then an irreducible hypersurface. Little is known about the special loci $\mathscr{C}^{(2)}_{22,d}$ (and even less about their intersections with $\mathscr{U}^{(2)}_{22}$) apart from the fact that $\mathscr{C}^{(2)}_{22,22}$ is empty (Remark 2.4) and that $\mathscr{C}^{(2)}_{22,2}$, $\mathscr{C}^{(2)}_{22,54}$, $\mathscr{C}^{(2)}_{22,62}$, $\mathscr{C}^{(2)}_{22,2(m^2+m+3)}$, $\mathscr{C}^{(2)}_{22,6(3m^2+9m+4)}$ are non-empty for all $m\geq 0$ (see Sections 5.3 and 5.8).

Example 2.7 (Degree 38, divisibility 2)**.** General elements of $\mathscr{M}^{(2)}_{38}$ are *varieties of sums of powers* $\operatorname{VSP}(W,10)$, as constructed and analyzed in [IR1, IR2]. Here, $W\subset \mathbf{P}(V_6)$ is a cubic hypersurface and $\operatorname{VSP}(W,10)$ is naturally embedded in $\operatorname{Gr}(4,\bigwedge^2 V_6)$; the degree and type of the Plücker polarization on $\operatorname{VSP}(W,10)$ are computed in [Mo, Proposition 1.4.16].

By Proposition 2.3, the locus $\mathscr{D}^{(2)}_{38,d}$ is non-empty if and only if d is positive, even, and $d\equiv 0,2,8,10,12,14,18,22,32,\text{ or }34 \pmod{38}$; it is then an irreducible hypersurface. We also know that $\mathscr{C}^{(2)}_{38,38}$ is empty (Remark 2.4) and that $\mathscr{C}^{(2)}_{38,2}$, $\mathscr{C}^{(2)}_{38,46}$, $\mathscr{C}^{(2)}_{38,2(m^2+m+5)}$, $\mathscr{C}^{(2)}_{38,6(3m^2+9m+2)}$ are non-empty for all $m\geq 0$ (see Sections 5.4 and 5.9).

The structures of $\mathscr{D}^{(1)}_{2n,d}$ and $\mathscr{C}^{(1)}_{2n,d}$ can be worked out similarly.

Proposition 2.8. *Let n and d be positive numbers.*

a) *The locus $\mathscr{D}^{(1)}_{2n,d}$ is non-empty if and only if d is even and either $d/2$ or $d/2-n$ is a square modulo $4n$. If this is the case, $\mathscr{C}^{(1)}_{2n,d}$ either has pure codimension 1 in $\mathscr{M}^{(1)}_{2n}$ or is empty, the latter occurring only for finitely many values of d.*

b) *If* n *is* prime *and* d *satisfies the conditions in* a), $\mathscr{D}^{(1)}_{2n,d}$ *is an irreducible hypersurface except when* $n \equiv 1 \pmod 4$ *and* $d/2 \equiv 1 \pmod 4$, *or when* $n \equiv -1 \pmod 4$ *and* $d/2 \equiv 0 \pmod 4$, *in which case* $\mathscr{D}^{(1)}_{2n,d}$ *has two irreducible components.*

Proof. Let (u, v) be a standard basis for a hyperbolic plane U contained in $L_{\mathrm{K3}^{[2]}}$ and let ℓ be a basis for the $I_1(-2)$ factor. We may take $h_0 := u + nv$, in which case $h_0^\perp = \mathbf{Z}(u - nv) \oplus \mathbf{Z}\ell \oplus M$, where $M := \{u, v, \ell\}^\perp = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}$ is unimodular. The discriminant group $D(h_0^\perp) \simeq \mathbf{Z}/2n\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ is generated by $(u - nv)_* = (u - nv)/2n$ and $\ell_* = \ell/2$, and $\mathbf{q}((u - nv)_*) = -1/2n$ and $\mathbf{q}(\ell_*) = -1/2$.

Let κ be a generator for $K \cap h_0^\perp$. We write

$$\kappa = a(u - nv) + b\ell + cw,$$

where $w \in M$ is primitive. We have $\kappa^2 < 0$ and Formula (4) gives

$$d = \left|\operatorname{disc}(K^\perp)\right| = \left|\frac{\kappa^2 \operatorname{disc}(h_0^\perp)}{\delta^2}\right| = \frac{8n(na^2 + b^2 + mc^2)}{\delta^2} \equiv \frac{8n(na^2 + b^2)}{\delta^2} \pmod{8n}, \tag{6}$$

where $m := -\frac{1}{2}w^2$ and $\delta := \operatorname{div}(\kappa) = \gcd(2na, 2b, c)$. If $\delta \mid b$, we obtain $d \equiv 2\left(\frac{2na}{\delta}\right)^2 \pmod{8n}$, which is the first case of the conclusion. Assume $\delta \nmid b$ and, for any integer x, write $x = 2^{v_2(x)}x'$, where x' is odd. One has then

$$\nu_2(b) < \nu_2(\delta) = \min\{\nu_2(na) + 1, \nu_2(b) + 1, \nu_2(c)\},$$

hence $\nu_2(\delta) = \nu_2(b) + 1$ and

$$d \equiv 2\left(\frac{2na}{\delta}\right)^2 + 2n\left(\frac{b'}{\delta'}\right)^2 \equiv 2\left(\frac{2na}{\delta}\right)^2 + 2n \pmod{8n},$$

which is the second case of the conclusion. It is then easy, taking suitable integers a, b, and c, to construct examples that show that these necessary conditions on d are also sufficient.

We now assume that n is prime and prove b). As in the proof of Proposition 2.3, we want to know to what extent the value of d determines the $\widetilde{O}(h_0^\perp)$-orbit of the primitive vector $\kappa = a(u - nv) + b\ell + cw \in h_0^\perp$, where $\gcd(a, b, c) = 1$. Set $\delta := \operatorname{div}(\kappa) = \gcd(2na, 2b, c) \in \{1, 2, n, 2n\}$. By Eichler's criterion ([GHS2, Lemma 3.5]), this orbit is characterized by the number $\kappa^2 = -d\delta^2/4n$ and the vector

$$\kappa_* = (2na/\delta, 2b/\delta) \in \mathbf{Z}/2n\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}.$$

We distinguish several cases according to the value of δ, using (6).

If $\delta = 1$, we have $d \equiv 0 \pmod{8n}$ and $\kappa_* = 0$.

If $\delta = 2$, the integer c is even and a and b cannot be both even. We have

$$\begin{cases} d \equiv 2n^2 \pmod{8n} \quad \text{and } \kappa_* = (n, 0) & \text{if } b \text{ is even;} \\ d \equiv 2n \pmod{8n} \quad \text{and } \kappa_* = (0, 1) & \text{if } b \text{ is odd and } a \text{ is even;} \\ d \equiv 2n(n+1) \pmod{8n} \quad \text{and } \kappa_* = (n, 1) & \text{if } b \text{ and } a \text{ are odd.} \end{cases}$$

If $\delta = n$ (and n is odd), we have $n \mid b$, $n \mid c$, $n \nmid a$, and

$$d \equiv 8a^2 \pmod{8n} \quad \text{and } \kappa_* = (2a, 0).$$

If $\delta = 2n$, the integer c is even, a and b cannot be both even, $n \mid b$, and $n \nmid a$. We have

$$\begin{cases} d \equiv 2a^2 \pmod{8n} \quad \text{and } \kappa_* = (a,0) & \text{if } 2n \mid b \text{ (and } a \text{ is odd);} \\ d \equiv 2a^2 + 2n \pmod{8n} \quad \text{and } \kappa_* = (a,1) & \text{if } b \text{ is odd (and } n \text{ is odd);} \\ d \equiv 2a^2 + 4 \pmod{16} \quad \text{and } \kappa_* = (a,1) & \text{if } 4 \nmid b \text{ is odd and } n = 2. \end{cases}$$

When $n = 2$, one then checks that the class of d modulo 16 (which is in $\{0, 2, 4, 6, 8, 12\}$) completely determines δ and κ_* up to sign. The corresponding divisors $\mathscr{D}^{(1)}_{4,d}$ are therefore all irreducible.

When $n \equiv 1 \pmod 4$, we have $2n^2 \equiv 2n \pmod{8n}$ and $2a^2 \equiv 2(n-a)^2 + 2n \pmod{8n}$ when a is odd. When $n \equiv -1 \pmod 4$, we have $2n(n+1) \equiv 0 \pmod{8n}$ and $2a^2 \equiv 2(n-a)^2 + 2n \pmod{8n}$ when a is even. Together with changing a into $-a$ (which does not change the lattice K), these are the only coincidences: the corresponding divisors $\mathscr{D}^{(1)}_{2n,d}$ therefore have two components and the others are irreducible. □

Example 2.9 (Degree 2 (divisibility 1))**.** This case, which was studied in detail in [DIM], is not completely covered by Proposition 2.8 (although we could easily analyze the irreducibility of the divisors $\mathscr{D}^{(1)}_{2,d}$ from our proof). We translate their results in our setting. The group $O(L_{\mathrm{K3}^{[2]}}, h_0) = \widetilde{O}(L^{(1)}_{\mathrm{K3}^{[2]},2})$ has index 2 in the full orthogonal group $O(L^{(1)}_{\mathrm{K3}^{[2]},2})$ ([O6, (4.1.7)]). We denote by r the (non-trivial) involution of the period domain corresponding to the double cover

$$O(L_{\mathrm{K3}^{[2]}}, h_0)\backslash\Omega_{h_0} = \widetilde{O}(L^{(1)}_{\mathrm{K3}^{[2]},2})\backslash\Omega_{h_0} \longrightarrow O(L^{(1)}_{\mathrm{K3}^{[2]},2})\backslash\Omega_{h_0}.$$

By [DIM, Proposition 6.2] (or Proposition 2.8), the locus $\mathscr{D}^{(1)}_{2,d}$ is non-empty if and only if d is positive and $d \equiv 0, 2, \text{ or } 4 \pmod 8$; it is irreducible if $d \equiv 0 \pmod 4$ and has two irreducible components, $\mathscr{D}^{(1)\prime}_{2,d}$ and $\mathscr{D}^{(1)\prime\prime}_{2,d} = r(\mathscr{D}^{(1)\prime}_{2,d})$, if $d \equiv 2 \pmod 8$.[3]

Double EPW sextics are defined in [O5] as ramified double covers $\widetilde{Y}$ of certain singular sextic hypersurfaces $Y \subset \mathbf{P}^5$. When smooth, they are hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type and the polarization $\tilde{h}$ on $\widetilde{Y}$ coming from the composition $\widetilde{Y} \to Y \hookrightarrow \mathbf{P}^5$ satisfies $q_{BBJ}(\tilde{h}) = 2$. Smooth double EPW sextics fill out a dense open subset $\mathscr{U}^{(1)}_2$ in $\mathscr{M}^{(1)}_2$.

Gushel–Mukai (GM for short) fourfolds were defined and studied in [IMa], [DK1], [DK2]. Any GM fourfold X comes with a canonical morphism $\gamma_X \colon X \to \mathrm{Gr}(2,5)$. One can construct an associated double EPW sextic $\widetilde{Y}_X$ and all smooth double EPW sextics are obtained in this way. The main result of [DK2] is that when X and $\widetilde{Y}_X$ are smooth, there is an isomorphism

$$\alpha_{\mathrm{DK}} \colon H^4(X, \mathbf{Z})(-1)_{\Lambda_X} \xrightarrow{\sim} H^2(\widetilde{Y}_X, \mathbf{Z})_{\tilde{h}} \tag{7}$$

[3] We label these two hypersurfaces following [O6, Section 4.3]. Let (u,v) be a standard basis for a hyperbolic plane U contained in $L_{\mathrm{K3}^{[2]}}$, let ℓ be a basis for the $I_1(-2)$ factor (so that $\ell^2 = -2$), and set $h_0 := u + v$; write $d = 8m + 2$, let (u', v') be a standard basis for another hyperbolic plane orthogonal to U in $L_{\mathrm{K3}^{[2]}}$, and set $x' := u' - (4m+1)v' \in h_0^\perp$ and $x'' := 2(u' - mv') + \ell \in h_0^\perp$. We have $x'^2 = x''^2 = -d$ and the divisibility of x' (resp. x'') in $L_{\mathrm{K3}^{[2]}}$ is 1 (resp. 2). The hypersurface $\mathscr{D}^{(1)\prime}_{2,d}$ (resp. $\mathscr{D}^{(1)\prime\prime}_{2,d}$) is then the image in $O(L_{\mathrm{K3}^{[2]}}, h_0)\backslash\Omega_{h_0}$ of $\Omega_{h_0} \cap x'^\perp$ (resp. of $\Omega_{h_0} \cap x''^\perp$).

One can also arrange things so that this is the same labeling as in [DIM, Section 6.1] (see the discussion in [DK2, Remark 5.29]).

of polarized Hodge structures, analogous to (5), where $H^4(X,\mathbf{Z})_{\Lambda_X}$ is the orthogonal in $H^4(X,\mathbf{Z})$ of the rank-2 sublattice $\Lambda_X := \gamma_X^* H^4(\mathrm{Gr}(2,5),\mathbf{Z})$. In [DIM], X is said to be *special of discriminant* d if the lattice $H^4(X,\mathbf{Z})\cap H^{2,2}(X)$ contains a primitive rank-3 discriminant-d lattice containing Λ_X. Because of the isomorphism (7) and the unimodularity of the lattice $H^4(X,\mathbf{Z})$, this is equivalent to $\widetilde{Y}_X$ being special of discriminant d in our sense.

It was shown in [DIM, Theorem 8.1] that $\mathscr{C}^{(1)}_{2,d}$ is non-empty if $d \equiv 0 \pmod 4$ and $d \geq 12$, both loci $\mathscr{C}^{(1)\prime}_{2,d}$ and $\mathscr{C}^{(1)\prime\prime}_{2,d}$ are non-empty if $d \equiv 2 \pmod 8$ and $d \geq 18$, and $\mathscr{C}^{(1)\prime\prime}_{2,10}$ is non-empty; moreover, all these irreducible hypersurfaces meet $\mathscr{U}^{(1)}_2$.[4]

The loci $\mathscr{C}^{(1)}_{2,2}$, $\mathscr{C}^{(1)}_{2,8}$, and $\mathscr{C}^{(1)\prime}_{2,10}$ are empty[5] and one expects the image of the period map $\wp^{(1)}_2$ to be exactly the complement of $\mathscr{D}^{(1)}_{2,2} \cup \mathscr{D}^{(1)}_{2,8} \cup \mathscr{D}^{(1)\prime}_{2,10}$ in the period domain.

Example 2.10 (Degree 4 (divisibility 1))**.** One finds in [IKKR] geometric descriptions of the hyperkähler fourfolds corresponding to general points of a hypersurface $\mathscr{U}$ of $\mathscr{M}^{(1)}_4$. These fourfolds have a non-trivial involution and are special of discriminant 8 (see No. 2 in the proof of [OW, Proposition 2.5]). Since $\mathscr{D}^{(1)}_{4,8}$ is irreducible by Proposition 2.8, we have

[4]The missing case $\mathscr{C}^{(1)\prime\prime}_{2,18}$ from [DIM, Theorem 8.1] can be dealt with using [DK1, Theorem 3.27] and [DK2, (35)], which say that "dual" GM fourfolds have dual associated EPW sextics and "dual" period points: if one is in $\mathscr{C}^{(1)\prime}_{2,d}$, the other is in $\mathscr{C}^{(1)\prime\prime}_{2,d}$.

[5]This can be seen as follows. In [O6, Section 5.2], it is shown that a general point x of $\mathscr{D}^{(1)\prime\prime}_{2,2}$ is the period point of a semi-polarized pair $(S^{[2]}, f)$, where (S, f_S) is a general polarized K3 surface of degree 2. One then uses the fact (which is a consequence of the Torelli Theorem; see [Dr, Theorem 1.1.26]) that if (smooth) semi-polarized hyperkähler fourfolds (F_1, h_1) and (F_2, h_2) of K3$^{[2]}$-type have the same period point and h_1 is ample, then h_2 is also ample. Since f is nef but not ample on $S^{[2]}$, this proves that x cannot be the period point of a polarized pair (F, h). Similarly, in the notation of Section 3.2 and still by [O6, Section 5.2], a general point of $\mathscr{D}^{(1)\prime}_{2,2}$ is the period point of a pair $(M((0, f, 0)), h)$, where $h := \theta(1, 0, -1)$ is nef but not ample (this last statement follows from [BM1, Example 9.8, Case 1]).

Analogously, a general point of $\mathscr{D}^{(1)}_{2,8}$ is the period point of a pair $(\widetilde{Y}', \varepsilon^*\tilde{h})$, where $\varepsilon\colon \widetilde{Y}' \to \widetilde{Y}$ is the minimal desingularization of a (singular) double EPW sextic $\widetilde{Y}$, so that $\varepsilon^*\tilde{h}$ is nef but not ample ([O2, Section 5.4]; in O'Grady's articles, this divisor is denoted by $\mathbb{S}^\star_2$ and is the image by the period map of a divisor denoted by Σ (see [DK2, Remark 5.29])). Finally, a general point of $\mathscr{D}^{(1)\prime\prime}_{2,10}$ is the period point of a pair $(S^{[2]}, f - 2\delta)$, where (S, f_S) is a general polarized K3 surface of degree 10 and $f - 2\delta$ is nef but not ample (this divisor $\mathscr{D}^{(1)\prime\prime}_{2,10}$ is the image by the period map of a divisor denoted by Δ in O'Grady's articles (see [DK2, Remark 5.29])).

The K3 surface S is obtained as the intersection of the Fano threefold $T := \mathrm{Gr}(2, 5) \cap \mathbf{P}^6 \subset \mathbf{P}^9$ with a general quadric. The variety $F(T)$ of lines contained in T is isomorphic to $\mathbf{P}^2$ ([O5, Proposition 5.2]) and this $\mathbf{P}^2$ embeds into $S^{[2]}$ via the map $L \mapsto L \cap S$ ([O2, (3.2.4)]).

The linear system $|f - 2\delta|$ defines a morphism $\varphi\colon S^{[2]} \to \mathbf{P}^5$ which can be described as follows ([O2, Lemma 3.7]): taking for $\mathbf{P}^5$ the dual of the linear system $|I_S(2)|$ of quadrics in $\mathbf{P}^6$ containing S, the map $\varphi\colon S^{[2]} \to |I_S(2)|^\vee$ is defined by sending a pair of points $(p, q) \in S^2$ to the hyperplane of $|I_S(2)|$ consisting of quadrics containing S and the line $\langle pq\rangle$. Its image $Y \subset |I_S(2)|$ is an EPW sextic and it contracts the plane $F(T)$ to the singular point of Y corresponding to the hyperplane $|I_T(2)| \subset |I_S(2)|$. The morphism φ has degree 2 onto Y and the corresponding birational involution on $S^{[2]}$ was described geometrically in [O1, Proposition 4.21]. It acts on $H^2(S^{[2]}, \mathbf{Z})$ as the reflection about the line spanned by the square-2 class $f - 2\delta$.

Finally, still in the notation of Section 3.2, $\mathscr{C}^{(1)}_{2,4}$ consists of pairs $(S^{[2]}, f - \delta)$, where S is a polarized K3 surface of degree 4 ([O6, Section 5.3] or [F, Proposition 3.4]).

$\mathscr{U} = \mathscr{C}_{4,8}^{(1)}$. We will see that $\mathscr{C}_{4,44}^{(1)}$, $\mathscr{C}_{4,6(3m(m+1)+1)}^{(1)}$, and $\mathscr{C}_{4,2(m^2+2)}^{(1)}$ are non-empty for all $m \geq 1$ (Propositions 5.1, 5.5, and 5.18).

In the notation of Section 3.2, general elements of $\mathscr{C}_{4,6}^{(1)}$ are pairs $(S^{[2]}, f - \delta)$, where $S \subset \mathbf{P}^4$ is a general K3 surface of degree 6. The linear system $|f - \delta|$ defines the finite birational morphism $S^{[2]} \to G(2,5) \hookrightarrow \mathbf{P}^9$ that takes a subscheme of S of length 2 to the line that it spans. There is at the moment no geometric description of the hyperkähler fourfolds corresponding to general points of $\mathscr{M}_4^{(1)}$.

2.3. **Automorphisms of very general polarized hyperkähler fourfolds.** We determine below the birational automorphism groups of very general polarized hyperkähler fourfolds F (of $\mathrm{K3}^{[2]}$-type). In Sections 3.4 and 4.3, we will do the same when F is special.

Let F be a hyperkähler fourfold. There are natural morphisms

$$\Psi_A\colon \operatorname{Aut}(F) \to O(H^2(F,\mathbf{Z}), q_{BBF}) \qquad \Psi_B\colon \operatorname{Bir}(F) \to O(H^2(F,\mathbf{Z}), q_{BBF}) \tag{8}$$

which send a (birational) automorphism φ of F to its action φ^* on cohomology (see [GHJ, Proposition 25.14] for Ψ_B). Elements of $\operatorname{Im}(\Psi_A)$ preserve the nef cone $\operatorname{Nef}(F)$, elements of $\operatorname{Im}(\Psi_B)$ preserve the movable cone $\operatorname{Mov}(F)$, and both preserve the Picard lattice and the Hodge structure. We will also consider the restrictions

$$\bar\Psi_A\colon \operatorname{Aut}(F) \to O(\operatorname{Pic}(F)) \qquad \bar\Psi_B\colon \operatorname{Bir}(F) \to O(\operatorname{Pic}(F)) \tag{9}$$

induced from (8).

The kernel of Ψ_B (resp. of $\bar\Psi_B$) is contained in $\operatorname{Aut}(F)$, hence in the kernel of Ψ_A (resp. of $\bar\Psi_A$) ([Og1, Proposition 2.4]). The group $\operatorname{Ker}(\Psi_A)$ is a finite group which is invariant by smooth deformations ([HT4, Theorem 2.1]) and is trivial for the Hilbert square of a K3 surface ([B2, Proposition 10]). It follows that for a hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type, both Ψ_A and Ψ_B are injective.

Proposition 2.11. *Let F be a hyperkähler fourfold corresponding to a very general point of a moduli space $\mathscr{M}_{2n}^{(\gamma)}$. The group* $\operatorname{Bir}(F)$ *of birational automorphisms of F is trivial, unless $n = 1$, in which case* $\operatorname{Aut}(F) = \operatorname{Bir}(F) \simeq \mathbf{Z}/2\mathbf{Z}$.

Proof. As we saw in Section 2.2, the Picard group of F is generated by the class h of the polarization. Any birational automorphism leaves this class fixed, hence is in particular biregular of finite order. Let φ be a non-trivial automorphism of F. Since φ extends to small deformations of F, the restriction of φ^* to $h^\perp$ is a homothety[6] whose ratio is, by [B2, Proposition 7], a root of unity; since it is real and non-trivial (by injectivity of Ψ_A), it must be $-\operatorname{Id}$. We will prove that such an isometry of $\mathbf{Z}h \oplus h^\perp$ does not extend to an isometry Φ of $H^2(F,\mathbf{Z})$ unless $h^2 = 2n = 2$.

[6]The argument is classical: let F be a hyperkähler fourfold, let ω_F be a symplectic form on F, let φ be an automorphism of F, and write $\varphi^*\omega_F = \xi\omega_F$, where $\xi \in \mathbf{C}^\star$. Assume that (F,φ) deforms along a subvariety of the moduli space; the image of this subvariety by the period map consists of period points which are eigenvectors for the action of φ^* on $H^2(F,\mathbf{C})$ and the eigenvalue is necessarily ξ. In our case, the span of the image by the period map is $h^\perp$, which is therefore contained in the eigenspace $H^2(F,\mathbf{C})_\xi$.

When $\gamma = 1$, we may take $h = u + nv$, where (u, v) is a standard basis for a hyperbolic plane U contained in $H^2(F, \mathbf{Z})$ (it has the correct square and divisibility). Then, $u - nv$ is in $h^\perp$, hence the isometry Φ, if it exists, must satisfy

$$\Phi(u + nv) = u + nv \quad \text{and} \quad \Phi(u - nv) = -u + nv,$$

which yields $2n\Phi(v) = 2u$. This is possible only when $n = 1$. Conversely, in the case $n = 1$, the fourfold F is a double EPW sextic and does carry a non-trivial involution (Example 2.9). Moreover, this involution is the only non-trivial automorphism of a very general EPW sextic (see the end of the proof of [DK1, Proposition B.9]).

When $\gamma = 2$ (so that $n \equiv -1 \pmod 4$), we let ℓ be an element of $H^2(F, \mathbf{Z})$ orthogonal to U and such that $\ell^2 = -2$. We may take (as in the proof of Proposition 2.3) $h = 2u + \frac{n+1}{2}v + \ell$, and $h^\perp$ contains $v + \ell$ and $u - \frac{n+1}{4}v$. The isometry Φ must then satisfy

$$\Phi\big(2u + \tfrac{n+1}{2}v + \ell\big) = 2u + \tfrac{n+1}{2}v + \ell\ , \quad \Phi(v + \ell) = -v - \ell\ , \quad \Phi\big(u - \tfrac{n+1}{4}v\big) = -u + \tfrac{n+1}{4}v,$$

hence $n\Phi(v) = 4u + v + 2\ell$; this is absurd since $n \geq 3$. □

Remark 2.12. The conclusion of the proposition does not necessarily hold if we only assume that the Picard number of F is 1. In fact, Proposition 2.11 is also proved in [BCS1, Theorem 3.1] and the proof given there implies that $\mathrm{Bir}(F)$ is trivial when the Picard number of F is 1, unless $n \in \{1, 3, 23\}$. These three cases are actual exceptions: all fourfolds corresponding to points of $\mathscr{M}_2^{(1)}$ carry a non-trivial involution (Example 2.9); there is a 10-dimensional subfamily of $\mathscr{M}_6^{(2)}$ whose elements consists of fourfolds that have an automorphism of order 3 and whose very general elements have Picard number 1 ([BCS1, Section 7.1]); there is a (unique) fourfold in $\mathscr{M}_{46}^{(2)}$ with Picard number 1 and an automorphism of order 23 ([BCMS, Theorem 1.1]).

3. The geometry of Hilbert squares of K3 surfaces

We study in this section Hilbert squares of K3 surfaces, which are hyperkähler fourfolds (obviously of $\mathrm{K3}^{[2]}$-type). These fourfolds carry no canonical polarizations but their nef and movable cones were precisely described by Bayer and Macrì. These cones are very simple when the Picard number is 2 (Theorem 3.4) and we describe the contractions of their two extremal rays (Proposition 3.6). Finally, following ideas of Boissière, Cattaneo, Nieper-Wißkirchen, and Sarti, we describe the birational automorphism group of a general Hilbert square (Proposition 3.11) and characterize *ambiguous* Hilbert squares (Proposition 3.14).

3.1. Pell-type equations.

Given non-zero integers m and t with $m > 0$, we follow the notation of [BCNS] and call $\mathscr{P}_m(t)$ the Pell-type equation

$$a^2 - mb^2 = t, \tag{10}$$

where a and b are integers. A solution (a,b) of this equation is called positive if $a>0$ and $b>0$. If $x:=a+b\sqrt{m}$, this is equivalent to $x>\sqrt{|t|}$.[7] If m is not a perfect square, (a,b) is a solution if and only if the norm of $a+b\sqrt{m}$ in the quadratic number field $\mathbf{Q}(\sqrt{m})$ is t.

A positive solution with minimal a is called the minimal solution; it is also the positive solution (a,b) for which the ratio a/b is minimal when $t<0$, maximal when $t>0$. Since the function $x\mapsto x+\frac{t}{x}$ is increasing on the interval $(\sqrt{|t|},+\infty)$, the minimal solution is also the one for which the real number $a+b\sqrt{m}$ is $>\sqrt{|t|}$ and minimal.

Assume that m is not a perfect square. There is always a minimal solution (a_1,b_1) of the Pell equation $\mathscr{P}_m(1)$ and if $x_1:=a_1+b_1\sqrt{m}$, all the solutions of the equation $\mathscr{P}_m(1)$ correspond to the "nth powers" $\pm x_1^n$, for $n\in\mathbf{Z}$, in $\mathbf{Z}[\sqrt{m}]$. If an equation $\mathscr{P}_m(t)$ has a solution (a,b), the elements $\pm(a+b\sqrt{m})x_1^n$ of $\mathbf{Z}[\sqrt{m}]$, for $n\in\mathbf{Z}$, all give rise to solutions of $\mathscr{P}_m(t)$ which are said to be *associated with* (a,b). The set of all solutions of $\mathscr{P}_m(t)$ associated with each other form a *class of solutions.* A class and its conjugate (generated by $(a,-b)$) may be distinct or equal (see [N, Section 58]).

Example 3.1. Let m be a positive integer. The minimal solution of the equation $\mathscr{P}_{m^2+1}(-1)$ is $(m,1)$ and the minimal solution of the equation $\mathscr{P}_{m^2+1}(1)$ is its "square" $(2m^2+1,2m)$. The equations $\mathscr{P}_{m^2+1}(\pm t)$ are not solvable when $m\geq t>1$, and the equations $\mathscr{P}_{m^2+1}(\pm 3)$ and $\mathscr{P}_{m^2+1}(\pm 5)$ are not solvable except for $\mathscr{P}_5(\pm 5)$.[8]

We will be also concerned with the following generalization of the equation $\mathscr{P}_m(t)$. We let $m>0$ and t be integers as before and we introduce another positive integer n. We denote by $\mathscr{P}_{n,m}(t)$ the equation

$$na^2-mb^2=t,$$

where a and b are integers. Given a solution (a,b) to $\mathscr{P}_{n,m}(t)$, we obtain a solution (na,b) to $\mathscr{P}_{nm}(nt)$; if n is square-free, any solution to $\mathscr{P}_{nm}(nt)$ arises in this way.

As in the previous case, a positive solution (a,b) to $\mathscr{P}_{n,m}(t)$ is called minimal if a is minimal. If nm is not a perfect square, solutions (a,b) and (a',b') of $\mathscr{P}_{n,m}(t)$ are associated if (na,b) and (na',b') are associated solutions of $\mathscr{P}_{nm}(nt)$.[9]

The following result is an extension of [N, Theorem 110].

Lemma 3.2. *Let r be a positive square-free integer, let n be a positive integer such that $\gcd(r,n)=1$, and let m be a positive integer such that nm is not a perfect square. Let*

[7] Let $\varepsilon\in\{-1,1\}$ be the sign of t and set $x':=\varepsilon(a-b\sqrt{m})$, so that $xx'=|t|$. If both a and b are positive, we have $0<x'<x$, hence $x>\sqrt{|t|}$. Conversely, if $x>\sqrt{|t|}$, we have $0<x'<x$ and both $a=(x+\varepsilon x')/2$ and $b=(x-\varepsilon x')/2\sqrt{m}$ are positive.

[8] Since the equation $\mathscr{P}_{m^2+1}(-1)$ is solvable, it is enough to prove that the equation $\mathscr{P}_{m^2+1}(t)$ is not solvable. If (a,b) is a positive solution of the equation $\mathscr{P}_{m^2+1}(t)$, we have $\left|\frac{a}{b}-\sqrt{m^2+1}\right|=\frac{t}{b(a+b\sqrt{m^2+1})}<\frac{t}{2b^2\sqrt{m^2+1})}<\frac{1}{2b^2}$, because $m\geq t$. The fraction $\frac{a}{b}$ must then be a convergent $\frac{p_n}{q_n}$ to $\sqrt{m^2+1}$; since its continued fraction is $[m,\overline{2m}]$, the possible values for $p_n^2-(m^2+1)q_n^2$ are obtained for $n\in\{0,1\}$: they are -1 and 1; this contradicts the assumption $t>1$. The (finitely many) remaining statements ($3=t>m$ and $5=t>m$) are checked directly.

[9] Assume that nm is not a perfect square. If (a,b) is a solution to $\mathscr{P}_{n,m}(t)$ and (a_1,b_1) is a solution to $\mathscr{P}_{nm}(1)$, and if we set $x_1:=a_1+b_1\sqrt{nm}$, then $(na+b\sqrt{nm})x_1=na'+b'\sqrt{nm}$, where (a',b') is again a solution to $\mathscr{P}_{n,m}(t)$.

$\varepsilon \in \{-1, 1\}$. *If the equation* $\mathscr{P}_{n,m}(\varepsilon r)$ *is solvable, it has one or two classes of solutions according to whether* r *divides* $2m$ *or not.*

Proof. Let (a, b) be a solution to $\mathscr{P}_{n,m}(\varepsilon r)$. Since $\gcd(r, n) = 1$ and r is square-free, we get $r \nmid b$ and we can write $nm \equiv u^2 \pmod r$.

If (a', b') is any solution of the equation $\mathscr{P}_{n,m}(\varepsilon r)$, we have $n^2 a'^2 \equiv u^2 b'^2 \pmod r$, hence $na' \equiv \eta' u b' \pmod r$, with $\eta' \in \{-1, 1\}$. Consider the quotient

$$\begin{aligned}\frac{na' + b'\sqrt{nm}}{na + \eta\eta' b\sqrt{nm}} &= \frac{(na' + b'\sqrt{nm})(na - \eta\eta' b\sqrt{nm})}{\varepsilon nr} \\ &= \frac{n^2 aa' - \eta\eta' bb' nm + (ab' - \eta\eta' a'b)n\sqrt{nm}}{\varepsilon nr} \\ &= \frac{naa' - \eta\eta' bb' m + (ab' - \eta\eta' a'b)\sqrt{nm}}{\varepsilon r}.\end{aligned}$$

By reducing modulo r, we obtain

$$n^2 aa' - \eta\eta' bb' nm \equiv \eta\eta' bb' u^2 - \eta\eta' bb' nm \equiv 0 \pmod r$$

and

$$nab' - \eta\eta' a'bn \equiv \eta bb' u - \eta b'bu \equiv 0 \pmod r.$$

Since $\gcd(r, n) = 1$, this implies that the fraction above gives rise to an integral solution of the equation $\mathscr{P}_{nm}(1)$ and proves that there are at most two classes of solutions.

As explained in [N, p. 205], the solutions (na, b) and $(na, -b)$ of $\mathscr{P}_{nm}(\varepsilon nr)$ are associated if and only if nr divides $2nab$ and $2nmb^2 + nr$. Since $r \nmid b$, we get $r \mid 2nm$; since $\gcd(r, n) = 1$, we get $r \mid 2m$. Conversely, if $r \mid 2m$, we obtain $r \mid 2a$ and the conditions are satisfied. This finishes the proof of the lemma. □

Finally, we will use the following result several times.

Lemma 3.3. *Let* n *and* m *be positive integers. Assume that for some* $\varepsilon \in \{-1, 1\}$, *the equation* $\mathscr{P}_{n,m}(\varepsilon)$ *has a positive solution and let* (a, b) *be its minimal solution. The minimal solution of the equation* $\mathscr{P}_{nm}(1)$ *is either* (b, a) *if* $m = 1$ *and* $\varepsilon = -1$, *or* $(na^2 + mb^2, 2ab)$ *otherwise.*

Proof. Observe that nm is not a perfect square: if $nm = u^2$, with $u > 0$, we have $(na + ub)(na - ub) = \varepsilon n$, hence $na + ub \leq n$, which is absurd since $a, b > 0$.

Let (a_1, b_1) be the minimal solution of the equation $\mathscr{P}_{nm}(1)$ and set $x_1 := a_1 + b_1\sqrt{nm}$ and $x := na + b\sqrt{nm}$. We have $x > \sqrt{n}$ and $x_1 > 1$. The equation $\mathscr{P}_{n,m}(\varepsilon)$ has only one class of solutions (Lemma 3.2) and the map $\iota\colon z \mapsto \varepsilon\bar{z} = n/z$ defines an involution on the set of solutions viewed as elements of $\mathbf{Z}[\sqrt{nm}]$. The solutions giving rise to positive numbers are

$$\cdots < xx_1^{-2} < xx_1^{-1} \leq \sqrt{r} < x < xx_1 < xx_1^2 < \cdots$$

There are two cases:

- either $xx_1^{-1} = \iota(x) = n/x$ and we are in the second case of the lemma;

- or $xx_1^{-1} = \sqrt{n}$, in which case we have

$$a + b\sqrt{nm} = x = x_1\sqrt{n} = a_1\sqrt{n} + nb_1\sqrt{m}.$$

In the second case, $s := \sqrt{m}$ is an integer, $a = nsb_1$, and $a_1 = bs$. Since $a_1^2 - ns^2b_1^2 = 1$, the last equality implies $s = 1$ and $a^2 - nmb^2 = n^2b_1^2 - na_1^2 = -n$, hence $\varepsilon = -1$ and we are in the first case of the lemma. This ends the proof of the lemma. □

3.2. **The nef and movable cones of a very general Hilbert square.** Let (S, f_S) be a polarized K3 surface such that $\mathrm{Pic}(S) = \mathbf{Z}f_S$. Then $\mathrm{Pic}(S^{[2]}) = \mathbf{Z}f \oplus \mathbf{Z}\delta$, where f the class on $S^{[2]}$ induced by f_S and 2δ is the class of the divisor $E_S \subset S^{[2]}$ parametrizing non-reduced length-2 subschemes of S ([B1, Remarque, p. 768]). For the Beauville–Bogomolov–Fujiki form q_{BBJ} on $H^2(S^{[2]}, \mathbf{Z})$, we have the following products

$$f^2 = 2e \ , \quad \delta^2 = -2 \ , \quad f \cdot \delta = 0,$$

where $e := \frac{1}{2}f_S^2$ (self-intersection on S). Since f is nef non-ample, it spans one of the two extremal rays of the nef cone $\mathrm{Nef}(S^{[2]}) \subset \mathrm{Pic}(S^{[2]}) \otimes \mathbf{R}$. The other extremal ray is spanned by a class $f - \nu_S\delta$, where ν_S is a positive real number. Similarly, the extremal rays of the (closed) movable cone $\mathrm{Mov}(S^{[2]})$ are spanned by f and $f - \mu_S\delta$, with $\mu_S \geq \nu_S$.

The following result comes from [BM2, Proposition 13.1 and Lemma 13.3] and proves [HT1, Conjecture 3.3] in the case under consideration.[10] It shows that the slopes ν_S and μ_S are rational numbers and only depend on the positive integer e.

Theorem 3.4 (Bayer–Macrì)**.** *Let (S, f_S) be a polarized K3 surface of degree $2e$ with Picard group $\mathbf{Z}f_S$. The slopes ν_S and μ_S are respectively equal to the rational numbers ν_e and μ_e defined as follows.*

- *Assume that the equation $\mathscr{P}_{4e}(5)$ has no solutions.*
  a) *If e is a perfect square, we have $\nu_e := \mu_e := \sqrt{e}$.*
  b) *If e is not a perfect square and (a_1, b_1) is the minimal solution of the equation $\mathscr{P}_e(1)$, we have $\nu_e := \mu_e := e\frac{b_1}{a_1}$.*
- *Assume that equation $\mathscr{P}_{4e}(5)$ has a solution.*
  c) *If (a_5, b_5) is its minimal solution, we have $\nu_e := 2e\frac{b_5}{a_5}$ and $\mu_e := e\frac{b_1}{a_1} > \nu_e$.*[11]

The inequality $\mu_e > \nu_e$ in c) is a consequence of the results in [BM2], although it is not stated explicitly there. It can also easily be checked directly.

The values of ν_e and μ_e will be computed for small values of e in Example 3.16.

3.3. **Contractions of extremal rays.** The extremal rays of the nef cone (or rather, the dual rays in the dual Mori cone of curves of $S^{[2]}$) can be contracted: let G be a divisor on $S^{[2]}$ whose class g is a primitive integral generator of the ray. Then G is nef and

[10] The inequality $\nu_S \geq \nu_e$ was first proved in [HT2, Theorem 22] and the rationality of ν_S was also proved by very different methods in [Og1, Corollary 5.2].

[11] There is a typo in [BM2, Lemma 13.3(b)]: one should replace d with $2d$.

- either $g^2 > 0$ (cases b) and c)),[12] G is big, and the linear system $|mG|$ is base-point-free for $m \gg 0$ by Kawamata's base-point-free theorem and defines a birational contraction $S^{[2]} \to M$ of the ray $g^\perp$;
- or $g^2 = 0$ (case a)) and the linear system $|G|$ is already base-point-free and defines a Lagrangian fibration $S^{[2]} \to \mathbf{P}^2$ ([BM2, Theorem 1.5]).

The (divisorial) contraction of the ray $\mathbf{R}_{\geq 0} f$ is the Hilbert–Chow morphism $S^{[2]} \to S^{(2)}$. The contraction of the other ray was described in [BM2]. We summarize the results in Proposition 3.6 below. Before stating them, we define the Mukai lattice of a K3 surface S as the free abelian group

$$H^\bullet(S, \mathbf{Z}) := H^0(S, \mathbf{Z}) \oplus H^2(S, \mathbf{Z}) \oplus H^4(S, \mathbf{Z})$$

endowed with the non-degenerate quadratic form

$$(r, c, s)^2 := c^2 - 2rs.$$

We denote by

$$H^\bullet_{\mathrm{alg}}(S, \mathbf{Z}) := H^0(S, \mathbf{Z}) \oplus \mathrm{Pic}(S) \oplus H^4(S, \mathbf{Z})$$

its algebraic part. Given a coherent sheaf E on S, we define its Mukai vector by

$$\mathbf{v}(E) := \mathrm{ch}(E)\sqrt{\mathrm{td}(S)} = \mathrm{rk}(E) + c_1(E) + \big(\mathrm{rk}(E) + \tfrac{1}{2}c_1(E)^2 - c_2(E)\big) \in H^\bullet_{\mathrm{alg}}(S, \mathbf{Z}).$$

For any primitive vector $\mathbf{v} \in H^\bullet_{\mathrm{alg}}(S, \mathbf{Z})$ with $\mathbf{v}^2 \geq -2$, the moduli space $M_h(\mathbf{v})$ of sheaves E on S such that $\mathbf{v}(E) = \mathbf{v}$ and which are stable for a fixed very general polarization on S is a smooth projective hyperkähler manifold of dimension $\mathbf{v}^2 + 2$. Moreover, when $\mathbf{v}^2 > 0$, there is an isomorphism

$$\theta\colon H^\bullet(S, \mathbf{Z})_{\mathbf{v}} \xrightarrow{\sim} H^2(M(\mathbf{v}), \mathbf{Z}) \tag{11}$$

of polarized Hodge structures, where $H^\bullet(S, \mathbf{Z})_{\mathbf{v}} := \mathbf{v}^\perp \subset H^\bullet(S, \mathbf{Z})$. These statements were proved by Mukai, Huybrechts, O'Grady, and Yoshioka and appear in [BM2, Theorem 3.6]. For example, we have $S^{[2]} = M((1, 0, -1))$, $f = \theta(0, -f_S, 0)$, and $\delta = \theta(-1, 0, -1)$.

Remark 3.5. The Mukai lattice can also be defined for "twisted" K3 surfaces. Let S be a K3 surface and let $\alpha \in \mathrm{Br}(S)$ be a Brauer class. By [HS], there exists a weight-2 Hodge structure on $H^\bullet(S, \mathbf{Z})$ that takes into account the Brauer class α. We will use the notation $H^\bullet(S, \alpha, \mathbf{Z})$ for this Hodge structure and $H^\bullet_{\mathrm{alg}}(S, \alpha, \mathbf{Z})$ for its algebraic part. When $\alpha = 1$, this coincides with the previous definition. A twisted Chern character and a Mukai vector can be defined for α-twisted sheaves. Moduli spaces of twisted sheaves with Mukai vector $\mathbf{v}$ behave as in the untwisted case ([Y2]). In particular, if $\mathbf{v} \in H^\bullet_{\mathrm{alg}}(S, \alpha, \mathbf{Z})$ is primitive with $\mathbf{v}^2 \geq -2$, the scheme $M(\mathbf{v})$ is a smooth projective hyperkähler manifold of dimension $\mathbf{v}^2 + 2$ and, when $\mathbf{v}^2 > 0$, its second cohomology group can be identified with the orthogonal $H^\bullet(S, \alpha, \mathbf{Z})_{\mathbf{v}}$ as in (11).

Proposition 3.6. *Let (S, f_S) be a polarized K3 surface of degree $2e$ with Picard group $\mathbf{Z}f_S$.*

[12] In case b), we have $g = a_1 f - eb_1\delta$ and $g^2 = 2e$. In case c), we have $g = a_5 f - 2eb_5\delta$ and $g^2 = 10e$ if $5 \nmid e$ and $g = (a_5 f - 2eb_5\delta)/5$ and $g^2 = 2e/5$ otherwise.

- *Assume that the equation* $\mathscr{P}_{4e}(5)$ *is not solvable and let* (a_1, b_1) *be the minimal solution of the equation* $\mathscr{P}_e(1)$.

  There exists a polarized K3 surface (T, f_T)*, a Brauer class* $\alpha \in \mathrm{Br}(T)$*, and a Fourier–Mukai isomorphism* $\Phi\colon \mathrm{D^b}(S) \xrightarrow{\sim} \mathrm{D^b}(T,\alpha)$ *such that we are in one of the following cases.*

  a) *If* e *is a perfect square, we have* $f_T^2 = 2$ *and* Φ *induces an isomorphism*

  $$\varphi\colon S^{[2]} \xrightarrow{\sim} M_{f_T}(\mathbf{v}),$$

  where $\mathbf{v} = (0, f_T, *)$*. The contraction of the extremal ray* $\varphi_*(f - \nu_S\delta)^\perp$ *on* $M_{f_T}(\mathbf{v})$ *is the Lagrangian fibration* $M_{f_T}(\mathbf{v}) \to \mathbf{P}^2$ *mapping a torsion sheaf to its support (Beauville integrable system).*

  b′) *If* e *is not a perfect square and* b_1 *is odd,* Φ *induces an isomorphism*

  $$\varphi\colon S^{[2]} \xrightarrow{\sim} M_{f_T}(\mathbf{v}),$$

  where $\mathbf{v} = (2, c, *)$*. The contraction of the extremal ray* $\varphi_*(f - \nu_S\delta)^\perp$ *on* $M_{f_T}(\mathbf{v})$ *is the divisorial contraction of the locus of non-locally-free torsion-free slope-stable sheaves (LGU divisorial contraction). The exceptional locus is a* $\mathbf{P}^1$*-bundle over* T*.*

  b″) *If* e *is not a perfect square and* b_1 *is even, we have* $\alpha = 1$ *and* Φ *induces an isomorphism*

  $$\varphi\colon S^{[2]} \xrightarrow{\sim} T^{[2]}.$$

  The contraction of the extremal ray $\varphi_*(f - \nu_S\delta)^\perp$ *on* $T^{[2]}$ *is the divisorial contraction given by the Hilbert–Chow morphism* $T^{[2]} \to T^{(2)}$*.*

- *Assume that the equation* $\mathscr{P}_{4e}(5)$ *is solvable.*

  c) *The contraction of the extremal ray* $(f - \nu_S\delta)^\perp$ *is the small contraction of a Lagrangian plane in* $S^{[2]}$*.*[13]

In the proof of the proposition, we will need the notion of Bridgeland stability conditions on the derived category $\mathrm{D^b}(S)$ of the K3 surface S. We refer to [Br1] for the basic definitions and to [Br2] for the case of K3 surfaces. We will only be concerned with stability conditions of the form $\sigma_{s\cdot f_S, t\cdot f_S}$, where $s \in \mathbf{R}_{>0}$ and $t \in \mathbf{R}$; their central charges are given by $Z_{s\cdot f_S, t\cdot f_S}(\mathbf{v}) := e^{(t+\sqrt{-1}s)\cdot f_S} \cdot \mathbf{v}$. Moduli spaces of Bridgeland semistable objects make sense as in the usual case of sheaves; given a Mukai vector $\mathbf{v}$ and a stability condition σ, the corresponding moduli space will be denoted by $M_\sigma(\mathbf{v})$. The case of Bridgeland stability conditions on twisted K3 surfaces is similar and was studied in [HMS].

Proof of Proposition 3.6. Assume first that the equation $\mathscr{P}_{4e}(5)$ is not solvable. If e is a perfect square, we consider the Mukai vector $\mathbf{w} := (-\sqrt{e}, f_S, -\sqrt{e})$. As in [BM2, Section 11], in the space of Bridgeland stability conditions for $\mathrm{D^b}(S)$, we consider stability conditions of the form $\sigma_s := \sigma_{s\cdot f_S, -1/\sqrt{e}\cdot f_S}$. The moduli space $M_{\sigma_s}(\mathbf{w})$ does not change for all s sufficiently small: it is a smooth K3 surface which we denote by T and comes with a Brauer class $\alpha \in \mathrm{Br}(T)$ which detects whether the moduli space is fine or not. We let $\Phi\colon \mathrm{D^b}(S) \xrightarrow{\sim} \mathrm{D^b}(T,\alpha)$ be the corresponding Fourier–Mukai functor.

[13] In that case, there is a Mukai flop $S^{[2]} \dashrightarrow F$, where F is a hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type.

Since the equation $\mathscr{P}_{4e}(5)$ is not solvable, the moduli space $M_{\sigma_s}((1,0,-1))$ is still isomorphic to the Hilbert scheme $S^{[2]}$. Via the functor Φ, we get an isomorphism

$$M_{\sigma_s}((1,0,-1)) \xrightarrow{\sim} M_{\Phi(\sigma_s)}(\Phi(1,0,-1)).$$

It is an easy consequence of [BM2, Lemmas 11.1 and 11.2] that in our case, we have $\Phi(1,0,-1) = (0, f_T, *)$, where f_T is the ample generator of the Picard group of T and $f_T^2 = 2$. Finally, since stable objects with Mukai vector $\mathbf{w}$ become skyscraper sheaves on T, we have an isomorphism between $M_{\Phi(\sigma_s)}(\Phi(1,0,-1))$ and the moduli space of torsion sheaves on (T,α) with Mukai vector $(0, f_T, *)$. This gives part a) of the proposition.

If e is not a perfect square, we set $\mathbf{s} := (a_1, -b_1 f_S, a_1) \in H^\bullet_{\mathrm{alg}}(S, \mathbf{Z})$ and $\mathbf{w} := (1,0,-1) - \mathbf{s} = (1-a_1, b_1 f_S, -1-a_1)$. Then, $\mathbf{s}\cdot\mathbf{w} = 2$, $\mathbf{w}^2 = 0$, and $(1,0,-1)\cdot\mathbf{w} = 2$. Moreover, the divisibility of $\mathbf{w}$ is 2 or 1, according to whether b_1 is even or odd.

Assume first that b_1 is even. Let $\mathbf{w}_0 := \mathbf{w}/2$ and let $\sigma = \sigma_{s_0 f_S, t_0 f_S}$ be a Bridgeland stability condition on $\mathrm{D^b}(S)$ lying on the wall for $(1,0,-1)$ corresponding to the divisor $a_1 f - eb_1\delta$.[14] Since $-b_1/a_1 > -1/\sqrt{e}$, the class $\mathbf{s}$ is effective (in the sense of [BM2, Proposition 5.5]) on the wall; this means that the wall is given by a σ-stable spherical *vector bundle* on S with Mukai vector $\mathbf{s}$. As before, we look at the moduli space $M_\sigma(\mathbf{w}_0)$. Since $\mathbf{s}$ is effective and $\mathbf{s}\cdot\mathbf{w}_0 \geq 0$, there are no properly σ-semistable objects of class $\mathbf{w}_0$ ([Br2, Theorem 12.1]). Therefore, the scheme $M_\sigma(\mathbf{w}_0)$ is a smooth projective K3 surface, denoted again by T, and it is a fine moduli space since $\mathbf{w}_0\cdot(1,0,-1) = 1$. Let $\Phi\colon \mathrm{D^b}(S) \xrightarrow{\sim} \mathrm{D^b}(T)$ be the corresponding Fourier–Mukai functor. Let $\sigma' := \sigma_{s\cdot f_S, t_0\cdot f_S}$, for $s > s_0$ sufficiently close to s_0. As in the previous case, we have isomorphisms

$$S^{[2]} \xrightarrow{\sim} M_{\sigma'}((1,0,-1)) \xrightarrow{\sim} M_{\Phi(\sigma')}(\Phi(1,0,-1)).$$

But now, since $\mathbf{w}_0\cdot(1,0,-1) = 1$, we must have $\Phi(1,0,-1) = (-1,*,*)$. Upon composing Φ with the tensor product by a line bundle on T, we have $\Phi(1,0,-1) = (-1,0,1)$. Since stable objects with Mukai vector $\mathbf{w}_0$ become skyscraper sheaves on T, we have an isomorphism between $M_{\Phi(\sigma')}(\Phi(1,0,-1))$ and the moduli space of shifts by 1 of ideal sheaves of length-2 subschemes of T, namely $T^{[2]}$. Moreover, the wall corresponds exactly to the wall giving the Hilbert–Chow morphism $T^{[2]} \to T^{(2)}$ in the space of geometric stability conditions for $\mathrm{D^b}(T)$. This proves b$''$).

If b_1 is odd, $\mathbf{w}$ is primitive. Keeping the notation as above, we consider the moduli space $M_\sigma(\mathbf{w})$, which is still a smooth projective K3 surface, denoted again by T. We again have a Brauer class $\alpha \in \mathrm{Br}(T)$ which may be non-trivial. The Fourier–Mukai transform $\Phi\colon \mathrm{D^b}(S) \xrightarrow{\sim} \mathrm{D^b}(T,\alpha)$ now maps $S^{[2]}$ isomorphically onto a moduli space of rank-2 slope-stable[15] α-twisted torsion-free sheaves on T. The induced contraction is the morphism contracting all non-locally-free sheaves studied in [Li], thus proving b$'$).

Finally, c) follows from [BM2, Theorem 5.7]. □

[14] The wall is given by the locus of those $(s,t) \in \mathbf{R}_{>0}\times\mathbf{R}$ for which $Z_{s\cdot f_S, t\cdot f_S}(\mathbf{s})$ and $Z_{s\cdot f_S, t\cdot f_S}((1,0,-1))$ are parallel.

[15] The fact that there are no properly slope-semistable sheaves follows from a direct computation.

Remark 3.7. In the case b′) of Proposition 3.6, the induced contraction is not a Hilbert–Chow morphism. Indeed, as observed in [M3, Example 10.15 and Lemma 10.16],[16] the class of the divisor consisting of non-locally-free sheaves is not divisible by 2, as it is in the Hilbert–Chow case.

3.4. **Automorphisms of very general Hilbert squares.** Since the extremal rays of the movable cone of the Hilbert square $S^{[2]}$ of a very general K3 surface S of given degree $2e$ are rational (Theorem 3.4), its group $\mathrm{Bir}(S^{[2]})$ of birational automorphisms is finite ([Og1, Theorem 1.3(2)]). Using the Torelli Theorem 2.2, one can determine its group $\mathrm{Aut}(S^{[2]})$ of biregular automorphisms ([BCNS, Theorem 1.1]).

Theorem 3.8 (Boissière–Cattaneo–Nieper-Wißkirchen–Sarti)**.** *Let (S, f_S) be a polarized K3 surface of degree $2e$ with Picard group $\mathbf{Z}f_S$. The variety $S^{[2]}$ has a non-trivial automorphism if and only if either $e = 1$, or e is not a perfect square, the equation $\mathscr{P}_e(-1)$ is solvable, and the equation $\mathscr{P}_{4e}(5)$ is not.*

This automorphism is then a non-symplectic involution. When $e \geq 2$, this involution acts on $H^2(S^{[2]}, \mathbf{Z})$ as the symmetry s_D about the line spanned by the class $D := b_{-1}f - a_{-1}\delta$, of square 2, where (a_{-1}, b_{-1}) is the minimal solution of the equation $\mathscr{P}_e(-1)$. When $e = 1$, this involution is induced by an involution of (S, f_S) and its acts on $H^2(S^{[2]}, \mathbf{Z})$ as the symmetry about the plane $\mathrm{Pic}(S^{[2]})$.

Remark 3.9. When the hypotheses of Theorem 3.8 are realized, the extremal rays of the nef cone of $S^{[2]}$ are spanned (with the notation of Theorem 3.4) by f and

$$s_D(f) = -f + (f, D)D = a_1(f - \nu_e\delta) = a_1 f - eb_1\delta,$$

where $(a_1, b_1) = (2a_{-1}^2 + 1, 2a_{-1}b_{-1})$ is the minimal solution of the equation $\mathscr{P}_e(1)$ ([BCNS, Lemma 2.1]). In particular, the class $D = \frac{1}{2eb_{-1}}(f + s_D(f))$ is always ample ($\frac{a_{-1}}{b_{-1}} < \nu_e$).

Example 3.10. Theorem 3.8 applies for example for $e = m^2 + 1$ with $m \neq 2$, or $e = 13$ (see Examples 3.1 and 3.16).

When $e = 2$, the surface S is a quartic in $\mathbf{P}^3$ which contains no lines nor conics, and the involution σ of $S^{[2]}$ is the Beauville involution, with $D = f - \delta$ and $s_D(f) = 3f - 4\delta$ ([D, Théorème 4.1], [BCNS, Section 6.1]); the quotient $S^{[2]}/\sigma$ is a triple cover of the Plücker quadric $G(2, 4) \subset \mathbf{P}^5$. This is the only case where the involution on $S^{[2]}$ has been described geometrically (see however Corollary 5.11 for an indirect description which explains what the quotient $S^{[2]}/\sigma$ is for $e \geq 3$).

When $e = 10$, one has $D = f - 3\delta$ and $s_D(f) = 19f - 60\delta$ (compare with [HT1, Example 7.8]). When $e = 13$, one has $D = 5f - 18\delta$ and $s_D(f) = 649f - 2340\delta$ (compare with [HT1, Example 7.9]).

Proposition 3.11. *Let (S, f_S) be a polarized K3 surface of degree $2e \geq 4$ with Picard group $\mathbf{Z}f_S$. The groups $\mathrm{Bir}(S^{[2]})$ and $\mathrm{Aut}(S^{[2]})$ are equal (and described in Theorem 3.8) except when $e = 5$, where $\mathrm{Aut}(S^{[2]})$ is trivial but $\mathrm{Bir}(S^{[2]})$ has two elements.*

[16] Although only the untwisted case is considered in [M3], the proof in the twisted rank-2 case follows similar ideas.

Proof. Consider the morphism $\bar\Psi_B$ defined in (9). If $\varphi \in \mathrm{Bir}(S^{[2]})$ is not biregular, $\bar\varphi^* = \bar\Psi_B(\varphi)$ acts on the movable cone $\mathrm{Mov}(S^{[2]})$ in such a way that $\bar\varphi^*(\mathrm{Amp}(S^{[2]})) \cap \mathrm{Amp}(S^{[2]}) = \varnothing$ (if the pull-back by φ of an ample class were ample, φ would be regular). This implies $\mathrm{Mov}(S^{[2]}) \neq \mathrm{Nef}(S^{[2]})$ hence, by Theorem 3.4, the equation $\mathscr{P}_{4e}(5)$ has a minimal solution (a_5, b_5). By Theorem 3.8, the group $\mathrm{Aut}(S^{[2]})$ is then trivial. Moreover, with the notation of Theorem 3.4, we have

$$\bar\varphi^*_{\mathbf{R}}(\mathbf{R}_{\geq 0} f) = \mathbf{R}_{\geq 0}\Big(f - e\frac{b_1}{a_1}\delta\Big) \quad \text{and} \quad \bar\varphi^*_{\mathbf{R}}\Big(\mathbf{R}_{\geq 0}\big(f - 2e\frac{b_5}{a_5}\delta\big)\Big) = \mathbf{R}_{\geq 0}\Big(f - 2e\frac{b_5}{a_5}\delta\Big).$$

Since $\bar\varphi^*$ is defined over $\mathbf{Z}$ and both f and $a_1 f - eb_1\delta$ are primitive, we have $\bar\varphi^*(f) = a_1 f - eb_1\delta$ and, by applying this relation to φ^{-1}, also $\bar\varphi^*(a_1 f - eb_1\delta) = f$. This implies that $\bar\varphi^*$ is a completely determined involution of $\mathrm{Pic}(S^{[2]})$. In particular, φ^2 is an automorphism, hence is trivial: φ is an involution.

The transcendental lattice $\mathrm{Pic}(S^{[2]})^\perp \subset H^2(S^{[2]}, \mathbf{Z})$ carries a simple rational Hodge structure.[17] Since the eigenspaces of the involution φ^* of $H^2(S^{[2]}, \mathbf{Z})$ are sub-Hodge structures, the restriction of φ^* to $\mathrm{Pic}(S^{[2]})^\perp$ is $\varepsilon\,\mathrm{Id}$, with $\varepsilon \in \{\pm 1\}$. On $\mathrm{Pic}(S^{[2]})$, we saw that φ^* has matrix $\left(\begin{smallmatrix} a_1 & b_1 \\ -eb_1 & -a_1 \end{smallmatrix}\right)$ in the basis (f, δ). The extension from $\mathrm{Pic}(S^{[2]}) \oplus \mathrm{Pic}(S^{[2]})^\perp$ to the overlattice $H^2(S^{[2]}, \mathbf{Z})$ of such an involution was studied in [BCNS, Lemma 5.2] when $\varepsilon = -1$. The reasoning is the same when $\varepsilon = 1$ and the conclusion is that in both cases, there exist positive integers α_ε and β_ε such that $\alpha_\varepsilon^2 - e\beta_\varepsilon^2 = \varepsilon$ and $a_1 + b_1\sqrt{e} = (\alpha_\varepsilon + \beta_\varepsilon\sqrt{e})^2$. The case $\varepsilon = 1$ is impossible since it would contradict the minimality of the solution (a_1, b_1) of the equation $\mathscr{P}_e(1)$. It follows that $\varepsilon = -1$, and $(\alpha_{-1}, \beta_{-1})$ is the minimal solution of the equation $\mathscr{P}_e(-1)$. In particular, φ^* is a completely determined involution of $H^2(S^{[2]}, \mathbf{Z})$ and, since Ψ_B is injective (Section 2.3), $\mathrm{Bir}(S^{[2]})$ has at most 2 elements.

The invariant subspace of the involution φ^* is spanned, on the one hand by $a_5 f - 2eb_5\delta$, and on the other hand by $f + \varphi^*(f) = (a_1+1)f - eb_1\delta = 2e\beta_{-1}(\beta_{-1}f - \alpha_{-1}\delta)$. Since $\beta_{-1}f - \alpha_{-1}\delta$ is primitive, this implies $a_5 = m\beta_{-1}$ and $2eb_5 = m\alpha_{-1}$, where $m := \gcd(a_5, 2eb_5) \in \{1, 5\}$. Plugging these values into the equation $\mathscr{P}_{4e}(5)$, we obtain $5e = m^2$, hence $m = e = 5$.

When $e = 5$, there is indeed a non-trivial birational involution on $S^{[2]}$ (footnote 5). This finishes the proof of the proposition. ☐

3.5. **Ambiguous Hilbert squares.** Recall the following definition ([H1, Definition 6.2.1]).

Definition 3.12 (Hassett)**.** Let S be a K3 surface. We say that $S^{[2]}$ is *ambiguous* if there exist a K3 surface $\bar S$ and an isomorphism $S^{[2]} \xrightarrow{\sim} \bar S^{[2]}$ which is not induced by an isomorphism $S \xrightarrow{\sim} \bar S$. We say that $S^{[2]}$ is *strongly ambiguous* if there exists such a K3 surface $\bar S$ which is in addition not isomorphic to S.

An isomorphism $r\colon S^{[2]} \xrightarrow{\sim} \bar S^{[2]}$ is induced by an isomorphism $S \xrightarrow{\sim} \bar S$ if and only if $r^*\delta_{\bar S} = \delta_S$.[18]

[17]This is a classical fact found for example in [Hu2, Lemma 3.1].

[18]For this, one can follow the proof of [BS, Theorem 1].

Remark 3.13. If S has Picard group $\mathbf{Z}f_S$ and $r\colon S^{[2]} \xrightarrow{\sim} \bar{S}^{[2]}$ is an isomorphism as in Definition 3.12, $\mathrm{Pic}(\bar{S})$ has rank 1, with ample generator $\bar{f}_{\bar{S}}$ of the same degree $2e$,[19] and the extremal rays of the nef cone of $S^{[2]}$ are, on the one hand $\mathbf{R}_{\geq 0}f$ and $\mathbf{R}_{\geq 0}(f-\nu_e\delta)$, and, on the other hand $\mathbf{R}_{\geq 0}(r^*\bar{f})$ and $\mathbf{R}_{\geq 0}(r^*\bar{f}-\nu_e r^*\delta_{\bar{S}})$ (with obvious notation). Since $r^*\delta_{\bar{S}} \neq \delta_S$ and r^* is an isometry, $r^*\bar{f}$ and f are not colinear, hence $r^*\bar{f}$ is the unique positive multiple of $f-\nu_e\delta$ with square $2e$. Therefore, if $r'\colon S^{[2]} \xrightarrow{\sim} \bar{S}'^{[2]}$ is another isomorphism such that $r'^*\delta_{\bar{S}'} \neq \delta_S$, we have $r^*\bar{f} = r'^*\bar{f}'$, hence $(r' \circ r^{-1})^*(\bar{f}') = \bar{f}$ and $(r' \circ r^{-1})^*(\delta_{\bar{S}'}) = \delta_{\bar{S}}$ (because δ spans $f^\perp$). Therefore, there exists an isomorphism $\varphi\colon \bar{S} \xrightarrow{\sim} \bar{S}'$ such that $r' = \varphi^{[2]} \circ r$.

If $S^{[2]}$ is ambiguous with Picard number 2, there is therefore a K3 surface $\bar{S}$, unique up to isomorphism, that satisfies the conditions of the definition and

- either $\bar{S}$ is not isomorphic to S and $S^{[2]}$ is strongly ambiguous;
- or $\bar{S}$ is isomorphic to S and $S^{[2]}$ has a non-trivial automorphism which, by Theorem 3.8, is an involution; $S^{[2]}$ is not strongly ambiguous.

Conversely, for any K3 surface S satisfying the conditions of Theorem 3.8, the variety $S^{[2]}$ is ambiguous (take $\bar{S} = S$ and for r the non-trivial involution of $S^{[2]}$) but not strongly ambiguous.

We obtain the following characterization of (strongly) ambiguous Hilbert squares of K3 surfaces as a direct consequence of Proposition 3.6.

Proposition 3.14. *Let (S, f_S) be a polarized K3 surface of degree $2e$ with Picard group $\mathbf{Z}f_S$. The fourfold $S^{[2]}$ is ambiguous if and only if the following conditions are all realized:*

1) *e is not a perfect square;*
2) *the equation $\mathscr{P}_{4e}(5)$ is not solvable;*
3) *the minimal solution (a_1, b_1) of the equation $\mathscr{P}_e(1)$ has b_1 even.*

The fourfold $S^{[2]}$ is strongly ambiguous if and only if, in addition, the equation $\mathscr{P}_e(-1)$ is not solvable.

Proof. Assume that $S^{[2]}$ is ambiguous, so that there is an isomorphism $S^{[2]} \xrightarrow{\sim} \bar{S}^{[2]}$ as in Definition 3.12. If we use r to identify $\bar{S}^{[2]}$ with $S^{[2]}$, the nef cone of $S^{[2]}$ is, with obvious notation, spanned by f and $\bar{f}$ (Remark 3.13). Comparing with Theorem 3.4, we see that

- either the equation $\mathscr{P}_{4e}(5)$ is not solvable and
  - either e is a perfect square and $\bar{f}$ is a multiple of $f - \sqrt{e}\delta$; this is absurd since $(f-\sqrt{e}\delta)^2 = 0$ and $\bar{f}^2 = 2e > 0$;
  - or e is not a perfect square and $\bar{f}$ is a multiple of $a_1 f - eb_1\delta$, where (a_1, b_1) is the minimal solution of the equation $\mathscr{P}_e(1)$; taking squares, we obtain $\bar{f} = a_1 f - eb_1\delta$;
- or $\mathscr{P}_{4e}(5)$ has a minimal solution (a_5, b_5) and $\bar{f}$ is a multiple of $a_5 f - 2eb_5\delta$; this is absurd since $(a_5 f - 2eb_5\delta)^2 = 10e$ and $\bar{f}^2 = 2e$.

Conditions 1) and 2) are therefore satisfied and, by Theorem 3.4, the two extremal rays of the nef cone are generated by f and $\bar{f} = a_1 f - eb_1\delta$ respectively.

[19] This is because the discriminant of the lattice $(\mathrm{Pic}(S^{[2]}), q_{BBJ})$ is equal to $-2q_{BBJ}(f) = -2f_S^2$.

The divisorial contraction associated with this second ray is therefore the Hilbert–Chow morphism $\bar{S}^{[2]} \to \bar{S}^{(2)}$. As observed in Remark 3.7, b_1 must be even. This proves condition 3).

Conversely, if conditions 1), 2), and 3) are all satisfied, $S^{[2]}$ is ambiguous by Proposition 3.6.b″). □

Remark 3.15. Let n be a square-free positive integer and let e be a positive integer divisible by n, distinct from n, and such that the equation $\mathscr{P}_{4e}(-n)$ (or equivalently, the equation $\mathscr{P}_{n,4e/n}(-1)$) is solvable but the equation $\mathscr{P}_{4e}(5)$ is not. If (S, f_S) is a polarized K3 surface of degree $2e$ with Picard group $\mathbf{Z}f_S$, the fourfold $S^{[2]}$ is ambiguous: e is not a perfect square (because the equation $\mathscr{P}_{n,4e/n}(-1)$ is solvable) and condition 3) of Proposition 3.14 follows from Lemma 3.3 (because the equation $\mathscr{P}_{n,e/n}(-1)$ is solvable and $e/n \neq 1$). A slightly weaker result was proved in [H1, Proposition 6.2.2] when $n = 3$.

Example 3.16. Using Propositions 3.4, 3.8, and 3.14, and Theorem 3.8 we compute, for very general Hilbert squares of K3 surfaces of small degrees $2e$, the values of the slopes ν_e and μ_e of their nef and movable cones, their biregular and birational automorphism groups, and whether they are (strongly) ambiguous (the case $e = 6$ was considered in [Y1, Example 7.2]).

e	1	2	3	4	5	6	7	8	9	10	11	12	13
$\mathscr{P}_e(1)$	–	$(3,2)$	$(2,1)$	–	$(9,4)$	$(5,2)$	$(8,3)$	$(3,1)$	–	$(19,6)$	$(10,3)$	$(7,2)$	$(649,180)$
$\mathscr{P}_{4e}(5)$	$(3,1)$	–	–	–	$(5,1)$	–	–	–	–	–	$(7,1)$	–	–
$\mathscr{P}_e(-1)$	–	$(1,1)$	–	–	$(2,1)$	–	–	–	–	$(3,1)$	–	–	$(18,5)$
ν_e	1	$\frac{4}{3}$	$\frac{3}{2}$	2	2	$\frac{12}{5}$	$\frac{21}{8}$	$\frac{8}{3}$	3	$\frac{60}{19}$	$\frac{22}{7}$	$\frac{24}{7}$	$\frac{2340}{649}$
μ_e	1	$\frac{4}{3}$	$\frac{3}{2}$	2		$\frac{12}{5}$	$\frac{21}{8}$	$\frac{8}{3}$	3	$\frac{60}{19}$	$\frac{33}{10}$	$\frac{24}{7}$	$\frac{2340}{649}$
$\mathrm{Aut}(S^{[2]})$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	Id	Id	Id	Id	Id	Id	Id	$\mathbf{Z}/2$	Id	Id	$\mathbf{Z}/2$
$\mathrm{Bir}(S^{[2]})$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	Id	Id	$\mathbf{Z}/2$	Id	Id	Id	Id	$\mathbf{Z}/2$	Id	Id	$\mathbf{Z}/2$
$S^{[2]}$ ambiguous	–	✓	–	–	–	✓	–	–	–	✓	–	✓	✓
$S^{[2]}$ strongly ambiguous	–	–	–	–	–	✓	–	–	–	–	–	✓	–

Data for $1 \leq e \leq 13$

e	14	15	16	17	18	19	20	21	22
$\mathscr{P}_e(1)$	$(15,4)$	$(4,1)$	–	$(33,8)$	$(17,4)$	$(170,39)$	$(9,2)$	$(55,12)$	$(197,42)$
$\mathscr{P}_{4e}(5)$	–	–	–	–	–	$(9,1)$	–	–	–
$\mathscr{P}_e(-1)$	–	–	–	$(4,1)$	–	–	–	–	–
ν_e	$\frac{56}{15}$	$\frac{15}{4}$	4	$\frac{136}{33}$	$\frac{72}{17}$	$\frac{38}{9}$	$\frac{40}{9}$	$\frac{252}{55}$	$\frac{924}{197}$
μ_e	$\frac{56}{15}$	$\frac{15}{4}$	4	$\frac{136}{33}$	$\frac{72}{17}$	$\frac{741}{170}$	$\frac{40}{9}$	$\frac{252}{55}$	$\frac{924}{197}$
$\mathrm{Aut}(S^{[2]}) = \mathrm{Bir}(S^{[2]})$	Id	Id	Id	$\mathbf{Z}/2$	Id	Id	Id	Id	Id
$S^{[2]}$ ambiguous	–	–	–	✓	✓	–	✓	✓	✓
$S^{[2]}$ strongly ambiguous	–	–	–	–	✓	–	✓	✓	✓

Data for $14 \leq e \leq 22$

e	23	24	25	26	27	28	29	30	31
$\mathscr{P}_e(1)$	$(24,5)$	$(5,1)$	–	$(51,10)$	$(26,5)$	$(127,24)$	$(9801,1820)$	$(11,2)$	$(1520,273)$
$\mathscr{P}_{4e}(5)$	–	–	–	–	–	–	$(11,1)$	–	$(657,59)$
$\mathscr{P}_e(-1)$	–	–	–	$(5,1)$	–	–	$(70,13)$	–	–
ν_e	$\frac{115}{24}$	$\frac{24}{5}$	5	$\frac{260}{51}$	$\frac{135}{26}$	$\frac{672}{127}$	$\frac{58}{11}$	$\frac{60}{11}$	$\frac{3658}{657}$
μ_e	$\frac{115}{24}$	$\frac{24}{5}$	5	$\frac{260}{51}$	$\frac{135}{26}$	$\frac{672}{127}$	$\frac{52780}{9801}$	$\frac{60}{11}$	$\frac{3658}{657}$
$\mathrm{Aut}(S^{[2]}) = \mathrm{Bir}(S^{[2]})$	Id	Id	Id	$\mathbf{Z}/2$	Id	Id	Id	Id	Id
$S^{[2]}$ ambiguous	–	–	–	✓	–	✓	–	✓	–
$S^{[2]}$ strongly ambiguous	–	–	–	–	–	✓	–	✓	–

Data for $23 \le e \le 31$

4. The geometry of special hyperkähler fourfolds with Picard number 2

We study in this section the geometry of hyperkähler fourfolds F of $\mathrm{K3}^{[2]}$-type with a polarization h of square $2n$ and divisibility 2 (in particular, we must have $n \equiv -1 \pmod 4$), when the Picard number is 2; we let $2e$ be the discriminant (see Section 2.2). For simplicity, we will always assume that n divides e.

4.1. **The Markman–Mukai lattice.** We recall a construction of Markman's (see [M2, Section 9] and [BHT, Section 1]) which generalizes to arbitrary hyperkähler manifolds F of $\mathrm{K3}^{[r]}$-type the Mukai lattice for moduli spaces of sheaves on a K3 surface. We define the *extended K3 lattice*

$$\widetilde{\Lambda}_{\mathrm{K3}} := U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}.$$

There is an extension $H^2(F,\mathbf{Z}) \subset \widetilde{\Lambda}_F$ of lattices and weight-2 Hodge structures, where the lattice $\widetilde{\Lambda}_F$ is isomorphic to $\widetilde{\Lambda}_{\mathrm{K3}}$, characterized as follows:

- the orthogonal $H^2(F,\mathbf{Z})^\perp \subset \widetilde{\Lambda}_F$ is generated by a primitive vector of square $2r-2$;
- there is a natural extension of the monodromy action on $H^2(F,\mathbf{Z})$ to $\widetilde{\Lambda}_F$;
- the Torelli Theorem has the following form: hyperkähler manifolds F_1 and F_2 of $\mathrm{K3}^{[r]}$-type are birationally isomorphic if and only if there is a Hodge isometry $\widetilde{\Lambda}_{F_1} \xrightarrow{\sim} \widetilde{\Lambda}_{F_2}$ mapping $H^2(F_1,\mathbf{Z})$ isomorphically to $H^2(F_2,\mathbf{Z})$;
- if F is a moduli space of sheaves on a K3 surface S with Mukai vector $\mathbf{v} \in H^\bullet_{\mathrm{alg}}(S,\mathbf{Z})$, the extension $H^2(F,\mathbf{Z}) \subset \widetilde{\Lambda}_F$ is naturally identified with the extension $\mathbf{v}^\perp \subset H^\bullet(S,\mathbf{Z})$ coming from the isomorphism (11).

In the case $r = 2$ (or more generally when $r-1$ is a prime power), any two primitive embeddings of $H^2(F,\mathbf{Z})$ into $\widetilde{\Lambda}_F$ differ by an automorphism of $\widetilde{\Lambda}_F$ ([M1, Section 4.1]).

We denote by $\widetilde{\Lambda}_{\mathrm{alg},F}$ the algebraic part of $\widetilde{\Lambda}_F$.

Proposition 4.1. *Let (F,h) be a polarized hyperkähler fourfold of* $\mathrm{K3}^{[2]}$*-type, degree $2n$ and divisibility 2, Picard number 2, and discriminant $2e$. Assume moreover that n divides e and set $e' := e/n$. The algebraic Markman–Mukai lattice $\widetilde{\Lambda}_{\mathrm{alg},F}$ can be identified with the lattice*

$$\mathbf{Z}\lambda_1 \oplus \mathbf{Z}\lambda_2 \oplus \mathbf{Z}\tau$$

with intersection form

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & \frac{n+1}{2} & 0 \\ 0 & 0 & -2e' \end{pmatrix}.$$

The class of the polarization is $h = \lambda_1 + 2\lambda_2$ and the lattice $(\mathrm{Pic}(F), q_{BBF})$ is identified with $\lambda_1^\perp = \mathbf{Z}h \oplus \mathbf{Z}\tau$, with intersection matrix $\left(\begin{smallmatrix} 2n & 0 \\ 0 & -2e' \end{smallmatrix}\right)$.

Proof. As observed in Section 2.1, we have $n \equiv -1 \pmod 4$. Consider the primitive embedding $I_1(-2) \subset U$ given by mapping the generator ℓ of $I_1(-2)$ to $u_2 - v_2$, where (u_2, v_2) is a standard basis of U. Using the description of $L_{\mathrm{K3}^{[2]}}$ given in Section 2.1, this extends to a primitive embedding $L_{\mathrm{K3}^{[2]}} \subset \widetilde{\Lambda}_{\mathrm{K3}}$ in the obvious way. The orthogonal complement of $L_{\mathrm{K3}^{[2]}}$ in $\widetilde{\Lambda}_{\mathrm{K3}}$ is then generated by $\lambda_1 := u_2 + v_2$, and $\lambda_1^2 = 2$. As in the proof of Proposition 2.3, let (u_1, v_1) be a standard basis of another copy of U. Setting $\lambda_2 := u_1 + \frac{n+1}{4} v_1 - v_2$, we have $\lambda_1 \cdot \lambda_2 = -1$, and $\lambda_2^2 = \frac{n+1}{2}$. Moreover, we can identify h with $\lambda_1 + 2\lambda_2$.

Since, as observed before, any two primitive embeddings of $H^2(F, \mathbf{Z})$ into $\widetilde{\Lambda}_{\mathrm{K3}}$ differ by an automorphism of $\widetilde{\Lambda}_{\mathrm{K3}}$, the only thing left to prove is that there is a primitive class $\tau \in \mathrm{Pic}(F)$ with $h \cdot \tau = 0$ and $\tau^2 = -2e'$. This can be proved as in the proof of Proposition 2.3. □

4.2. **The nef and movable cones.** Our first result is a description of the nef and movable cones for our polarized fourfolds (F,h) when their Picard number is 2, as we did in Theorem 3.4 for Hilbert squares of K3 surfaces. It follows from a simple computation which uses the dual statement of [BHT, Theorem 1].

We keep the notation of Proposition 4.1: the lattice $\mathrm{Pic}(F)$ is $\mathbf{Z}h \oplus \mathbf{Z}\tau$, with intersection matrix $\left(\begin{smallmatrix} 2n & 0 \\ 0 & -2e/n \end{smallmatrix}\right)$. Since the class h is ample on F, the extremal rays of the nef cone $\mathrm{Nef}(F)$ are spanned by $h - \nu_F^- \tau$ and $h + \nu_F^+ \tau$, where ν_F^- and ν_F^+ are positive real numbers. Similarly, the extremal rays of the movable cone $\mathrm{Mov}(F)$ are spanned by $h - \mu_F^- \tau$ and $h + \mu_F^+ \tau$, with $\nu_F^- \le \mu_F^-$ and $\nu_F^+ \le \mu_F^+$. Our result generalizes [HT3, Proposition 7.2], which dealt with the case $n = 3$ and $e = 6$.[20]

Proposition 4.2. *Let F be a hyperkähler fourfold of* $\mathrm{K3}^{[2]}$*-type with a polarization of square $2n$ and divisibility 2, Picard number 2, and discriminant $2e$. Assume that n divides e and set $e' := e/n$. We have then $e > n$, $\nu_F^- = \nu_F^+ = \nu_{n,e}$, and $\mu_F^- = \mu_F^+ = \mu_{n,e}$, where the real numbers $\nu_{n,e}$ and $\mu_{n,e}$ are defined as follows.*[21]

[20]The notation in [HT3] is different from ours: their g is our h and their τ is our $h - \tau$.

[21]If e is a perfect square, we are in case a). If $n \equiv 2, 3 \pmod 5$, the equations $\mathscr{P}_{n,e'}(-1)$ and $\mathscr{P}_{n,4e'}(-5)$ cannot be both solvable at the same time and case d) cannot occur; however, when $n = 11$ and $e' = 4$, both equations have solutions, and we are in case d).

a) *If both equations* $\mathscr{P}_{n,e'}(-1)$ *and* $\mathscr{P}_{n,4e'}(-5)$ *are not solvable, we have* $\nu_{n,e} = \mu_{n,e} := n/\sqrt{e}$.
b) *If the equation* $\mathscr{P}_{n,e'}(-1)$ *is not solvable but the equation* $\mathscr{P}_{n,4e'}(-5)$ *is, and we denote by* (a_{-5}, b_{-5}) *its minimal solution, we have* $\nu_{n,e} = \frac{na_{-5}}{2e'b_{-5}}$ *and* $\mu_{n,e} = n/\sqrt{e} > \nu_{n,e}$.
c) *If the equation* $\mathscr{P}_{n,4e'}(-5)$ *is not solvable but the equation* $\mathscr{P}_{n,e'}(-1)$ *is, and we denote by* (a_{-1}, b_{-1}) *its minimal solution, we have* $\nu_{n,e} = \mu_{n,e} = \frac{na_{-1}}{e'b_{-1}}$.
d) *If the equations* $\mathscr{P}_{n,e'}(-1)$ *and* $\mathscr{P}_{n,4e'}(-5)$ *have respective minimal solutions* (a_{-1}, b_{-1}) *and* (a_{-5}, b_{-5})*, we have* $\nu_{n,e} = \frac{na_{-5}}{2e'b_{-5}}$ *and* $\mu_{n,e} = \frac{na_{-1}}{e'b_{-1}} > \nu_{n,e}$.

When n is square-free, we can consider equivalently the equation $\mathscr{P}_e(n)$ instead of $\mathscr{P}_{n,e'}(1)$ and the equation $\mathscr{P}_{4e}(-5n)$ instead of $\mathscr{P}_{n,4e'}(-5)$.

Proof. Consider the component of the positive cone $\{\mathbf{w} \in \mathrm{Pic}(F) \otimes \mathbf{R} \mid \mathbf{w}^2 \geq 0\}$ containing the class h and its chamber decomposition given by hyperplanes $\mathbf{s}^\perp$, where $\mathbf{s} \in \widetilde{\Lambda}_{\mathrm{alg},F}$ is such that $\mathbf{s}^2 = -2$ and $\mathbf{s} \cdot \lambda_1 \in \{0, 1\}$. The dual statement of [BHT, Theorem 1] shows that the nef cone is one of the chambers of this decomposition and the movable cone is one of the chambers when we consider only those classes $\mathbf{s}$ with $\mathbf{s} \cdot \lambda_1 = 0$. A straightforward computation shows that having a class $\mathbf{s}$ with $\mathbf{s}^2 = -2$ and $\mathbf{s} \cdot \lambda_1 = 1$ corresponds to the solvability of the equation $\mathscr{P}_{n,4e'}(-5)$. Similarly, having a class $\mathbf{s}$ with $\mathbf{s}^2 = -2$ and $\mathbf{s} \cdot \lambda_1 = 0$ corresponds to the solvability of the equation $\mathscr{P}_{n,e'}(-1)$. The proposition then follows from a computation as in [BM2, Proposition 13.1 and Lemma 13.3].

If $e = n$, the class h cannot be ample: we have then $\tau \in \widetilde{\Lambda}_{\mathrm{alg},F}$, $\tau^2 = -2$, $\tau \cdot \lambda_1 = 0$, and h lies on the hyperplane $\tau^\perp$. □

Example 4.3. In the case $n = 3$ (so that F is the variety of lines on a cubic fourfold), the following table gives the slopes $\nu_{3,e}$ and $\mu_{3,e}$ for small values of e (case d) of the proposition does not occur when $n = 3$).

e'	2	3	4	5	6	7	8	9	10	11	12	13	14	15
e	6	9	12	15	18	21	24	27	30	33	36	39	42	45
$\mathscr{P}_{3,e'}(-1)$	–	–	$(1,1)$	–	–	$(3,2)$	–	–	–	–	–	$(2,1)$	–	–
$\mathscr{P}_{3,4e'}(-5)$	$(1,1)$	–	–	$(5,2)$	–	–	$(3,1)$	–	–	–	–	–	–	–
Case	b)	a)	c)	b)	a)	c)	b)	a)	a)	a)	a)	c)	a)	a)
$\nu_{3,e}$	$\frac{3}{4}$	1	$\frac{3}{4}$	$\frac{3}{4}$	$\sqrt{\frac{1}{2}}$	$\frac{9}{14}$	$\frac{9}{16}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{3}{11}}$	$\sqrt{\frac{1}{4}}$	$\frac{6}{13}$	$\sqrt{\frac{3}{14}}$	$\sqrt{\frac{1}{5}}$
$\mu_{3,e}$	$\sqrt{\frac{3}{2}}$	1	$\frac{3}{4}$	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{1}{2}}$	$\frac{9}{14}$	$\sqrt{\frac{3}{8}}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{3}{11}}$	$\sqrt{\frac{1}{4}}$	$\frac{6}{13}$	$\sqrt{\frac{3}{14}}$	$\sqrt{\frac{1}{5}}$

As in the Hilbert square case (Proposition 3.6), we can describe the contractions corresponding to the extremal rays of the nef cone (when their slopes are rational).

Proposition 4.4. *Let* F *be a hyperkähler fourfold of* $\mathrm{K3}^{[2]}$*-type with a polarization of square* $2n$ *and divisibility 2, of Picard number* 2 *and discriminant* $2e$*. Assume that* n *divides* e *and set* $e' := e/n$*. The contractions associated with the extremal rays of the nef cone are:*

a) *Lagrangian fibrations* $F \to \mathbf{P}^2$ *if* e *is a perfect square;*
b) *small contractions of Lagrangian planes if the equation* $\mathscr{P}_{n,4e'}(-5)$ *is solvable;*
c) *divisorial contractions if the equation* $\mathscr{P}_{n,4e'}(-5)$ *is not solvable but the equation* $\mathscr{P}_{n,e'}(-1)$ *is.*

In cases a) and c), we can describe explicitly the contractions: by [A, Proposition 4] (generalized to the twisted case in [Hu1, proof of Proposition 4.1]), the fourfold F is isomorphic to a moduli space of sheaves on a (maybe twisted) K3 surface (S, α). As in Proposition 3.6, in case a), F is isomorphic to a moduli space of torsion sheaves on (S, α) (and the Lagrangian fibration is the Beauville integrable system); in case c), F is either isomorphic to a Hilbert scheme (and the contraction is the Hilbert–Chow morphism), or to a moduli space of rank-2 slope-stable sheaves (and the contraction is the LGU contraction). All these statements follow directly from [BM2] as in the proof of Proposition 3.6.

In case b), the induced contraction is small by [Ma, Proposition 2.1] (see also [HT2, Theorem 7]) and we do have a composition $F \dashrightarrow F'$ of Mukai flops (with respect to numerically equivalent Lagrangian planes) associated with this small contraction, where F' is another hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type ([WW, Theorem 1.1]).

Proposition 4.4 follows from this discussion.

Remark 4.5. In case b) of Proposition 4.2, the nef cones of the hyperkähler fourfolds birational to F can be explicitly described as follows. If the equation $\mathscr{P}_{n,e'}(-1)$ is not solvable, all positive solutions (a, b) to the equation $\mathscr{P}_{n,4e'}(-5)$ determine an infinite sequence of rays $h \pm \frac{na}{2e'b}\tau$ in the movable cone of F. If the equation $\mathscr{P}_{n,e'}(-1)$ is solvable, only finitely many rays occur. In either case, by [BHT, Theorem 1], the nef cone of any hyperkähler fourfold F' birational to F can be identified with one of the chambers in $\mathrm{Mov}(F)$ defined by this sequence of rays. Moreover, as in Proposition 4.4, all induced contractions are small contractions and all such F' can be obtained from F by compositions of Mukai flops.

Remark 4.6. If n does not divide e, a similar analysis can be performed. By Proposition 2.3, e is a square modulo n and we write $e = m^2 + e'n$, with $m, e' \geq 0$. The algebraic Markman–Mukai lattice can be identified with the rank-3 lattice with intersection matrix $\begin{pmatrix} 2 & -1 & 0 \\ -1 & (n+1)/2 & -m \\ 0 & -m & -2e' \end{pmatrix}$. The statement of Proposition 4.2 becomes more involved because the equations regulating the nef and movable cones are now $na^2 - 4mab - 4e'b^2 = -5$ and $na^2 - 2mab - e'b^2 = -1$, respectively. With these two equations, Proposition 4.4 still holds.

4.3. **Automorphisms of polarized hyperkähler fourfolds of degree $2n$, divisibility 2, and Picard number 2.** A very general hyperkähler fourfold F of $\mathrm{K3}^{[2]}$-type with a polarization of degree $2n \geq 4$ has no non-trivial birational automorphisms (Proposition 2.11; when $n \in \{3, 23\}$, the condition $\mathrm{Pic}(F) \simeq \mathbf{Z}$ is not enough!). As we did in Section 3.4 for Hilbert squares of very general polarized K3 surfaces, we determine, in some cases, the groups $\mathrm{Aut}(F)$ and $\mathrm{Bir}(F)$ when F has Picard number 2 and the divisibility of the polarization is 2.

Proposition 4.7. *Let F be a hyperkähler fourfold of* $\mathrm{K3}^{[2]}$*-type with a polarization of square $2n$ and divisibility 2. Assume that the lattice* $(\mathrm{Pic}(F), q_{BBF})$ *has rank 2 and discriminant $2e$ divisible by n, with $e' := e/n \geq 2$.*

a) *If both equations $\mathscr{P}_{n,e'}(-1)$ and $\mathscr{P}_{n,4e'}(-5)$ are not solvable and e is not a perfect square, the groups* $\mathrm{Aut}(F)$ *and* $\mathrm{Bir}(F)$ *are equal. They are infinite cyclic, except when the equation $\mathscr{P}_{n,e'}(1)$ is solvable, in which case these groups are isomorphic to the infinite dihedral group.*[22]

[22]This is the group $\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$, also isomorphic to the free product $\mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z}$.

b) *If the equation* $\mathscr{P}_{n,e'}(-1)$ *is not solvable but the equation* $\mathscr{P}_{n,4e'}(-5)$ *is, the group* $\mathrm{Aut}(F)$ *is trivial and the group* $\mathrm{Bir}(F)$ *is infinite cyclic, except when the equation* $\mathscr{P}_{n,e'}(1)$ *is solvable, in which case it is infinite dihedral.*

c) *If the equation* $\mathscr{P}_{n,e'}(-1)$ *is solvable or if* e *is a perfect square, the group* $\mathrm{Bir}(F)$ *is trivial.*

When n is square-free, we can consider equivalently the equation $\mathscr{P}_e(\pm n)$ instead of $\mathscr{P}_{n,e'}(\pm 1)$ and the equation $\mathscr{P}_{4e}(-5n)$ instead of $\mathscr{P}_{n,4e'}(-5)$.

Proof. We saw in Section 2.3 that the map $\Psi_A\colon \mathrm{Aut}(F) \to O(H^2(F,\mathbf{Z}))$ is injective. Its image consists of isometries which preserve $\mathrm{Pic}(F)$ and the ample cone and, since $b_2(F) - \rho(F)$ is odd, restrict to $\pm\,\mathrm{Id}$ on $\mathrm{Pic}(F)^\perp$ ([Og2, proof of Lemma 4.1][23]). Conversely, by Theorem 2.2, any isometry with these properties is in the image of Ψ_A. We begin with some general remarks on the group G of isometries of $H^2(F,\mathbf{Z})$ which preserve $\mathrm{Pic}(F)$ and the components of the positive cone, and restrict to $\varepsilon\,\mathrm{Id}$ on $\mathrm{Pic}(F)^\perp$, with $\varepsilon = \pm 1$.

The orthogonal group of the rank-2 lattice $\mathrm{Pic}(F)$, with intersection matrix $\left(\begin{smallmatrix} 2n & 0 \\ 0 & -2e' \end{smallmatrix}\right)$, is easily determined: if we let $\delta := \gcd(n,e')$ and we write $n = \delta n'$ and $e' = \delta e''$, we have ([BCNS, Section 4])

$$O(\mathrm{Pic}(F)) = \left\{ \begin{pmatrix} a & \alpha e'' b \\ n' b & \alpha a \end{pmatrix} \;\middle|\; a,b \in \mathbf{Z},\ a^2 - n'e''b^2 = 1,\ \alpha = \pm 1 \right\}.$$

Notice that α is the determinant of the isometry and

- such an isometry preserves the components of the positive cone if and only if $a > 0$; we denote the corresponding subgroup by $O^+(\mathrm{Pic}(F))$;
- when e is not a perfect square, the group $SO^+(\mathrm{Pic}(F))$ is infinite cyclic, generated by the isometry R corresponding to the minimal solution of the equation $\mathscr{P}_{n'e''}(1)$ and the group $O^+(\mathrm{Pic}(F))$ is infinite dihedral;
- when e is a perfect square, so is $n'e'' = e/\delta^2$, and $O^+(\mathrm{Pic}(F)) = \{\mathrm{Id}, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}$.

As we saw during the proof of Proposition 2.3, there exist standard bases (u_1, v_1) and (u_2, v_2) for two orthogonal hyperbolic planes in $L_{\mathrm{K3}^{[2]}}$, a generator ℓ for the $I_1(-2)$ factor, and an isometric identification $H^2(F,\mathbf{Z}) \xrightarrow{\sim} L_{\mathrm{K3}^{[2]}}$ such that

$$h = 2u_1 + \frac{n+1}{2} v_1 + \ell \quad \text{and} \quad \tau = u_2 - e' v_2.$$

The elements Φ of G must then satisfy $a > 0$ and

$$\begin{aligned}
\Phi(2u_1 + \tfrac{n+1}{2} v_1 + \ell) &= a(2u_1 + \tfrac{n+1}{2} v_1 + \ell) + n'b(u_2 - e'v_2) \\
\Phi(u_2 - e'v_2) &= \alpha e'' b(2u_1 + \tfrac{n+1}{2} v_1 + \ell) + \alpha a(u_2 - e'v_2) \\
\Phi(v_1 + \ell) &= \varepsilon(v_1 + \ell) \\
\Phi(u_1 - \tfrac{n+1}{4} v_1) &= \varepsilon(u_1 - \tfrac{n+1}{4} v_1) \\
\Phi(u_2 + e'v_2) &= \varepsilon(u_2 + e'v_2)
\end{aligned}$$

[23]The argument uses again the trick mentioned in footnote 17.

(the last three lines correspond to vectors in $\mathrm{Pic}(F)^\perp$). From this, we deduce

$$\begin{aligned} n\Phi(v_1) &= 2(a-\varepsilon)u_1 + \big((a+\varepsilon)\tfrac{n+1}{2} - \varepsilon\big)v_1 + (a-\varepsilon)\ell + n'b(u_2 - e'v_2) \\ 2\Phi(u_2) &= 2\alpha e''bu_1 + \alpha e''b\tfrac{n+1}{2}v_1 + \alpha e''b\ell + (\varepsilon+\alpha a)u_2 + e'(\varepsilon-\alpha a)v_2 \\ 2e'\Phi(v_2) &= -2\alpha e''bu_1 - \alpha e''b\tfrac{n+1}{2}v_1 - \alpha e''b\ell + (\varepsilon-\alpha a)u_2 + e'(\varepsilon+\alpha a)v_2. \end{aligned}$$

From the first equation, we get $\delta \mid b$ and $a \equiv \varepsilon \pmod n$; from the second equation, we deduce that $e''b$ and $\varepsilon + \alpha a$ are even; from the third equation, we get $2\delta \mid b$ and $a \equiv \alpha\varepsilon \pmod{2e'}$. All this is equivalent to $a > 0$ and

$$2\delta \mid b \quad , \quad a \equiv \varepsilon \pmod n \quad , \quad a \equiv \alpha\varepsilon \pmod{2e'}. \tag{12}$$

Conversely, if these conditions are realized, one may define Φ uniquely on $\mathbf{Z}u_1 \oplus \mathbf{Z}v_1 \oplus \mathbf{Z}u_2 \oplus \mathbf{Z}v_2 \oplus \mathbf{Z}\ell$ using the formulas above, and extend it by $\varepsilon\,\mathrm{Id}$ on the orthogonal of this lattice in $L_{\mathrm{K3}^{[2]}}$ to obtain an element of G.

The first congruence in (12) tells us that the identity on $\mathrm{Pic}(F)$ extended by $-\mathrm{Id}$ on its orthogonal does not lift to an isometry of $H^2(F,\mathbf{Z})$. This means that the restriction $G \to O^+(\mathrm{Pic}(F))$ is injective. Moreover, the two congruences in (12) imply $a \equiv \varepsilon \equiv \alpha\varepsilon \pmod\delta$. If $\delta > 1$, since n, hence also δ, is odd, we get $\alpha = 1$, hence the image of G is contained in $SO^+(\mathrm{Pic}(F))$.

Assume $\alpha = 1$**.** Using the relations (12), the equality $a^2 - n'e''b^2 = 1$, and the fact that $n \equiv -1 \pmod 4$, we write $b = 2\delta b'$ and $a = 2\delta n'e''a' + \varepsilon$, and obtain

$$4\delta^2 n'^2 e''^2 a'^2 + 4\varepsilon\delta n'e''a' = 4\delta^2 n'e''b'^2,$$

hence

$$\delta n'e''a'^2 + \varepsilon a' = \delta b'^2.$$

In particular, $a'' := a'/\delta$ is an integer and $b'^2 = a''(ea'' + \varepsilon)$.

Since $a > 0$ and a'' and $ea'' + \varepsilon$ are coprime, both are perfect squares and there exist coprime integers r and s, with $r > 0$, such that

$$a'' = s^2 \quad , \quad ea'' + \varepsilon = r^2 \quad , \quad b' = rs.$$

Since $n \mid e$ and -1 is not a square modulo n, we obtain $\varepsilon = 1$; the pair (r,s) satisfies the Pell equation $r^2 - es^2 = 1$, and $a = 2es^2 + 1$ and $b = 2\delta rs$. In particular, either e is not a perfect square and there are always infinitely many solutions, or e is a perfect square and we get $r = 1$ and $s = 0$, so that $\Phi = \mathrm{Id}$.

Assume $\alpha = -1$**.** As observed before, we have $\delta = 1$, i.e., n and e' are coprime. Using (12), we may write $b = 2b'$ and $a = 2a'e' - \varepsilon$. Since $2 \nmid n$ and $a \equiv \varepsilon \pmod n$, we deduce $\gcd(a', n) = 1$. Substituting into the equation $a^2 - ne'b^2 = 1$, we obtain

$$a'(e'a' - \varepsilon) = nb'^2,$$

hence there exist coprime integers r and s, with $r \geq 0$, such that $b' = rs$, $a' = s^2$, and $e'a' - \varepsilon = nr^2$. The pair (r,s) satisfies the equation $nr^2 - e's^2 = -\varepsilon$, and $a = 2e's^2 - \varepsilon$ and $b = 2rs$. In particular, there are solutions if and only if one of the two equations $\mathscr{P}_{n,e'}(\pm1)$ is solvable. Note that at most one of these equations may be solvable: if $\mathscr{P}_{n,e'}(\varepsilon)$ is solvable,

$-\varepsilon e'$ is a square modulo n, while -1 is not. These isometries are all involutions and, since $n \geq 2$ and $e' \geq 2$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is not one of them. In particular, if e is a perfect square, $G = \{\mathrm{Id}\}$.

We now go back to the proof of the proposition. We proved that the composition $\mathrm{Aut}(F) \to G \to O^+(\mathrm{Pic}(F))$ is injective and, by the discussion in Section 2.3, so is the morphism $\mathrm{Bir}(F) \to G \to O^+(\mathrm{Pic}(F))$ (any element of its kernel is in $\mathrm{Aut}(F)$).

Under the hypotheses of a), both slopes of the nef cone are irrational (Proposition 4.2). By [Og1, Theorem 1.3], the groups $\mathrm{Aut}(F)$ and $\mathrm{Bir}(F)$ are then equal and infinite. The calculations above allow us to be more precise: in this case, the ample cone is just one component of the positive cone and the groups $\mathrm{Aut}(F)$ and G are isomorphic. The conclusion follows from the discussions above.

Under the hypotheses of c), the slopes of the extremal rays of the nef and movable cones are rational (Proposition 4.2) hence, by [Og1, Theorem 1.3] again, $\mathrm{Bir}(F)$ is a finite group. By [Og1, Proposition 3.1(2)], any non-trivial element Φ of its image in $O^+(\mathrm{Pic}(F))$ is an involution which satisfies $\Phi(\mathrm{Mov}(F)) = \mathrm{Mov}(F)$, hence switches the two extremal rays of this cone. This means $\Phi(h \pm \mu_{n,e}\tau) = h \mp \mu_{n,e}\tau$, hence $\Phi(h) = h$, so that $\Phi = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Since we saw that this is impossible, the group $\mathrm{Bir}(F)$ is trivial.

Under the hypotheses of b), the slopes of the nef cone are both rational and the slopes of the movable cone are both irrational (Proposition 4.2). By [Og1, Theorem 1.3] again, $\mathrm{Aut}(F)$ is a finite group and $\mathrm{Bir}(F)$ is infinite. The same reasoning as in case c) shows that the group $\mathrm{Aut}(F)$ is in fact trivial; moreover, the group $\mathrm{Bir}(F)$ is a subgroup of $\mathbf{Z}$, except when the equation $\mathscr{P}_{n,e'}(1)$ is solvable, where it is a subgroup of $\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$.

In the latter case, such an infinite subgroup is isomorphic either to $\mathbf{Z}$ or to $\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$ and we exclude the first case by showing that there is indeed a regular involution on a birational model of F (this generalizes the case $n = 3$ and $e = 6$ treated in [HT3]).

As observed in Remark 4.5, the set of all positive solutions (x', y') to the equation $\mathscr{P}_{n,e'}(-5)$ determine an infinite sequence of rays $\mathbf{R}_{\geq 0}(2e'y'h \pm nx'\tau)$ in $\mathrm{Mov}(F)$. The nef cones of hyperkähler fourfolds birational to F can be identified with the chambers with respect to this collection of rays. In order to apply Lemma 3.2 and show that the equation $\mathscr{P}_{n,e'}(-5)$ has two classes of solutions, we need to check that 5 divides neither n nor e'. We will use quadratic reciprocity and, given integers u and v, we denote by $\left(\frac{u}{v}\right)$ their Jacobi symbol.

Assume first $5 \mid e'$. Since the equation $\mathscr{P}_{n,e'}(1)$ is solvable, we have $\left(\frac{n}{5}\right) = 1$; moreover, since $n \equiv -1 \pmod 4$, we have $\left(\frac{e'}{n}\right) = -1$. The solvability of the equation $\mathscr{P}_{n,4e'}(-5)$ implies $\left(\frac{5}{n}\right) = \left(\frac{e'}{n}\right)$; putting all that together contradicts quadratic reciprocity.

Assume now $5 \mid n$ and set $n' := n/5$. Since the equation $\mathscr{P}_{n,e'}(1)$ is solvable, we have $\left(\frac{e'}{5}\right) = 1$; moreover, since $n' \equiv -1 \pmod 4$, we have $\left(\frac{e'}{n'}\right) = -1$. Since $5 \nmid e'$, the equation $\mathscr{P}_{n',20e'}(-1)$ is solvable, hence $\left(\frac{5e'}{n'}\right) = 1$; again, this contradicts quadratic reciprocity.

The assumptions of Lemma 3.2 are therefore satisfied and the equation $\mathscr{P}_{n,4e'}(-5)$ has two classes of solutions. We can reinterpret this as follows. Let (x, y) be the minimal solution to the equation $\mathscr{P}_{n,e'}(1)$; by Lemma 3.3, the minimal solution to the equation $\mathscr{P}_{ne'}(1)$ is

$(a,b) := (nx^2 + e'y^2, 2xy)$ and it corresponds to the generator $R = \left(\begin{smallmatrix} a & e'b \\ nb & a \end{smallmatrix}\right)$ of $SO^+(\mathrm{Pic}(F))$ previously defined (we are in the case $\gcd(n, e') = 1$).

The two extremal rays of the nef cone of F are spanned by $\alpha_0 := 2e'b_{-5}h - na_{-5}\tau$ and $\alpha_1 := 2e'b_{-5}h + na_{-5}\tau$, where (a_{-5}, b_{-5}) is the minimal solution to $\mathscr{P}_{n,4e'}(-5)$. If we set $\alpha_{i+2} := R(\alpha_i)$, the fact the $\mathscr{P}_{n,4e'}(-5)$ has two classes of solutions means exactly that the ray $\mathbf{R}_{\geq 0}\alpha_2$ is "above" the ray $\mathbf{R}_{\geq 0}\alpha_1$; in other words, we get an "increasing" infinite sequence of rays

$$\cdots < \mathbf{R}_{\geq 0}\alpha_{-1} < \mathbf{R}_{\geq 0}\alpha_0 < \mathbf{R}_{\geq 0}\alpha_1 < \mathbf{R}_{\geq 0}\alpha_2 < \cdots .$$

The involution $R\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ belongs to G and preserves the nef cone of the birational model F' of F whose nef cone is generated by α_1 and α_2. It is therefore induced by a biregular involution of F' which defines a birational involution of F. This concludes the proof of the proposition. □

Remark 4.8. It follows from the proof above that in case b), if both equations $\mathscr{P}_{n,4e'}(-5)$ and $\mathscr{P}_{n,e'}(1)$ are solvable, F has exactly one non-trivial birational model. It is obtained from F by a composition of Mukai flops with respect to numerically equivalent Lagrangian planes.

Example 4.9 ($n = 3$)**.** We list, for $2 \leq e' \leq 15$, the biregular and birational automorphism groups of very general hyperkähler fourfolds of degree 6 and discriminant $6e'$.

e'	2	3	4	5	6	7	8	9	10	11	12	13	14	15
e	6	9	12	15	18	21	24	27	30	33	36	39	42	45
$\mathscr{P}_{3,e'}(-1)$	–	–	$(1,1)$	–	–	$(3,2)$	–	–	–	–	–	$(2,1)$	–	–
$\mathscr{P}_{3,4e'}(-5)$	$(1,1)$	–	–	$(5,2)$	–	–	$(3,1)$	–	–	–	–	–	–	–
$\mathscr{P}_{3,e'}(1)$	$(1,1)$	–	–	–	–	–	–	–	–	$(2,1)$	–	–	–	–
Case	b)	c)	c)	b)	a)	c)	b)	a)	a)	a)	c)	c)	a)	a)
$\mathrm{Aut}(F)$	Id	Id	Id	Id	$\mathbf{Z}$	Id	Id	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z} \rtimes \mathbf{Z}/2$	Id	Id	$\mathbf{Z}$	$\mathbf{Z}$
$\mathrm{Bir}(F)$	$\mathbf{Z} \rtimes \mathbf{Z}/2$	Id	Id	$\mathbf{Z}$	$\mathbf{Z}$	Id	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z} \rtimes \mathbf{Z}/2$	Id	Id	$\mathbf{Z}$	$\mathbf{Z}$

5. Unexpected isomorphisms between hyperkähler fourfolds

We proved in Proposition 2.3 that the loci of special hyperkähler fourfolds of fixed degree, divisibility, and discriminant are empty or of pure dimension 19. It is therefore enough to find an irreducible 19-dimensional family of such polarized fourfolds to get a description of fourfolds corresponding to the general point of one component of these special loci.

We apply this remark to the family of Hilbert squares of K3 surfaces (Proposition 5.1) and prove that infinitely many explicit hypersurfaces in each of the moduli spaces $\mathscr{M}_{2n}^{(1)}$ and $\mathscr{M}_{2n}^{(2)}$ consist of Hilbert squares (Proposition 5.5).

This result applies to varieties of lines on cubic fourfolds (degree 6 and divisibility 2; see Corollary 5.7, which extends earlier results of Hassett's), hyperkähler fourfolds of Debarre–Voisin type (degree 22 and divisibility 2; see Section 5.3), varieties of sums of powers (degree 38 and divisibility 2; see Section 5.4), hyperkähler fourfolds of IKKR type (degree 4 and divisibility 1; see Section 5.5), but also to double EPW sextics (degree 2 and divisibility 1; see Proposition 5.10). The latter case allows us to characterize Hilbert squares (of K3 surfaces) of Picard number 2 with a non-trivial automorphism as being exactly those which are isomorphic to double EPW sextics (Corollary 5.11).

We also apply the above remark to the varieties of lines $F(W)$ on special cubic fourfolds W of fixed discriminant (Corollary 5.14) and deduce that infinitely many hypersurfaces of certain moduli spaces consist of fourfolds isomorphic to $F(W)$ (Proposition 5.18). We also introduce and study *ambiguous* cubic fourfolds in Section 5.11.

Finally, in the last section, when n divides e, we determine when the fourfold corresponding to a general element in $\mathscr{C}^{(2)}_{2n,2e}$ is a double EPW sextic.

5.1. **Hyperkähler fourfolds and Hilbert squares of K3 surfaces.** If a hyperkähler fourfold is isomorphic to the Hilbert square of a K3 surface, it is special in the sense defined in Section 2.2. We formalize this remark in the following proposition, where we use the standard notation for cohomology classes on a Hilbert square (see Section 3.2). The slope ν_e was defined in Theorem 3.4 and the loci $\mathscr{C}^{(\gamma)}_{2n,2e} \subset \mathscr{M}^{(\gamma)}_{2n}$ in Section 2.2.

Proposition 5.1. *Let n and e be positive integers. Assume that the equation $\mathscr{P}_e(-n)$ has a positive solution (a, b) that satisfies the conditions*

$$\frac{a}{b} < \nu_e \qquad \text{and} \qquad \gcd(a, b) = 1. \tag{13}$$

If $\mathscr{K}_{2e}$ is the moduli space of polarized K3 surfaces of degree $2e$, the rational map

$$\begin{array}{rcl} \varpi\colon \mathscr{K}_{2e} & \dashrightarrow & \mathscr{M}^{(\gamma)}_{2n} \\ (S, f_S) & \longmapsto & (S^{[2]}, bf - a\delta), \end{array}$$

where $\gamma = 2$ if b is even and $\gamma = 1$ if b is odd, induces a birational isomorphism onto an irreducible component of $\mathscr{C}^{(\gamma)}_{2n,2e}$. In particular, if n is prime and b is even, it induces a birational isomorphism

$$\mathscr{K}_{2e} \overset{\sim}{\dashrightarrow} \mathscr{C}^{(2)}_{2n,2e}.$$

Proof. If (S, f_S) is a polarized K3 surface of degree $2e$ and $K := \mathbf{Z}f \oplus \mathbf{Z}\delta \subset H^2(S^{[2]}, Z)$ (see Section 3.2 for the notation), the lattice $K^\perp$ is the orthogonal in $H^2(S, \mathbf{Z})$ of the class f_S. Since the lattice $H^2(S, \mathbf{Z})$ is unimodular, $K^\perp$ has discriminant $-2e$, hence $S^{[2]}$ is special of discriminant $2e$.

The class $h = bf - a\delta$ has divisibility γ and square $2n$. It is primitive, because $\gcd(a, b) = 1$, and, if S is very general, ample on $S^{[2]}$ because of the inequality in (13). Therefore, the pair $(S^{[2]}, h)$ corresponds to a point of $\mathscr{C}^{(\gamma)}_{2n,2e}$.

The map ϖ therefore sends a very general point of $\mathscr{K}_{2e}$ to $\mathscr{C}^{(\gamma)}_{2n,2e}$. To prove that ϖ is generically injective, it is enough to prove that when S is very general, any automorphism φ of $S^{[2]}$ which preserves h comes from an automorphism of (S, f_S). By Theorem 3.8, the condition $\varphi^*(h) = h$ implies that φ is trivial when $e > 1$; when $e = 1$, the only non-trivial automorphism of $S^{[2]}$ is indeed induced by an involution of (S, f_S).

The map ϖ is therefore generically injective and since $\mathscr{K}_{2e}$ is of dimension 19, its image is a component of $\mathscr{C}^{(\gamma)}_{2n,2e}$. When n is prime and b is even, the conclusion follows from the irreducibility of $\mathscr{C}^{(2)}_{2n,2e}$ (Proposition 2.3). □

Remark 5.2. When e is a perfect square, the positive solutions (a, b) to the equation $\mathscr{P}_e(-n)$ satisfy $a - b\sqrt{e} = -n''$ and $a + b\sqrt{e} = n'$, with $n = n'n''$. This implies $a = \frac{1}{2}(n' - n'')$ and $b\sqrt{e} = \frac{1}{2}(n' + n'')$, hence $0 < n'' < n'$. We then have $\frac{a}{b} = \frac{n'-n''}{n'+n''}\sqrt{e} < \sqrt{e} = \nu_e$ hence the proposition applies to all positive solutions (a, b) of the equation $\mathscr{P}_e(-n)$ with $\gcd(a, b) = 1$. In particular, when n is odd, $\mathscr{C}^{(\gamma)}_{2n,2}$ is non-empty, where $\gamma = 1$ if $n \equiv 1 \pmod 4$ and $\gamma = 2$ if $n \equiv -1 \pmod 4$ (take $b = (n + 1)/2$) and when $n \equiv -1 \pmod 4$, the locus $\mathscr{C}^{(2)}_{2n,(n+1)^2/2}$ is non-empty (take $b = 2$).

Remark 5.3. Under the hypotheses of Proposition 5.1, *all* polarized hyperkähler fourfolds (F, h) with Picard number 2 which are in the component of $\mathscr{C}^{(\gamma)}_{2n,2e}$ dominated by $\mathscr{K}_{2e}$ are actually isomorphic to a Hilbert square $S^{[2]}$; however, *some* generality condition on F is needed: the varieties F of lines of some smooth cubic fourfolds of discriminant 14 ($n = 3$) are not isomorphic to the Hilbert square of a K3 surface ([H2, p. 41]).

Indeed, by [A, Proposition 4], F is always birational to $S^{[2]}$, where (S, f_S) is a polarized K3 surface of degree $2e$. By [BM2, Theorem 1.2], (F, h) is isomorphic to a moduli space $(M_\sigma(\mathbf{v}), \ell_\sigma)$ of Bridgeland stable objects in $\mathrm{D^b}(S)$ (together with its natural polarization ℓ_σ defined in [BM1]) and its nef cone can be identified with a chamber in the movable cone of $S^{[2]}$ with respect to the wall structure given by solutions to the equation $\mathscr{P}_{4e}(5)$. But this wall and chamber decomposition is invariant under deformations of (S, f_S) as long as the Picard number of S remains 1 (equivalently, the Picard number of F is 2).

Hence, the pair $(M_\sigma(\mathbf{v}), \ell_\sigma)$ defines a generically injective map from $\mathscr{K}_{2e}$ to $\mathscr{M}^{(2)}_{2n}$ as in Proposition 5.1. By Proposition 5.1, for a very general K3 surface (S, f_S), $M_\sigma(\mathbf{v})$ is isomorphic to $S^{[2]}$ and the polarization ℓ_σ corresponds to the class $bf - a\delta$. This means there are no walls in between ℓ_σ and f. Since, as remarked before, the wall and chamber structure is invariant, this means that there are no walls for *all* (S, f_S) with Picard rank 1, i.e., that (F, h) is actually always isomorphic to $(S^{[2]}, bf - a\delta)$.

Remark 5.4. The varieties $\mathscr{K}_{2e}$ are known to be of general type for $e > 61$, by celebrated work by Gritsenko, Hulek, and Sankaran ([GHS1]). Proposition 5.1 implies that for any prime number p satisfying its hypotheses, the Noether–Lefschetz divisors $\mathscr{C}^{(2)}_{2p,2e}$ are also of general type. There are more precise results on the geometry of the hypersurfaces $\mathscr{C}^{(2)}_{6,2e}$ in [Nu, TV, L] (see Remark 5.17).

Proposition 5.5. *Let n be a positive integer. Inside the moduli space $\mathscr{M}^{(1)}_{2n}$, the general points of some component of each of the infinitely many distinct hypersurfaces $\mathscr{C}^{(1)}_{2n,2(m^2+n)}$, where m describes the set of all positive integers and $(n, m) \neq (1, 2)$, correspond to Hilbert squares of K3 surfaces. In particular, $\mathscr{M}^{(1)}_{2n}$ is not empty.*

Assume moreover $n \equiv -1 \pmod 4$. Inside the moduli space $\mathscr{M}^{(2)}_{2n}$, the general points of some component of each of the infinitely many distinct hypersurfaces $\mathscr{C}^{(2)}_{2n,2\left(m^2+m+\frac{n+1}{4}\right)}$, *where m describes the set of all nonnegative integers and $(n,m)\neq(3,1)$, correspond to Hilbert squares of K3 surfaces. In particular, $\mathscr{M}_{2n}^{(2)}$ is not empty.*

In both cases, the union of these hypersurfaces is dense in the moduli space for the euclidean topology.

Proof. For the first statement (case $\gamma=1$), the pair $(m,1)$ is a solution of the equation $\mathscr{P}_e(-n)$, with $e=m^2+n$. We need to check that the inequality $m<\nu_e$ in (13) holds.

If e is a perfect square, this was done in Remark 5.2. If the equation $\mathscr{P}_e(5)$ is not solvable, we have $\nu_e=e\frac{b_1}{a_1}$ (Theorem 3.4). If the inequality (13) fails, since $\nu_e^2=e^2\frac{b_1^2}{a_1^2}=e-\frac{e}{a_1^2}=m^2+n-\frac{e}{a_1^2}$, we have $a_1^2\leq e/n$. Since $a_1^2=eb_1^2+1\geq e+1$, this is absurd and (13) holds in this case.

If the equation $\mathscr{P}_{4e}(5)$ has a solution (a_5,b_5), we have $\nu_e=2e\frac{b_5}{a_5}$ (Theorem 3.4). If the inequality (13) fails, we have again $\nu_e^2=4e^2\frac{b_5^2}{a_5^2}=m^2+n-\frac{5e}{a_5^2}\leq m^2$, hence $a_5^2\leq 5e/n$. Since $a_5^2=4eb_5^2+5$, this is possible only if $n=b_5=1$, in which case $a_5^2=4e+5=4m^2+9$. This implies $a_5>2m$, hence $a_5^2\geq 4m^2+4m+1$ and $m\leq 2$. If $m=1$, the integer $4m^2+9$ is not a perfect square. If $m=2$, we have $a_5=e=5$ and $m=\nu_e$, but this is a case that we have excluded. The inequality (13) therefore holds in this case.

For the second statement (case $\gamma=2$), the pair $(2m+1,2)$ is a solution of the equation $\mathscr{P}_e(-n)$, with $e=m^2+m+\frac{n+1}{4}$ and we need to check that the inequality $m+\frac12<\nu_e$ in (13) holds.

If e is a perfect square, this was checked in Remark 5.2. If the equation $\mathscr{P}_e(5)$ is not solvable, we have $\nu_e=e\frac{b_1}{a_1}$ (Theorem 3.4). If the inequality (13) fails, since $\nu_e^2=e^2\frac{b_1^2}{a_1^2}=e-\frac{e}{a_1^2}=m^2+m+\frac{n+1}{4}-\frac{e}{a_1^2}$, we have $a_1^2\leq 4e/n$. Since $a_1^2=eb_1^2+1$ and $n\equiv -1 \pmod 4$, this is possible only if $n=3$ and $b_1=1$, in which case $a_1^2=m^2+m+2$. This implies $a_1>m$, hence $a_1^2\geq m^2+2m+1$ and $m\leq 1$. If $m=0$, the integer m^2+m+2 is not a perfect square. If $m=1$, we have $a_1=2$, $e=3$, and $m=\nu_e$, but this is a case that we have excluded (and indeed, $\mathscr{C}_{6,6}^{(2)}$ is empty as noted in Remark 2.4). The inequality (13) therefore holds in this case.

If the equation $\mathscr{P}_{4e}(5)$ has a solution (a_5,b_5), we have $\nu_e=2e\frac{b_5}{a_5}$ (Theorem 3.4). If the inequality (13) fails, we have again $\nu_e^2=4e^2\frac{b_5^2}{a_5^2}=m^2+m+\frac{n+1}{4}-\frac{5e}{a_5^2}\leq\left(m+\frac12\right)^2$, hence $a_5^2\leq 20e/n$. Since $a_5^2=4eb_5^2+5$, this is possible only if $n=3$ and $b_5=1$, in which case $a_5^2=4e+5=4m^2+4m+9$. This implies $a_5>2m+1$, hence $a_5^2\geq 4m^2+8m+4$ and $m\leq 1$. If $m=1$, the integer $4m^2+4m+9$ is not a perfect square. If $m=0$, we have $a_5=3$, $e=1$, and $m=\nu_e=\frac23>m+\frac12$, so the inequality (13) always holds.

Finally, the density of the union of the special hypersurfaces in the moduli space follows from a powerful result of Clozel and Ullmo: the target space $S_{2n}^{(\gamma)}:=O(L_{\mathrm{K3}^{[2]}},h_0)\backslash\Omega_{h_0}$ of the period map $\wp_{2n}^{(\gamma)}$ is a (component of a) Shimura variety and each special divisor $\mathscr{D}_{2n,2e}^{(\gamma)}$ is a "strongly special" subvariety, hence is endowed with a canonical probability measure $\mu_{\mathscr{D}_{2n,2e}^{(\gamma)}}$; given any infinite family $(\mathscr{D}_{e_m})_m$ of special divisors, there exists a subsequence $(m_k)_k$, a strongly special subvariety Z which contains $\mathscr{D}_{e_{m_k}}$ for all $k \gg 0$ such that $(\mu_{\mathscr{D}_{e_{m_k}}})_k$ converges weakly to μ_Z ([CU, th. 1.2]). For dimensional reasons, we have $Z = S_{2n}^{(\gamma)}$; this implies that $\bigcup_m \mathscr{D}_{e_m}$ is dense in $S_{2n}^{(\gamma)}$. $\square$

Assume $n = 3$. When $e = 13$, two positive solutions, $(7, 1)$ and $(137, 19)$, of the equation $\mathscr{P}_{4e}(-3)$ satisfy the conditions (13) of Proposition 5.1 (see Remark 5.9). Let $u\colon F \xrightarrow{\sim} S^{[2]}$ be the isomorphism such that the polarization h on F corresponds to $2f - 7\delta$. By Theorem 3.8 or Example 4.3, there exists a non-trivial involution σ on $S^{[2]}$. Under the isomorphism $\sigma \circ u\colon F \xrightarrow{\sim} S^{[2]}$, the polarization h corresponds to the other class $38f - 137\delta$ (compare with Example 3.10). When $e = 21$, Proposition 5.1 also applies, but only one positive solution, $(9, 1)$, of the equation $\mathscr{P}_{4e}(-3)$ satisfies the conditions (13), although the corresponding fourfold $S^{[2]}$ is again ambiguous (Proposition 3.14). If $\bar{S}^{[2]}$ is another Hilbert square which is non-trivially isomorphic to $S^{[2]}$, we have $h = 2f - 9\delta = 2\bar{f} - 9\bar{\delta}$, with $\bar{f} = 55f - 252\delta$ and $\bar{\delta} = 12f - 55\delta$.

These examples are two instances of a more general phenomenon.

Proposition 5.6. *Let n be a square-free positive integer and let e be a positive integer. Assume that the equation $\mathscr{P}_{4e}(-n)$ has a solution that satisfies the conditions* (13)*.*

a) *The equation $\mathscr{P}_{4e}(-n)$ has several positive solutions that satisfy the conditions* (13) *if and only if $n \nmid e$ and the Hilbert square of a very general K3 surface of degree $2e$ is ambiguous (see Proposition 3.14). It then has exactly two such solutions.*

b) *If $n \mid e$ and $e \neq n$, and the equation $\mathscr{P}_{4e}(5)$ is not solvable, the Hilbert square of a very general K3 surface of degree $2e$ is ambiguous but the positive solution of the equation $\mathscr{P}_{4e}(-n)$ that satisfies the conditions* (13) *is unique.*

Proof. Assume that the equation $\mathscr{P}_{4e}(-n)$ has distinct positive solutions (a, b) and $(\bar{a}, \bar{b})$ that satisfy the conditions (13). The corresponding Hilbert squares provided by Proposition 5.1 are then ambiguous. Indeed, there is an isomorphism $F \xrightarrow{\sim} S^{[2]}$ for which the polarization h on F corresponds to $2bf - a\delta$, and another isomorphism $F \xrightarrow{\sim} \bar{S}^{[2]}$ for which h corresponds to $2\bar{b}\bar{f} - \bar{a}\bar{\delta}$. In particular, if $r\colon S^{[2]} \xrightarrow{\sim} \bar{S}^{[2]}$ is the resulting isomorphism, r^* does not send $\bar{\delta}$ to δ, and $S^{[2]}$ and $\bar{S}^{[2]}$ are ambiguous.

If, as in the proof of Proposition 3.14, we use r to identify $\bar{S}^{[2]}$ with $S^{[2]}$, we have $\bar{f} = s(f)$ and $\bar{\delta} = s(\delta)$, where (a_1, b_1) is the minimal solution of the equation $\mathscr{P}_e(1)$ and s is the involution of $\mathrm{Pic}(S^{[2]})$ with matrix $\begin{pmatrix} a_1 & b_1 \\ -eb_1 & -a_1 \end{pmatrix}$ in the basis (f, δ). Moreover, $h = 2bf - a\delta = 2\bar{b}\bar{f} - \bar{a}\bar{\delta} = s(2\bar{b}f - \bar{a}\delta)$. This proves that at most two solutions of the equation $\mathscr{P}_{4e}(-n)$ satisfy the inequality (13): if one is (a, b), the other is $(\bar{a}, \bar{b}) = (2ebb_1 - aa_1, a_1b - ab_1/2)$ (note that b_1 is even by Proposition 3.14).

We now want to show that e is not divisible by n. Assume to the contrary $n \mid e$. Then $n \mid a$ and we may assume that $(a/n, b)$ is the minimal solution of the equation $\mathscr{P}_{n,4e/n}(-1)$. Since $(a_1, b_1/2)$ is the minimal solution of the equation $\mathscr{P}_{4e}(1)$, we obtain, by Lemma 3.3, $n(a/n)^2 + (4e/n)b^2 = a_1$ and $2(a/n)b = b_1/2$, from which we deduce

$$\bar{a} = 2ebb_1 - aa_1 = 8eb(a/n)b - an(a/n)^2 - a(4e/n)b^2 = (a/n)(4eb^2 - a^2) = a$$

and similarly, $\bar{b} = nb(a/n)^2 + (4e/n)b^3 - 2a(a/n)b = (b/n)(4eb^2 - a^2) = b$. Therefore, only one solution of the equation $\mathscr{P}_{4e}(-n)$ satisfies the inequality (13), contradicting our original assumption. This proves a).

Under the hypotheses of b), the fourfold $S^{[2]}$ is ambiguous by Remark 3.15, and, as discussed above, the nef cone of $S^{[2]}$ is invariant under the involution s. Moreover, we just saw that $s(2bf - a\delta) = 2bf - a\delta$, hence the the positive solution of the equation $\mathscr{P}_{4e}(-n)$ that satisfies the condition s(13) is unique. This proves b). □

5.2. **Varieties of lines contained in a cubic fourfold and Hilbert squares of K3 surfaces.** Putting together Example 2.5 and Proposition 5.1, we obtain the following.

Corollary 5.7. *Let e be an integer such that $e \geq 2$. Assume that the equation $\mathscr{P}_{4e}(-3)$ has a positive solution (a, b) that satisfies the inequality $\frac{a}{2b} < \nu_e$. If W is a general cubic fourfold of discriminant $2e$, there exists a polarized K3 surface (S, f_S) of degree $2e$ such that the variety $F(W)$ of lines contained in W is isomorphic to the Hilbert square $S^{[2]}$ and the Plücker polarization of $F(W)$ corresponds to the class $2bf - a\delta$.*

Remark 5.8. By [A, Theorem 2], the variety $F(W)$ of lines contained in a smooth cubic fourfold W is birationally isomorphic to the Hilbert square of a K3 surface if and only if W is special of discriminant $2e$, where e is any positive integer such that the equation $\mathscr{P}_{4e}(-3)$ is solvable. This condition holds when $e = 19$, although Proposition 5.1 does not apply (see Remark 5.9 below): the isomorphism is therefore not biregular in that case. This also shows that the inequality (13) in Proposition 5.1 is not superfluous.[24]

Remark 5.9. As shown in Proposition 5.5, Corollary 5.7 applies when $e = m^2+m+1$ (with $(a, b) = (2m + 1, 1)$), for all $m \geq 2$ (when $m = 1$, the class $2f - 3\delta$ is nef but not ample[25]). This particular case was shown in [H1, Theorem 6.1.4] by a different method. Corollary 5.7 also applies in cases where $b > 1$: when $e = 129$, the minimal solution $(a, b) := (159, 7)$ of the equation $\mathscr{P}_{4e}(-3)$ satisfies $\frac{a}{2b} < \nu_e = \frac{191436}{16855}$.

We gather in the following table what happens for the positive integers $e \leq 31$ for which the equation $\mathscr{P}_{4e}(-3)$ is solvable, using the values of ν_e given in Example 3.16. We list all positive solutions (a, b) to $\mathscr{P}_{4e}(-3)$ that satisfy the inequality $\frac{a}{2b} < \nu_e$; a crossed out

[24]The case $e = 19$ is part of the infinite series of positive integers e for which $(a, 7)$ is a solution of the equation $\mathscr{P}_{4e}(-3)$ (the smallest possible value for b after 1). One finds $a = 98m \pm 37$ and $e = 49m^2 \pm 37m + 7$. Bounds similar to the ones worked out in the proof of Proposition 5.5 imply that if condition (13) is *not* satisfied, one has $b_1 \leq 8$ if the equation $\mathscr{P}_{4e}(5)$ has no solutions and $b_5 \leq 9$ if the equation $\mathscr{P}_{4e}(5)$ has a solution. This happens when $e = 19$ (the equation $\mathscr{P}_{4e}(5)$ has the solution $(9, 1)$) but should be quite rare in this series.

[25]As explained in [H2, Section 4.2], cubic fourfolds W with a single node p can be considered as having discriminant 6. Lines in W through p are parametrized by a K3 surface $S \subset \mathbf{P}^4$ of degree 6 along which $F(W)$ is singular. If S contains no lines, the map $S^{[2]} \to F(W)$ which sends a pair of lines (ℓ, ℓ') in W through p to the residual line of the intersection $\langle \ell, \ell' \rangle \cap W$ is a desingularization of $F(W)$ which is the morphism associated with the linear system $|2f - 3\delta|$ (we are in case b′) of Proposition 3.6).

pair is a minimal solution that does not satisfy that inequality.

e	1	3	7	13	19	21	31
$\mathscr{P}_{4e}(-3)$	$(1,1)$	~~$(3,1)$~~	$(5,1)$	$(7,1),(137,19)$	~~$(61,7)$~~	$(9,1)$	$(11,1)$
ν_e	1	$\frac{3}{2}$	$\frac{21}{8}$	$\frac{2340}{649}$	$\frac{38}{9}$	$\frac{252}{55}$	$\frac{3658}{657}$
$F(W)\simeq S^{[2]}$	–	–	✓	✓	–	✓	✓

5.3. **Hyperkähler fourfolds of Debarre–Voisin type and Hilbert squares of K3 surfaces.** These fourfolds are briefly described in Example 2.6; they have degree 22 and divisibility 2. Proposition 5.5 says that the loci $\mathscr{C}^{(2)}_{22,2(m^2+m+3)}$ are (non-empty) hypersurfaces for all $m\geq 0$ and their general points correspond to fourfolds that are isomorphic to Hilbert squares of K3 surfaces.

As we did in Remark 5.9 (and with the same notation), we gather in the following table what happens for the positive integers $e\leq 31$ such that the equation $\mathscr{P}_{4e}(-11)$ is solvable.

e	1	3	5	9	11	15	23	27	31
$\mathscr{P}_{4e}(-11)$	$(5,3)$	$(1,1)$	$(3,1)$	$(5,1)$	~~$(33,5)$~~	$(7,1)$	$(9,1)$	$(31,3)$	$(167,15)$
ν_e	1	$\frac{3}{2}$	2	3	$\frac{22}{7}$	$\frac{15}{4}$	$\frac{115}{24}$	$\frac{135}{26}$	$\frac{3658}{657}$
$DV\simeq S^{[2]}$	✓	✓	✓	✓	–	✓	✓	✓	✓

5.4. **Varieties of sums of powers and Hilbert squares of K3 surfaces.** These fourfolds are briefly described in Example 2.7; they have degree 38 and divisibility 2. Proposition 5.5 says that the loci $\mathscr{C}^{(2)}_{22,2(m^2+m+5)}$ are (non-empty) hypersurfaces for all $m\geq 0$ and their general points correspond to fourfolds that are isomorphic to Hilbert squares of K3 surfaces (when $m=2$, this was obtained in [IR1] by an explicit geometric construction; see also [Mo, Proposition 1.4.16]).

As we did in Remark 5.9 (and with the same notation), we gather in the following table what happens for the positive integers $e\leq 31$ such that the equation $\mathscr{P}_{4e}(-19)$ is solvable.

e	1	5	7	11	17	19	23	25
$\mathscr{P}_{4e}(-19)$	$(9,5)$	$(1,1)$	$(3,1)$	$(5,1)$	$(7,1)$	~~$(741,85)$~~	$(67,7)$	$(9,1)$
ν_e	1	2	$\frac{21}{8}$	$\frac{22}{7}$	$\frac{136}{33}$	$\frac{38}{9}$	$\frac{115}{24}$	5
$VSP\simeq S^{[2]}$	✓	✓	✓	✓	✓	–	✓	✓

5.5. **Hyperkähler fourfolds of IKKR type and Hilbert squares of K3 surfaces.** These fourfolds are mentioned in Example 2.10; they have degree 4. Proposition 5.5 says that the loci $\mathscr{C}^{(1)}_{4,2(m^2+2)}$ are (non-empty irreducible) hypersurfaces for all $m\geq 1$ and their general points correspond to fourfolds that are isomorphic to Hilbert squares of K3 surfaces.

As we did in Remark 5.9 (and with the same notation), we gather in the following table what happens for the positive integers $e\leq 31$ such that the equation $\mathscr{P}_{e}(-2)$ is solvable.

e	2	3	6	11	18	19	22	27
$\mathscr{P}_{e}(-2)$	~~$(4,3)$~~	$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	~~$(13,3)$~~	$(14,3)$	$(5,1)$
ν_e	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{12}{5}$	$\frac{22}{7}$	$\frac{72}{17}$	$\frac{38}{9}$	$\frac{924}{197}$	$\frac{135}{26}$
$IKKR\simeq S^{[2]}$	–	✓	✓	✓	✓	–	✓	✓

5.6. **Double EPW sextics and Hilbert squares of K3 surfaces.** In the situation of Example 2.9 ($n = \gamma = 1$), the conclusion of Proposition 5.1 can be made a bit more precise.

Proposition 5.10. *Let e be an integer such that $e \geq 3$. Assume that the equation $\mathscr{P}_e(-1)$ has a positive solution (a, b) such that $\frac{a}{b} < \nu_e$. If $\mathscr{K}_{2e}$ is the moduli space of polarized K3 surfaces of degree $2e$, there is a birational isomorphism*

$$\begin{aligned} \varpi\colon \mathscr{K}_{2e} \quad &\overset{\sim}{\dashrightarrow} \quad \begin{cases} \mathscr{C}^{(1)}_{2,2e} & \text{if } e \text{ is even} \\ \mathscr{C}^{(1)\prime\prime}_{2,2e} & \text{otherwise} \end{cases} \\ (S, f_S) \quad &\longmapsto \quad (S^{[2]}, bf - a\delta). \end{aligned}$$

Proof. The solvability of the equation $\mathscr{P}_e(-1)$ implies $e \equiv 1$ or $2 \pmod 4$. We saw in Example 2.9 that the hypersurface $\mathscr{C}^{(1)}_{2,2e}$ of $\mathscr{M}^{(1)}_2$ is irreducible when e is even and $e \geq 6$; it is therefore the image of ϖ. When $e \equiv 1 \pmod 4$, one checks that the image of ϖ is the component $\mathscr{C}^{(1)\prime\prime}_{2,2e}$ of $\mathscr{C}^{(1)}_{2,2e}$.[26] □

Corollary 5.11. *Let e be an integer such that $e \geq 3$ and let (S, f_S) be a general polarized K3 surface of degree $2e$. The following conditions are equivalent:*

(i) *the equation $\mathscr{P}_e(-1)$ is solvable and the equation $\mathscr{P}_{4e}(5)$ is not;*
(ii) *the equation $\mathscr{P}_e(-1)$ has a positive solution (a, b) such that $\frac{a}{b} < \nu_e$;*
(iii) *the Hilbert square $S^{[2]}$ is isomorphic to a double EPW sextic of discriminant $2e$;*
(iv) *the variety $S^{[2]}$ has a non-trivial automorphism.*

When these conditions are realized, $S^{[2]}$ has a non-trivial involution σ, the quotient $S^{[2]}/\sigma$ is an EPW sextic $Y \subset \mathbf{P}^5$, and the complete linear system $|bf - a\delta|$ defines a morphism which factors as $S^{[2]} \twoheadrightarrow S^{[2]}/\sigma = Y \hookrightarrow \mathbf{P}^5$.

Proof. The equivalence (i) $\Leftrightarrow$ (iv) is Theorem 3.8. The implication (iv) $\Rightarrow$ (ii) is Remark 3.9. The implication (ii) $\Rightarrow$ (iii) is Proposition 5.10. The implication (iii) $\Rightarrow$ (iv) is obvious.[27] The consequences stated at the end follow from Proposition 5.10 and [O3, Section 4], which explains why $\dim(|\tilde{h}|) = 5$, where $\tilde{h}$ is the canonical polarization on the double EPW sextic. □

Remark 5.12. When $e = 2$, all the conditions of Corollary 5.11 hold, except for (iii). The fourfold $S^{[2]}$ carries the non-trivial Beauville involution σ (Example 3.10) and the complete linear system $|f - \delta|$ defines a morphism which factors as $S^{[2]} \twoheadrightarrow S^{[2]}/\sigma \overset{3:1}{\twoheadrightarrow} G(2, 4) \hookrightarrow \mathbf{P}^5$.

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[26]We refer to footnote 5 for the definition of this component. To see that this is indeed the component that we get, note that the class $bf - a\delta$ is orthogonal to the vector $x := af - be\delta$, of square $-2e$. Since $e \equiv 1 \pmod 4$, we have $a^2 - b^2 \equiv -1 \pmod 4$, hence a is even (and b odd); this implies that x has divisibility 2 in $H^2(S^{[2]}, \mathbf{Z})$, hence we are in $\mathscr{C}^{(1)\prime\prime}_{2,2e}$ (footnote 5).

[27]The implication (ii) $\Rightarrow$ (i) can also be seen directly as follows. Let (a, b) be the minimal solution of $\mathscr{P}_e(-1)$ and assume that $\mathscr{P}_{4e}(5)$ has a minimal solution (a_5, b_5), so that $\nu_e = 2e\frac{b_5}{a_5} > \frac{a}{b}$. Then, $(2a^2+1, 2ab)$ is a solution of $\mathscr{P}_e(1)$ and $(a'_5, b'_5) := ((2a^2 + 1)a_5 - 4eabb_5, (2a^2 + 1)b_5 - aba_5)$ is a solution of $\mathscr{P}_{4e}(5)$. However, $a'_5 = a_5 + 2a(aa_5 - 2ebb_5) < a_5$ and $a'_5 = -a_5 + (2a^2 + 2)a_5 - 4eabb_5 = -a_5 + 2eb^2a_5 - 4eabb_5 = -a_5 + 2eb(ba_5 - 2ab_5)$. Since $2ab_5 < 4eb\frac{b_5^2}{a_5} = ba_5\frac{4eb_5^2}{a_5^2} = ba_5\frac{a_5^2-5}{a_5^2} < ba_5$, we obtain $|a'_5| < a_5$, which contradicts the minimality of the solution (a_5, b_5).

In some sense, this fits with the fact that $3G(2,4)$ is a (degenerate) EPW sextic ([O4, Claim 2.14], [F, Proposition 3.4]).

Remark 5.13. The equivalent conditions of Corollary 5.11 hold in particular in the following cases: when $e = m^2+1$ for $m \geq 3$ (the minimal solution of the equation $\mathscr{P}_e(-1)$ is $(m,1)$ and one checks that condition (ii) holds[28]),[29] but also when $e = 13$; in the latter case, the Hilbert square of a general polarized K3 surface of degree $2e = 26$ is isomorphic to the variety of lines contained in a smooth cubic fourfold and to a double EPW sextic (see Example 3.16 and Remark 5.9). The same phenomenon occurs for $e = 157$.

5.7. **Hyperkähler fourfolds and varieties of lines contained in cubic fourfolds.** We now apply the idea explained in the introduction to Section 5 to the families $\mathscr{C}^{(2)}_{6,2e}$ of lines contained in special cubic fourfolds, for which we know exactly when they are non-empty (see Example 2.5). The same proof as that of Proposition 5.1 shows that some special hyperkähler fourfolds are isomorphic to these varieties of lines. The slope $\nu_{3,e}$ was defined in Proposition 4.2 and the loci $\mathscr{C}^{(\gamma)}_{2n,2e} \subset \mathscr{M}^{(\gamma)}_{2n}$ in Section 2.2.

Proposition 5.14. *Let e be an integer divisible by 3 such that $e \geq 6$ and let n be a positive integer. Assume that the equation $\mathscr{P}_{3,e/3}(n)$ has a solution (a,b) with $a > 0$ that satisfies the conditions*

$$\frac{|b|}{a} < \nu_{3,e} \qquad \text{and} \qquad \gcd(a,b) = 1. \tag{14}$$

The rational map

$$\begin{aligned} \varpi\colon \mathscr{C}^{(2)}_{6,2e} &\dashrightarrow \mathscr{M}^{(\gamma)}_{2n} \\ (F,h) &\longmapsto (F, ah + b\tau), \end{aligned}$$

where $\gamma = 2$ if b is even and $\gamma = 1$ if b is odd, induces a birational isomorphism onto an irreducible component of $\mathscr{C}^{(\gamma)}_{2n,2e}$. In particular, if n is prime and b is even, it induces a birational isomorphism

$$\mathscr{C}^{(2)}_{6,2e} \stackrel{\sim}{\dashrightarrow} \mathscr{C}^{(2)}_{2n,2e}.$$

Proof. The proof is the same as that of Proposition 5.1 and is based on the fact that if F is a very general hyperkähler fourfold of degree 6 and discriminant $2e$ divisible by 3, we have $\operatorname{Pic}(F) = \mathbf{Z}h \oplus \mathbf{Z}\tau$, with intersection matrix $\left(\begin{smallmatrix} 6 & 0 \\ 0 & -2e/3 \end{smallmatrix}\right)$. The class $ah + b\tau$ is primitive, has square $2n$ and divisibility γ (this follows for example from the isomorphism (5)), and is ample on F because of the inequality in (14). Therefore, the pair $(F, ah+b\tau)$ corresponds to a point of $\mathscr{C}^{(\gamma)}_{2n,2e}$. The hypothesis $e > 3$ ensures that the locus $\mathscr{C}^{(2)}_{6,2e}$ is not empty (Example 2.5).

[28]If the equation $\mathscr{P}_{4e}(5)$ is not solvable, we have $\frac{a_1^2}{e} = b_1^2 + \frac{1}{e} > 1$ and $\nu_e^2 = e\big(1 - \frac{1}{a_1^2}\big) = m^2 + 1 - \frac{e}{a_1^2} > m^2$ (see the proof of Proposition 5.5).

If the equation $\mathscr{P}_{4e}(5)$ has a solution (a,b), we have $\frac{a^2}{e} = 4b^2 + \frac{5}{e} > 16$ (if $b = 1$, we have $5 = a^2 - 4e = a^2 - 4m^2 - 4$ and $(a-2m)(a+2m) = 9$, which implies $a = 5$ and $m = 2$; this contradicts our assumption $m \geq 3$) and $\nu_e = e\big(1 - \frac{5}{a^2}\big) = m^2 + 1 - \frac{5e}{a^2} > m^2$.

[29]When m is moreover even, it was proved in [IM] that $S^{[2]}$ is birationally isomorphic to a double EPW sextic. We see that they are in fact isomorphic.

To prove that ϖ is birational onto its image, it is enough to check that any automorphism φ of F such that $\varphi^*(ah+b\tau)=ah+b\tau$ is the identity. The only isometries of $\mathrm{Pic}(F)$ that fix a non-zero vector are the reflections. By Proposition 4.7 and its proof, a reflection comes from an automorphism of F if and only if it is given as $\left(\begin{smallmatrix} a & -eb/3 \\ 3b & -a\end{smallmatrix}\right)$ and there exists a solution $(3r,s)$ to the equation $\mathscr{P}_e(3)$ such that $a=2s^2e/3+1$ and $b=2rs$. The axis of such a reflection is spanned by the class $rh+s\tau$, which has square 2, hence no integral classes of square 6 are fixed by these reflections. $\square$

Remark 5.15. Given a pair (a,b) that satisfies the conditions in Proposition 5.14, we can construct two maps ϖ by sending (F,h) to either $(F,ah+b\tau)$ or $(F,ah-b\tau)$. We saw in the proof of Proposition 4.7 that the involution $h\mapsto h$, $\tau\mapsto-\tau$ never comes from an automorphism of F. This means that these two maps are different. When $\mathscr{C}^{(\gamma)}_{2n,2e}$ is irreducible, it therefore carries a non-trivial involution.

Remark 5.16. When the slope $\nu_{3,e}$ is computed as in case a) of Proposition 4.2, we have

$$\left(\frac{|b|}{a}\right)^2=\frac{3a^2-n}{(e/3)a^2}<\frac{9}{e}=\nu^2_{3,e},$$

hence the inequality in (14) holds for all of the solutions of the equation $\mathscr{P}_{3,e/3}(n)$. When non-empty, the set of these solutions is infinite and this fits with the fact that the automorphism group of F is infinite and acts non-trivially on the set of ample classes of square $2n$ (see Proposition 4.7).

Remark 5.17. The spaces $\mathscr{C}^{(2)}_{6,2e}$ are known to be of general type for $e>96$ ([TV]) and unirational for $e\le 19$ ([Nu]).

Proposition 5.18. *Let n be a positive integer. In the moduli space $\mathscr{M}^{(1)}_{2n}$, the general points of some component of each of the infinitely many distinct hypersurfaces $\mathscr{C}^{(1)}_{2n,6(3m^2-n)}$, where m describes the set of all integers such that $3m^2\ge n+2$ and, if $n=1$, $m\not\equiv\pm1 \pmod 5$, correspond to varieties of lines contained in a cubic fourfold.*

Assume moreover $n\equiv-1 \pmod 4$. In the moduli space $\mathscr{M}^{(2)}_{2n}$, the general points of some component of each of the infinitely many distinct hypersurfaces $\mathscr{C}^{(2)}_{2n,6\left(3m(m+1)-\frac{n-3}{4}\right)}$, where m describes the set of all integers such that $3m(m+1)\ge\frac{n+5}{4}$, correspond to varieties of lines contained in a cubic fourfold.

In both cases, the union of these hypersurfaces is dense in the moduli space for the euclidean topology.

Proof. For the first statement (case $\gamma=1$), the pair $(a,b):=(m,1)$ is a solution of the equation $\mathscr{P}_{3,e'}(n)$, with $e':=3m^2-n\ge 2$. We need to check that the inequality $\frac{b}{a}<\nu_{3,e}$ (with $e:=3e'$) in (14) holds.

In case a) of Proposition 4.2, this was checked in Remark 5.16; in the case $n=1$, the additional assumption $m^2\not\equiv 1 \pmod 5$ ensures that we are in that case so we may now assume $n\ge 2$. If the equation $\mathscr{P}_{3,4e'}(-5)$ has minimal solution (a_{-5},b_{-5}), we have $\nu_{3,e}=\frac{3a_{-5}}{2e'b_{-5}}$ (Proposition 4.2). If the inequality in (14) fails, we have $\nu^2_{3,e}=\frac{9a^2_{-5}}{4e'^2b^2_{-5}}=\frac{3}{e'}-\frac{15}{4e'^2b^2_{-5}}\le$

$\left(\frac{b}{a}\right)^2 = \frac{3}{e'} - \frac{n}{e'a^2}$, hence $\frac{4e'}{15} \le \frac{4e'b_{-5}^2}{15} \le \frac{a^2}{n} = \frac{e'b^2}{3n} + \frac{1}{3}$, or equivalently, $e'\left(1 - \frac{5b^2}{4n}\right) \le \frac{5}{4}$. Since $b = 1$, $n \ge 2$, and $e' \ge 2$, one checks that this is possible only if $(e', n) \in \{(2,2),(3,2),(2,3)\}$. These values are however incompatible with the relation $e' = 3m^2 - n$.

If the equation $\mathscr{P}_{3,4e'}(-5)$ is not solvable but the equation $\mathscr{P}_{3,e'}(-1)$ has minimal solution (a_{-1}, b_{-1}), we have $\nu_{3,e} = \frac{3a_{-1}}{e'b_{-1}}$ (Proposition 4.2). If the inequality in (14) fails, we have $\nu_{3,e}^2 = \frac{9a_{-1}^2}{e'^2b_{-1}^2} = \frac{3}{e'} - \frac{3}{e'^2b_{-1}^2} \le \left(\frac{b}{a}\right)^2 = \frac{3}{e'} - \frac{n}{e'a^2}$, hence $\frac{e'}{3} \le \frac{e'b_{-1}^2}{3} \le \frac{a^2}{n} = \frac{e'b^2}{3n} + \frac{1}{3}$, or equivalently, $e'\left(1 - \frac{b^2}{n}\right) \le 1$. Since $b = 1$, $n \ge 2$, and $e' \ge 2$, the only possibility is $(e', n) = (2, 2)$, which, as already noted, does not occur.

For the second statement (case $\gamma = 2$), the case $n = 3$ is empty, so we asssume $n \ge 7$. The pair $(2m + 1, 2)$ is a solution of the equation $\mathscr{P}_{3,e'}(n)$, with $e' := 3m(m + 1) - \frac{n-3}{4} \ge 2$ and we need to check that the inequality $m + \frac{1}{2} < \nu_{3,e}$ in (14) holds.

In case a) of Proposition 4.2, this was checked in Remark 5.16. If the equation $\mathscr{P}_{3,4e'}(-5)$ has minimal solution (a_{-5}, b_{-5}), we have as in the first case $e'\left(1 - \frac{5}{n}\right) \le \frac{5}{4}$. Since $n \equiv -1 \pmod 4$, $n \ge 7$, and $e \ge 6$, this is possible only if $(e', n) \in \{(2, 7), (3, 7), (4, 7), (2, 11)\}$. These values are however incompatible with the relation $e' = 3m(m + 1) - \frac{n-3}{4}$.

If the equation $\mathscr{P}_{3,4e'}(-5)$ is not solvable but the equation $\mathscr{P}_{3,e'}(-1)$ has minimal solution (a_{-1}, b_{-1}), we have $\nu_{3,e} = \frac{3a_{-1}}{e'b_{-1}}$ (Proposition 4.2). If the inequality in (14) fails, we have as in the first case $e'\left(1 - \frac{4}{n}\right) \le 1$. Since $n \equiv -1 \pmod 4$, $n \ge 7$, and $e' \ge 2$, the only possibility is $(e', n) = (2, 7)$, which, as already noted, does not occur.

The density statement follows from [CU] as in the proof of Proposition 5.5. □

5.8. **Hyperkähler fourfolds of Debarre–Voisin type and varieties of lines contained in a cubic fourfold.** By Proposition 5.18, the loci $\mathscr{C}^{(2)}_{22,6(3m(m+1)-2)}$ are (non-empty) hypersurfaces for all $m \ge 1$ and their general points correspond to fourfolds that are isomorphic to varieties of lines contained in a cubic fourfold. There are no further values of $e' \le 15$ for which the equation $\mathscr{P}_{3,4e'}(11)$ is solvable.

5.9. **Varieties of sums of powers and varieties of lines contained in a cubic fourfold.** By Proposition 5.18, the loci $\mathscr{C}^{(2)}_{38,6(3m(m+1)-4)}$ are (non-empty) hypersurfaces for all $m \ge 1$ and their general points correspond to fourfolds that are isomorphic to varieties of lines contained in a cubic fourfold. There are no further values of $e' \le 15$ for which the equation $\mathscr{P}_{3,4e'}(19)$ is solvable.

5.10. **Hyperkähler fourfolds of IKKR type and varieties of lines contained in a cubic fourfold.** By Proposition 5.18, the loci $\mathscr{C}^{(1)}_{4,6(3m^2-2)}$ are (non-empty) hypersurfaces for all $m \ge 2$ and their general points correspond to fourfolds that are isomorphic to varieties of lines contained in a cubic fourfold. There are no further values of $e' \le 15$ for which the equation $\mathscr{P}_{3,e'}(2)$ is solvable.

5.11. **Ambiguous cubic fourfolds.** It is elementary to check that given smooth cubic fourfolds W and $\bar{W}$ in $\mathbf{P}^5$, an isomorphism between $F(W)$ and $F(\bar{W})$ is induced by a projective isomorphism between W and $\bar{W}$ if and only if it maps the Plücker class g to the Plücker class $\bar{g}$. By analogy with Definition 3.12, we make the following definition.

Definition 5.19. Let W be a smooth cubic fourfold. We say that $F(W)$ (or W) is *ambiguous* if there exist a smooth cubic fourfold $\bar{W}$ and an isomorphism $r\colon F(W) \xrightarrow{\sim} F(\bar{W})$ such that $r^*\bar{g} \neq g$. We say that $F(W)$ is *strongly ambiguous* if there exists such cubic $\bar{W}$ which is in addition not isomorphic to W.

In the situation of the definition, $F(W)$ needs to be a special fourfold (of square 6 and divisibility 2); we determine in some cases whether a very general $F(W)$ of given discriminant is ambiguous.

Proposition 5.20. *Let e be a positive integer such that $e' := e/3$ is an integer. Let W be a general cubic fourfold of discriminant $2e$. Then the variety $F(W)$ of lines contained in W is ambiguous if and only if the equation $\mathscr{P}_{3,4e'}(3)$ has a positive solution, both equations $\mathscr{P}_{3,e'}(-1)$ and $\mathscr{P}_{3,4e'}(-5)$ are not solvable, and e is not a perfect square. If these conditions are realized, $F(W)$ is then strongly ambiguous.*

Proof. We may assume that $F(W)$ has Picard number 2.

Assume that $F(W)$ is ambiguous. If r is an isomorphism as in Definition 5.19, $r^*\bar{g}$ is an ample class of square 6 and divisibility 2 hence can be written (in the usual notation) as $ag + 2b\tau$, with $a > 0$, $b \neq 0$, $3a^2 - 4e'b^2 = 3$, and $\frac{2|b|}{a} < \nu_{3,e}$.

We will rule out cases b) and c) of Proposition 4.2 (case d) does not occur). In both cases, $3 \nmid e'$, hence $b' := |b|/3$ is an integer and $a^2 - 4eb'^2 = 1$.

In case b), the equation $\mathscr{P}_{3,4e'}(-5)$ has a minimal solution (a_{-5}, b_{-5}) and $\nu_{3,e} = \frac{3a_{-5}}{2e'b_{-5}}$. Set

$$x_1 := a + 2b'\sqrt{e}\ , \quad x_{-5} := 3a_{-5} + 2b_{-5}\sqrt{e}\ , \quad x'_{-5} := x_{-5}x_1^{-1} = x_{-5}\bar{x}_1 > 0.$$

In the field $\mathbf{Q}[\sqrt{e}]$, we have $N(x_1) = 1$ and $N(x_{-5}) = N(x'_{-5}) = -15$. We compute

$$x'_{-5} = 3(aa_{-5} - 4e'b'b_{-5}) + 2(ab_{-5} - 3b'a_{-5})\sqrt{e} =: 3a'_{-5} + b'_{-5}\sqrt{e}.$$

The inequality $\frac{2|b|}{a} < \nu_{3,e}$ means exactly $a'_{-5} > 0$. Since $-15 = x'_{-5}\bar{x}'_{-5}$ and $x'_{-5} > 0$, we have $\bar{x}'_{-5} < 0$, hence $b'_{-5} > 0$: the pair (a'_{-5}, b'_{-5}) is a positive solution of the equation $\mathscr{P}_{4e}(-15)$. However (see Section 3.1), we have $x_1 > 1$ hence $x'_{-5} < x_{-5}$ and this contradicts the minimality of the solution (a_{-5}, b_{-5}) (see Section 3.1).

In case c), the equation $\mathscr{P}_{3,e'}(-1)$ has a minimal solution (a_{-1}, b_{-1}) and $\nu_{3,e} = \frac{3a_{-1}}{e'b_{-1}}$. Set $x_{-1} := 3a_{-1} + b_{-1}\sqrt{e}$ and

$$x'_{-1} := x_{-1}x_1^{-1} = 3(aa_{-1} - 2e'b'b_{-1}) + (ab_{-1} - 2b'a_{-1})\sqrt{e}.$$

Similar considerations lead to a contradiction as above.

Finally, the solvability of the equation $\mathscr{P}_{3,4e'}(3)$ implies that e is not a perfect square.

For the converse, assume that the conditions of the proposition are realized. We are then in case a) of Proposition 4.2, hence the inequality in (14) holds by Remark 5.16. The class $ag+2b\tau$ is therefore ample on $F(W)$ and $(F(W), ag+2b\tau)$ is isomorphic to $(F(\bar{W}), \bar{g})$ for another cubic fourfold $\bar{W}$.

The same then holds for the pair $(F(W), ag-2b\tau)$. Since we saw in the proof of Proposition 4.7 that the involution $g \mapsto g$, $\tau \mapsto -\tau$ never comes from an automorphism of $F(W)$, this implies that $F(W)$ is strongly ambiguous. □

Example 5.21. We describe in the following table the situation for $2 \le e' \le 15$.

e'	2	3	4	5	6	7	8	9	10	11	12	13	14	15
e	6	9	12	15	18	21	24	27	30	33	36	39	42	45
$\mathscr{P}_{3,4e'}(3)$	–	–	–	–	$(3,1)$	–	–	$(7,2)$	–	–	–	–	–	$(9,2)$
$\mathscr{P}_{3,e'}(-1)$	–	–	$(1,1)$	–	–	$(3,2)$	–	–	–	–	–	$(2,1)$	–	–
$\mathscr{P}_{3,4e'}(-5)$	$(1,1)$	–	–	$(5,2)$	–	–	$(3,1)$	–	–	–	–	–	–	–
$F(W)$ (strongly) ambiguous	–	–	–	–	✓	–	–	✓	–	–	–	–	–	✓

5.12. **Hyperkähler fourfolds and double EPW sextics.** We next determine which special hyperkähler fourfolds are double EPW sextics.

Proposition 5.22. *Let n be a positive integer such that $n \equiv -1 \pmod 4$ and let e be a positive integer divisible by n. Set $e' := e/n$. Let F be a general hyperkähler fourfold in any irreducible component of $\mathscr{C}^{(2)}_{2n,2e}$. Then F is isomorphic to a double EPW sextic if and only if the equation $\mathscr{P}_{n,e'}(1)$ is solvable but the equation $\mathscr{P}_{n,4e'}(-5)$ is not.*

Proof. We may assume that F has Picard number 2. Let (h,τ) be a basis for $\mathrm{Pic}(F)$ as in Section 4.

If F is isomorphic to a double EPW sextic, its ample cone contains a class $ah+b\tau$ of square 2. This means that (a,b) is a solution of the equation $\mathscr{P}_{n,e'}(1)$. Moreover, the automorphism group of F is non-trivial, hence the equation $\mathscr{P}_{n,4e'}(-5)$ is not solvable by Proposition 4.7.

For the converse, assume that the equation $\mathscr{P}_{n,e'}(1)$ has a positive solution (a,b) (so that $\left(\frac{-e'}{n}\right)=1$) and the equation $\mathscr{P}_{n,4e'}(-5)$ is not solvable. Since $\left(\frac{-1}{n}\right)=-1$, we have $\left(\frac{e'}{n}\right)=-1$ and the equation $\mathscr{P}_{n,e'}(-1)$ is not solvable. The class $ah+b\tau$ has square 2. By Proposition 4.2, the slope of the nef cone is $\nu_{n,e}=n/\sqrt{e}$ and

$$\left(\frac{b}{a}\right)^2=\frac{n^2}{e}-\frac{n}{ea^2}<\nu_{n,e}^2$$

hence the class $ah+b\tau$ is ample. The rest of the proof proceeds as in Proposition 5.10. □

Under the hypotheses of the proposition, the automorphism group of F is isomorphic to $\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$ (Proposition 4.7) hence contains infinitely many involutions $(\iota_m)_{m\in\mathbf{Z}}$. All the quotients F/ι_m are EPW sextics.

As in Proposition 5.10, this construction yields a birational isomorphism from any component of $\mathscr{C}^{(2)}_{2n,2e}$ to some component of $\mathscr{C}^{(1)}_{2,2e}$. When e is even, $\mathscr{C}^{(1)}_{2,2e}$ is irreducible; when e is odd, this component is $\mathscr{C}^{(1)''}_{2,2e}$.[30]

Corollary 5.23. *Let n be a positive integer such that $n \equiv -1 \pmod 4$. There are infinitely many distinct hypersurfaces in the moduli space $\mathscr{M}^{(2)}_{2n}$ whose general points correspond to double EPW sextics. Their union is dense in $\mathscr{M}^{(2)}_{2n}$.*

Proof. When $e = n(nm^2 - 1)$, the equation $\mathscr{P}_{n,e'}(1)$ has solution $(m, 1)$.

If $5 \mid n$, we showed in the proof of Proposition 4.7 that $\mathscr{P}_{n,4e'}(-5)$ is not solvable. If $5 \nmid n$, we can choose m such that $n(nm^2 - 1) \equiv \pm 2 \pmod 5$ and by reducing modulo 5, we see that the equation $\mathscr{P}_{n,4e'}(-5)$ is then not solvable. Since there are infinitely many such m and the loci $\mathscr{C}^{(2)}_{2n,2n(nm^2-1)}$ are empty for only finitely many m (Proposition 2.8), this concludes the proof. $\square$

Remark 5.24. Assume that both equations $\mathscr{P}_{n,4e'}(-5)$ and $\mathscr{P}_{n,e'}(1)$ are solvable. As in Remark 4.8, let F' be the other birational model of a general F in an irreducible component of $\mathscr{C}^{(2)}_{2n,2e}$. Then F' is isomorphic to a double EPW sextic.

Indeed, by Proposition 4.7, F' has a non-trivial involution which leaves invariant the square-2 class $ah + b\tau$, where (a, b) is a solution to the equation $\mathscr{P}_{n,e'}(1)$. That class is therefore ample on F' and the rest of the argument proceeds as in Proposition 5.10.

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