HYPERBOLICITY OF COMPLEX VARIETIES

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Summary

1. Ample line bundles

1.1. **Algebraic definition.** Let X be a variety (i.e., a scheme of finite type over a field $\mathbf{k}$). An invertible sheaf L on X is *ample* if, for any coherent sheaf F on X, the sheaf $F \otimes L^m$ is globally generated for all $m \gg 0$.

Examples 1. 1) On an affine scheme, any invertible sheaf is ample, because every coherent sheaf is globally generated.

2) The sheaf $\mathscr{O}_{\mathbf{P}^n}(m)$ on the projective space $\mathbf{P}^n$ is ample if and only if $m > 0$.

3) On a complete connected curve, an invertible sheaf is ample if and only if it has positive degree.

An invertible sheaf L on X is ample if and only if for some (or for any) $m > 0$, the invertible sheaf L^m is ample.

The restriction of an ample invertible sheaf to a (closed) subvariety is still ample.

An invertible sheaf L on X is *very ample* (over $\mathrm{Spec}(\mathbf{k})$) if there is a closed embedding $\iota : X \hookrightarrow \mathbf{P}^n$ such that $L = \iota^* \mathscr{O}_{\mathbf{P}^n}(1)$. It follows from Serre's theorem (applied to the closure of $\iota(X)$) that a very ample invertible sheaf is ample. Conversely, all sufficiently high powers of an ample line bundle are very ample.

On a *complete* variety, ample invertible sheaves can be characterized cohomologically.

Theorem 1. *Let X be a complete variety. The following conditions are equivalent:*

(i) *L is ample;*
(ii) *for each coherent sheaf F on X, we have $H^i(X, F \otimes L^m) = 0$ for all $i > 0$ and all $m \gg 0$.*

This criterion implies in particular that if $f : Y \to X$ is a morphism between proper varieties and L is an ample invertible sheaf on X, the pull-back f^*L is ample if and only if f is finite.

Definition 1. *A line bundle L on a proper algebraic variety X is* nef *if it has nonnegative degree on any curve $C \to X$.*

Ampleness implies nefness (but the converse is false!).

There are many properties of ample and nef invertible sheaves that I will not prove here. Here are some of them:

- If M is an invertible sheaf and L an ample invertible sheaf, $L^m \otimes M$ is very ample for all $m \gg 0$.
- The restriction of a nef invertible sheaf on a complete variety to a (closed) subvariety is still nef.
- The tensor product of two ample invertible sheaves is ample.
- On a complete variety, the tensor product of an ample invertible sheaf with a nef invertible sheaf is ample. In particular, ampleness is a numerical property: if P is a numerically trivial line bundle (i.e., of degree 0 on any curve), L is ample if and only if $L \otimes P$ is.
- We can therefore talk about ample (or nef) classes in the Néron–Severi group $\mathrm{NS}(X)$ and define ample rational, and even real, classes in $\mathrm{NS}(X)_{\mathbf{R}}$.
- The set of ample classes in $\mathrm{NS}(X)_{\mathbf{R}}$ is an open cone.
- A class in $\mathrm{NS}(X)_{\mathbf{R}}$ is ample if and only if it is nonnegative on the closed convex cone generated by classes of curves in X (Kleiman).
- An ample invertible sheaf L on a proper irreducible variety X is *big*: if n is the dimension of X, this means $h^0(X, L^{\otimes m}) > \alpha m^n$ for $m \gg 0$, for some $\alpha > 0$ (this follows from the Riemann–Roch theorem and item (ii) of Theorem 1).

1.2. **Forms of type (p, q) on a complex manifold.** Let X be a complex manifold of dimension n. At each point x of X, the complex tangent space $T_{X,x}$ is a complex vector space of dimension n. It can also be viewed as a real vector space $T^{\mathbf{R}}_{X,x}$ of (real) dimension $2n$, equipped with an endomorphism J_x such that $J_x^2 = -\operatorname{Id}_{T^{\mathbf{R}}_{X,x}}$.

Complex valued $\mathbf{R}$-linear forms on $T^{\mathbf{R}}_{X,x}$ form a complex vector space $\Omega^{\mathbf{R}}_{X,x}$ of dimension $2n$. It splits as a direct sum

$$\Omega_{X,x} \oplus \bar{\Omega}_{X,x}$$

of complex vector subspaces, where $\Omega_{X,x}$ is the space of $\mathbf{C}$-linear forms (also said to be of type $(1,0)$), and $\bar{\Omega}_{X,x}$ is the space of $\mathbf{C}$-antilinear forms (also said to be of type $(0,1)$).

A *real* 2-form $\omega : \bigwedge^2 T^{\mathbf{R}}_X \to \mathbf{R}$ is said to be *of type* $(1,1)$ if

$$\omega_x(J_x\xi, J_x\xi') = \omega_x(\xi, \xi')$$

One can associate to such a form a Hermitian form on T_X by the formula

$$h(\xi,\xi') = \omega(\xi, J\xi') - i\omega(x,y)$$

and conversely, to any Hermitian form on T_X, one can associate the $(1,1)$-form $\omega_h = -\operatorname{Im} h$. If $z_1,\dots,z_n$ are local holomorphic coordinates on X, we have

$$\begin{aligned} \omega &= i\sum_{j,k} h_{jk} dz_j \wedge d\bar z_k \\ h &= \sum_{j,k} h_{jk} dz_j \otimes d\bar z_k \end{aligned}$$

where (h_{jk}) is a Hermitian matrix of $\mathbf{C}$-valued $\mathscr{C}^\infty$ local functions on X.

A Hermitian metric on X is a Hermitian form on T_X that is everywhere positive definite. In terms of the associated $(1,1)$-form ω_h, this means

$$\omega_x(J_x\xi,\xi) > 0$$

for all nonzero $x \in X$ and $\xi \in T_{X,x}$. In local coordinates, with the notation above, this means that the Hermitian matrix (h_{jk}) is positive definite.

We say that the metric h is Kähler if the form ω_h is *closed,* i.e., $d\omega_x = 0$.

1.3. **Metric characterizations of ampleness.** Let X be a complex manifold of dimension n and L be a holomorphic line bundle on X endowed with a Hermitian metric h.

The curvature form $\Theta_h(L)$ is a closed $(1,1)$-form on X that has the property that $\Theta_h(L)$ represents $c_1(L)$. It can be defined locally as[1]

$$\Theta_h(L) = -\frac{i}{\pi}\,\partial\bar\partial \log \|s\|_h$$

for any nonvanishing local holomorphic section s of L.

Examples 2. 1) On the unit disk Δ, the *Poincaré metric* is defined as

$$h_P = \frac{dz\otimes d\bar z}{(1-|z|^2)^2} \quad , \quad \omega_P = \frac{i\,dz\wedge d\bar z}{(1-|z|^2)^2} \qquad (1)$$

[1]Different normalizations exist. This is the one used by Demailly in [De2]. In [De1], he uses $\Theta_h(L) = -\frac{i}{2\pi}\partial\bar\partial \log \|s\|_h$. Griffiths and Harris use $\Theta_h(L) = -2\partial\bar\partial \log \|s\|_h$ in [GH].

Its curvature

$$\begin{aligned}\Theta_{h_P} &= -\frac{i}{\pi}\,\partial\bar\partial \log \frac{1}{(1-|z|^2)^2}\\ &= \frac{2i}{\pi}\,\frac{\partial^2}{\partial z\partial\bar z}\,\log(1-|z|^2)\;dz\wedge d\bar z\\ &= -\frac{2i}{\pi}\,\frac{1}{(1-|z|^2)^2}\;dz\wedge d\bar z\\ &= -\frac{2}{\pi}\,\omega_P\end{aligned}$$

is constant negative. We will see later that it is invariant by biholomorphisms of Δ, hence descends to any quotient. For instance, this endows any compact Riemann surface of genus ≥ 2 with a metric with negative constant curvature.

2) The metric $(1+|z|^2)dz\otimes d\bar z$ on $\mathbf{C}$ has negative curvature.

3) The metric $\frac{1}{(\operatorname{Im} z)^2}\,dz\otimes d\bar z$ on the upper half-plane $\mathscr{H}$ has negative curvature $-1/2\pi$. Since it is invariant by the action of $\mathrm{SL}_2(\mathbf{Z})$, it descends to any (smooth) quotient by a subgroup of $\mathrm{SL}_2(\mathbf{Z})$, such as the one given by the modular function $j:\mathscr{H}\to\mathbf{C}$.

We say that a line bundle L on a compact Kähler manifold X is very ample if there exists an embedding $\iota: X\hookrightarrow \mathbf{P}^n_{\mathbf{k}}$ such that $L=\iota^*\mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(1)$. By Chow's theorem, X is then algebraic and L is very ample on X in the sense already defined in 1.1. In the same context, we say that L is ample if some positive tensor power of L is very ample.

Theorem 2. *A line bundle L on a compact Kähler manifold X is ample if and only if it carries a Hermitian metric whose curvature form is positive.*

This is the Kodaira embedding theorem. One direction is easy: if L is ample, the Fubini–Study metric induces, via the corresponding embedding into a projective space, a metric on X with positive curvature. The converse is more difficult: one needs to prove that if the curvature form is positive, some power of L has enough sections to define a projective embedding of X. This is usually done by proving a vanishing theorem for higher cohomology.

Nefness is more difficult to generalize to arbitrary compact complex manifolds, because X may contain no algebraic curves.

We need to introduce several concepts.

1.4. **Currents.** Let X be a complex manifold of dimension n. A *current* T on X is a differential form on X whose coefficients are distributions. To the newcomer, currents may seem to behave in many ways just like differential forms, but this is not the case! For example, although one can wedge a current with a smooth differential form, the exterior product of two currents is in general not defined since the product of two distributions may not define a distribution.

The pairing

$$\langle T, \omega \rangle \longmapsto \int_X T \wedge \omega$$

identifies the vector space of currents of type (p, q) on X with the topological dual of the vector space of smooth $(n-p, n-q)$-forms with compact support on X. The differential dT is then defined as usual by

$$\langle dT, \omega \rangle = -\langle T, d\omega \rangle$$

Obviously, a smooth form η of type (p, q) defines a current by the formula

$$\langle \eta, \omega \rangle \longmapsto \int_X \eta \wedge \omega$$

and the corresponding inclusion of complexes defines an isomorphism in cohomology. One can therefore associate a cohomology class $\{T\} \in H^{p+q}_{\mathrm{DR}}(X)$ to a current T of type (p, q).

Any smooth subvariety Y of X of *codimension* m defines a current $[Y]$ of type (m, m) by the formula

$$\langle [Y], \omega \rangle = \int_Y \omega|_Y$$

When Y is singular, a theorem of Lelong shows that the integral

$$\omega \longmapsto \int_{Y_{\mathrm{reg}}} \omega|_{Y_{\mathrm{reg}}}$$

converges (even if the support of ω meets Y_{sing}) and defines a current on X that we still denote by $[Y]$. The cohomology class $\{[Y]\} \in H^{2m}_{\mathrm{DR}}(X)$ is the fundamental class of Y.

This is the beauty of the theory of currents: the same formalism includes subvarieties and smooth differential forms.

A $(1, 1)$-current T, written in coordinates as $T = i \sum_{j,k} T_{jk} dz_j \wedge d\bar{z}_k$, where the T_{jk} are distributions, is *semipositive* if $\sum_{j,k} T_{jk} \alpha_j \bar{\alpha}_k$ is a nonnegative (real) measure for all functions $\alpha_1, \dots, \alpha_n$ on X with compact support. It implies that T is real and that its coefficients are measures.

Examples 3. 1) If D is an effective divisor on X, the current $[D]$ is semipositive.

2) (**The Lelong–Poincaré equation**) If $f : X \to \mathbf{C}$ is holomorphic, with zero divisor D,

$$\frac{i}{\pi}\,\partial\bar\partial \log|f| = [D]$$

as currents.

1.5. **Plurisubharmonic functions.** Let φ be a real locally integrable function on the complex manifold X. The complex Hessian of φ is the $(1,1)$-current

$$i\partial\bar\partial\varphi = i\sum_{j,k}\frac{\partial^2\varphi}{\partial z_j\partial\bar z_k}\,dz_j\wedge d\bar z_k$$

We say that φ is *plurisubharmonic* if this current is semipositive.

Examples 4. 1) If the f_{jk} are holomorphic functions and the m_{jk} positive real numbers, the function

$$\varphi = \log\max_j \sum_k |f_{jk}|^{m_{jk}}$$

is plurisubharmonic.

2) A plurisubharmonic function can always be (uniquely) modified on a negligible subset so that it becomes an upper semicontinuous function $X \to [-\infty, +\infty)$ that satisfies the inequality

$$\varphi(h(0)) \le \frac{1}{2\pi}\int_0^{2\pi}\varphi(h(e^{i\theta}))d\theta$$

for any holomorphic map $h : \overline{\Delta} \to X$. Conversely, any function that satisfies these two properties is plurisubharmonic.

1.6. **Singular Hermitian metrics.** Let L be a line bundle on a complex manifold X. A singular (Hermitian) metric on L is a function $\|\cdot\|_h : L \to \mathbf{R}^+$ that, read in any local trivialization $U\times\mathbf{C} \xrightarrow{u} L|_U$, has the form

$$\|u(x,\xi)\|_h = |\xi|e^{-\varphi}$$

where $\varphi \in L^1_{\mathrm{loc}}(U)$. Its *curvature current* is defined as the closed real $(1,1)$-current

$$\Theta_h(L) = \frac{i}{\pi}\,\partial\bar\partial\varphi$$

One checks that its cohomology class in $H^2_{\mathrm{DR}}(X)$ is independent of the choice of h and equals the first Chern class $c_1(L)$.

By definition, the curvature current is semipositive if the weight φ is plurisubharmonic.

In general, we define the *singular set* $\mathrm{Sing}(h)$ of a (singular) metric h as the closed subset of X of points x such that φ is unbounded on any neighborhood of x.

Examples 5. 1) Let h be a (smooth) Hermitian matric on the complex manifold X and let $f : \Delta \to X$ be a holomorphic map. If the derivative of f vanishes nowhere, f^*h is a (smooth) metric on the disk Δ. However, this rarely happens. However, whenever f is nonconstant, f^*h still defines a singular Hermitian metric: choose local holomorphic coordinates $(z_1, \dots, z_n)$ for X, so that f becomes a holomorphic function to $\mathbf{C}^n$. We have locally

$$f^*h = u(z)|f'(z)|^2\, dz \otimes d\bar{z}$$

where u is a real positive $\mathscr{C}^\infty$ function. This is a singular metric on Δ with weight

$$\varphi = \log|f'| + \frac{1}{2}\log u$$

2) Let D be an effective divisor on the complex manifold X. Sections of $L = \mathscr{O}_X(D)$ correspond to meromorphic functions on X with pole divisor $\leq D$. Equip L with the singular metric h given by the modulus of these functions. If $f = 0$ is a local (holomorphic) equation for D, the weight φ is $\log|f|$ and using the Poincaré–Lelong equation (Example 3.2)), we obtain

$$\Theta_h(L) = [D] - \frac{i}{\pi}\,\partial\bar{\partial}\log|f|$$

where $[D]$ is the current of integration over the divisor D. One sees in particular that the cohomology class of D is that of $\Theta_h(L)$, i.e., $c_1(L)$.

3) If $s_0, \dots, s_N$ are nonzero holomorphic sections of L, define a singular metric h on L by setting

$$\|s\|_h^2 = \frac{|u^{-1}\circ s|^2}{|u^{-1}\circ s_0|^2 + \cdots + |u^{-1}\circ s_N|^2}$$

The associated weight function is the plurisubharmonic function

$$\varphi = \frac{1}{2}\log\Big(\sum_{j=0}^{N}|u^{-1}\circ s_i|^2\Big)$$

This metric is also the pull-back of the Fubini–Study metric by the rational map $\psi_L : X \dashrightarrow \mathbf{P}^N$ defined by $s_0, \dots, s_N$.

The union of the base locus $\mathrm{Base}\,|L^m|$ and the set of points of $X \smallsetminus \mathrm{Base}\,|L^m|$ where ψ_{L^m} has positive fiber-dimension is a closed subset of X that does not depend on m for m sufficiently large and divisible.

We denote it by $\Sigma(L)$. It is different from X if and only if L is big, and the induced map

$$\psi_{L^m} : X - \Sigma(L) \longrightarrow \mathbf{P}^N$$

is an embedding. If this is the case, taking the mth root of the metric defined above yields a singular metric h on L, smooth outside $\Sigma(L)$, with semipositive curvature current.

One can then see ampleness, nefness, and bigness in terms of these singular metrics. Pick a smooth metric on the compact complex manifold X, with positive associated $(1,1)$-form ω.

- A line bundle L is ample if there is a *smooth* metric h on L such that $\Theta_h(L) \geq \varepsilon\omega_h$ for some $\varepsilon > 0$.
- A line bundle L is nef if for any $\varepsilon > 0$, there is a smooth metric h on L such that $\Theta_h(L) \geq -\varepsilon\omega_h$.
- A line bundle L is big if there is a (singular) metric h on L such that $\Theta_h(L) \geq \varepsilon\omega_h$ for some $\varepsilon > 0$.

The first item is just Theorem 2. The second is a definition. The third is new. In one direction, it follows from Examples 5.3) above (the singular set of the metric h on L can be chosen to be contained in the locus $\Sigma(L)$). The other direction is harder and uses Hörmander L^2-estimates to construct many sections of L^m.

This is a very important point that illustrates the flexibility brought by allowing singular metrics: a *big* line bundle has a (singular) metric with positive curvature current, whereas smooth metrics with positive curvature only exist on *ample* line bundles.

2. Ample vector bundles

2.1. **Algebraic definition.** Let E be a coherent sheaf on a scheme X. One can construct the projective X-scheme

$$\pi : \mathbf{P}(E) \longrightarrow X$$

of one-dimensional quotients of E. More algebraically, $\mathbf{P}(E)$ is realized as the scheme

$$\mathbf{P}(E) = \operatorname{Proj}\Big(\bigoplus_{m\geq 0} \mathbf{S}^m(E)\Big)$$

It comes equipped with a line bundle $\mathscr{O}_{\mathbf{P}(E)}(1)$ arising as the tautological quotient of π^*E:

$$\pi^*E \longrightarrow \mathscr{O}_{\mathbf{P}(E)}(1) \longrightarrow 0$$

(this is a relative version of the fact that on $\mathbf{P}^n$, the sheaf $\mathscr{O}_{\mathbf{P}^n}(1)$ is globally generated by its sections $x_0, \dots, x_n$). Also, we have for all $m \geq 0$

$$\pi_* \mathscr{O}_{\mathbf{P}(E)}(m) \simeq \mathbf{S}^m(E)$$

and $R^j \pi_* \mathscr{O}_{\mathbf{P}(E)}(m) = 0$ for all $j > 0$.

We say that *E is ample on X if the line bundle $\mathscr{O}_{\mathbf{P}(E)}(1)$ is ample on $\mathbf{P}(E)$.*

The restriction of an ample sheaf on X to a subscheme of X is still ample. A quotient $E \twoheadrightarrow F$ of coherent sheaves induces a closed embedding $\mathbf{P}(F) \hookrightarrow \mathbf{P}(E)$. It follows that a quotient of an ample sheaf is ample.

Following the analogous proofs for line bundles, one obtains the following characterizations of ample vector bundles on a proper variety.

Theorem 3. *Let X be a proper variety and let E be a vector bundle on X. The following conditions are equivalent:*

(i) *E is ample;*
(ii) *for each coherent sheaf F on X, the sheaf $F \otimes \mathbf{S}^m E$ is globally generated for all $m \gg 0$;*
(iii) *for each coherent sheaf F on X, we have $H^i(X, F \otimes \mathbf{S}^m E) = 0$ for all $i > 0$ and all $m \gg 0$;*
(iv) *for any ample line bundle L on X, such that, for all $m \gg 0$, the sheaf $\mathbf{S}^m E$ is a quotient of a direct sum of copies of L;*
(v) *for some $m > 0$, the vector bundle $\mathbf{S}^m E$ is ample.*

It follows from this theorem that given a finite and surjective morphism $f : Y \to X$, a vector bundle on X is ample if and only if its pull-back to Y is. Also, any extension of two ample vector bundles is ample. Finally, it is also true that the tensor product of two ample vector bundles is again ample. This is not easy to prove, especially in nonzero characteristic ([L], Corollary 6.1.16 and Remark 6.1.17).

It is easier to grasp the meaning of ampleness for a globally generated vector bundle. Assume that there is a finite-dimensional vector space V and a surjection

$$V \otimes \mathscr{O}_X \longrightarrow E \longrightarrow 0$$

This gives rise to a morphism

$$\varphi : \mathbf{P}(E) \longrightarrow \mathbf{P}(V)$$

defined as the composition

$$\mathbf{P}(E) \longrightarrow \mathbf{P}(V \otimes \mathscr{O}_X) = \mathbf{P}(V) \times X \longrightarrow \mathbf{P}(V)$$

that satisfies $\varphi * \mathscr{O}_{\mathbf{P}(V)}(1) = \mathscr{O}_{\mathbf{P}(E)}(1)$. It follows that E is ample if and only if φ is finite. More concretely, consider for each $x \in X$ the fiber

E_x, considered as a quotient of V, so that $\mathbf{P}(E_x)$ is a linear subspace of $\mathbf{P}(V)$. Then E is ample if and only if through any point of $\mathbf{P}(V)$, there are only finitely many subspaces $\mathbf{P}(E_x)$.

Examples 6. 1) Let $G = G(V, k)$ be the Grassmannian of k-dimensional quotients of a vector space V and let Q be the tautological quotient bundle on G, of rank k. We are in the situation above:

$$V \otimes \mathscr{O}_G \longrightarrow Q \longrightarrow 0$$

but Q is not ample for $k \geq 2$ because through any point of $\mathbf{P}(V)$, there are infinitely many $\mathbf{P}^{k-1}$.

2) Let X be a smooth subvariety of dimension d of an abelian variety A of dimension n. There is a surjection

$$\Omega_A|_X \longrightarrow \Omega_X \longrightarrow 0$$

For each $x \in X$, consider, using a translation by $-x$, the tangent space $T_{X,x}$ as a subspace of $T_{A,0}$. It follows that Ω_X is ample if and only if, for any nonzero $t \in T_{A,0}$, there are only finitely many $T_{X,x}$ through t. This is possible only if $2d \leq n$.

3) Let X be a smooth subvariety of dimension d of $\mathbf{P} = \mathbf{P}(V)$, let $\gamma : X \to G = G(V, d+1)$ be the Gauss map, and let Q the tautological rank $d+1$ quotient bundle on G. We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & N^*_{X/\mathbf{P}}(1) & = & N^*_{X/\mathbf{P}}(1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega_{\mathbf{P}^n}(1)|_X & \longrightarrow & V \otimes \mathscr{O}_X & \longrightarrow & \mathscr{O}_X(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega_X(1) & \longrightarrow & \gamma^* Q & \longrightarrow & \mathscr{O}_X(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{2}$$

Again, the vector bundle $\gamma^* Q$ is ample if and only if, for any point $p \in \mathbf{P}$, there are only finitely many (projective) tangent spaces $\mathbf{T}_{X,x}$ through p. This is possible only if $2d \leq n$.

We now introduce $\mathbf{Q}$-twisted vector bundles. This is a very useful formalism for questions of positivity and ampleness. For any vector bundle E on a variety X and rational class δ, introduce a symbol $E\langle\delta\rangle$ and declare that $E\langle\delta\rangle$ and $(E \otimes L)\langle\delta - [L]\rangle$ are equal for all line bundles L on X. Since $\mathbf{P}(E \otimes L) \simeq \mathbf{P}(E)$ under an isomorphism that carries $\mathscr{O}_{\mathbf{P}(E\otimes L)}(1)$ to $\mathscr{O}_{\mathbf{P}(E)}(1) \otimes \pi^* L$, it makes sense to say that $E\langle\delta\rangle$ is ample (or nef) if the rational class $[\mathscr{O}_{\mathbf{P}(E}(1)] + \pi^*\delta$ is ample on $\mathbf{P}(E)$. One defines tensor products

$$E_1\langle\delta_1\rangle \otimes E_2\langle\delta_2\rangle = (E_1 \otimes E_2)\langle\delta_1 + \delta_2\rangle$$

The beauty of these definitions is that most theorems (that still make sense) remain valid.

Theorem 4. *Let X be a proper variety, let E be a vector bundle on X, and let $\delta \in \mathrm{NS}(X)_{\mathbf{Q}}$.*

- *If $f : Y \to X$ is a finite and surjective morphism, $E\langle\delta\rangle$ is ample if and only if its pull-back $f^*E\langle f^*\delta\rangle$ is.*
- *The $\mathbf{Q}$-twisted vector bundle $E\langle\delta\rangle$ is ample if and only if, for some $m > 0$, the $\mathbf{Q}$-twisted vector bundle $\mathbf{S}^m E\langle m\delta\rangle$ is ample.*
- *If the $\mathbf{Q}$-twisted vector bundle $E\langle\delta\rangle$ is nef, $E\langle\delta + a\rangle$ is ample for any ample class $a \in \mathrm{NS}(X)_{\mathbf{Q}}$.*

This is not a futile generalization, and the proof of this theorem is not obvious. It becomes easier if one uses a construction of Bloch and Gieseker that says that there always exists a finite flat surjective morphism $f : Y \to X$ such that $f^*\delta$ is an integral class.

Here is an application of this formalism.

Proposition 1. *A vector bundle E on a projective variety X is ample if and only if, for any ample line bundle A on X, there exists $\varepsilon > 0$ such that, for every finite map $f : C \to X$ from a smooth projective curve and quotient line bundle $f^*E \twoheadrightarrow L$,*

$$\deg(L) \geq \varepsilon\, C \cdot f^*A$$

Proof. Giving a quotient line bundle $f^*E \twoheadrightarrow L$ is the same as giving a map $g : C \to \mathbf{P}(E)$ such that $\pi \circ g = f$ and $L = g^*\mathscr{O}_{\mathbf{P}(E)}(1)$. Since E is nef if and only if $\deg(g^*\mathscr{O}_{\mathbf{P}(E)}(1)) \geq 0$ for any such map g, it is clear that E is nef if and only if $\deg(L) \geq 0$ for all quotients line bundles L.

If the condition in the theorem is satisfied, the $\mathbf{Q}$-twisted vector bundle $E\langle-\varepsilon A\rangle$ is therefore nef, hence E is ample. The converse is done in the same way. □

2.2. **Varieties with ample cotangent bundle.** Let X be a smooth projective variety with ample cotangent bundle and let Y be a (possibly singular) subvariety of dimension d of X. Any desingularization $f : \tilde{Y} \to Y$ induces a generically surjective morphism $f^*\Omega_X^d \to \Omega_{\tilde{Y}}^d = \omega_{\tilde{Y}}$ hence, as above, a rational map $\rho : \tilde{Y} \dashrightarrow \mathbf{P}(f^*\Omega_X^d)$ that is birational onto its image, because f is. It follows that $\omega_{\tilde{Y}} = \rho^*\mathscr{O}_{\mathbf{P}(f^*\Omega_X^d)}(1)$ is big.

Any subvariety of X is therefore of general type. In particular, X contains no rational or elliptic curves. More precisely, if $f : C \to X$

is a nonconstant map from a smooth projective curve, it follows from Proposition 1 that there exists $\varepsilon > 0$ such that for *any* quotient line bundle $f^*\Omega_X \twoheadrightarrow L$, we have

$$\deg(L) \geq \varepsilon\, C \cdot K_X$$

In particular, taking for L the image of $f^*\Omega_X \to \Omega_C$, we obtain

$$2g(C) - 2 \geq \varepsilon\, C \cdot K_X$$

In particular, it is *algebraically hyperbolic* in the sense of Definition 3.

It is clear that smooth projective curves of genus at least 2 have ample cotangent bundle, but it is not so easy to construct higher-dimensional examples.

- The cotangent bundle of a nontrivial product $X \times Y$ is never ample, because it restricts to $X \times \{y\}$ as $\Omega_X \oplus \Omega_{Y,y}$. This vector bundle has a trivial quotient hence cannot be ample.
- It follows from Example 6.3) that if X is a subvariety of $\mathbf{P}^n$ of dimension $n/2$ (for example, a hypersurface in $\mathbf{P}^n$, with $n \geq 3$), $\Omega_X(1)$ cannot be ample. In particular, Ω_X cannot be even nef.

We will present a construction of varieties with ample cotangent bundle of any dimension in §4.1.

2.3. **Metric approach.** Assume now that X is a complex manifold of dimension n and let E be a vector bundle on X, equipped with a Hermitian metric h. These data determine a unique compatible Hermitian connexion with curvature

$$\Theta_h(E) \in \mathscr{C}^\infty(X, \bigwedge^{1,1} \Omega_X \otimes \mathrm{Hom}(E,E))$$

a $(1,1)$-form with values in $\mathrm{Hom}(E,E)$. If $z_1, \dots, z_n$ are local analytic coordinates on X and $(e_1, \dots, e_r)$ a local orthogonal frame on E, one can write

$$\Theta_h(E) = i \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}\, dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu$$

where $c_{kj\mu\lambda} = \bar{c}_{jk\lambda\mu}$ are $\mathscr{C}^\infty$ functions on X. To $\Theta_h(E)$ one associates the Hermitian form

$$\theta_h(E) = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}\, dz_j \otimes d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu$$

on $T_X \otimes E$. The pair (E,h) has *positive holomorphic bisectional curvature, or is Griffiths-positive,* if, for all $x \in X$ and all nonzero $s \in E_x$, the Hermitian form

$$\theta_h(E)(s \otimes \bar{s}) = \theta_h(E)(\cdot \otimes s, \cdot \otimes s)$$

is positive definite on $T_{X,x}$.

Our main interest will be the case where $E = \Omega_X$. We say in this case that X has (a metric with) *negative* holomorphic bisectional curvature (because the metric on the dual T_X has this property).

Examples 7. 1) On the unit ball Δ_n, the Bergman (Kähler) metric, given by

$$\frac{dz \otimes d\bar{z}}{(1 - |z_1|^2 - \cdots - |z_n|^2)^{n+1}}$$

where $|(z_1, \ldots, z_n)|^2 = |z_1|^2 + \cdots + |z_n|^2$, has negative holomorphic bisectional curvature (for $n = 1$, this is the Poincaré metric, defined in Example 2.1)). In particular, any bounded domain in $\mathbf{C}^n$ has a metric with negative holomorphic bisectional curvature.

It is invariant by biholomorphisms of Δ_n, hence endows any smooth quotient of Δ_n with a metric with negative constant curvature.

2) There is also a Bergman metric on the Siegel upper half-space

$$\mathscr{H}_g = \{M \in \mathscr{M}_{g\times g}(\mathbf{C}) \mid \operatorname{Im}(M) > 0\}$$

that generalizes the metric defined in Example 2.3). It is invariant by automorphisms and has negative holomorphic bisectional curvature, hence so does the metric induced on any smooth quotient of $\mathscr{H}_g$ such as $\mathscr{A}_g$, the moduli space of principally polarized abelian varieties of dimension g.

3) Any submanifold of a manifold with negative holomorphic bisectional curvature has the same property (curvature increases in quotients).

There is an important difference between Examples 2.2) or 3), and Example 7.1). In the latter, the holomorphic bisectional curvature is bounded away from zero: there is $\varepsilon > 0$ such that

$$\theta_h(T_X)(\xi \otimes \xi', \xi \otimes \xi') \le -\varepsilon \|\xi\|_h^2 \|\xi'\|_h^2 \tag{3}$$

for all $x \in X$ and all $\xi, \xi' \in T_{X,x}$. On a *compact* manifold, this is of course automatic once the holomorphic bisectional curvature is negative.

Proposition 2. *On a complex manifold, a vector bundle that has a metric with positive holomorphic bisectional curvature is ample.*

The converse (does an ample vector bundle always carry a metric with positive holomorphic bisectional curvature?) is unknown.

Sketch of proof. Let E be a vector bundle on the complex manifold X and let $\pi : \mathbf{P}(E) \to X$ be the associated projective bundle. A metric h on E with positive holomorphic bisectional curvature induces a metric π^*h on π^*E. As a $\mathscr{C}^\infty$ vector bundle, the latter splits as the direct sum of its quotient $\mathscr{O}_{\mathbf{P}(E)}(1)$ and the associated kernel. This defines by restriction a metric h' on $\mathscr{O}_{\mathbf{P}(E)}(1)$.

Take $y \in \mathbf{P}(E)$, set $x = \pi(y)$, and consider a nonzero $s \in \pi^*E_y = (\pi^*E)_x$. The Hermitian form $\theta_{\pi^*h}(\pi^*E)(s' \otimes \bar{s}')$ is positive on tangent vectors to $\mathbf{P}(E)$ at y that are not vertical, hence (curvature increases in quotients) so is the Hermitian form $\theta_{h'}(\mathscr{O}_{\mathbf{P}(E)}(1))(s' \otimes \bar{s}')$, where s' is the image of s in $\mathscr{O}_{\mathbf{P}(E)}(1)_y$. On the tangent space to the fiber $\pi^{-1}(x)$, the metric induced by h' is a multiple of the Fubini–Study metric, hence the Hermitian form $\theta_{h'}(\mathscr{O}_{\mathbf{P}(E)}(1))(s' \otimes \bar{s}')$ is also positive definite there. □

3. Analytic hyperbolicity

3.1. **Entire curves in complex manifolds.** An entire curve of a complex manifold (or more generally a complex space) X is a nonconstant holomorphic map $\mathbf{C} \to X$.

When X is an algebraic variety, we say that an entire curve is *algebraically degenerate* if its image is contained in a (closed) algebraic subvariety of X different from X.

Definition 2. *A complex manifold (or more generally a complex space) is* analytically hyperbolic *if it has no entire curves.*

Examples 8. 1) The varieties $\mathbf{C}$ and $\mathbf{C}^*$ are not analytically hyperbolic, but $\mathbf{C} - \{0, 1\}$ is. This is the Picard Theorem, which will follow from Example 2.3) and Corollary 1.

2) Any compact Riemann surface C of genus at least 2 is analytically hyperbolic, because any holomorphic map $\mathbf{C} \to C$ lifts to its universal cover Δ, and a bounded holomorphic function on $\mathbf{C}$ is constant. Since the product of two analytically hyperbolic manifolds is analytically hyperbolic, we obtain examples in any dimensions.

3) The complement of 4 lines in $\mathbf{P}^2$ is not analytically hyperbolic, because it contains a copy of $\mathbf{C}^*$ (why?). There are configurations of 5 lines in $\mathbf{P}^2$ whose complement is analytically hyperbolic (find one!).

4) There are smooth plane curves of arbitrary degree whose complement is not analytically hyperbolic. Find such curves! (*Hint:* there is a line that meets the Fermat curve $x_0^d + x_1^d + x_2^d = 0$ in exactly one point).

5) More generally, the complement in $\mathbf{P}^n$ of $2n$ hyperplanes is not analytically hyperbolic (why?) and there are configurations of $2n + 1$ hyperplanes in $\mathbf{P}^n$ whose complement is analytically hyperbolic (find

one!). It is known that the complement in $\mathbf{P}^n$ of $2n+1$ *general* hyperplanes is analytically hyperbolic ([G]).

6) The complement in $\mathbf{P}^n$ of a hypersurface of degree $2n$ that contains no lines is analytically hyperbolic (*Hint:* count parameters to show that there is a line that meets the hypersurface in at most 2 points; see [Z]).

3.2. **The Schwarz lemma.** We begin with a very classical lemma.

Lemma 1 (Schwarz's Lemma). *For any holomorphic map $f:\Delta\to\Delta$, one has*

$$\frac{|f'(z)|}{1-|f(z)|^2}\le\frac{1}{1-|z|^2}$$

for all $z\in\Delta$. If there is equality at one point, f is a biholomorphism.

Proof. Let us first do the case when $z=0$ and $f(0)=0$. Write $f(z)=\sum_{m=1}^{\infty}a_m z^m$. The Parseval formula yields

$$\sum_{m=1}^{\infty}|a_m|^2r^{2m}=\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2\,d\,\theta\le 1$$

for all $r\in(0,1)$. We obtain $\sum_{m=1}^{\infty}|a_m|^2\le 1$ hence in particular $|a_1|\le 1$, and if there is equality, $a_m=0$ for all $m\ge 2$ and $f(z)=a_1z$.

To do the general case, consider the automorphism $g_z:\Delta\to\Delta$ defined by

$$g_z(u)=\frac{u+z}{1+\bar{z}u} \tag{4}$$

One has $g_z(0)=z$ and $g_z^{-1}=g_{-z}$. Set $F=g_{-f(z)}\circ f\circ g_z$, so that $F(0)=0$. Writing $|F'(0)|\le 1$ yields the lemma. □

Endow Δ with the Poincaré metric (Example 2.1))

$$h_P=\frac{dz\otimes d\bar{z}}{(1-|z|^2)^2}$$

The Schwarz lemma can be restated as: *any holomorphic map $f:\Delta\to\Delta$ is distance-decreasing: $f^*h_P\le h_P$. There is equality at one point if and only if f is an automorphism, in which case f is an isometry.*

3.3. **The Ahlfors–Schwarz lemma.** We will need a generalization of the Schwarz lemma that deals with singular metrics, as introduced in 1.3.

Recall that the Poincaré metric has constant curvature $-2/\pi$ (Example 2.1)). More precisely, if we write $h_P=u_Pdz\otimes d\bar{z}$, we have $i\partial\bar{\partial}\log u_P=2\omega_P$.

Lemma 2 (Ahlfors–Schwarz lemma)**.** *Let $h(z) = u(z)dz \otimes d\bar{z}$ be a singular Hermitian metric on Δ such that $i\partial\bar{\partial}\log u \geq \varepsilon\omega_h$, in the sense of currents, for some positive constant c. Then $h \leq \frac{2}{\varepsilon}h_P$.*

Proof. Assume first that h is smooth. Replacing h by $h_r(z) = h(rz)$ (and letting $r \to 1$), we may assume that h extends to some bigger disk. Let $a : \Delta \to \mathbf{R}^+$ be the function defined by $h = ah_P$. It is continuous and goes to 0 on $\partial\Delta$, hence attains its maximum at some interior point z_0, so that

$$\begin{aligned} 0 &\geq i\partial\bar{\partial}\log a(z_0) \\ &= i\partial\bar{\partial}\log u(z_0) - i\partial\bar{\partial}\log u_P(z_0) \\ &\geq \varepsilon\omega_h(z_0) - 2\omega_P(z_0) \\ &= (\varepsilon a(z_0) - 2)\omega_P(z_0) \end{aligned}$$

The maximum of a is therefore at most $2/\varepsilon$, hence the lemma in this case. If h is not smooth, one uses a regularization argument (see [De1], p. 300, for details). □

3.4. **Ample cotangent bundle implies analytic hyperbolicity.** Let X be a complex manifold with a Hermitian metric h and let $f : \Delta \to X$ be a holomorphic map. Consider the metric f^*h on Δ (it is singular where $f' = 0$) and write $f^*h(z) = u(z)dz \otimes d\bar{z}$ as above. If the metric h satisfies (3), so does f^*h, so that $-i\partial\bar{\partial}\log u \leq -\varepsilon\omega_{f^*h}$. The Ahlfors–Schwarz lemma implies that *any holomorphic map $f : \Delta \to X$ satisfies*

$$f^*h \leq \frac{2}{\varepsilon}h_P \tag{5}$$

Corollary 1. *A complex manifold with a Hermitian metric that satisfies (3) (e.g., a compact complex manifold with negative bisectional holomorphic curvature) is analytically hyperbolic.*

Proof. Let h be a Hermitian metric on a complex manifold X and assume that (3) holds. Given an entire curve $f : \mathbf{C} \to X$, we set $f_R(z) = f(Rz)$. Since $f_R^*h = Rf^*h$, we obtain by (5)

$$Rf^*h \leq \frac{2}{\varepsilon}h_P$$

hence $f^*h = 0$ by letting $R \to \infty$. □

We can do better. We use the notation introduced in Example 5.3): if L is a line bundle on an algebraic variety or complex manifold X, the locus $\Sigma(L)$ is the smallest closed subset of X such that the sections of L^m, for $m \gg 0$, induce an embedding $X - \Sigma(L) \to \mathbf{P}^N$. Note that

$\Sigma(L) \neq X$ if and only if L is big, and $\Sigma(L) = \varnothing$ if and only if L is ample.

Proposition 3. *Let X be a compact complex manifold, let $\pi : \mathbf{P}(\Omega_X) \to X$ be the associated projective bundle, and let A be an ample line bundle on X. Let $f : \mathbf{C} \to X$ be an entire curve, with lift $\bar{f} : \mathbf{C} \to \mathbf{P}(\Omega_X)$. For any positive integer m, the image $\bar{f}(\mathbf{C})$ is contained in $\Sigma(\mathscr{O}_{\mathbf{P}(\Omega_X)}(m) \otimes \pi^* A^{-1})$.*

Corollary 2. *A compact complex manifold with ample cotangent bundle is analytically hyperbolic.*

Proof. If $\mathscr{O}_{\mathbf{P}(\Omega_X)}(1)$ is ample, $\mathscr{O}_{\mathbf{P}(\Omega_X)}(m) \otimes \pi^* A^{-1}$ is ample for $m \gg 0$, hence its Σ locus is empty. ☐

Proof of the proposition. Choose a smooth metric h_A on A with positive curvature. Following Example 5.3), we define a singular metric h on $L_m = \mathscr{O}_{\mathbf{P}(\Omega_X)}(m) \otimes \pi^* A^{-1}$ as the pth root of the pull-back of the Fubini–Study metric by the map

$$\psi_{L_m^p} : \mathbf{P}(\Omega_X) - \Sigma(L_m) \longrightarrow \mathbf{P}^N$$

for $p \gg 0$. It has positive curvature (in the sense of currents). Consider the metric $(h\pi^* h_A)^{-1/m}$ on $\mathscr{O}_{\mathbf{P}(\Omega_X)}(-1)$ and, if $\bar{f}(\mathbf{C})$ is not contained in $\Sigma(L_m)$, its pull-back $h_0 = \bar{f}^*(h\pi^* h_A)^{-1/m} = u\, dz \otimes d\bar{z}$ on Δ. Since h has positive curvature, we have

$$\frac{i}{\pi}\, \partial\bar{\partial} \log u = -\Theta_{\bar{f}^*(h\pi^* h_A)^{-1/m}} \geq f^* \Theta_{h_A^{1/m}}$$

Moreover, for any choice of a Hermitian metrix h_X on X, we have $f^*\Theta_{h_A^{1/m}}(z) \geq c\|f'(z)\|^2_{h_X} dz \otimes d\bar{z}$ for some positive constant c because h_A is a smooth metric with positive curvature. Since both $\|f'(z)\|^2_{h_X}$ and $u(z)$ depend quadratically[2] on $f'(z)$ and X is compact, there is another positive constant c' such that $\|f'(z)\|^2_{h_X} \geq c' u(z)$ for all z. We have therefore

$$\frac{i}{\pi}\, \partial\bar{\partial} \log u \geq cc' u(z)$$

and the Ahlfors–Schwarz lemma applies to give $h_0 \leq \frac{2}{cc'} h_P$. As in the proof of Corollary 1, we obtain a contradiction by considering the

[2]If $(s_0, \ldots, s_N)$ is a basis of $H^0(\mathbf{P}(\Omega_X), \mathscr{O}_{\mathbf{P}(\Omega_X)}(m) \otimes \pi^* A^{-1})$,

$$u(z) = \sum_{j=0}^{N} \|s_j(f(z)) \cdot f'(z)^m\|^{2/m}_{h_A^{-1}}$$

maps $z \mapsto f(Rz)$ and letting $R \to \infty$. Therefore, $\bar{f}(\mathbf{C})$ is contained in $\Sigma(L_m)$. □

3.5. **Brody's Lemma.** The following reparametrization result is very useful ([Br]).

Lemma 3 (Brody's Lemma)**.** *Let X be a complex manifold with a Hermitian metric h and let $f : \Delta \to X$ be a holomorphic map. For every $r \in (0,1)$, there exist $R \geq r\|f'(0)\|_h$ and a biholomorphism*

$$\psi : D(0,R) \longrightarrow D(0,r)$$

such that

$$\|(f\circ\psi)'(0)\|_h = 1 \quad , \quad \|(f\circ\psi)'(t)\|_h \leq \frac{1}{1-|t/R|^2}$$

for all $t \in D(0,R)$.

Proof. Pick $z_0 \in \Delta$ such that $(1-|z|^2)\|f'(rz)\|_h$ reaches its maximum at z_0 (i.e., the norm of the derivative of $z \mapsto f(rz)$ with respect to the Poincaré metric and h is maximal). Set $\psi(t) = rg_{z_0}(t/R)$ (the function g_{z_0} is defined in (4)), so that $\psi(0) = rz_0$. We have

$$\|(f\circ\psi)'(0)\|_h = |\psi'(0)|\|f'(rz_0)\|_h = (1-|z_0|^2)\,\frac{r}{R}\,\|f'(rz_0)\|_h$$

so we must take

$$R = r(1-|z_0|^2)\|f'(rz_0)\|_h \geq r\|f'(0)\|_h$$

Finally, the norm of the derivative of $f\circ\psi$ at a point t with respect to the Poincaré metric and h is

$$(1-|t/R|^2)\|(f\circ\psi)'(t)\|_h$$

Since g_{z_0} is an isometry for the Poincaré metric, this quantity is, up to a constant, the norm of the derivative of $z \mapsto f(rz)$ at the point $g_{z_0}(t/R)$ with respect to the same metrics. It is therefore maximal for $t = 0$. □

In particular, any sequence of holomorphic maps $f_m : \Delta \to X$ can be reparametrized to yield a sequence of holomorphic maps

$$g_m : D(0,R_m) \to X$$

with

$$R_m \geq \frac{1}{2}\|f_m'(0)\|_h$$

and

$$\|g_m'(z)\|_h \leq \frac{1}{1-|z/R_m|^2} \quad , \quad \|g_m'(0)\|_h = 1$$

for all m and $z \in D(0, R_m)$. If X is compact and the sequence $(\|f'_m(0)\|_h)_m$ is unbounded, passing to a subsequence, we obtain[3] an entire curve $g : \mathbf{C} \to X$ that satisfies

$$\|g'(z)\|_h \leq \|g'(0)\|_h = 1$$

for all $z \in \mathbf{C}$. In particular, g is nonconstant.

Corollary 3. *Let X be a* compact *complex manifold endowed with a Hermitian metric h. The following properties are equivalent:*

(i) *X is analytically hyperbolic;*

(ii) *there exist a constant c such that*

$$\|f'(0)\|_h \leq c$$

for all holomorphic maps $f : \Delta \to X$;

(iii) *there exist a constant c such that*

$$f^*h \leq c^2 h_P$$

for all holomorphic maps $f : \Delta \to X$.

Proof. If X is analytically hyperbolic, there is no map g as above, hence no sequence of holomorphic maps $f_m : \Delta \to X$ such that the sequence $(\|f'_m(0)\|_h)_m$ is unbounded.

Conversely, if X satisfies this property and $f : \Delta \to X$ is holomorphic, by composing f with the automorphism g_z, we obtain

$$\|f'(z)\|_h \leq \frac{c}{1 - |z|^2}$$

for all $z \in \Delta$, which is the same as saying $f^*h \leq c^2 h_P$. As we saw in the proof of Corollary 1, this inequality, applied to the maps $f_R : \Delta \to X$ defined by $z \mapsto f(Rz)$, implies $f^*h = 0$, hence f is constant. □

We now show that analytic hyperbolicity is open, which will be very useful for the construction of examples (often, degenerate reducible examples are easier to construct).

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[3]We may use Ascoli's theorem, or (re)prove it: consider a sequence $(z_k)_k$ of points that is dense in $\mathbf{C}$. Since X is compact, we may find increasing maps $\varphi_k : \mathbf{N} \to \mathbf{N}$ such that the sequence $(g_{\varphi_1 \circ \cdots \circ \varphi_k(m)}(z_k))_m$ converges for each k. Setting $\varphi(k) = \varphi_1 \circ \cdots \circ \varphi_k(k)$, the sequence $(g_{\varphi(m)}(z_k))_m$ converges for each k. Call $g(z_k)$ its limit. Because of the uniform bound on the derivatives of the g_m on any compact, the function g is uniformly continuous on compacts and extands to a continuous function $g : \mathbf{C} \to X$ towards which the sequence $(g_{\varphi(m)})_m$ converges, uniformly on compacts. It is therefore holomorphic.

Proposition 4 (Analytic hyperbolicity is open)**.** *Let $X \to S$ be a proper holomorphic map between complex manifolds. The set of $s \in S$ such that the fiber X_s is analytically hyperbolic is open in S for the Euclidean topology.*

Note that we do not assume anything on the singularities, or the dimensions, of the fibers.

Proof. Fix a Hermitian metric on X and consider a sequence $f_m : \mathbf{C} \to X_{s_m}$ of entire curves, where (s_m) is a sequence a points of S that converges to $s \in S$. By Brody's lemma, we may assume $\|f'_m\|_h \leq \|f'_m(0)\|_h = 1$ for all m. By Ascoli's theorem (see note 3), there is a subsequence of (f_m) that converges uniformly to a holomorphic map $f : \mathbf{C} \to X$. Its image is necessarily contained in X_s and f is not constant since $\|f'(0)\|_h = 1$. □

3.6. **Algebraic hyperbolicity.** This property, defined purely in terms of algebraic curves on say a smooth projective variety, is conjectured to be equivalent to analytic hyperbolicity.

Definition 3. *A compact complex manifold X with a metric h is* algebraically hyperbolic *if there exists $\varepsilon > 0$ such that, for every finite morphism $f : C \to X$ from a smooth projective curve,*

$$2g(C) - 2 \geq \varepsilon \int_C \omega_{f^*h}$$

This property is independent of the choice of the metric h. Using the Riemann–Hurwitz formula, one sees that it is enough to check it for morphisms f that are birational onto their image. On a projective variety (not necessarily smooth), one may replace the right-hand side with $C \cdot f^*A$, where A is an ample line bundle on X.

Proposition 5. *An analytically hyperbolic compact complex manifold is algebraically hyperbolic.*

Proof. Let X be an analytically hyperbolic compact complex manifold, endow it with a Hermitian metric h, and let $f : C \to X$ be a nonconstant map from a smooth projective curve. Since X is analytically hyperbolic, the curve C has genus at least 2 hence is covered by the unit disk by $\rho : \Delta \to C$. It inherits from the Poincaré metric h_P a metric h_C with constant negative curvature $-2/\pi$ (Example 2.1)).

It follows from Corollary 3 that there is a constant c such that $(f \circ \rho)^* h \leq c^2 h_P$, hence $f^*h \leq c^2 h_C$. The Gauss–Bonnet formula yields

$$2g(C) - 2 = -\int_C \Theta_{h_C} = \frac{2}{\pi}\int_C \omega_{h_C} \geq \frac{2}{\pi c^2}\int_C f^*\omega_h$$

This proves the proposition. □

Proposition 6. *Let X be an algebraically hyperbolic projective manifold and let V be a complex torus. Any holomorphic map $f : V \to X$ is constant.*

Proof. Any holomorphic map from a complex torus to an algebraic variety factors through an abelian variety, so we may assume that V is an abelian variety. Let C be a smooth curve in V. For any positive integer m, consider the composition

$$f_m : C \subset V \xrightarrow{m_V} V \xrightarrow{f} X$$

where m_V is the multiplication by m in V. Let A be an ample line bundle on X. By definition, there is a positive constant ε such that for all m,

$$2g(C) - 2 \geq \varepsilon\, C \cdot f_m^* A = \varepsilon m^2\, C \cdot f^* A$$

By letting m go to infinity, we obtain $C \cdot f^* A = 0$. This shows that f is constant on all curves in V, hence is constant on V. This proves the proposition. □

4. Construction of varieties with ample cotangent bundle

4.1. Bogomolov's construction.

Proposition 7 (Bogomolov)**.** *Let $X_1, \dots, X_m$ be smooth projective varieties with big cotangent bundle, all of dimension at least $d > 0$. Let V be a general linear section of $X_1 \times \cdots \times X_m$. If $\dim(V) \leq \frac{d(m+1)+1}{2(d+1)}$, the cotangent bundle of V is ample.*

Proof. Since Ω_{X_i} is big, there exist a proper closed subset B_i of $\mathbf{P}(\Omega_{X_i})$ and an integer q such that for each i, the sections of $\mathscr{O}_{\mathbf{P}(\Omega_{X_i})}(q)$, i.e., the sections of $\mathbf{S}^q\, \Omega_{X_i}$, define an *injective* morphism

$$f_i : \mathbf{P}(\Omega_{X_i}) \smallsetminus B_i \longrightarrow \mathbf{P}^{n_i}$$

Lemma 4. *Let X be a smooth subvariety of a projective space and let B be a subvariety of $\mathbf{P}(\Omega_X)$. A general linear section V of X of dimension at most $\frac{1}{2}\operatorname{codim}(B)$ satisfies*

$$\mathbf{P}(\Omega_V) \cap B = \varnothing$$

Proof. Let $\mathbf{P}^n$ be the ambiant projective space. Consider the variety

$$\{((t,x),\Lambda) \in B \times G(n-c, \mathbf{P}^n) \mid x \in X \cap \Lambda,\ t \in T_{X,x} \cap T_{\Lambda,x})$$

The fibers of its projection to B have codimension $2c$, hence it does not dominate $G(n-c, \mathbf{P}^n)$ as soon as $2c > \dim(B)$. This is equivalent to $2(\dim(X)-\dim(V))-1 \geq 2\dim(X)-1-\operatorname{codim}(B)$ and the lemma is proved. $\square$

Let B'_i be the (conical) inverse image of B_i in the total space of the tangent bundle of X_i. Let V be a general linear section of $X_1\times\cdots\times X_m$ and set $a = m+1-2\dim(V)$.

The image B of the product $B'_1 \times \cdots \times B'_m$ in $\mathbf{P}(\Omega_{X_1\times\cdots\times X_m})$ has codimension $\geq m$. Assume $a > 0$. By the lemma, any nonzero tangent vector $\xi = (\xi_1, \dots, \xi_m)$ to V, with $\xi_i \in T_{X_i,x_i}$, is not in $B'_1 \times \cdots \times B'_m$ hence at least one of the ξ_i is not in B'_i. The lemma actually says more: there are at least a values of the index i for which $\xi_i \notin B'_i$. If, say, ξ_1 is not in B'_1, there exists a section of $\mathbf{S}^q\,\Omega_{X_1}$ that does not vanish at ξ_1. This section induces, via the projection $V \to X_1$, a section of $\mathbf{S}^q\,\Omega_V$ that does not vanish at ξ. It follows that when a is positive, $\mathscr{O}_{\mathbf{P}(\Omega_V)}(q)$ is base-point-free and its sections define a morphism $f : \mathbf{P}(\Omega_V) \longrightarrow \mathbf{P}^n$.

We need to show that f is *finite*. Assume to the contrary that a curve C in $\mathbf{P}(\Omega_V)$ through t is contracted. Since the restriction of the projection $\pi : \mathbf{P}(\Omega_V) \to V$ to any fiber of f is injective, and since f_i is injective, the argument above proves that the curve $\pi(C)$ is contracted by each projection $p_i : V \to X_i$ such that $t_i \notin B'_i$.

The following lemma leads to a contradiction when $2\dim(V) \leq ad + 1$. This proves the proposition. $\square$

Lemma 5. *Let V be a general linear section of a product $X \times Y$ in a projective space. If $2\dim(V) \leq \dim(X) + 1$, the projection $V \to X$ is finite.*

Proof. Let $\mathbf{P}^n$ be the ambiant projective space. Consider the variety that is the closure of

$$\{(x, y, y', \Lambda) \in X\times Y\times Y\times G(n-c, \mathbf{P}^n) \mid y \neq y',\ (x, y) \in \Lambda,\ (x, y') \in \Lambda)\}$$

The fibers of its projection to $X \times Y \times Y$ have codimension $2c$, hence the general fiber of its projection to $G(n-c, \mathbf{P}^n)$, which is the closure of

$$\overline{\{((x, y), (x, y')) \in V \times V \mid y \neq y',\ (x, y) \in V,\ (x, y') \in V)\}}$$

has dimension at most 1 as soon as $2c \geq \dim(X \times Y \times Y) - 1$. This implies that the projection $V \to X$ is finite and is equivalent to $2\dim(V) \leq \dim(X) + 1$: the lemma is proved. $\square$

Using this construction, the Lefschetz theorem, and the fact that smooth projective surfaces of general type with $c_1^2 > c_2$ have big cotangent bundle (§7.2), it is not hard to prove the following.

Corollary 4. *Given any smooth projective variety X, there exists a smooth projective surface with ample cotangent bundle and same fundamental group as X.*

4.2. **Subvarieties of abelian varieties.** Let X be a smooth subvariety of an abelian variety A. By translation, we identify the tangent space $T_{A,x}$ at a point x of A with the tangent space $T_{A,0}$ at the origin. The natural surjection $(\Omega_A)|_X \to \Omega_X$ shows that the cotangent bundle of X is globally generated. As we saw in §2.1, it is ample if and only if the map

$$f: \begin{array}{ccc} \mathbf{P}(\Omega_X) & \longrightarrow & \mathbf{P}(\Omega_{A,0}) \\ (x,\xi) & \longmapsto & \xi \end{array}$$

is finite. The following result is taken from [D] ; unfortunately, the proof is incomplete (Lemma 12 is wrong) and I state it as a conjecture.

Conjecture 1. *Let L be a very ample line bundle on a simple abelian variety A of dimension n. Consider general divisors $H_1, \dots, H_c \in |L^m|$. If $m > n$ and $c \geq n/2$, the cotangent bundle of $H_1 \cap \cdots \cap H_c$ is ample.*

An abelian variety is *simple* if it contains no abelian subvarieties other than itself and 0.

One way to check the conjecture would be to prove that the fibers of the map f above have dimension at most $e = n - 2c$ for $c \leq n/2$. This means that for H_i general in $|L^m|$ and *any* nonzero constant vector field ∂ on A, the dimension of the set of points x in $X = H_1 \cap \cdots \cap H_c$ such that $\partial(x) \in T_{X,x}$ is at most e; in other words, that

$$\dim(H_1 \cap \partial H_1 \cap \cdots \cap H_c \cap \partial H_c) \leq e$$

A related result is proved in [DI]: *let (A, Θ) be a* general *principally polarized abelian fourfold.*[4] *For $a \in A$ general, the smooth surface $\Theta \cap \Theta_a$ has ample cotangent bundle.*

4.3. **Ball quotients.** Let G be a discontinuous group of biholomorphisms of the ball Δ_2 acting without fixed points and with compact quotient $X = \Delta_2/G$. According to [H3], the Chern numbers of X are proportional to those of the compact symmetric space dual to Δ_2, i.e., $\mathbf{P}^2$. This construction therefore yields (the actual construction is not easy, see [H1] and [H2]) surfaces of general type with $c_1^2 = 3c_2$.

According to [M], Proposition 5, for any surface X of general type with $c_1^2 = 3c_2$, the canonical bundle K_X is ample and the vector bundle $\Omega_X\langle -\frac{1}{3}K_X\rangle$ is nef, so that Ω_X is ample. Moreover, X is covered by the unit ball ([Y]).

[4]This means that Θ is an ample divisor on A such that $h^0(A,\Theta) = 1$. We write $\Theta_A = \Theta + a$.

5. Construction of analytically hyperbolic hypersurfaces

5.1. **An analytically hyperbolic octic surface in $\mathbf{P}^3$.** This is an example of Duval and Fujimoto. Consider the surface $X \subset \mathbf{P}^3$ with equation

$$P(z_0, z_1, z_2)^2 - Q(z_2, z_3) = 0$$

where P and Q are general polynomials of respective degrees $d \geq 4$ and $2d$. It is smooth outside the finite set $\Sigma = \{(z_0, z_1, 0, 0) \in \mathbf{P}^3 \mid P(z_0, z_1, 0) = 0\}$, which is also the indeterminacy set of the projection

$$\begin{array}{ccc} X & \dashrightarrow & \mathbf{P}^1 \\ (z_0, z_1, z_2, z_3) & \longmapsto & (z_2, z_3) \end{array}$$

If $\tilde{X} \to X$ is a minimal desingularization, we obtain a holomorphic map $\tilde{X} \to \mathbf{P}^1$ which factorizes as $\tilde{X} \xrightarrow{u} C \xrightarrow{p} \mathbf{P}^1$, where C is the hyperelliptic curve with inhomogeneous equation $t^2 = Q(1, z_3)$, the map u is given in inhomogeneous coordinates by

$$u(z_0, z_1, 1, z_3) = (P(z_0, z_1, 1), z_3)$$

and p is the double cover $(t, z_3) \mapsto z_3$. Any entire curve $f : \mathbf{C} \to X$ lifts to $\tilde{f} : \mathbf{C} \to \tilde{X}$. Since C has genus $d - 1 \geq 3$, the image of $\tilde{f}$ is contained in a fiber $u^{-1}(t, z_3)$, which is isomorphic to the plane curve with inhomogeneous equation $P(z_0, z_1, 1) = t$. This is a plane curve of degree d with at most one singular point, which is a node; it has therefore genus ≥ 2, hence $\tilde{f}$ is constant. The degree $2d$ surface X is therefore analytically hyperbolic, and so is any small deformation of X (Proposition 4). So we obtain examples of *smooth* analytically hyperbolic surfaces in $\mathbf{P}^3$ of any even degree ≥ 8.

A more intricate construction of a *sextic* surface in $\mathbf{P}^3$ can be found in [Du1]. Other examples appear in [SZ1] and [CZ]. It is also known that very general surfaces of degree ≥ 21 in $\mathbf{P}^3$ are analytically hyperbolic ([DEG], [Mc]).

5.2. **Analytically hyperbolic hypersurfaces in $\mathbf{P}^n$.** There are examples of analytically hyperbolic hypersurfaces of degree $4(n-1)^2$ in $\mathbf{P}^n$ in [SZ2].

6. Algebraic hyperbolicity of hypersurfaces in the projective space

Let X be a hypersurface of degree d in $\mathbf{P}^n$. When $d \leq 2n-3$, it is easy to show that X contains lines, hence cannot be algebraic hyperbolic. When $n = 3$, quartic surfaces always contain rational curves. On the other hand, there exist smooth hypersurfaces of any degree that contain lines.

The aim of this section is to prove the following theorem, due to Pacienza ([P]).

Theorem 5. *All subvarieties of a very general hypersurface of degree* $\geq 2n$ *in* $\mathbf{P}^n$ *are of general type.*

Here "very general" means outside of a countable union of proper subvarieties of the parameter space $H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(d))$ for hypersurfaces of degree d in $\mathbf{P}^n$.

The case $n = 3$ of this result is crucial in the proof given by in [DEG] and [Mc] of the Kobayashi conjecture for very general surfaces of high degree in $\mathbf{P}^3$. A slightly different version of this theorem was first proved in [C1], whose result was later generalized in [E]. The proof presented here follows [V]. It is closer in its spirit to the analytic approach to the Kobayashi conjecture for surfaces in $\mathbf{P}^3$, and it allows further (algebraic) improvements (see [C2]). We actually prove the following stronger result.

Theorem 6. *Let* X *be a very general hypersurface of degree* $\geq 2n$ *in* $\mathbf{P}^n$ *and let* Y *be a subvariety of* X*, with desingularization* $\nu : \tilde{Y} \to Y$*. We have*

$$H^0(\tilde{Y}, \omega_{\tilde{Y}} \otimes \nu^* \mathscr{O}_{\mathbf{P}^n}(-1)) \neq 0$$

Corollary 5. *A a very general hypersurface of degree* $\geq 2n$ *in* $\mathbf{P}^n$ *is algebraically hyperbolic.*

Indeed, if $f : C \to X$ is a morphism from a smooth projective curve to such a hypersurface X that is birational onto its image, we have

$$0 \leq \deg(\omega_C \otimes f^* \mathscr{O}_{\mathbf{P}^n}(-1)) = 2g(C) - 2 - \deg(f(C))$$

Proof of the theorem. We fix the following notation:

$$\begin{aligned} S^d &= H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(d)) \simeq \mathbf{C}^N; \\ \mathscr{X} &\subset \mathbf{P}^n \times S^d \quad \text{the universal hypersurface of degree } d; \\ X_F &\subset \mathbf{P}^n \quad \text{the hypersurface defined by } F \in S^d. \end{aligned}$$

Let $\mathscr{Y} \subset \mathscr{X}$ be a subvariety such that the projection map $\mathscr{Y} \to S^d$ is dominant of relative dimension k (to be more precise we should allow an étale base change $U \to S^d$, but we will omit this point). Let $\tilde{\mathscr{Y}} \to \mathscr{Y}$ be a desingularization. Both $\tilde{\mathscr{Y}}$ and $\mathscr{X}$ are smooth over a dense open subset U of S^d. We will show that for $F \in U$ general,

$$H^0(\tilde{Y}_F, \omega_{\tilde{Y}_F}(-1)) \neq 0 \qquad (6)$$

The idea is to produce holomorphic forms on the tangent space to the family $\mathscr{X}$ along a fibre X_F. Then, by pulling them back to $\tilde{Y}_F$ and using the adjunction formula, we will obtain a nonzero element of $H^0(\tilde{Y}_F, \omega_{\tilde{Y}_F}(-1))$. To make this more precise, first recall the following two elementary facts:

a) $\omega_{\tilde{Y}_F} \simeq \omega_{\tilde{\mathscr{Y}}}|_{\tilde{Y}_F} = \Omega^{N+k}_{\tilde{\mathscr{Y}}}|_{\tilde{Y}_F}$;
b) $\Omega^{N+k}_{\mathscr{X}}|_{X_F} \simeq (\bigwedge^{n-1-k} T_{\mathscr{X}}|_{X_F}) \otimes \omega_{X_F}$.

The first fact is just the adjunction formula, since the normal bundle of a fiber in a family is trivial. The second one is again the adjunction formula, and standard linear algebra. So, using a), we must show that $\Omega^{N+k}_{\tilde{\mathscr{Y}}}|_{\tilde{Y}_F}(-1)$ has a nonzero section. The generically surjective restriction $\Omega_{\mathscr{X}} \to \Omega_{\tilde{\mathscr{Y}}}$ induces a map

$$\Omega^{N+k}_{\mathscr{X}}|_{X_F}(-1) \to \Omega^{N+k}_{\tilde{\mathscr{Y}}}|_{\tilde{Y}_F}(-1)$$

that is nonzero for F general in U. Using b), it is enough to show that

$$\Big(\bigwedge^{n-1-k} T_{\mathscr{X}}|_{X_F} \Big) \otimes \omega_{X_F}(-1)$$

is globally generated. Since

$$\omega_{X_F} = \mathscr{O}_{X_F}(d-n-1) = \mathscr{O}_{X_F}((n-k-1)+(d-2n+k))$$

we have

$$\Big(\bigwedge^{n-1-k} T_{\mathscr{X}} \Big) \otimes \omega_{X_F}(-1) = \Big(\bigwedge^{n-1-k} T_{\mathscr{X}}(1) \Big) \otimes \mathscr{O}_{X_F}(d-2n-1+k)$$

Since we are assuming $d \geq 2n \geq 2n+1-k$, condition (6) is implied by the following result.

Proposition 8. *For all smooth hypersurfaces $X_F \subset \mathbf{P}^n$, the sheaf $T_{\mathscr{X}}|_{X_F}(1)$ is generated by its global sections.*

Before giving a proof of the proposition let us see how this implies the theorem: we have shown that (6) holds for the general fibre of the family $\mathscr{Y}$. Letting the family $\mathscr{Y}$ vary, that is, varying the Hilbert polynomial, we obtain that the same statement holds for all subvarieties of X_F for F outside a *countable* union of proper closed subvarieties of U. □

Proof of the proposition. For any $x \in X_F$, denote by $\mathscr{I}_x$ the ideal sheaf of x. We want to prove that the evaluation map

$$H^0(X_F, T_{\mathscr{X}}|_{X_F}(1)) \to T_{\mathscr{X},x}(1)$$

is surjective, that is, equivalently, that $H^0(X_F, T_{\mathscr{X}}|_{X_F}(1) \otimes \mathscr{I}_x)$ has codimension $\dim \mathscr{X}$ inside $H^0(X_F, T_{\mathscr{X}}|_{X_F}(1))$. To do so, we will describe these cohomology groups as kernels of appropriate multiplication maps of polynomials. First note

$$H^0(X_F, T_{X_F}(1)) = 0 \tag{7}$$

Indeed, we have

$$T_{X_F}(1) \simeq \Omega_{X_F}(1) \otimes \omega_{X_F}^{-1} \hookrightarrow \Omega_{X_F}$$

(the inclusion follows from the fact that $\omega_{X_F}(-1)$ has a nonzero section, as insured by the hypothesis on d), and $h^0(X_F, \Omega_{X_F}) = h^1(X_F, \mathscr{O}_{X_F}) = 0$.

The tangent exact sequences relative to the inclusions $X_F \subset \mathscr{X}$ and $X_F \subset \mathbf{P}^n$ yield a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{X_F}(1) & \longrightarrow & T_{\mathscr{X}}|_{X_F}(1) & \longrightarrow & S^d \otimes \mathscr{O}_{X_F}(1) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \mathrm{pr}_{1*}} & & \downarrow{\scriptstyle \mathrm{ev}} & & \\ 0 & \longrightarrow & T_{X_F}(1) & \longrightarrow & T_{\mathbf{P}^n}|_{X_F}(1) & \xrightarrow{\alpha} & \mathscr{O}_{X_F}(d+1) & \longrightarrow & 0 \end{array}$$

With (7), we obtain

$$H^0(X_F, T_{\mathscr{X}}|_{X_F}(1)) = \operatorname{Ker}\big(S^d \otimes S^1 \xrightarrow{\mu} H^1(X_F, T_{X_F}(1))\big)$$

where μ is the coboundary map in the long cohomology exact sequence associated to the first line of the diagram. Since $H^0(\mathrm{ev})$ is surjective in that same diagram, the map μ takes values in

$$\operatorname{Ker}(H^1(X_F, T_{X_F}(1)) \to H^1(X_F, T_{\mathbf{P}^n}(1)|_{X_F})$$

which is isomorphic to $H^0(X_F, \mathscr{O}_{X_F}(d+1))/\operatorname{Im} H^0(\alpha)$. All in all, we get

$$H^0(X_F, T_{\mathscr{X}}|_{X_F}(1)) = \operatorname{Ker}\big(S^d \otimes S^1 \xrightarrow{\mu} H^0(X_F, \mathscr{O}_{X_F}(d+1))/\operatorname{Im} H^0(\alpha)\big)$$

Twisting all the exact sequences above by the ideal sheaf $\mathscr{I}_x$ and using exactly the same arguments, we also get

$$H^0(X_F, T_{\mathscr{X}}|_{X_F}(1) \otimes \mathscr{I}_x) = \\ \operatorname{Ker}(S^d \otimes S^1_x \xrightarrow{\mu_x} H^0(X_F, \mathscr{O}_{X_F}(d+1) \otimes \mathscr{I}_x)/\operatorname{Im} H^0(\alpha_x))$$

where $S^1_x = H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(1)) \otimes \mathscr{I}_x) \subset S^1$ is the set of hyperplanes that contain x. Now it follows from the Euler exact sequence that $T_{\mathbf{P}^n}(1)|_{X_F}$ is globally generated, thus we have an inclusion

$$H^1(X_F, T_{\mathbf{P}^n}(1)|_{X_F} \otimes \mathscr{I}_x) \hookrightarrow H^1(X_F, T_{\mathbf{P}^n}(1)|_{X_F})$$

Using this and, again, the vanishing (7), we get the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T_{X_F,x} & \to & \operatorname{Im}\mu_x & \to & \operatorname{Im}\mu & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & T_{X_F,x} & \to & H^1(X_F, T_{X_F}(1)\otimes \mathscr{I}_x) & \to & H^1(X_F, T_{X_F}(1)) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
 & & 0 & \to & H^1(X_F, T_{\mathbf{P}^n}(1)|_{X_F}\otimes \mathscr{I}_x) & \to & H^1(X_F, T_{\mathbf{P}^n}(1)|_{X_F}) & &
\end{array}$$

from which we obtain $\operatorname{rank}(\mu_x) = \operatorname{rank}(\mu) + \dim X_F$. We obtain

$$\begin{aligned}
\dim \operatorname{Ker}(\mu_x) &= \dim(S^d \otimes S^1_x) - \operatorname{rank}\mu_x \\
&= \dim(S^d \otimes S^1) - \dim S^d - \operatorname{rank}(\mu) - \dim X_F \\
&= \dim \operatorname{Ker}(\mu) - \dim S^d - \dim X_F \\
&= \dim \operatorname{Ker}(\mu) - \dim \mathscr{X}
\end{aligned}$$

as we wanted to prove. □

7. Algebraic and entire curves on surfaces of general type

7.1. **Foliations on algebraic surfaces.** We gather here some results on foliations that we will need in the next subsection.

Definition 4. *A* foliation *on a smooth complex surface X is a rank 1 coherent subsheaf $\mathscr{F}$ of T_X such that $T_X/\mathscr{F}$ is torsion-free.*[5]

Given a rank 1 coherent subsheaf $\mathscr{F}$ of T_X, the sheaf $\mathscr{F}^{**}$ defines a foliation. In particular, any rank 1 subbundle of the restriction of T_X to an open dense subset of X defines a foliation.

Let $\mathscr{F}$ be a foliation on the smooth complex surface X. There is an exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow T_X \longrightarrow N_{\mathscr{F}} \otimes \mathscr{I}_{\operatorname{Sing}\mathscr{F}} \longrightarrow 0$$

[5]On a higher-dimensional complex manifold, one can make a similar definition. The foliation is then called *involutive*, or *integrable,* if $\mathscr{F}$ is closed under Lie bracket (this is automatic for surfaces since $\mathscr{F}$ has rank 1). This is the condition needed to ensure that at least on the locus where $\mathscr{F}$ is a subbundle of T_X, it is locally analytically integrable (Liouville's theorem).

where $\operatorname{Sing}\mathscr{F}$ is a discrete analytic subset of X called the *singular set* of $\mathscr{F}$. The invertible sheaf $N_{\mathscr{F}}$ is called the *normal bundle* to $\mathscr{F}$. The dual exact sequence is

$$0 \longrightarrow N_{\mathscr{F}}^* \longrightarrow \Omega_X \longrightarrow \mathscr{F}^* \otimes \mathscr{I}_{\operatorname{Sing}\mathscr{F}} \longrightarrow 0$$

When X is algebraic, any line bundle on X has nonzero meromorphic sections, and one can view a foliation as given by a meromorphic vector field, or a meromorphic 1-form $\omega_{\mathscr{F}}$, with $N_{\mathscr{F}}^* = \mathscr{O}_X(\operatorname{div}(\omega_{\mathscr{F}}))$. Locally around a point of X that is not singular for $\mathscr{F}$, we can write in local coordinates $\omega_{\mathscr{F}}(z_1, z_2) = f(z_1, z_2)dz_1$, where f is a local meromorphic function.

An *integral curve* of $\mathscr{F}$ is a Riemann surface C with a holomorphic map $\iota : C \to X$ such that the image of the tangent map $T_C \to \iota^* T_X$ lies in $\mathscr{F}$. Alternatively $\iota^*\omega_{\mathscr{F}} = 0$. There is a local analytic integral curve through any nonsingular point of $\mathscr{F}$ (defined in the notation above by $z_1 = 0$).[6]

Examples 9. 1) The meromorphic 1-form $\omega = \frac{dz_1}{z_1} + i\,\frac{dz_2}{z_2}$ on the surface $X = \mathbf{P}^1 \times \mathbf{P}^1$ defines a foliation

$$0 \longrightarrow \mathscr{O}_X(-D) \longrightarrow \Omega_X \longrightarrow \mathscr{F}^* \otimes \mathscr{I}_{\operatorname{Sing}\mathscr{F}} \longrightarrow 0$$

with

$$D = \{0, \infty\} \times \mathbf{P}^1 + \mathbf{P}^1 \times \{0, \infty\}$$

and

$$\operatorname{Sing}\mathscr{F} = \{0, \infty\} \times \{0, \infty\}$$

The leaves are $z \mapsto (e^z, \lambda e^{iz})$. Only 4 of them are algebraic: they correspond to the 4 components of D.

2) A rational map $X \dashrightarrow C$ onto a curve defines the foliation whose conormal bundle is the image of $\Omega_C \to \Omega_X$ (equivalently, $\mathscr{F}$ is the relative tangent sheaf). The leaves are the (connected components of the) fibers. A foliation $\mathscr{F}$ is of this type if and only if there exists a nonconstant meromorphic function f such that $\omega_{\mathscr{F}} \wedge df = 0$; it is then the relative tangent sheaf to the rational map $X \dashrightarrow \mathbf{P}^1$ induced by f.

Nonalgebraic leaves do not occur when the foliation is "positive".

Theorem 7 (Miyaoka)**.** *Let X be a smooth projective surface and let $\mathscr{F} \subset T_X$ be a foliation on X. If there exists an ample line bundle A on X such that $\mathscr{F} \cdot A > 0$, the leaves of $\mathscr{F}$ are algebraic.*

We now prove a general structure theorem for foliations on smooth complex projective surfaces.

---

[6] Many more details on foliations can be found in the excellent book [Bru].

Theorem 8 (Jouanolou). *Let X be a smooth complex projective surface with a foliation $\mathscr{F}$. Either $\mathscr{F}$ is the relative tangent sheaf of a rational map from X to a curve, or $\mathscr{F}$ has only finitely many algebraic integral curves.*

Proof. Let $\mathscr{M}_X$ be the sheaf of meromorphic functions on X, let $\operatorname{Div}(X) = H^0(X, \mathscr{M}_X^*/\mathscr{O}_X^*)$ be the group of (Cartier) divisors of X, and let $\operatorname{Div}^0(X)$ be the subgroup of numerically trivial divisors. Define a morphism

$$\psi : \operatorname{Div}^0(X) \otimes \mathbf{C} \longrightarrow H^0(X, \Omega_X \otimes \mathscr{M}_X^*)/H^0(X, \Omega_X)$$

as follows. Let D be a numerically trivial divisor on X. This means that its image under

$$c_1 : H^1(X, \mathscr{O}_X^*) \xrightarrow{d\log} H^1(X, \Omega_X) \subset H^2(X, \mathbf{C})$$

vanishes: if D is given on a covering (U_α) of X by meromorphic functions $f_\alpha \in \mathscr{M}_{U_\alpha}^*$, we have $d\log f_\alpha - d\log f_\beta = \omega_\alpha - \omega_\beta$ on $U_\alpha \cap U_\beta$, with $\omega_\alpha \in \Omega_{U_\alpha}$, and the $\big(d\log f_\alpha - \omega_\alpha\big)$ patch together to define a global meromorphic 1-form $\psi(D)$ on X. Because the ω_α are only defined up to addition of a global holomorphic 1-form on X, so is $\psi(D)$.

Let M be the subgroup of $\operatorname{Div}(X)$ generated by classes of (algebraic) integral curves of $\mathscr{F}$ and let $M^0 = M \cap \operatorname{Div}^0(X)$. If D is an element of M^0, keeping the notation above, the 1-form $\omega_{\mathscr{F}}$ on U_α is proportional to df_α, hence

$$\omega_{\mathscr{F}} \wedge \psi(C)|_{U_\alpha} = \omega_{\mathscr{F}} \wedge \big(d\log f_\alpha - \omega_\alpha\big) = -\omega_{\mathscr{F}} \wedge \omega_\alpha \in H^0(U_\alpha, \omega_X \otimes N_{\mathscr{F}})$$

because $\omega_{\mathscr{F}}$ is a global section of $\Omega_X \otimes N_{\mathscr{F}}$. Since $\psi(C)$ is defined only modulo $H^0(X, \Omega_X)$, this defines a morphism

$$\begin{array}{rccc} \psi_{\mathscr{F}} : & M^0 \otimes \mathbf{C} & \longrightarrow & H^0(X, \omega_X \otimes N_{\mathscr{F}})/\omega_{\mathscr{F}} \wedge H^0(X, \Omega_X) \\ & D & \longmapsto & \omega_{\mathscr{F}} \wedge \psi(D) \end{array}$$

I claim that *if* $\dim \operatorname{Ker} \psi_{\mathscr{F}} > 1$, *there is a nonconstant meromorphic function f such that* $\omega_{\mathscr{F}} \wedge df = 0$, so that $\mathscr{F}$ is the relative tangent sheaf to the rational map $X \dashrightarrow \mathbf{P}^1$ induced by f (see Example 9). This proves the theorem, because if $\mathscr{F}$ is not the relative tangent sheaf to such a map, the vector space $M^0 \otimes \mathbf{C}$ then has finite dimension. Since its codimension in $M \otimes \mathbf{C}$ is at most $\dim H^{1,1}(X) < \infty$, the vector space $M \otimes \mathbf{C}$ also has finite dimension, and this dimension is the number of distinct algebraic (irreducible) curves of $\mathscr{F}$.

We now prove the claim. Let $C_1, \dots, C_r$ be distinct irreducible curves on X and let $\lambda_1, \dots, \lambda_r$ be nonzero complex numbers such that $c = \lambda_1 C_1 + \cdots + \lambda_r C_r$ is in the kernel of $\psi_{\mathscr{F}}$. Pick an open covering (U_α) such that C_j is defined in U_α by a holomorphic equation $f_{j,\alpha} = 0$, with $d\log f_{j,\alpha} - d\log f_{j,\beta} = \omega_{j,\alpha} - \omega_{j,\beta}$ on $U_\alpha \cap U_\beta$, and set $\omega_{c,\alpha} =$

$\lambda_1\omega_{1,\alpha} + \cdots + \lambda_r\omega_{r,\alpha}$, holomorphic 1-form on U_α. Since $\psi_{\mathscr{F}}(c) = 0$, there is $\omega \in H^0(X, \Omega_X)$ such that

$$\omega_{\mathscr{F}} \wedge \omega_{c,\alpha} = \omega_{\mathscr{F}} \wedge \omega|_{U_\alpha}$$

Therefore, we can write $\omega_{c,\alpha} - \omega = g_\alpha \omega_{\mathscr{F}}$ for some meromorphic section g_α of $N^*_{\mathscr{F}}$ on U_α. Since holomorphic forms are closed,[7] we obtain

$$0 = d(g_\alpha \omega_{\mathscr{F}}) = dg_\alpha \wedge \omega_{\mathscr{F}} + g_\alpha \wedge d\omega_{\mathscr{F}}$$

If c' is another element of the kernel of $\psi_{\mathscr{F}}$, we obtain similarly

$$0 = dg'_\alpha \wedge \omega_{\mathscr{F}} + g'_\alpha \wedge d\omega_{\mathscr{F}}$$

hence $d(g_\alpha/g'_\alpha) \wedge \omega_{\mathscr{F}} = 0$. If the claim fails, $\mu = g_\alpha/g'_\alpha$ is constant, hence

$$\psi(c - \mu c')|_{U_\alpha} = -\omega_{c,\alpha} + \mu\omega_{c',\alpha} = (\mu - 1)\omega = 0$$

modulo $H^0(X, \Omega_X)$. The claim will follow if we prove that ψ is injective. We keep the same notation as above and assume

$$\psi(\lambda_1 C_1 + \cdots + \lambda_r C_r) = 0$$

where $\lambda_1, \ldots, \lambda_r$ are complex numbers. This means that

$$\lambda_1 \frac{df_{1,\alpha}}{f_{1,\alpha}} + \cdots + \lambda_r \frac{df_{r,\alpha}}{f_{r,\alpha}}$$

is a holomorphic 1-form in U_α. Pick a smooth point x of $C_1 - (C_2 \cup \cdots \cup C_r)$ and a U_α that contains it. We obtain that $\lambda_1 \frac{df_{1,\alpha}}{f_{1,\alpha}}$ is holomorphic, which implies $\lambda_1 = 0$ since $\frac{df_{1,\alpha}}{f_{1,\alpha}}$ has a simple pole at x. □

7.2. **Surfaces of general type with $c_1^2 > c_2$.** Let X be a smooth projective surface of general type. Let $\pi : \mathbf{P} = \mathbf{P}(\Omega_X) \to X$ and set $L = \mathscr{O}_{\mathbf{P}}(1)$. The Riemann–Roch theorem gives

$$h^0(\mathbf{P}, L^m) + h^2(\mathbf{P}, L^m) \geq \chi(\mathbf{P}, L^m) = \frac{m^3}{6}(c_1^2 - c_2) + O(m^2)$$

and by Serre duality,

$$\begin{aligned} h^2(\mathbf{P}, L^m) &= h^0(X, \mathbf{S}^m\,\Omega_X \otimes \mathscr{O}_X(-(m-1)K_X)) \\ &\leq h^0(X, \mathbf{S}^m\,\Omega_X) = h^0(\mathbf{P}, L^m) \end{aligned}$$

because, X being of general type, $h^0(X, \mathscr{O}_X((m-1)K_X))$ is nonzero for $m \gg 0$.

If $c_1^2 > c_2$, the line bundle L is big, and for any ample line bundle A on X, the linear system $|L^m \otimes A^{-1}|$ is not empty for $m \gg 0$. Let D be

[7] This is true on any compact complex surface: a holomorphic 1-form ω satisfies $\bar\partial\omega = 0$, so that $\overline{d\omega} = \overline{\partial\omega} = \bar\partial\bar\omega = d\bar\omega$. Since $d(d\omega \wedge \bar\omega) = d\omega \wedge d\bar\omega = d\omega \wedge \overline{d\omega}$, the Stokes theorem gives $\int_X d\omega \wedge \overline{d\omega} = 0$, hence $d\omega = 0$.

an element of $|L^m \otimes A^{-1}|$. Proposition 3 implies that the lift to $\mathbf{P}(\Omega_X)$ of any entire curve $f : \mathbf{C} \to X$ lands in the support of D. If $D \to X$ has degree $k > 0$, this means that at a general point x of X, we have k tangent directions and that every entire curve through x must be tangent to one of these directions.

Pulling the situation back on some desingularization $\tilde{D} \to D$, we obtain a *foliation* on $\tilde{D}$ and $f(\mathbf{C})$ is an integral curve of that foliation. Unfortunately, there is not much we can say at this point about transcendental leaves of foliations. However, rational or elliptic curves on X must map to *algebraic* leaves of the foliation on $\tilde{D}$. Since $\tilde{D}$, being a surface of general type, is not ruled by rational or elliptic curves, there are at most finitely many such curves by Jouanolou's theorem 8. We have proved the following.[8]

Theorem 9 (Bogomolov)**.** *On a smooth projective surface of general type with $c_1^2 > c_2$, there are only finitely many rational or elliptic curves.*

The same ideas, pushed a bit further, prove the following more general result.

Theorem 10 (Bogomolov)**.** *On a smooth projective surface of general type with $c_1^2 > c_2$, curves with fixed geometric genus form a bounded family.*[9]

Examples 10. 1) In $\mathbf{P}^2$, there are rational curves of arbitrarily high degree. The family of rational curves in $\mathbf{P}^2$ is not bounded.

2) In the surface $E \times E$, let E_m be the image of the morphism $E \to E \times E$ that sends x to (x, mx), $m \in \mathbf{Z}$. The curves E_m are all elliptic but have distinct classes. The family of elliptic curves in $E \times E$ is not bounded.

Proof of Theorem 11. We keep the same notation as above. Since $L = \mathscr{O}_{\mathbf{P}(\Omega_X)}(1)$ is big, it defines, for $m \gg 0$, an embedding

$$\psi_m : \mathbf{P} - \Sigma(L) \longrightarrow \mathbf{P}H^0(\mathbf{P}, L^m) = \mathbf{P}^N$$

Let $f : C \to X$ be a morphism from a smooth projective curve, with lift $\bar{f} : C \to \mathbf{P}$.

If $\bar{f}(C) \not\subset \Sigma(L)$, we obtain a *morphism* $\psi_m \circ \bar{f} : C \to \mathbf{P}^N$ associated with a linear subsystem of $|\bar{f}^* L^m|$. By construction, $\bar{f}^* L$ is the image

[8]The argument presented in the proof of Theorem 10 shows that we do not need Proposition 3 in this case.

[9]This means that these curves are parametrized by a finite number of irreducible algebraic varieties, or equivalently, that their degrees in some (or any) fixed embedding are bounded.

of the differential $df : f^*\Omega_X \to \Omega_C$, hence

$$\deg(\psi_m \circ \bar{f}(C)) \leq \deg \bar{f}^* L^m \leq m \deg \Omega_C = m(2g(C) - 2)$$

so that curves on X with fixed geometric genus that satisfy this property form a bounded family.

Assume now that $\bar{f}(C)$ lies in some irreducible component D of $\Sigma(L)$. If D does not dominate X, there are only finitely many such curves. It D dominates X and $\tilde{D} \to D$ is a desingularization, as above, the map $\tilde{D} \to D \subset \mathbf{P}$ defines a foliation on $\tilde{D}$ of which C is an integral curve. By Jouanolou's theorem, all such curves, except finitely many, are fibers of some rational map $\tilde{D} \to C'$ onto a smooth curve C'. Such a map becomes a morphism upon composing with a birational morphism $\tilde{D}' \to \tilde{D}$. Let $\tilde{f}' : C \to \tilde{D}'$ and $\tilde{f} : C \to \tilde{D}$ be the induced morphisms. Since the genus of C is fixed, so is $\big(K_{\tilde{D}'} + \tilde{f}'(C)\big) \cdot \tilde{f}'(C)$ by the genus formula. The smooth curve $\tilde{f}'(C)$ being a fiber of the morphism $\tilde{D}' \to C'$, we have $\tilde{f}'(C) \cdot \tilde{f}'(C) = 0$, hence $K_{\tilde{D}'} \cdot \tilde{f}'(C)$ is also fixed. Finally,

$$K_{\tilde{D}} \cdot \tilde{f}(C) \leq K_{\tilde{D}'} \cdot \tilde{f}'(C)$$

so the $K_{\tilde{D}}$-degree of $\tilde{f}(C)$ is bounded. Because $K_{\tilde{D}}$ is big, this bounds the family of all such curves C. □

Using analogous ideas, Bogomolov proves the following, valid on *any* smooth projective surface of general type. The key extra ingredient is the *stability* of the cotangent bundle (see [R], [Des]).

Theorem 11. *On a smooth projective surface of general type, curves with fixed geometric genus and negative self-intersection form a bounded family.*

Example 11. Let $X \to B$ be a nonisotrivial elliptic fibration. The self-intersection of the image of any section $B \to X$ is negative. If there is a section with infinite order, we obtain infinitely many curves with same genus and negative self-intersection. Hence the conclusion of Theorem 11 may not hold on surfaces that are not of general type.

7.3. **Surfaces of general type with $c_1^2 > 2c_2$.** When $c_1^2 > 2c_2$, we can say more. Note however that surfaces of general type with this property (positive index) are much less common. For example, a complete intersection surface in $\mathbf{P}^n$, $n \geq 3$, satisfies $c_1^2 < \frac{2(n-2)}{n-1} c_2 < 2c_2$.

Theorem 12 (Lu–Yau)**.** *Let X be a smooth algebraic surface of general type satisfying $c_1^2 > 2c_2$. There are only finitely many rational or elliptic curves on X and any entire curve of X maps to one of these.*

Proof. We may assume that X is minimal, i.e., that K_X is nef. We keep the same notation as above: $\pi : \mathbf{P} = \mathbf{P}(\Omega_X) \to X$ and $L = \mathscr{O}_{\mathbf{P}}(1)$. We know that L is big.

Let D be an irreducible effective divisor. There exist a positive integer k and a divisor F on X such that $D \in |L^k(-\pi^* F)|$. It induces an injection $\mathscr{O}_X(F) \hookrightarrow \mathbf{S}^k \Omega_X$. Since Ω_X, hence also $\mathbf{S}^k \Omega_X$, is K_X-semistable (this is a theorem of Bogomolov), we obtain

$$F \cdot K_X \leq \frac{1}{k+1}\big(c_1(\mathbf{S}^k \Omega_X) \cdot K_X\big) = \frac{k}{2} c_1^2$$

hence

$$L \cdot L \cdot D = L \cdot L \cdot (L^k(-\pi^* F)) = k(c_1^2 - c_2) - K_X \cdot F \geq \frac{k}{2}(c_1^2 - 2c_2) > 0$$

(here we used $L \cdot L = L \cdot \pi^* c_1 - \pi^* c_2$, hence $L \cdot L \cdot \pi^* G = K_X \cdot G$). Again, we obtain from the Riemann–Roch theorem applied on D the inequality

$$h^0(D, L^m) + h^2(D, L^m) \geq \chi(D, L^m) = \frac{m^2}{2} L \cdot L \cdot D + O(m)$$

and by Serre duality, since $\omega_D = \mathscr{O}_D(K_{\mathbf{P}} + D)$,

$$h^2(D, L^m) = h^0(D, L^{-m}(K_{\mathbf{P}} + D))$$

Since

$$L|_D \cdot \pi^* K_X|_D = L \cdot \pi^* K_X \cdot (kL - \pi^* F) = k c_1^2 - F \cdot K_X \geq \frac{k}{2} c_1^2 > 0$$

we have

$$L^{-m}(K_{\mathbf{P}} + D)|_D \cdot \pi^* K_X|_D < 0$$

for $m \gg 0$. Since $\pi^* K_X|_D$ is nef, $h^0(D, L^{-m}(K_X + D))$ must vanish for $m \gg 0$, hence $L|_D$ is big.

Let now $f : \mathbf{C} \to X$ be an entire curve and let $\bar{f} : \mathbf{C} \to \mathbf{P}(\Omega_X)$ be its lift. Let A be an ample line bundle on X and let D' be an element of $|L^m \otimes A^{-1}|$, for $m \gg 0$. As we saw above $\bar{f}(\mathbf{C})$ is contained in some irreducible component $D \in |L^k(-\pi^* F)|$ of D'. If $k = 0$, the divisor D projects onto an algebraic (necessarily rational or elliptic) curve in X and we are done. If $k > 0$, we have k tangent directions at a general point of $f(\mathbf{C})$ and f is tangent to one of these directions.

Since $L|_D$ is big, we can find a singular metric on $L|_D$ (or more precisely, on the pull-back of L to a desingularization $\tilde{D}$ of D) that has positive curvature current. Now if we look at the proof of Proposition 3, we see that such a metric is enough to let the argument go through, since we need only know that its curvature be positive at the points of $\bar{f}(\mathbf{C})$ in the direction tangent to this curve. So again, the image of

$\bar{f}$ must be contained in the support of any curve in the linear system $|L^m \otimes \pi^* A^{-1}|_D|$, for $m \gg 0$. This proves the proposition.[10] □

It is a theorem of McQuillan that an entire curve tangent to a foliation on a smooth projective surface of general type is not Zariski dense ([Bru], [Mc]). It follows that the conclusion of the theorem still holds under the weaker hypothesis $c_1^2 > c_2$.

8. Entire curves in complex tori

There is a complete description of analytically hyperbolic subvarieties of a complex torus, thanks to the following theorem.

Theorem 13 (Bloch)**.** *Let A be a complex torus and let $f : \mathbf{C} \to A$ be an entire curve. The Zariski closure of $f(\mathbf{C})$ is a translated subtorus of A.*

Corollary 6. *A subvariety of a complex torus is analytically hyperbolic if and only if it contains no translated nontrivial subtorus.*

Corollary 7. *Let X be a compact complex Kähler manifold such that $h^0(X, \Omega_X) > \dim(X)$. Every entire curve in X is algebraically degenerate. More precisely, there exists a (fixed) proper analytic subvariety of X that contains the image of every entire curve in X.*

Proof. The Albanese variety $\mathrm{Alb}(X)$ has dimension $h^0(X, \Omega_X)$. Since $h^0(X, \Omega_X) > \dim(X)$, the image of the Albanese map $\alpha_X : X \to \mathrm{Alb}(X)$ is a proper subvariety of $\mathrm{Alb}(X)$ that generates $\mathrm{Alb}(X)$, hence is particular not a translated subtorus. If $f : \mathbf{C} \to X$ is an entire curve, it follows that the Zariski closure of $\alpha(f(\mathbf{C}))$, begin a translated subtorus, is a proper subvariety Y of $\alpha(X)$, and $f(\mathbf{C})$ is contained in the proper subvariety $\alpha^{-1}(Y)$ of X.

In fact, one can prove that any subvariety of a complex torus contains only finitely many maximal nontrivial translated subtori. This implies the stronger statement at the end of the corollary. □

Sketch of proof of the theorem. Let X be the Zariski closure of $f(\mathbf{C})$, let X_{reg} be its smooth locus, and let $\mathbf{P}(\Omega_X)$ be the closure in $\mathbf{P}(\Omega_A)$ of $\mathbf{P}(\Omega_{X_{\mathrm{reg}}})$. The line bundle $\mathscr{O}_{\mathbf{P}(\Omega_X)}(1)$ defines the projection

$$p_1 : \mathbf{P}(\Omega_X) \subset \mathbf{P}(\Omega_A) = \mathbf{P}(\Omega_{A,0}) \times A \longrightarrow \mathbf{P}(\Omega_{A,0})$$

(see §2). The variety X might be singular, but the curve $f(\mathbf{C})$, being dense in it, meets it smooth locus; we assume $f(0) \in X_{\mathrm{reg}}$. A slight

[10] The argument presented by Demailly in [De1], p. 349, relies on his Proposition 7.2.iv), which is unfortunately wrong.

generalization of Proposition 3 implies that $f(\mathbf{C})$ lies in the locus where the fibers of p_1 are positive-dimensional.

For any $k > 0$, one can define inductively the space P_Y^k of k-jets on a complex manifold Y (see [De1], p. 306). These spaces fit into a sequence

$$P_Y^k \xrightarrow{\pi_k} P_Y^{k-1} \longrightarrow \cdots \longrightarrow P_Y^1 = \mathbf{P}(\Omega_Y) \xrightarrow{\pi} Y$$

of $\mathbf{P}^{\dim(Y)-1}$-bundles. The entire curve f lifts as $f^{(k)} : \mathbf{C} \to P_A^k$, whose image lies in the closure P_X^k of $P_{X_{\mathrm{reg}}}^k$. A similar reasoning shows that it actually lies in the locus where the fibers of

$$p_k : P_X^k \subset P_A^k = P_{A,0}^k \times A \longrightarrow P_{A,0}^k$$

are positive-dimensional. The closed analytic set

$$A_k = \{a \in A \mid f^{(k)}(0) \in a + P_X^k\} \simeq p_k^{-1}(p_k(f^{(k)}(0)))$$

is therefore positive-dimensional at 0 and since fibers of p_k are contained in fibers of p_{k-1}, we have a chain of inclusions

$$A_k \subset A_{k-1} \subset \cdots \subset A_1$$

The Noetherian property implies that these must stabilize to a closed analytic subset $A_\infty = A_k$ of A that has positive dimension at 0. But the translation by A_∞ of the germ of f at 0 is then contained in X, hence

$$A_\infty + f(\mathbf{C}) \subset X$$

Since $f(\mathbf{C})$ is dense in X, we obtain $A_\infty + X \subset X$, and, since X is irreducible, there is equality and X is invariant by translation by the (nonzero) subtorus B of A generated by A_∞. Proceeding by induction on the dimension of A, we conclude that the image X/B of X in A/B is also a translated subtorus, hence so is X. □

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