QUADRATIC LINE COMPLEXES

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Abstract. In this talk, a quadratic line complex is the intersection, in its Plücker embedding, of the Grassmannian of lines in an 4-dimensional projective space with a quadric. We study the moduli space of these Fano 5-folds in relation with that of EPW sextics.

1. Introduction

As explained in Dolgachev's recent book on classical algebraic geometry, a *quadratic line complex* is the proper intersection of a Grassmannian $G(2, V_n) \subset \mathbf{P}(\wedge^2 V_n)$ with a quadric.[1] It is a Fano variety of dimension $2n-5$ and index $n-2$.

A lot of classical works, dating back to Klein (1870), deal with the case $n = 4$ (see for example the last chapter of Griffiths and Harris' Principles of Algebraic Geometry or §10.3 of Dolgachev's book). In this talk, we will be interested in the case $n = 5$, already considered by Segre (1925; one of his first articles) and Roth (1951), who proved for example that they are rational.

Mukai proved that

- "almost all" smooth prime Fano fivefolds of degree 10 and index 3 are quadratic line complexes;
- any smooth prime Fano sixfold of degree 10 and index 4 is a double cover of $G(2, V_5)$ branched along a quadratic line complex.

Our aim is to discuss, for these 5-dimensional quadratic line complexes, the structure of their moduli space and period map.

Our notation will be

$$\psi : Z \to G := G(2, V_5),$$

a double cover branched along the smooth quadratic line complex

$$X = G \cap Q.$$

[1] In this talk we denote by V_n an n-dimensional complex vector space.

2. Moduli space

Theorem 1. *There exists an irreducible affine 25-dimensional coarse moduli space $\mathscr{X}$ for smooth quadratic line complexes.*

Proof. We follow a classical argument of Mumford's:

- the set of points in $\mathbf{P}(H^0(G,\mathscr{O}_G(2)))$ corresponding to sections whose zero-locus X in G is singular is a hypersurface;[2]
- this hypersurface is ample, hence its complement $\mathbf{P}(H^0(G,\mathscr{O}_G(2)))^0$ is affine and $\mathrm{SL}(V_5)$-invariant;
- the action of the reductive group $\mathrm{SL}(V_5)$ on this affine set is linearizable;
- the automorphism group of any smooth X is finite,[3] hence the stabilizers are finite at points of $\mathbf{P}(H^0(G,\mathscr{O}_G(2)))^0$, which is therefore contained in the stable locus $\mathbf{P}(H^0(G,\mathscr{O}_G(2)))^s$.

The moduli space $\mathscr{X}$ is therefore a dense affine open subset of the projective irreducible 25-dimensional GIT quotient

$$\overline{\mathscr{X}} := \mathbf{P}(H^0(G,\mathscr{O}_G(2)))/\!/\,\mathrm{SL}(V_5).$$

□

Local deformations of any smooth X are unobstructed ($H^2(X,T_X)=0$) hence the scheme $\mathscr{X}$ is, locally analytically around $[X]$, isomorphic to the quotient of the 25-dimensional vector space $H^1(X,T_X)$ by the finite group $\mathrm{Aut}(X)$.

Of course, $\mathscr{X}$ is also

- a coarse moduli space for smooth prime Fano sixfolds of degree 10 and index 4;[4]
- a dense open set in the moduli stack $\mathscr{X}^5_{10}$ for smooth prime Fano fivefolds of degree 10 and index 3.

Indeed, according to Mukai, all such fivefolds are either quadratic line complexes or degenerations thereof.[5] However, it is not clear that $\mathscr{X}^5_{10}$ is a scheme, or even an algebraic space (the problem is separatedness). It would be interesting to construct $\mathscr{X}^5_{10}$ as a subscheme of some blow-up of the projective scheme $\overline{\mathscr{X}}$.

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[2] The projective dual of the image of the embedding $G\hookrightarrow\mathbf{P}^{49}$ given by $|\mathscr{O}_G(2))|$ is a hypersurface, because it is easy to produce a quadric Q such that $Q\cap G$ has a single singular point (just choose Q general singular at a point of G).

[3] One computes $H^0(X,T_X)=0$.

[4] Because the double cover $\psi : Z\to G$ is uniquely determined up to an automorphism of V_5.

[5] More precisely, they are double covers of a smooth hyperplane section of G branched along its smooth intersection with a quadric.

3. Hodge structures and period map

Tedious (nontrivial) calculations show that in middle dimensions, the Hodge numbers are, for X,

$$0 \quad 0 \quad 10 \quad 10 \quad 0 \quad 0$$

so that the intermediate Jacobian $J(X)$ is a 10-dimensional principally polarized abelian variety, and for Z,

$$0 \quad 0 \quad 1 \quad 22 \quad 1 \quad 0 \quad 0$$

so that the Hodge structure is of K3 type.

We will define a period map by considering the Hodge structure of the sixfold Z attached to the quadratic line complex X.

We first define the vanishing cohomology $H^6(Z, \mathbf{Z})_{\mathrm{van}}$ as the orthogonal complement (for the intersection form) of the lattice

$$\psi^* H^6(G, \mathbf{Z}) \subset H^6(Z, \mathbf{Z}).$$

Standard lattice-theoretic considerations show that this lattice (of rank 22) is isomorphic to

$$\Lambda := 2E_8(-1) \oplus 2U \oplus 2A_1(-1).$$

As usual, we define a diagram of period maps

$$\begin{array}{ccc} \mathbf{P}(H^0(G, \mathscr{O}_G(2)))^0 & \xrightarrow{P} & \mathscr{Q} := \{\omega \in \mathbf{P}(\Lambda \otimes \mathbf{C}) \mid (\omega \cdot \omega) = 0 \ , (\omega \cdot \bar{\omega}) < 0\}^+ \\ \big\downarrow & & \big\downarrow \\ \mathscr{X} & \xrightarrow{\qquad p \qquad} & \mathscr{D} := \widetilde{O}(\Lambda) \backslash \mathscr{Q}, \end{array}$$

where $\widetilde{O}(\Lambda)$ is some subgroup of the group of isometries of Λ, and the period domain $\mathscr{D}$ is an irreducible quasi-projective variety of dimension 20.

Theorem 2. *The tangent map to P is everywhere surjective. In particular, p is dominant.*

Proof. One computes that the kernel of the natural map

$$\begin{aligned} H^1(Z, T_Z) &\to \mathrm{Hom}(H^{4,2}(Z), H^{4,2}(Z)^\perp / H^{4,2}(Z)) \\ &\simeq \mathrm{Hom}(H^2(Z, \Omega_Z^4), H^3(Z, \Omega_Z^3)) \end{aligned}$$

defined by the natural pairing $H^1(Z, T_Z) \otimes H^2(Z, \Omega_Z^4) \to H^3(Z, \Omega_Z^3)$ (recall that $H^2(Z, \Omega_Z^4)$ is one-dimensional) has dimension 5. ☐

Question 3. Describe

- the image of p;
- the fibers of p.

4. Associated sextics

Let $X = G \cap Q \subset \mathbf{P}(\wedge^2 V_5)$ be a quadratic line complex. Quadrics in $\mathbf{P}(\wedge^2 V_5)$ that contain G are called Plücker quadrics. They are of the type $P_v = v \wedge \cdot \wedge \cdot$, and the map $v \mapsto P_v$ defines an isomorphism $V_5 \simeq I_G(2)$. We have

$$I_X(2) \simeq I_G(2) \oplus \mathbf{C}Q \simeq V_5 \oplus \mathbf{C}Q.$$

Following O'Grady and Iliev-Manivel, we consider in the 5-plane $|I_X(2)|$ the locus of singular quadrics containing X. *Assume that X is contained in a smooth quadric.*[6] This locus is then a hypersurface of degree 10 and, since Plücker quadrics have corank 4, it decomposes as

$$4H_X + Y_X,$$

where $H_X := |I_G(2)| \simeq \mathbf{P}(V_5)$ is the *Plücker hyperplane* and where $Y_X \subset \mathbf{P}^5$ is a (possibly non-reduced or reducible) sextic, called an *EPW sextic.* We now describe these sextics.

Remark 4. One classically defines the "singular variety" $\Delta_X \subset \mathbf{P}(V_5)$ of X : it consists of all $x \in \mathbf{P}(V_5)$ such that $\mathbf{P}^3_x \cap Q = \mathbf{P}^3_x \cap X$ is singular, where $\mathbf{P}^3_x \subset G$ is the set of lines passing through x. This is (in general) a sextic hypersurface,[7] in fact equal to $Y_X \cap H_X$.

4.1. EPW sextics.

They were first defined by Eisenbud-Popescu-Walter and then studied extensively by O'Grady. We recall their definition and some of their properties.

Let V_6 be a 6-dimensional complex vector space and let $\mathrm{LG}(\wedge^3 V_6)$ be the (55-dimensional) Grassmannian of (10-dimensional) Lagrangian subspaces of $\wedge^3 V_6$ for the symplectic form given by wedge product. Given $[A] \in \mathrm{LG}(\wedge^3 V_6)$, we set

$$Y_A := \{v \in \mathbf{P}(V_6) \mid (\wedge^2 V_6 \wedge v) \cap A \neq 0\}.$$

Endowed with a natural scheme structure, it is either the whole of $\mathbf{P}(V_6)$ or a sextic hypersurface called an *EPW sextic.*

Set

$$Y_A[k] := \{v \in \mathbf{P}(V_6) \mid \dim\big((\wedge^2 V_6 \wedge v) \cap A\big) \geq k\},$$

so that $Y_A = Y_A[1]$.

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[6] There are cases when this does not hold, for example when Q has rank ≤ 3. But it holds when X is smooth.

[7] This was known to Segre, together with the fact that the singular locus of Δ_X is a curve of degree 40 and genus 81.

When $[A]$ is general in $\mathrm{LG}(\wedge^3 V_6)$, the singular set of Y_A is exactly $Y_A[2]$, which is a smooth surface (of degree 40). Here, "general" means "in the complement

$$\mathrm{LG}(\wedge^3 V_6)^0 = \mathrm{LG}(\wedge^3 V_6) - \Sigma - \Delta$$

of two irreducible hypersurfaces."

More generally, if $[A] \in \mathrm{LG}(\wedge^3 V_6) - \Sigma$, the singular set of Y_A is still $Y_A[2]$, a surface singular exactly along the finite set $Y_A[3]$ (which is empty exactly when $[A] \notin \Delta$).

4.2. **Dual EPW sextic.** Given a Lagrangian $A \subset \wedge^3 V_6$, we define the *dual EPW sextic*

$$Y_A^\vee := \{[V_5] \in \mathbf{P}(V_6^\vee) \mid \wedge^3 V_5 \cap A \neq 0\}.$$

When A is general, this is the projective dual of Y_A and it is again an EPW sextic in the dual projective space $\mathbf{P}(V_6^\vee)$.

4.3. **Moduli space for EPW sextics.** There is a natural action of $\mathrm{SL}(V_6)$ on $\mathrm{LG}(\wedge^3 V_6)$ and a projective irreducible 20-dimensional GIT quotient

$$\overline{\mathscr{EPW}} := \mathrm{LG}(\wedge^3 V_6) /\!/ \mathrm{SL}(V_6).$$

The hypersurface Σ is $\mathrm{SL}(V_6)$-invariant and its (affine) complement consists of stable points. We denote by $\Sigma_{\mathscr{EPW}}$ its image in $\overline{\mathscr{EPW}}$. Its (affine) complement $\overline{\mathscr{EPW}} - \Sigma_{\mathscr{EPW}}$ is a coarse moduli space for Lagrangians A which contain no decomposable trivector.

4.4. **The sextic associated with a quadratic line complex is an EPW sextic.** We fix a decomposition $V_6 = V_5 \oplus \mathbf{C}v_0$ and an isomorphism $\wedge^5 V_5 \simeq \mathbf{C}$. We have isomorphisms

$$\wedge^3 V_6 \simeq \wedge^3 V_5 \oplus (\wedge^2 V_5 \wedge v_0) \simeq \wedge^2 V_5^\vee \oplus (\wedge^2 V_5 \wedge v_0).$$

It is classical that the morphism

$$\begin{array}{rcl} \mathbf{A}_{v_0} : \mathrm{Sym}^2(\wedge^2 V_5^\vee) & \hookrightarrow & \mathrm{LG}(\wedge^3 V_6) \\ Q & \longmapsto & \{Q(x,\cdot) + x \wedge v_0 \mid x \in \wedge^2 V_5\} \end{array}$$

parametrizes the dense open subset of $\mathrm{LG}(\wedge^3 V_6)$ consisting of Lagrangian subspaces A transverse to $\wedge^3 V_5$; in other words, those for which $[V_5] \notin Y_A^\vee$.

Given $Q \in \mathrm{Sym}^2(\wedge^2 V_5^\vee)$, the connection with the previous definition of the sextic Y_{X_Q} is as follows: if we identify V_6 with $I_{X_Q}(2)$ by sending v_0 to $-Q$, then $Y_{\mathbf{A}_{v_0}(Q)} \subset \mathbf{P}(V_6)$ is isomorphic to $Y_X \subset \mathbf{P}(I_{X_Q}(2))$. The latter is therefore an EPW sextic.

Plücker quadrics go to Lagrangian subspaces of the type $\wedge^2 V_5 \wedge v$, and one checks that this parametrization induces a bijection

$$\mathbf{P}(H^0(G, \mathscr{O}_G(2)))/\operatorname{SL}(V_5) \simeq \\ \{A \in \operatorname{LG}(\wedge^3 V_6) \mid A \text{ not of type } \wedge^2 V_5 \wedge v,\ [V_5] \notin Y_A^\vee\}/G,$$

where $G := \{g \in \operatorname{SL}(V_6) \mid g(V_5) = V_5\}$. This means that there is a canonical map

$$\mathbf{P}(H^0(G, \mathscr{O}_G(2)))/\operatorname{SL}(V_5) \to \operatorname{LG}(\wedge^3 V_6)/\operatorname{SL}(V_6)$$

whose fiber over a class $[A]$ such that A is not of type $\wedge^2 V_5 \wedge v$ is isomorphic to the quotient of $\mathbf{P}(V_6^\vee) - Y_A^\vee$ by the subgroup of $\operatorname{PGL}(V_6^\vee)$ leaving $Y_A^\vee$ fixed (this group is trivial for A general).

Theorem 5. *This map induces a rational dominant map*

$$\overline{\mathscr{X}} \dashrightarrow \overline{\mathscr{EPW}}$$

which restricts to a surjective morphism

$$\Phi : \mathscr{X} \to \overline{\mathscr{EPW}} - \Sigma_{\mathscr{EPW}}.$$

The fiber $\Phi^{-1}([A])$ is isomorphic to the quotient of $\mathbf{P}(V_6^\vee) - Y_A^\vee$ by the subgroup of $\operatorname{PGL}(V_6^\vee)$ leaving $Y_A^\vee$ fixed.[8]

Proof. The main two facts used in the proof are:

- Lagrangians of the type $\wedge^2 V_5 \wedge v$ are in Σ and they correspond to nonsemistable points of $\operatorname{LG}(\wedge^3 V_6)$;
- given a non-Plücker quadric Q, the quadratic line complex X_Q is smooth if and only if $\mathbf{A}_{v_0}(Q) \notin \Sigma$.

□

Example 6 (The tangential quadratic line complex)**.** This is a singular classical example. Consider a smooth quadric q in $\mathbf{P}(V_5)$. The variety $X \subset G(2, V_5)$ of projective lines in $\mathbf{P}(V_5)$ tangent to q is a quadratic line complex. Its singular locus is the set of lines contained in q (this is the orthogonal Grassmannian $OG(2, V_5)$); it is also the image of a Veronese map $v_2 : \mathbf{P}^3 \to \mathbf{P}^9 \simeq \mathbf{P}(\wedge^2 V_5)$.

The associated EPW sextic Y_X is 3 times a smooth quadric (and the "singular variety" Δ_X is also 3 times a smooth quadric), a degenerate EPW sextic considered by O'Grady (whose dual is of the same type).

All tangential quadratic line complexes define the same point in $\overline{\mathscr{X}}$; it is semistable but not stable.[9]

[8] This group is trivial for A very general.

[9] It is also the only point corresponding to semistable quadratic line complexes with singular locus of dimension ≥ 3.

Example 7. O'Grady showed that twice the discriminant cubics are also (degenerate) EPW sextics. They correspond to two distinct points (dual one to another) in $\overline{\mathscr{EPW}}$; they are semistable but not stable. They correspond to certain quadratic line complexes with singular loci isomorphic to the images of Veronese maps $v_3 : \mathbf{P}^2 \to \mathbf{P}^9 \simeq \mathbf{P}(\wedge^2 V_5)$.

5. Period maps

Why are we interested in these (singular) EPW sextics? One reason is that, given a Lagrangian A such that $Y_A \neq \mathbf{P}(V_6)$, there is a canonical (finite) double cover

$$f_A : \widetilde{Y}_A \to Y_A$$

which is a topological cover of degree 2 outside of $Y_A[2]$. The scheme $\widetilde{Y}_A$ is smooth and irreducible if and only if $[A] \notin \Sigma \cup \Delta$. It is then *an irreducible symplectic fourfold.*

More generally, if $[A] \notin \Sigma$, the fourfold $\widetilde{Y}_A$ is integral, singular exactly along the finite set $f^{-1}(Y_A[3])$ (if singular, it is obtained from the Hilbert square of a (smooth) K3 surface of degree 10 by contracting Lagrangian 2-planes to points).

We can then define a period map

$$\overline{q} : \overline{\mathscr{EPW}} \dashrightarrow \mathscr{D},$$

where *the period domain $\mathscr{D}$ is the same as before.* O'Grady proved that this map is regular on $\overline{\mathscr{EPW}} - \Sigma_{\mathscr{EPW}}$.

Theorem 8 (Verbitsky, Markman, O'Grady)**.** *The restriction*

$$q : \overline{\mathscr{EPW}} - \Sigma_{\mathscr{EPW}} \to \mathscr{D}$$

of $\overline{q}$ is an open embedding.

The image of q (which is an affine open subset of $\mathscr{D}$) was investigated by O'Grady. There are naturally defined divisors $\mathscr{D}_r \subset \mathscr{D}$ indexed by $r \in \mathbf{N}^*$, $r \equiv 0, 2, \text{or } 4 \pmod 8$[10] and one has (O'Grady)

$$q(\overline{\mathscr{EPW}} - \Sigma_{\mathscr{EPW}}) \subset \mathscr{D} - \mathscr{D}_8.$$

O'Grady and I conjecture the equality

$$q(\overline{\mathscr{EPW}} - \Sigma_{\mathscr{EPW}}) = \mathscr{D} - \mathscr{D}_2 - \mathscr{D}_4 - \mathscr{D}_8.$$

The period map $\overline{q}$

- blows up the two points of $\overline{\mathscr{EPW}}$ corresponding to double discriminant cubics and sends them onto the two components of the divisor $\mathscr{D}_2$;

[10]These divisors are known as Heegner divisors in the theory of modular forms. If $r \equiv 2 \pmod 8$, the divisor $\mathscr{D}_r$ has two components; otherwise, it is prime.

- blows up the point of $\overline{\mathscr{EPW}}$ corresponding to triple quadrics and sends it onto the divisor $\mathscr{D}_4$;
- sends the divisor $\Sigma_{\mathscr{EPW}}$ onto the divisor $\mathscr{D}_8$.

Let us go back to quadratic line complexes. We have the following situation:

$$\mathscr{X} \xrightarrow{\Phi} \overline{\mathscr{EPW}} - \Sigma_{\mathscr{EPW}} \xrightarrow{q} \mathscr{D}.$$

Question 9. Is $q \circ \Phi$ the period map p for quadratic line complexes?

Actually, I expect $q \circ \Phi$ to be rather the composition of p with a naturally defined involution of $\mathscr{D}$.

This amounts to showing an isomorphism of Hodge structures

$$H^6(Z, \mathbf{Z})_{\mathrm{van}} \simeq H^2(\widetilde{Y}_X, \mathbf{Z})_{\mathrm{prim}}$$

reminiscent of the Beauville-Donagi isomorphism for cubic fourfolds.

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