Problem set 3

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Problem 1. Let $\mathbf{k}$ be a field. We consider two copies $U_1 := \mathrm{Spec}(\mathbf{k}[T_1])$ and $U_2 := \mathrm{Spec}(\mathbf{k}[T_2])$ of the affine line $\mathbf{A}^1_{\mathbf{k}}$.

a) Compute the Picard groups of $\mathbf{A}^1_{\mathbf{k}}$ and $\mathbf{A}^1_{\mathbf{k}} \smallsetminus \{0\}$ (*Hint:* you may use without proof the fact that if A is a unique factorization domain, the Picard group of $\mathrm{Spec}(A)$ is trivial).

Proof. Since $\mathbf{A}^1_{\mathbf{k}} = \mathrm{Spec}(\mathbf{k}[T])$ and $\mathbf{A}^1_{\mathbf{k}} \smallsetminus \{0\} := \mathrm{Spec}(\mathbf{k}[T, T^{-1}])$ and both rings $\mathbf{k}[T]$ and $\mathbf{k}[T, T^{-1}]$ are unique factorization domains, their Picard groups are trivial.

b) Let X be the scheme obtained by glueing U_1 and U_2 along the open subsets $U_1 \smallsetminus \{0\} = \mathrm{Spec}(\mathbf{k}[T_1, T_1^{-1}])$ and $U_2 \smallsetminus \{0\} = \mathrm{Spec}(\mathbf{k}[T_2, T_2^{-1}])$ by the isomorphism $\mathbf{k}[T_1, T_1^{-1}] \xrightarrow{\sim} \mathbf{k}[T_2, T_2^{-1}]$ of $\mathbf{k}$-algebras sending T_1 to T_2^{-1}. Which scheme is X?

Asnwer. The scheme X is the projective line $\mathbf{P}^1_{\mathbf{k}}$.

c) Compute the Picard group of X (*Hint:* explain that you may use Leray's theorem to compute $H^1(X, \mathscr{O}_X^*)$).

Proof. The scheme X is covered by the affine subsets U_1 and U_2. Moreover, $H^q(U_i, \mathscr{O}_X^*) = 0$ for $q \geq 2$ because U_i has dimension 1, and for $q = 1$ because $\mathrm{Pic}(U_i) = \mathrm{Pic}(\mathbf{A}^1_{\mathbf{k}}) = 0$ by question a). Similarly, $H^q(U_1 \cap U_2, \mathscr{O}_X^*) = 0$ for $q > 0$ for the same reasons. We may therefore apply Leray's theorem to compute $H^1(X, \mathscr{O}_X^*)$ as the cokernel of the map

$$\begin{array}{rcl} \Gamma(U_1, \mathscr{O}_X^*) \times \Gamma(U_2, \mathscr{O}_X^*) & \longrightarrow & \Gamma(U_1 \cap U_2, \mathscr{O}_X^*) \\ (s_1, s_2) & \longmapsto & s_1/s_2. \end{array}$$

Since $\Gamma(U_i, \mathscr{O}_X^*) = \mathbf{k}[T_i]^* = \mathbf{k}^*$ and $\Gamma(U_1 \cap U_2, \mathscr{O}_X^*) = \mathbf{k}[T, T^{-1}]^* = \mathbf{k}^* \times \langle T \rangle \simeq \mathbf{k}^* \times \mathbf{Z}$, we obtain $\mathrm{Pic}(X) \simeq H^1(X, \mathscr{O}_X^*) \simeq \mathbf{Z}$.

d) Find the global sections of each invertible sheaf on X.

Proof. Let $\mathscr{L}_m$ be the invertible sheaf on X corresponding to $(1, T_1^m) \in \mathbf{k}^* \times \langle T_1 \rangle$. The sections of $\mathscr{L}_m$ on X correspond to pairs (P_1, P_2), where $P_i \in \Gamma(U_i, \mathscr{L}_m) \simeq \Gamma(U_i, \mathscr{O}_{U_i}) \simeq \mathbf{k}[T_i]$, with $P_1(T_1)/P_2(T_1^{-1}) = T_1^m$. It follows that $\Gamma(X, \mathscr{L}_m)$ is isomorphic to the space of polynomials in $\mathbf{k}[T]$ of degree $\leq m$: if $m \geq 0$, the sections are the pairs $(P(T_1), T_2^m P(T_2^{-1}))$, where $\deg(P) \leq m$.

e) Let Y be the scheme obtained by glueing U_1 and U_2 as in b), but using now the isomorphism $\mathbf{k}[T_1, T_1^{-1}] \xrightarrow{\sim} \mathbf{k}[T_2, T_2^{-1}]$ that sends T_1 to T_2. Compute the Picard group of Y (*Hint:* proceed as in c)).

Proof. The same proof as in c) gives $\mathrm{Pic}(Y) \simeq H^1(Y, \mathscr{O}_Y^*) \simeq \mathbf{Z}$.

f) Find the global sections of each invertible sheaf on Y.

Proof. Let $\mathscr{L}_m$ be the invertible sheaf on Y corresponding to $(1, T_1^m) \in \mathbf{k}^* \times \langle T_1 \rangle$. The sections of $\mathscr{L}_m$ on Y correspond to pairs (P_1, P_2), where $P_i \in \Gamma(U_i, \mathscr{L}_m) \simeq \Gamma(U_i, \mathscr{O}_{U_i}) \simeq \mathbf{k}[T_i]$, with $P_1(T_1)/P_2(T_1) = T_1^m$. It follows that $\Gamma(Y, \mathscr{L}_m) \simeq \mathbf{k}[T]$ for all m: if $m \geq 0$, the sections are the pairs $(T_1^m P(T_1), P(T_2))$; if $m \leq 0$, the sections are the pairs $(P(T_1), T_2^{-m} P(T_2))$. In all cases, $\Gamma(Y, \mathscr{L}_m)$ is isomorphic to $\mathbf{k}[T]$.

g) Prove that there are no ample invertible sheaves on Y.

Proof. If $m > 0$, the global sections of $\mathscr{L}_m$ all vanish at the origin on U_1; if $m < 0$, the global sections of $\mathscr{L}_m$ all vanish at the origin on U_2. Hence $\mathscr{L}_m$ is never generated by global sections if $m \neq 0$. It follows that $\mathscr{L}_m$ is not ample for any m (apply the definition of ampleness with $\mathscr{F} = \mathscr{L}_1$).

Problem 2. Prove that the scheme $Y_n := \mathbf{A}^n_{\mathbf{k}} \smallsetminus \{0\}$ is not an affine scheme for any $n \geq 2$ (*Hint:* use Leray's theorem to compute $H^1(Y_2, \mathscr{O}_{Y_2})$).

Proof. The scheme Y_2 is covered by the affine open subsets $U_1 := \mathbf{A}^1_{\mathbf{k}} \times_{\mathbf{k}} (\mathbf{A}^1_{\mathbf{k}} \smallsetminus \{0\}) = \operatorname{Spec}(\mathbf{k}[T_1, T_2, T_2^{-1}])$ and $U_2 := (\mathbf{A}^1_{\mathbf{k}} \smallsetminus \{0\}) \times_{\mathbf{k}} \mathbf{A}^1_{\mathbf{k}}) = \operatorname{Spec}(\mathbf{k}[T_1, T_1^{-1}, T_2])$. Since $\mathscr{O}_{Y_2}$ is a coherent sheaf, we can compute $H^1(Y_2, \mathscr{O}_{Y_2})$ using Leray's theorem as the cokernel of the map

$$\begin{aligned} \Gamma(U_1, \mathscr{O}^*_{Y_2}) \times \Gamma(U_2, \mathscr{O}^*_{Y_2}) &\longrightarrow \Gamma(U_1 \cap U_2, \mathscr{O}^*_{Y_2}) \\ (P_1, P_2) &\longmapsto P_1 - P_2. \end{aligned}$$

We have

$$\begin{aligned} \Gamma(U_1, \mathscr{O}^*_{Y_2}) &= \mathbf{k}[T_1, T_2, T_2^{-1}], \\ \Gamma(U_2, \mathscr{O}^*_{Y_2}) &= \mathbf{k}[T_1, T_1^{-1}, T_2], \\ \Gamma(U_1 \cap U_2, \mathscr{O}^*_{Y_2}) &= \mathbf{k}[T_1, T_1^{-1}, T_2, T_2^{-1}], \end{aligned}$$

hence $H^1(Y_2, \mathscr{O}_{Y_2})$ is an infinite-dimensional $\mathbf{k}$-vector space with basis $(T_1^m T_2^n)_{m,n<0}$. In particular, Y_2 is not affine. Since Y_2 is a closed subscheme of Y_n for all $n \geq 2$ and a closed subscheme of an affine scheme is affine, Y_n is also not an affine scheme for all $n \geq 2$.

Problem 3. Let X be a projective scheme over a field and let $\mathscr{L}$ and $\mathscr{M}$ be invertible sheaves on X.

a) If $\mathscr{L}$ is generated by global sections and $\mathscr{M}$ is very ample, the invertible sheaf $\mathscr{L} \otimes \mathscr{M}$ is very ample (*Hint:* use a Segre embedding).

Proof. Since $\mathscr{L}$ is generated by global sections, there exists a morphism $u \colon X \to \mathbf{P}^m_{\mathbf{k}}$ such that $u^* \mathscr{O}_{\mathbf{P}^m_{\mathbf{k}}}(1) = \mathscr{L}$. Since $\mathscr{M}$ is very ample, there exists a closed embedding $v \colon X \hookrightarrow \mathbf{P}^n_{\mathbf{k}}$ such that $v^* \mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(1) = \mathscr{M}$. The morphism $(u, v) \colon X \to \mathbf{P}^m_{\mathbf{k}} \times \mathbf{P}^n_{\mathbf{k}}$ is then also a closed embedding (because its composition with the second projection is) and so is the composition

$$w \colon X \xrightarrow{(u,v)} \mathbf{P}^m_{\mathbf{k}} \times \mathbf{P}^n_{\mathbf{k}} \xrightarrow{\text{Segre}} \mathbf{P}^{(m+1)(n+1)-1}_{\mathbf{k}}.$$

Since $w^* \mathscr{O}_{\mathbf{P}^{(m+1)(n+1)-1}_{\mathbf{k}}}(1) = \mathscr{L} \otimes \mathscr{M}$, this proves that $\mathscr{L} \otimes \mathscr{M}$ is very ample.

b) If $\mathscr{M}$ is ample, the invertible sheaf $\mathscr{L} \otimes \mathscr{M}^{\otimes r}$ is very ample for *all* sufficiently large integers r (*Hint:* we proved in class that $\mathscr{L} \otimes \mathscr{M}^{\otimes r}$ is ample for some integer $r > 0$).

Proof. Since $\mathscr{M}$ is ample, there exists an integer r_0 such that $\mathscr{L} \otimes \mathscr{M}^{\otimes r}$ is generated by global sections for all $r \geq r_0$, and there exists an integer s_0 such that $\mathscr{M}^{\otimes s_0}$ is very ample. For any $r \geq r_0 + s_0$, the invertible sheaf $\mathscr{L} \otimes \mathscr{M}^{\otimes r} = \mathscr{L} \otimes \mathscr{M}^{\otimes (r - s_0)} \otimes \mathscr{M}^{\otimes s_0}$ is then very ample by a).