Problem set 1

Olivier Debarre

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Problem 1. Recall that a topological space X is *irreducible* if it is non-empty and is not the union of two strict closed subsets. In other words, if X_1 and X_2 are closed subsets of X and $X = X_1 \cup X_2$, then $X = X_1$ or $X = X_2$.

a) Let X be a topological space and let $V \subset X$ be a subset (endowed with the induced topology). Prove that V is irreducible if and only if its closure $\overline{V}$ is irreducible.

b) Let X and Y be topological spaces and let $u : X \to Y$ be a continuous map. If X is irreducible, prove that $u(X)$ is irreducible

Problem 2. Let $\mathbf{k}$ be an *infinite* (not necessarily algebraically closed) field. Let $C \subset \mathbf{k}^2$ be the vanishing set $V(X^2 - Y^3)$.

a) Prove that the ideal of C is the ideal in $\mathbf{k}[X,Y]$ generated by $X^2 - Y^3$ and that C is irreducible (*Hint*: use the "parametrization" $\mathbf{k} \to C$ given by $t \mapsto (t^3, t^2)$ and express $A(C) = \mathbf{k}[X,Y]/I(C)$ as a subring of $\mathbf{k}[T]$).

b) Prove that C is not isomorphic to $\mathbf{k}$ (*Hint*: prove that $A(C)$ is not a principal ideal domain).

c) How do these these results generalize to the vanishing set $V(X^r - Y^s)$, where r and s are relatively prime positive integers?

Problem 3. Let $\mathbf{k}$ be an *infinite* (not necessarily algebraically closed) field, let $u\colon \mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}^3_{\mathbf{k}}$ be the regular map defined by $u(s,t) = (s^3, s^2t, st^2, t^3)$, and set $C := u(\mathbf{P}^1_{\mathbf{k}})$.

a) Prove that no 4 distinct points of C are contained in a hyperplane in $\mathbf{P}^3_{\mathbf{k}}$.

b) Prove that any quadric in $\mathbf{P}^3_{\mathbf{k}}$ (i.e., any subset of $\mathbf{P}^3_{\mathbf{k}}$ defined by a non-zero homogoneous polynomial of degree 2) that contains 7 distinct points of C contains C.

c) Prove that C is the vanishing set in $\mathbf{P}^3_{\mathbf{k}}$ of the (homogeneous) ideal I in $\mathbf{k}[T_0,T_1,T_2,T_3]$ generated by the homogeneous polynomials $T_0T_2 - T_1^2, T_2^2 - T_1T_3, T_1T_2 - T_0T_3$, which can be neatly expressed as the 2×2-minors of the matrix

$$\begin{pmatrix} T_0 & T_1 & T_2 \\ T_1 & T_2 & T_3 \end{pmatrix}.$$

d) Prove that the ideal of C is I (*Hint*: prove that any polynomial $P \in \mathbf{k}[T_0,T_1,T_2,T_3]$ is congruent modulo I to a polynomial of the type $A(T_0,T_1,T_3) + T_2B(T_3)$ and that if P vanishes on C, one has $B = 0$; then, use a similar method to show that A is divisible by $T_1^3 - T_0^2T_3$).

e) **(Extra credit)** How do these results generalize to the regular map $u\colon \mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}^n_{\mathbf{k}}$ ($n \geq 3$) defined by $u(s,t) = (s^n, s^{n-1}t, \dots, st^{n-1}, t^n)$?