
Problem set 2

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Everything is defined over an algebraically closed field $\mathbf{k}$.

Problem 1. Prove that a linear subspace L contained in a smooth hypersurface $X \subset \mathbf{P}^n$ of degree > 1 has dimension $\leq (n-1)/2$ and show by producing examples that for each integer $n \geq 1$, this bound is the best possible (*Hint*: look at the common zeroes on L of the partial derivatives of an equation of X).

Proof. Assume $L \subset X \subset \mathbf{P}^n$, where L is a linear subspace of dimension r and X is a hypersurface defined by a homogeneous polynomial F of degree $d > 1$. We may choose coordinates $x_0, \dots, x_n$ such that L is defined by the equations $x_{r+1} = \cdots = x_n = 0$. Since F vanishes identically on L, so do the partial derivatives $\frac{\partial F}{\partial x_i}$ for $0 \leq i \leq r$. It follows that any common zero on L of the (non-constant) $n-r$ homogeneous polynomials $\frac{\partial F}{\partial x_{r+1}}, \dots, \frac{\partial F}{\partial x_n}$ is singular on X. If X is smooth, we must therefore have $n - r > r$.

If $n = 2m$ is even, the smooth quadric in $\mathbf{P}^n$ with equation $x_0x_1 + x_2x_3 + \cdots + x_{n-2}x_{n-1} + x_n^2 = 0$ contains the $(m-1)$-dimensional linear space defined by the equations $x_0 = x_2 = \cdots = x_{n-2} = x_n = 0$.

If $n = 2m+1$ is odd, the smooth quadric in $\mathbf{P}^n$ with equation $x_0x_1 + x_2x_3 + \cdots + x_{n-2}x_{n-1} = 0$ contains the m-dimensional linear space defined by the equations $x_0 = x_2 = \cdots = x_{n-2} = 0$.

Problem 2. Let $\mathbf{A}$ be the affine space of $n \times n$-matrices with entries in $\mathbf{k}$, with $n > 1$. Given $M \in \mathbf{A}$, the n^2 entries of the matrix M^n are homogeneous polynomials of degree n in the n^2 entries of M. Let I be the ideal in the polynomial ring $A(\mathbf{A})$ generated by these n^2 polynomials. The n coefficients $\sigma_1, \dots, \sigma_n$ of the characteristic polynomial of M (not counting its leading coefficient 1) are homogeneous polynomials of degrees $1, \dots, n$ in the n^2 entries of M. Let J be the ideal in $A(\mathbf{A})$ generated by these n polynomials.

a) Show that $V(I) = V(J)$ and that the ideal I is not radical. Let $\mathscr{N} \subset \mathbf{A}$ be the subvariety defined by I (or J). How are the elements of $\mathscr{N}$ usually called?

Proof. Both $V(I)$ and $V(J)$ are equal to the set of nilpotent matrices. By the Nullstellensatz, $\sqrt{I} = \sqrt{J}$. Since the trace $a_{11} + \cdots + a_{nn}$ is in J, it is in $\sqrt{I}$; but it is not in I, since it is homogeneous of degree 1 and I is generated by homogeneous polynomials of degree n. Hence $I \neq \sqrt{I}$.

b) Show that every component of $\mathscr{N}$ has dimension $\geq n^2 - n$.

Proof. The variety $\mathscr{N}$ is defined by the n equations $\sigma_1, \dots, \sigma_n$ in $\mathbf{A}$. By Krull's theorem, every component of $\mathscr{N}$ has dimension $\geq \dim(\mathbf{A}) - n = n^2 - n$.

c) Let $\mathscr{N}^0 := \{M \in \mathbf{A} \mid M^n = 0,\ M^{n-1} \neq 0\}$. Prove that $\mathscr{N}^0$ is irreducible smooth of dimension $n^2 - n$ (*Hint*: use the fact that $\mathscr{N}^0$ is homogeneous under the action of a connected algebraic group).

Proof. An element M is in $\mathscr{N}^0$ if and only if it is similar to

$$N_n := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

The irreducible group $\mathrm{GL}(n,\mathbf{k})$ therefore acts transitively on $\mathscr{N}^0 \subset \mathbf{A}^{n^2}$ by conjugation, hence $\mathscr{N}^0$ is irreducible and smooth. Its dimension is the dimension n^2 of $\mathrm{GL}(n,\mathbf{k})$ minus the dimension of the stabilizer of N_n. This stabilizer consists of matrices of the type $\begin{pmatrix} t_1 & t_2 & \cdots & \cdots & t_n \\ 0 & t_1 & t_2 & \cdots & t_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & t_2 \\ 0 & \cdots & \cdots & 0 & t_1 \end{pmatrix}$, with $t_1,\dots,t_n \in \mathbf{k}$ and $t_1 \neq 0$ (all polynomials in N). It has dimension n, hence $\dim(\mathscr{N}^0) = n^2 - n$.

d) Prove that $\mathscr{N}^0$ is dense in $\mathscr{N}$ and that $\mathscr{N}$ is irreducible of dimension $n^2 - n$.

Proof. Any element M of $\mathscr{N}$ can be written as $M = P^{-1}NP$, where N is upper triangular with 0 diagonal and P invertible. For $t \in \mathbf{k}$, we have $N(t) := P^{-1}(N + tN_n)P \in \mathscr{N}$. Moreover, $(N + tN_n)^{n-1}$ is a polynomial of degree exactly $n-1$ in t (with matrix coefficients), hence there exists a finite set $F \subset \mathbf{k}$ such that $N + tN_n$ (hence also $N(t)$) is in $\mathscr{N}^0$ if and only if $t \in \mathbf{k} \smallsetminus F$. Since $N(0) = M$ and the Zariski closure of $\mathbf{k} \smallsetminus F$ is $\mathbf{k}$, we get $M \in \overline{\mathscr{N}^0}$. This proves $\mathscr{N} = \overline{\mathscr{N}^0}$, hence $\mathscr{N}$ is, as $\mathscr{N}^0$, irreducible of dimension $n^2 - n$.

e) Prove that the singular locus of $\mathscr{N}$ is exactly $\mathscr{N} \smallsetminus \mathscr{N}^0$.

Proof. It seems difficult to answer this question without knowing that the ideal of $\mathscr{N}$ is J, hence is generated by $\sigma_1,\dots,\sigma_n$.[1] Once this is known, one can compute the Jacobian matrix of these n equations at a point N of $\mathscr{N} \smallsetminus \mathscr{N}^0$: the rank of N is $\leq n-2$ and one sees that the differential of $\sigma_n = \pm\det$ at any such matrix (nilpotent or not) vanishes identically. The Jacobian matrix at N therefore has a zero row hence its rank is $< n$ and N is singular on $V(J) = \mathscr{N}$.

f) Show that the regular map

$$\begin{aligned} u\colon \mathbf{A} &\longrightarrow \mathbf{A}^n \\ M &\longmapsto (\sigma_1(M),\dots,\sigma_n(M)) \end{aligned}$$

is surjective, that general fibers are smooth irreducible of dimension $n^2 - n$, and that all fibers are irreducible of dimension $n^2 - n$.

[1] Proving this (i.e., that J is a radical ideal) without too much effort requires a bit of commutative algebra: since $V(J)$ has codimension in $\mathbf{A}$ equal to the number of its generators, it is enough, by the Unmixedness Theorem (Matsumura, H., *Commutative algebra,* W.A. Benjamin Co., New York, 1970, p. 107 and Theorem 31, p. 108) to check that J is "reduced" at some point of each component of $V(J)$; since $V(J)$ is irreducible, it is enough to check that the Jacobian matrix of $\sigma_1,\dots,\sigma_n$ has rank n at some point of $\mathscr{N}$. One checks that this is true at any point of $\mathscr{N}^0$.

Proof. Companion matrices give surjectivity. The fibers are stable by conjugation hence are unions of conjugacy classes.

If its characteristic polynomial has no multiple roots (this occur over the dense open subset of $\mathbf{A}$ defined by the non-vanishing of the discriminant of the characteristic polynomial), a matrix M is similar to a diagonal matrix D with distinct diagonal coefficients. Its fiber $u^{-1}(u(M))$ is therefore homogeneous under the action of $\mathrm{GL}(n, \mathbf{k})$ by conjugation, hence is smooth irreducible. Its dimension is given by a theorem from the course as $\dim(\mathbf{A}) - \dim(\mathbf{A}^n) = n^2 - n$.

The same reasoning as in question d) shows that matrices with the minimal number of Jordan blocks are dense in any fiber. Since $\mathrm{GL}(n, \mathbf{k})$ acts transitively on this set of matrices, the fiber is irreducible. Again, the centralizer of such a matrix consists of direct sums of matrices as in c), hence has dimension n. This implies that this set, hence also its closure the fiber, have dimension $n^2 - n$.

Problem 3. (Dual varieties) Let V be a $\mathbf{k}$-vector space of dimension $n + 1$ and let $X \subsetneq \mathbf{P}V$ be an irreducible (closed) proper subvariety. We let $X^0 \subset X$ be the dense open subset of smooth points of X. If $x \in X^0$, the *projective Zariski tangent space* $\mathbf{T}_{X,x} \subset \mathbf{P}V$ was defined in the class notes (Example 4.5.5)); it is a projective linear subspace passing through x of the same dimension as the Zariski tangent space $T_{X,x}$. If $\pi \colon V \smallsetminus \{0\} \to \mathbf{P}V$ is the canonical projection and $CX^0 := \pi^{-1}(X^0)$ (the affine cone over X), $\mathbf{T}_{X,x}$ is the image by π of $T_{CX^0,x'}$, for any $x' \in \pi^{-1}(x)$.

The set of hyperplanes in $\mathbf{P}V$ is the projective space $\mathbf{P}V^\vee$. We define the *dual variety* of X as the closure

$$X^\vee := \overline{\{H \in \mathbf{P}V^\vee \mid \exists x \in X^0 \quad H \supset \mathbf{T}_{X,x}\}} \subset \mathbf{P}V^\vee.$$

a) Show that $X^\vee \subset \mathbf{P}V^\vee$ is an irreducible variety of dimension $\leq n - 1$ and that for $H \in \mathbf{P}V^\vee \setminus X^\vee$, the intersection $X^0 \cap H$ is smooth (*Hint*: you might want to consider the variety $\overline{\{(x, H) \in X^0 \times \mathbf{P}V^\vee \mid \mathbf{T}_{X,x} \subset H\}} \subset \mathbf{P}V \times \mathbf{P}V^\vee$).

Proof. Let $\mathbf{I}_X := \{(x, H) \in X^0 \times \mathbf{P}V^\vee \mid \mathbf{T}_{X,x} \subset H\}$ and set $d := \dim(X)$. The fibers of the first projection $\mathbf{I}_X \to X^0$ are all isomorphic to $\mathbf{P}^{n-d-1}$, hence $\mathbf{I}_X$ is irreducible of dimension $d + n - d - 1 = n - 1$. Since the image of the second projection $\mathbf{I}_X \to \mathbf{P}V^\vee$ is $X^\vee$, this answers the first part of the question.

For the second part, let $x \in X^0 \cap H$. The projective Zariski tangent space $\mathbf{T}_{X^0 \cap H,x} \subset \mathbf{P}V$ is contained in $\mathbf{T}_{X^0,x}$ (which has dimension d) and in $\mathbf{T}_{H,x} = H$. Since $H \not\supset \mathbf{T}_{X,x}$, the intersection $\mathbf{T}_{X^0,x} \cap H$ has dimension $d - 1$. Since $X^0 \cap H$ has dimension $\geq d - 1$ by Krull's theorem, we have

$$d - 1 \leq \dim_x(X^0 \cap H) \leq \dim(\mathbf{T}_{X^0 \cap H,x}) \leq \dim(\mathbf{T}_{X^0,x} \cap H) = d - 1.$$

These numbers are all equal and $X^0 \cap H$ is smooth at x.

b) If $X \subset \mathbf{P}^n$ is a hypersurface whose ideal is generated by a homogeneous polynomial F, show that $X^\vee$ is the (closure of the) image of the so-called *Gauss map*

$$\begin{array}{ccc} X & \dashrightarrow & \mathbf{P}^n \\ x & \longmapsto & \Big(\dfrac{\partial F}{\partial x_0}(x), \dots, \dfrac{\partial F}{\partial x_n}(x)\Big). \end{array}$$

Proof. The projective Zariski tangent space to X at a smooth point $x \in X$ is the hyperplane with equation
$$\frac{\partial F}{\partial x_0}(x)a_0 + \cdots + \frac{\partial F}{\partial x_n}(x)a_n = 0.$$
The result follows by definition of $X^\vee$.

c) What is the dual of the plane conic curve $C \subset \mathbf{P}^2$ with equation $x_0^2 + x_1x_2 = 0$? (*Hint*: the answer is different in characteristic 2!).

Proof. Using b), we see that the dual $C^\vee \subset \mathbf{P}^2$ is the image of the Gauss map
$$\begin{array}{rcl} C & \longrightarrow & \mathbf{P}^2 \\ x & \longmapsto & (2x_0, x_2, x_1) \end{array}$$
(the curve C is smooth). If the characteristic of $\mathbf{k}$ is not 2, $C^\vee$ is the conic with equation $(a_0/2)^2 + a_1a_2 = 0$. If the characteristic of $\mathbf{k}$ is 2, $C^\vee$ is the line $a_0 = 0$.

d) Assume that V is the vector space of $2 \times (m+1)$-matrices with entries in $\mathbf{k}$. Recall that the set $X \subset \mathbf{P}V$ of matrices of rank 1 is a smooth variety of dimension $m+1$. What is the dual $X^\vee \subset \mathbf{P}V^\vee$? (*Hint*: find the orbits of the action of the group $\mathrm{GL}(2, \mathbf{k}) \times \mathrm{GL}(m+1, \mathbf{k})$ on $\mathbf{P}V$ or $\mathbf{P}V^\vee$.)

Proof. The group $G := \mathrm{GL}(2, \mathbf{k}) \times \mathrm{GL}(m+1, \mathbf{k})$ acts on $\mathbf{P}V$ (on the left) by $(P, Q) \cdot M = PMQ^{-1}$. The orbit are the equivalence classes of (non-zero) matrices: the orbit X of matrices of rank 1 and the orbit $\mathbf{P} \smallsetminus X$ of matrices of rank 2; the same description holds for the dual action of G on $\mathbf{P}V^\vee$. One checks that the dual variety $X^\vee \subset \mathbf{P}V^\vee$ is stable for the dual action of G on $\mathbf{P}V^\vee$. Since it has dimension $< 2m+1$, it has to be the closed orbit, which is of dimension $m+1$ (and non-canonically isomorphic to $\mathbf{P}^1 \times \mathbf{P}^m$ and to X).

e) The aim of this question is to show that if the characteristic of $\mathbf{k}$ is 0, one has $(X^\vee)^\vee = X$ (where we identify $\mathbf{P}V^{\vee\vee}$ with $\mathbf{P}V$). We introduce the variety
$$\begin{array}{rcl} I & := & \{(x, \ell) \in (V \smallsetminus \{0\}) \times (V^\vee \smallsetminus \{0\}) \mid \ell(x) = 0\}, \\ I_X & := & \{(x, \ell) \in I \mid x \in CX^0,\ \ell|_{T_{CX^0,x}} = 0\}. \end{array}$$

(i) What is the closure of the image of the projection $I_X \xrightarrow{p_2} V^\vee \smallsetminus \{0\} \xrightarrow{\pi^\vee} \mathbf{P}V^\vee$?

Proof. By definition, this is $X^\vee$ (note that I_X is just the inverse image in $(V \smallsetminus \{0\}) \times (V^\vee \smallsetminus \{0\})$ of the variety $\mathbf{I}_X \subset \mathbf{P}V \times \mathbf{P}V^\vee$ of question a)).

(ii) Let $(x, \ell) \in I_X$. Prove that the Zariski tangent space $T_{I_X,(x,\ell)}$ is contained in the vector space
$$T'_{I_X,(x,\ell)} := \{(a, m) \in V \times V^\vee \mid \ell(a) + m(x) = 0,\ a \in T_{CX^0,x}\}$$
and that $T'_{I_X,(x,\ell)}$ is also equal to $\{(a, m) \in T_{CX^0,x} \times V^\vee \mid m(x) = 0\}$.

Proof. Since $I_X \subset CX^0 \times (V^\vee \smallsetminus \{0\})$, we have $T_{I_X,(x,\ell)} \subset \{(a, m) \in V \times V^\vee \mid a \in T_{CX^0,x}\}$. Since $I_X \subset I$, we have $T_{I_X,(x,\ell)} \subset T_{I,(x,\ell)} = \{(a, m) \in V \times V^\vee \mid \ell(a) + m(x) = 0\}$. This answers the first part of the question. Now by definition of I_X, we have $\ell|_{T_{CX^0,x}} = 0$, hence $\ell(a) = m(x) = 0$.

(iii) Prove that in characteristic 0, one has $(X^\vee)^\vee = X$ (*Hint*: use generic smoothness for the map $\pi^\vee \circ p_2 \colon I_X \to X^\vee$).

Proof. In characteristic 0, generic smoothness tells us that the tangent map to $\pi^\vee \circ p_2 \colon I_X \to X^\vee$ at any point $(x, \ell) \in I_X$ is surjective for $[\ell] \in X^\vee$ general. This implies $m(x) = 0$ for all $(x, \ell) \in I_X$ and all $m \in \mathbf{T}_{X^\vee,[\ell]}$. If we identify $\mathbf{P}V^{\vee\vee}$ with $\mathbf{P}V$, this means exactly that $[x] \in \mathbf{P}V = \mathbf{P}V^{\vee\vee}$ is in $(X^\vee)^\vee$. In particular, the image by $\pi \circ p_1 \colon I_X \to \mathbf{P}V$ of an open dense subset of I_X is contained in $(X^\vee)^\vee$, hence $X \subset (X^\vee)^\vee$. This is not enough to conclude that there is equality. However, our argument actually shows that I_X is contained in $I_{X^\vee}$ or equivalently, $\mathbf{I}_X \subset \mathbf{I}_{X^\vee}$ (with the notation of the proof of question a)). We saw in that same proof that both are irreducible of the same dimension, hence $\overline{\mathbf{I}_X} = \overline{\mathbf{I}_{X^\vee}}$. This implies $(X^\vee)^\vee = p_1(\overline{\mathbf{I}_{X^\vee}}) = p_1(\overline{\mathbf{I}_X}) = X$.

(iv) Show that this result does not always hold in positive characteristics (*Hint*: use question c)).

Proof. If $C \subset \mathbf{P}^2$ is the conic of question c) in characteristic 2, the dual $C^\vee$ is the line $a_0 = 0$, hence $(C^\vee)^\vee$ is the point $(1, 0, 0)$: it is not even contained in C.