Singularities of divisors on abelian varieties

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This is joint work with Christopher Hacon.

We work over the complex numbers. Let D be an effective divisor on an abelian variety A of dimension g. If D is not ample, there exists a quotient abelian variety $A \to B$ such that D is the pull-back of a divisor on B. Since we are interested in the singularities of D, we will henceforth assume that D is *ample.*

1 Singularities of pairs

Let D be an effective $\mathbf{Q}$-divisor on a smooth projective variety A. A *log resolution* of the pair (A, D) is a proper birational morphism $\mu : A' \to A$ such that the union of $\mu^{-1}(D)$ and the exceptional locus of μ is a divisor with simple normal crossing support. Write

$$\mu^*(K_A + D) = K_{A'} + \sum a_i D_i$$

where the D_i are distinct prime divisors on A'. We define the *multiplier ideal sheaf* $\mathscr{I}(A, D) \subset \mathscr{O}_A$ by

$$\mathscr{I}(A, D) = \mu_*\big(\omega_{A'/A}(-\lfloor \mu^* D \rfloor)\big) = \mu_*\big(\mathscr{O}_{A'}(-\sum \lfloor a_i \rfloor D_i)\big)$$

and, if D is a prime divisor, the *adjoint ideal sheaf* $\mathscr{J}(A, D) \subset \mathscr{O}_A$ as fitting into an exact sequence

$$0 \to \omega_A \to \omega_A(D) \otimes \mathscr{J}(A, D) \to f_*\omega_X \to 0 \tag{1}$$

of sheaves on A, for any desingularization $f : X \to D$.

The pair (A, D) is

- *log canonical* if $a_i \leq 1$ for all i; this is equivalent to $\mathscr{I}(A, tD) = \mathscr{O}_A$ for all $t \in \mathbf{Q} \cap (0,1)$;
- *log terminal* if $a_i < 1$ for all i; this is equivalent to $\mathscr{I}(A, D) = \mathscr{O}_A$;
- (D prime) *canonical* if $a_i \leq 0$ for all i such that D_i is μ-exceptional; this is equivalent to $\mathscr{J}(A, D) = \mathscr{O}_A$.

These properties have consequences for the singularities of D: for any positive integers m and k,

- if $(A, \frac{1}{m}D)$ is log canonical, $\lfloor \frac{1}{m+1} D \rfloor = 0$ and $\operatorname{codim}_A(\operatorname{Sing}_{mk} D) \geq k$;
- if $(A, \frac{1}{m}D)$ is log terminal, $\lfloor \frac{1}{m} D \rfloor = 0$ and $\operatorname{codim}_A(\operatorname{Sing}_{mk} D) > k$;
- (D prime) if (A, D) is canonical, D is normal with rational singularities and $\operatorname{codim}_A(\operatorname{Sing}_k D) > k$ for $k \geq 2$.

Kollár was the first to use vanishing theorems to prove results on the singularities of divisors in abelian varieties.

2 The results

Let A be an abelian variety of dimension g. A *polarization* ℓ on A is an ample numerical equivalence class. Its *degree* is given by

$$\frac{1}{g!}\ell^g = h^0(A, L)$$

for any line bundle L on A that represents ℓ. A polarization of degree 1 is called *principal* and a divisor representing it is called a theta divisor. A polarized abelian variety (A, ℓ) is *indecomposable* if it is not the product of nonzero polarized abelian varieties.

Previous results were as follows:

- if $(A, [\Theta])$ is a principally polarized abelian variety and $D \sim m\Theta$,
  - the pair $(A, \frac{1}{m}D)$ is log canonical (Kollár, 1993 for $m = 1$; Ein–Lazarsfeld, 1997) and log terminal if $\lfloor \frac{1}{m} D \rfloor = 0$ (Hacon, 1999);
  - if Θ is irreducible, the pair (A, Θ) is canonical (Ein–Lazarsfeld, 1997).

- if (A, ℓ) is indecomposable of degree 2 and $D \sim m\ell$,
  - the pair $(A, \frac{1}{m}D)$ is log canonical (Hacon, 2000).

Here is Ein–Lazarsfeld's proof of the first point. Let Z be the subscheme of D defined by $\mathscr{I}(A, \frac{t}{m}D)$, for some $t \in \mathbf{Q} \cap (0,1)$. The Nadel–Kawamata–Viehweg vanishing theorem yields

$$H^i(A, \mathscr{O}_A(\Theta_a) \otimes \mathscr{I}_Z) = 0 \qquad \text{for all } i > 0 \text{ and all } a \in A.$$

If Z is nonempty, we have $Z \not\subset \Theta_a$ for a general, hence $H^0(A, \mathscr{O}_A(\Theta_a) \otimes \mathscr{I}_Z) = 0$. It follows that

$$\chi(A, \mathscr{O}_A(\Theta_a) \otimes \mathscr{I}_Z) = H^0(A, \mathscr{O}_A(\Theta_a) \otimes \mathscr{I}_Z) = 0$$

for $a \in A$ general hence for all a because the Euler characteristic is a numerical invariant. We conclude that

$$H^i(A, \mathscr{O}_A(\Theta) \otimes P \otimes \mathscr{I}_Z) = 0 \qquad \text{for all } i \text{ and all } P \in \operatorname{Pic}^0(A).$$

By the Fourier–Mukai theory, this implies $\mathscr{O}_A(\Theta) \otimes \mathscr{I}_Z = 0$, which is absurd.

The other proofs are more involved and use generic vanishing results of Green–Lazarsfeld. We'll come back to that. Here are our results.

Let (A, ℓ) be an indecomposable polarized abelian variety of degree d and dimension $g > (d+1)^2/4$ and let $D \sim m\ell$.

- If A is *simple,*
  - if $m = 1$, the divisor D is prime and the pair (A, D) is canonical;
  - if $m \geq 2$, the pair $(A, \frac{1}{m}D)$ is log terminal unless $D = mE$, with $E \in \ell$.
- If $d = 2$,
  - if $m = 1$ and D is prime, the pair (A, D) is canonical;
  - if $m \geq 2$, the pair $(A, \frac{1}{m}D)$ is log canonical and is log terminal if $\lfloor \frac{1}{m}D \rfloor = 0$.[1]

[1] This condition holds unless

* $D = mE$, with $E \in \ell$;

Remarks 1 1) The optimal bound seems to be $g > d$.

2) If A is not simple, but still indecomposable, ℓ may very well contain reducible elements (see footnote below). There are also examples, in any dimension ≥ 2, and for any $d \geq 3$ and $m \geq d-1$, of pairs $(A, \frac{1}{m}D)$ that are not log canonical on an indecomposable polarized abelian variety of degree d.[2]

3 The proofs

Skipping the proof of log terminality,[3] the point is to show that

- if $m = 1$, the ideal $\mathscr{J}(A, D)$ is trivial;
- if $m \geq 2$ and $t \in \mathbf{Q} \cap (0,1)$, the ideal $\mathscr{I}(A, tD)$ is trivial.

In both cases, let Z be the subscheme of D defined by the ideal and set

$$h = h^0(A, L \otimes \mathscr{I}_Z \otimes P) = h^0(A, L_a \otimes \mathscr{I}_Z) \in [0, d]$$

for P and a general in $\mathrm{Pic}^0(A)$.

If $h = d$, all sections of L contain all translates of Z, which must be empty.

$*$ or there are nonzero principally polarized abelian varieties $(B_1, [\Theta_1])$ and $(B_2, [\Theta_2])$, and an isogeny $p : A \to B_1 \times B_2$ of degree 2, such that

$$D = mp^*(\Theta_1 \times B_2) + p^*(B_1 \times D_2)$$

with $D_2 \in |m\Theta_2|$.

[2] Let $(A_1, [L_1])$ be a general polarized abelian variety of type $(d-1)$ and let E be an elliptic curve. Pick an isomorphism $\psi : K(L_1) \to E[d-1]$ and consider the quotient A of $A_1 \times E$ by the subgroup $\{(x, \psi(x)) \mid x \in K(L_1)\}$. There is a divisor Θ on A that defines a principal polarization and $\mathscr{O}_A(\Theta)$ restricts to L_1 on A_1. The line bundle $L = \mathscr{O}_A(\Theta + A_1)$ defines an indecomposable polarization of degree d on A that is indecomposable if $d \geq 3$. The linear system $|(d-1)\Theta - A_1|$ is nonempty, hence so is the linear system $|m\Theta - (m'-m)A_1|$ for $\frac{d}{d-1}m \geq m' > m \geq d-1$. If D' is in that linear system, $D = D' + m'A_1$ is in $|mL|$ and the pair $(A, \frac{1}{m}D)$ is not log canonical since it has a component with multiplicity > 1.

[3] The difficulties are first to prove that the Green–Lazarsfeld theory still applies in this case (this follows from previous work of Hacon), then to prove, in the notation below, $h > 0$ (this follows from the hypothesis $\lfloor \frac{1}{m}D \rfloor = 0$ and other work of Hacon).

Assume $h = 0$. In the case $m \geq 2$, we conclude as in the Ein–Lazarsfeld proof $\mathscr{O}_A(\Theta) \otimes \mathscr{I}_Z = 0$, which is absurd. In the case $m = 1$, this implies, by work of Ein and Lazarsfeld,[4] that D is fibered by nonzero abelian varieties, which is also absurd.

So we assume $0 < h < d$ (and Z nonempty). Set

$$J = \{(s, a) \in \mathbf{P}H^0(A, L) \times A \mid s|_{Z+a} \equiv 0\}$$

The fiber of a point a of A for the second projection $q : J \to A$ is $\mathbf{P}H^0(A, L \otimes \mathscr{I}_{Z+a}) \simeq \mathbf{P}H^0(A, L_a \otimes \mathscr{I}_Z)$ and a unique irreducible component I of J dominates A. It has dimension $g + h - 1$.

Let $p : I \to \mathbf{P}H^0(A, L)$ be the first projection. The fibers $F_s = q\big(p^{-1}(s)\big)$ are either empty or of dimension $\geq g + h - d$; they satisfy $Z + F_s \subset \operatorname{div}(s)$ hence $\dim(F_s) \leq g - 1$.

If $h = d - 1$ (this is the case when $d = 2$), F_s has dimension $g - 1$ and p is surjective. The divisor of a general section s being prime, the inclusion $z + F_s \subset \operatorname{div}(s)$ is an equality for all z in Z. This implies that Z is finite.

If A is simple, the inclusion $Z + F_s \subset \operatorname{div}(s)$ implies

$$\dim Z \leq g - 1 - \dim F_s \leq d - 1 - h$$

For a general in A, the subvariety $p(q^{-1}(a)) = \mathbf{P}H^0(A, L \otimes \mathscr{I}_{Z+a})$ of $\mathbf{P}H^0(A, L)$ is a linear subspace of dimension $h-1$. It must vary with a, because a nonzero s does not vanish on all translates of Z. It follows that the linear span of $p(I)$ has dimension at least h. For $s_1, \ldots, s_{h+1}$ general elements in $p(I)$, one has[5]

$$\dim(F_{s_1} \cap \cdots \cap F_{s_{h+1}}) \geq g - (h + 1)(d - h) \geq g - (d + 1)^2/4$$

For $a \in F_{s_1} \cap \cdots \cap F_{s_{h+1}}$, the sections $s_1, \ldots, s_{h+1}$ all vanish on $Z + a$, hence $h^0(A, L_a \otimes \mathscr{I}_Z) \geq h + 1$. Since the Euler characteristic $\chi(A, L_a \otimes \mathscr{I}_Z)$ is independent of a, this proves that

$$V_{>0} = \{P \in \operatorname{Pic}^0(A) \mid H^i(A, L \otimes \mathscr{I}_Z \otimes P) \neq 0 \text{ for some } i > 0\}$$

[4] By the Green–Lazarsfeld theory, the cohomological locus $V_{>0}$ defined below is not $\operatorname{Pic}^0(A)$. If $h = 0$, we have $V_0 \neq \operatorname{Pic}^0(A)$ as well, so that $\chi(X, \omega_X) = 0$ for any desingularization X of D (use (1)) and this implies that D is fibered by nonzero abelian varieties.

[5] As in the projective space, subvarieties of a simple abelian variety that should meet for dimensional reasons actually meet.

has dimension $\geq g - (d+1)^2/4 > 0$.

Now it follows from the Green–Lazarsfeld theory that in our case, the *cohomological loci*

$$V_i = \{P \in \operatorname{Pic}^0(A) \mid H^i(A, L \otimes \mathscr{I}_Z \otimes P) \neq 0\}$$

have striking properties: *every irreducible component of V_i is an abelian subvariety of codimension $\geq i$ of* $\operatorname{Pic}^0(A)$ *translated by a torsion point.* In our case, the inequality $h > 0$ means $V_0 = \operatorname{Pic}^0(A)$.

This immediatley finishes the proof when A is simple, since $V_{>0}$ must then be finite.

When $d = 2$ (and $h = 1$), since $V_{>0} \neq \operatorname{Pic}^0(A)$, we get that for a general, there is an exact sequence

$$0 \to H^0(A, L_a \otimes \mathscr{I}_Z) \to H^0(A, L_a) \to H^0(Z, L_a|_Z) \to 0$$

hence Z is a single point z. An element $\varphi_L(a) \in \operatorname{Pic}^0(A)$ is in V_1 if the restriction $H^0(A, L_a) \to H^0(Z, L_a|_Z)$ is zero. This happens exactly when $z + a$ is in the base locus of $|L|$, hence

$$V_1 = \varphi_L(\operatorname{Bs}(|L|) - z)$$

which has codimension ≤ 2 in A. A rather technical lemma proves that for any $i > 0$ such that V_i is nonempty, $\dim Z \geq i - 1 + \dim V_i$, which is a contradiction.

4 Base loci of ample linear systems

Not much seems to be known about the singularities, or even the dimension, of the base locus of an ample linear system $|L|$ on an abelian variety A.

4.1 Dimension

Let us begin with the following remark: let B be an abelian subvariety of A on which L has degree e and let C be an abelian subvariety of A such that the sum morphism $f : B \times C \to A$ is an isogeny. The base locus of $|L|$ contains

$$f(\operatorname{Bs}(|L|_B|) \times C) = \operatorname{Bs}(|L|_B|) + C$$

hence has codimension $\leq e$ if $e \leq \dim(B)$. Hence

$$\begin{array}{c}L \text{ has degree} \leq e \text{ on an abelian} \\ \text{subvariety of } A \text{ of dimension} \geq e\end{array} \Longrightarrow \operatorname{codim}(\operatorname{Bs}(|L|)) \leq e \qquad (2)$$

It is classical and easy to show that the converse holds for $e = 1$:

$$\begin{array}{ccc}\begin{array}{c}L \text{ has degree 1 on a nonzero} \\ \text{abelian subvariety of } A\end{array} & \Longleftrightarrow & \begin{array}{c}\text{the polarized abelian variety} \\ (A, [L]) \text{ is decomposable with} \\ \text{a principally polarized factor}\end{array} \\ & \Longleftrightarrow & \begin{array}{c}\text{all elements of } |L| \\ \text{are reducible}\end{array} \\ & \Longleftrightarrow & \operatorname{codim}(\operatorname{Bs}(|L|)) = 1\end{array}$$

I would like to conjecture that the converse holds for $e = 2$.[6]

Conjecture 1 *Let $(A, [L])$ be a polarized abelian variety. If* $\operatorname{Bs}(|L|)$ *has codimension* 2 *in A, the abelian variety A contains an abelian subvariety of dimension* ≥ 2 *on which L has degree* 2.

The conjecture holds in dimension $g \leq 3$.[7] I can also prove it if some codimension 2 component of the base locus is a *Cartier* divisor in some element of $|L|$.

[6]and perhaps for any e! Note that for $e \geq 2$, the property on the left-hand of (2) side does occur on some indecomposable polarized abelian varieties.

[7]We may assume that $\operatorname{Bs}(|L|)$ has codimension 2 and a general element of $|L|$ is irreducible.

If $g = 2$, the fixed part consists of at most $(L)^2 = 2d$ points. Since it is stable by translation by the subgroup $K(L)$ of order d^2, we get $d^2 \leq 2d$.

If $g = 3$ and $d > 2$, I showed 20 years ago in my thesis (p. 103) that every component of $\operatorname{Bs}(|L|)$ is a translated elliptic curve in A. More precisely, for any principally polarized quotient $p : (A, L) \to (A_0, L_0)$, either

- (A_0, L_0) is a product $E_1 \times E_2 \times E_3$ of principally polarized elliptic curves and one checks that, after reordering, $p(K(L))$ must be contained in $E_1 \times E_2 \times \{0\}$, so that E_3 embeds in A and L has degree 1 on it;
- or A_0 is the Jacobian of a bielliptic curve D that has a morphism $D \to E$ of degree 2 and $p(K(L)) \subset E$, so that a complementary surface $S \subset A_0$ embeds in A and L has degree 2 on it.

One could also include in the conjecture that all codimension 2 components of the base locus occur as in the construction above, so that, if $d > 2$, they are in particular stable by translation by a nonzero abelian subvariety of A.

4.2 Singularities

Let $(B, [\Theta])$ be an indecomposable principally polarized abelian variety and let $b \in B$ be a point of order 2. In all examples that I know of, the intersection $\Theta \cdot \Theta_b$ is *reduced.* For instance, if C is a hyperelliptic curve with hyperelliptic involution τ, the intersection $\Theta \cdot \Theta_{p-\tau p} = 2(W_{g-2} + p)$ is not reduced for any $p \in C$, but $p - \tau p$ is never of order 2. If one believes the conjecture above in its strong form, we should also have the following.

Conjecture 2 *Let $(A, [L])$ be a polarized abelian variety. Any component of* $\mathrm{Bs}(|L|)$ *that has codimension* 2 *in A is (generically?) reduced.*

4.3 Characterization of hyperelliptic Jacobians

The following conjecture has been around for some years. It was probably first explicitly formulated in 1977 by Beauville, who proved it, using his theory of generalized Prym varieties, in dimension $g \leq 5$.

Conjecture 3 *Let $(B, [\Theta])$ be a principally polarized abelian variety. If* $\mathrm{Sing}(\Theta)$ *has codimension* 3 *in B, the principally polarized abelian variety $(B, [\Theta])$ is a hyperelliptic Jacobian.*

Let $(A, [L])$ be a polarized abelian variety of dimension g such that the base locus of $|L|$ has a component Z of codimension 2 that is not generically reduced. Let $(A', [L'])$ be a polarized abelian variety of the same type and construct a principally polarized quotient $\pi : (A, L) \times (A', L') \to (B, \Theta)$ (we have to check that it is indecomposable). The equation of $\pi^*\Theta$ is

$$\sum_i s_i(x) s'_i(x') = 0$$

If $x \in Z$, we have $s_i(x) = 0$ and $\mathrm{rank}(D_j s_i(x)) \leq 1$. On $Z \times A'$, the singular locus of $\pi^*\Theta$ has equations

$$\sum_i D_j s_i(x) s'_i(x') = 0$$

for $j \in \{1, \ldots, g\}$. For fixed $x \in Z$, this determines a divisor in A'. It follows that $\mathrm{Sing}(\Theta)$ has codimension at most 3 in B. On the other hand, $(B, [\Theta])$ cannot be a hyperelliptic Jacobian: since Θ contains $Z + A'$, the intersection $\Theta \cap \Theta_{x'}$ is reducible for all $x' \in A'$; on a Jacobian, this can only happen if A' has dimension 1 and the curve is bielliptic, but a curve cannot be at the same time hyperelliptic and bielliptic.

This shows that Conjecture 3 implies Conjecture 2. In particular, Conjecture 2 holds in dimension ≤ 4 for polarizations of type (d).