Problem set 2

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Everything is defined over an algebraically closed field $\mathbf{k}$.

Problem 1. Prove that a linear subspace L contained in a smooth hypersurface $X \subset \mathbf{P}^n$ of degree > 1 has dimension $\leq (n-1)/2$ and show by producing examples that for each integer $n \geq 1$, this bound is the best possible (*Hint*: look at the common zeroes on L of the partial derivatives of an equation of X).

Problem 2. Let $\mathbf{A}$ be the affine space of $n \times n$-matrices with entries in $\mathbf{k}$, with $n > 1$. Given $M \in \mathbf{A}$, the n^2 entries of the matrix M^n are homogeneous polynomials of degree n in the n^2 entries of M. Let I be the ideal in the polynomial ring $A(\mathbf{A})$ generated by these n^2 polynomials. The n coefficients $\sigma_1, \dots, \sigma_n$ of the characteristic polynomial of M (not counting its leading coefficient 1) are homogeneous polynomials of degrees $1, \dots, n$ in the n^2 entries of M. Let J be the ideal in $A(\mathbf{A})$ generated by these n polynomials.

a) Show that $V(I) = V(J)$ and that the ideal I is not radical. Let $\mathscr{N} \subset \mathbf{A}$ be the subvariety defined by I (or J). How are the elements of $\mathscr{N}$ usually called?

b) Show that every component of $\mathscr{N}$ has dimension $\geq n^2 - n$.

c) Let $\mathscr{N}^0 := \{M \in \mathbf{A} \mid M^n = 0,\ M^{n-1} \neq 0\}$. Prove that $\mathscr{N}^0$ is irreducible smooth of dimension $n^2 - n$ (*Hint*: use the fact that $\mathscr{N}^0$ is homogeneous under the action of a connected algebraic group).

d) Prove that $\mathscr{N}^0$ is dense in $\mathscr{N}$ and that $\mathscr{N}$ is irreducible of dimension $n^2 - n$.

e) Prove that the singular locus of $\mathscr{N}$ is exactly $\mathscr{N} \smallsetminus \mathscr{N}^0$.

f) Show that the regular map

$$\begin{aligned} u\colon \mathbf{A} &\longrightarrow \mathbf{A}^n \\ M &\longmapsto (\sigma_1(M), \dots, \sigma_n(M)) \end{aligned}$$

is surjective, that general fibers are smooth irreducible of dimension $n^2 - n$, and that all fibers are irreducible of dimension $n^2 - n$.

Problem 3. (Dual varieties) Let V be a $\mathbf{k}$-vector space of dimension $n+1$ and let $X \subsetneq \mathbf{P}V$ be an irreducible (closed) proper subvariety. We let $X^0 \subset X$ be the dense open subset of smooth points of X. If $x \in X^0$, the *projective Zariski tangent space* $\mathbf{T}_{X,x} \subset \mathbf{P}V$ was defined in the class notes (Example 4.5.5)); it is a projective linear subspace passing through x of the same dimension as the Zariski tangent space $T_{X,x}$. If $\pi\colon V \smallsetminus \{0\} \to \mathbf{P}V$ is the canonical projection and $CX^0 := \pi^{-1}(X^0)$ (the affine cone over X), $\mathbf{T}_{X,x}$ is the image by π of $T_{CX^0,x'}$, for any $x' \in \pi^{-1}(x)$.

The set of hyperplanes in $\mathbf{P}V$ is the projective space $\mathbf{P}V^\vee$. We define the *dual variety* of X as the closure

$$X^\vee := \overline{\{H \in \mathbf{P}V^\vee \mid \exists x \in X^0 \quad H \supset \mathbf{T}_{X,x}\}} \subset \mathbf{P}V^\vee.$$

a) Show that $X^\vee \subset \mathbf{P}V^\vee$ is an irreducible variety of dimension $\leq n-1$ and that for $H \in \mathbf{P}V^\vee \setminus X^\vee$, the intersection $X^0 \cap H$ is smooth (*Hint*: you might want to consider the variety $\overline{\{(x,H) \in X^0 \times \mathbf{P}V^\vee \mid \mathbf{T}_{X,x} \subset H\}} \subset \mathbf{P}V \times \mathbf{P}V^\vee$).

b) If $X \subset \mathbf{P}^n$ is a hypersurface whose ideal is generated by a homogeneous polynomial F, show that $X^\vee$ is the (closure of the) image of the so-called *Gauss map*

$$\begin{array}{rcl} X & \dashrightarrow & \mathbf{P}^n \\ x & \longmapsto & \Big(\dfrac{\partial F}{\partial x_0}(x), \dots, \dfrac{\partial F}{\partial x_n}(x)\Big). \end{array}$$

c) What is the dual of the plane conic curve $C \subset \mathbf{P}^2$ with equation $x_0^2 + x_1 x_2 = 0$? (*Hint*: the answer is different in characteristic 2!).

d) Assume that V is the vector space of $2 \times (m+1)$-matrices with entries in $\mathbf{k}$. Recall that the set $X \subset \mathbf{P}V$ of matrices of rank 1 is a smooth variety of dimension $m+1$. What is the dual $X^\vee \subset \mathbf{P}V^\vee$? (*Hint*: find the orbits of the action of the group $\mathrm{GL}(2,\mathbf{k}) \times \mathrm{GL}(m+1,\mathbf{k})$ on $\mathbf{P}V$ or $\mathbf{P}V^\vee$.)

e) The aim of this question is to show that if the characteristic of $\mathbf{k}$ is 0, one has $(X^\vee)^\vee = X$ (where we identify $\mathbf{P}V^{\vee\vee}$ with $\mathbf{P}V$). We introduce the variety

$$\begin{array}{rcl} I & := & \{(x,\ell) \in (V \smallsetminus \{0\}) \times (V^\vee \smallsetminus \{0\}) \mid \ell(x) = 0\}, \\ I_X & := & \{(x,\ell) \in I \mid x \in CX^0,\ \ell|_{T_{CX^0,x}} = 0\}. \end{array}$$

(i) What is the closure of the image of the projection $I_X \xrightarrow{p_2} V^\vee \smallsetminus \{0\} \xrightarrow{\pi^\vee} \mathbf{P}V^\vee$?

(ii) Let $(x,\ell) \in I_X$. Prove that the Zariski tangent space $T_{I_X,(x,\ell)}$ is contained in the vector space

$$T'_{I_X,(x,\ell)} := \{(a,m) \in V \times V^\vee \mid \ell(a) + m(x) = 0,\ a \in T_{CX^0,x}\}$$

and that $T'_{I_X,(x,\ell)}$ is also equal to $\{(a,m) \in T_{CX^0,x} \times V^\vee \mid m(x) = 0\}$.

(iii) Prove that in characteristic 0, one has $(X^\vee)^\vee = X$ (*Hint*: use generic smoothness for the map $\pi^\vee \circ p_2 \colon I_X \to X^\vee$).

(iv) Show that this result does not always hold in positive characteristics (*Hint*: use question c)).