ON PRIME FANO VARIETIES OF DEGREE 10 AND COINDEX 3

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Abstract. We discuss the period maps of certain (complex) Fano fourfolds. The first part of this talk deals with cubic fourfolds. We recall their Hodge structure and results of Voisin, Looijenga, Hassett, and Laza about their (injective) period map. We also recall the Beauville-Donagi construction of an associated irreducible symplectic fourfold and discuss their rationality.

In the second part, we turn our attention to prime Fano fourfolds of degree 10 and index 2 (this is joint work in progress with Iliev and Manivel). According to Mukai, most of them are obtained as intersections of the Grassmannian $G(2,5)$ in its Plücker embedding with a hyperplane and a quadric. Their period map has a more complex behavior, because it has 4-dimensional fibers. We recall a construction of Iliev & Manivel of an associated irreducible symplectic fourfold called a double EPW-sextic and studied by O'Grady, as well as classical constructions of Fano, Roth, and Prokhorov of rational examples.

1. Cubic fourfolds

Let $X \subset \mathbf{P}^5_{\mathbf{C}}$ be a smooth complex cubic fourfold.

1.1. Hodge structure and periods.

The Hodge structure on $H^4(X)$ is as follows

$$\begin{array}{ccccccc} H^4(X,\mathbf{C}) & = & H^{1,3}(X) & \oplus & H^{2,2}(X) & \oplus & H^{3,1}(X) \\ \text{dimensions:} & & 1 & & 21 & & 1 \end{array}$$

and the primitive Hodge structure is therefore

$$\begin{array}{ccccccc} H^4(X,\mathbf{C})_0 & = & H^{1,3}(X) & \oplus & H^{2,2}(X)_0 & \oplus & H^{3,1}(X) \\ \text{dimensions:} & & 1 & & 20 & & 1 \end{array}$$

In terms of lattices, we have $H^4(X,\mathbf{Z}) \simeq I_{21,2}$ and

$$H^4(X,\mathbf{Z})_0 \simeq 2E_8 \oplus 2U \oplus A_2 =: \Lambda_0.$$

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The period domain is

$$\mathscr{D} = \{\omega \in \mathbf{P}(\Lambda_0 \otimes \mathbf{C}) \mid \omega \cdot \omega = 0, \omega \cdot \bar{\omega} < 0\}$$

and the period map is

$$\mathscr{M} \xrightarrow{p} \Gamma \backslash \mathscr{D},$$

where $\mathscr{M}$ is the GIT moduli spave of smooth cubic fourfolds and Γ a suitable group of isometries of Λ_0. It is an open injection (Voisin) between 20-dimensional quasi-projective manifolds. Laza and Looijenga extended this map to compactifications

$$\widetilde{\mathscr{M}} \xrightarrow{\overline{p}} (\Gamma \backslash \mathscr{D})^{\mathrm{BB}},$$

where $\widetilde{\mathscr{M}}$ is an explicit blow-up of the GIT compactification of $\mathscr{M}$ and $(\Gamma \backslash \mathscr{D})^{\mathrm{BB}}$ is the Baily-Borel compactification of $\Gamma \backslash \mathscr{D}$. They also described explicitly the (complement of the) image of p in $(\Gamma \backslash \mathscr{D})^{\mathrm{BB}}$.

1.2. **Special cubic fourfolds.** Since the Hodge conjecture holds for cubic fourfolds, the Noether-Lefschetz locus in $\mathscr{M}$ is also the locus of cubic fourfolds X containing a surface whose class is not proportional to h^2, i.e., for which

$$\mathrm{rank}(H^4(X, \mathbf{Z}) \cap H^{2,2}(X, \mathbf{Z})) \geq 2.$$

Following Hassett, consider rank-2 saturated lattices K such that

$$\mathbf{Z}h^2 \subset K \subset I_{21,2}.$$

Each such lattice defines an irreducible hypersurface

$$\mathscr{D}_K := \{\omega \in \mathscr{D} \mid \omega \cdot K = 0\}$$

whose image $\mathscr{C}_d$ in $(\Gamma \backslash \mathscr{D})^{\mathrm{BB}}$ depends only on the discriminant d of K. The only d for which $\mathscr{C}_d$ is non-empty are $d > 0$ and $d \equiv 0, 2 \pmod 6$.

Examples 1.1. 1) Voisin showed that $p(\mathscr{M}) \cap \mathscr{C}_2 = \varnothing$. In fact, the compactification $\widetilde{\mathscr{M}}$ mentioned above is the blow-up of the GIT compactification at the point corresponding to the (singular) determinantal cubic $\begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{vmatrix} = 0$ and $\mathscr{C}_2$ is the image by $\overline{p}$ of the corresponding exceptional divisor.

2) Hassett showed $p(\mathscr{M}) \cap \mathscr{C}_6 = \varnothing$. In fact, $\mathscr{C}_6$ corresponds to nodal cubics.

3) Laza showed $p(\mathscr{M}) = (\Gamma \backslash \mathscr{D})^{\mathrm{BB}} - \mathscr{C}_2 - \mathscr{C}_6$.

4) We have

$$\begin{aligned}\mathscr{C}_8 &= \{\text{periods of cubics containing a plane}\},\\ \mathscr{C}_{12} &= \{\text{periods of cubics containing a cubic surface scroll}\},\\ \mathscr{C}_{14} &= \{\text{periods of Pfaffian cubics}\}.\end{aligned}$$

1.3. **The associated irreducible symplectic fourfold.** Let $L(X) \subset G(1, \mathbf{P}^5)$ be the (smooth projective) variety that parametrizes lines contained in X. It is an irreducible symplectic fourfold of $\mathrm{Hilb}^{[2]}$K3-type (when X is a Pfaffian cubic, $L(X) \simeq \mathrm{Hilb}^{[2]}(K3)$).

Consider the incidence variety

$$I := \{(x, \ell) \in X \times L(X) \mid x \in \ell\}$$

with its projections $p : I \to X$ and $q : I \to L(X)$. Beauville & Donagi proved that the *Abel-Jacobi map*

$$a := q_* p^* : H^4(X, \mathbf{Z})_0 \to H^2(L(X), \mathbf{Z})_0(-1)$$

is an isomorphism of polarized Hodge structures, where $H^2(L(X), \mathbf{Z})_0$ *is endowed with the Beauville-Bogomolov quadratic form.*

The injectivity of the period map for cubic fourfolds therefore implies the injectivity of the period map for deformations of $L(X)$. The Torelli problem for irreducible symplectic varieties was recently solved in general by Verbitsky.

1.4. **Rationality.** All smooth cubic fourfolds are unirational: if ℓ is a line contained in X, any tangent line to X at a point of ℓ meets X in a third point. This defines a dominant rational map

$$\mathbf{P}(T_X|_\ell) \dashrightarrow X$$

which has degree 2.

Pfaffian cubic fourfolds (from the divisor $\mathscr{C}_{14}$) are rational. It is conjectured that a very general element of $\mathscr{C}_8$ is not rational.

2. Prime Fano fourfolds of degree 10 and index 2

Let X be a smooth projective fourfold with $\mathrm{Pic}(X) \simeq \mathbf{Z}[H]$, where H is ample, $H^4 = 10$, and $K_X \underset{\mathrm{lin}}{\equiv} -2K_X$. Mukai proved that

- either $X \simeq G(2, V_5) \cap H \cap Q$;
- or X is a double cover of $G(2, V_5) \cap H \cap H'$ branched along its intersection with a quadric.

The first case is the general case and depends on 24 parameters; the second case is a specialization of the first and depends on 22 parameters.

Let $\mathscr{X}$ be the (24-dimensional irreducible) moduli stack parametrizing all these fourfolds.

2.1. **Hodge structure and periods.** The Hodge structure on $H^4(X)$ is as follows

$$\begin{array}{ccccccc} H^4(X,\mathbf{C}) & = & H^{1,3}(X) & \oplus & H^{2,2}(X) & \oplus & H^{3,1}(X) \\ \text{dimensions:} & & 1 & & 22 & & 1 \end{array}$$

and the vanishing Hodge structure $H^4(G(2,V_5))^\perp$ is therefore

$$\begin{array}{ccccccc} H^4(X,\mathbf{C})_0 & = & H^{1,3}(X) & \oplus & H^{2,2}(X)_0 & \oplus & H^{3,1}(X) \\ \text{dimensions:} & & 1 & & 20 & & 1 \end{array}$$

In terms of lattices, we have $H^4(X,\mathbf{Z}) \simeq I_{22,2}$ and

$$H^4(X,\mathbf{Z})_0 \simeq 2E_8 \oplus 2U \oplus 2A_1.$$

Again, we have a (different) period domain $\mathscr{D}$ and a period map

$$\mathscr{X} \xrightarrow{p} \Gamma\backslash\mathscr{D}.$$

An infinitesimal calculations shows that p is dominant. Its general fibers are therefore 4-dimensional.

Problems: describe the fibers and the image of p, and extend it to suitable compactifications.

We conjecture that all fourfolds in any given fiber of p are birationally isomorphic.

2.2. **Special fourfolds.** Again, we define, for each rank-3 saturated lattice K such that

$$H^4(G(2,V_5)) \subset K \subset I_{22,2},$$

an irreducible hypersurface $\mathscr{D}_K \subset \mathscr{D}$. If $d := \operatorname{disc}(K)$, it may be non-empty only if $d > 0$ and $d \equiv 0,2,4 \pmod 8$. The corresponding hypersurface $\mathscr{D}_d \subset \Gamma\backslash\mathscr{D}$ is then irreducible if and only if $d \equiv 0 \pmod 4$; it has two irreducible components $\mathscr{D}'_d$ and $\mathscr{D}''_d$ when $d \equiv 2 \pmod 8$.

2.3. **The associated irreducible symplectic fourfold.** Iliev and Manivel proved that for X general, the family $C(X)$ of (possibly degenerate) conics contained in X is a smooth projective fivefold. The vector space $I_X(2)$ of quadrics containing X has dimension 6 and there is a geometrically defined rational map

$$C(X) \dashrightarrow \mathbf{P}(I_X(2)^\vee)$$

whose image $Z_X \subset \mathbf{P}^5$ is a sextic fourfold called an *EPW sextic.* Its Stein factorization is

$$C(X) \overset{\beta}{\dashrightarrow} Y_X \xrightarrow{\pi} Z_X,$$

where π is a quasi-étale double cover branched along the (2-dimensional) singular locus of Z_X and β is generically a $\mathbf{P}^1$-bundle.

O'Grady proved that Y_X (which he calls a double EPW sextic) is a (smooth) irreducible symplectic fourfold of $\mathrm{Hilb}^{[2]}$K3-type.

As in the case of cubic fourfolds, we may define an Abel-Jacobi map

$$a : H^4(X, \mathbf{Z}) \to H^2(C(X), \mathbf{Z})$$

which is a morphism of Hodge structures and very likely factors as

$$a : H^4_0(X, \mathbf{Z}) \xrightarrow{u} H^2_0(Y_X, \mathbf{Z}) \overset{\beta^*}{\hookrightarrow} H^2(C(X), \mathbf{Z}),$$

where u is an isomorphism of polarized Hodge structures.

O'Grady proved that the moduli space $\mathscr{EPW}$ of EPW sextics is quasi-projective and has dimension 20. There is a period map $q : \mathscr{EPW} \to \Gamma\backslash\mathscr{D}$ which is an open immersion by recent work of Verbitsky, and a commutative diagram

$$\begin{array}{ccc} \mathscr{X} & \overset{\mathrm{epw}}{\dashrightarrow\!\!\!\rightarrow} & \mathscr{EPW} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q,\ 1:1} & \\ & \Gamma\backslash\mathscr{D}. & \end{array}$$

The general (4-dimensional) fibers of p and epw are therefore the same (birationally).

2.4. **Rationality.** All our fourfolds X are unirational (they have a rational double cover). Fano and Roth already knew that some of them are rational.

Examples 2.1. 1) The irreducible family $\mathscr{X}_{\rho\text{-plane}}$ of fourfolds containing a ρ-plane has codimension 3 in $\mathscr{X}$. The period map p induces a dominant map $\mathscr{X}_{\rho\text{-plane}} \to \mathscr{D}_{12}$.

The unique irreducible component of $p^{-1}(\mathscr{D}_{12})$ that contains $\mathscr{X}_{\rho\text{-plane}}$ is the codimension-1 family $\mathscr{X}_{\text{cubic scroll}}$ of fourfolds containing a cubic scroll surface.

A general member of $\mathscr{X}_{\rho\text{-plane}}$ is birationally isomorphic to a cubic hypersurface containing a cubic scroll surface, hence is not expected to be rational.

2) The irreducible family $\mathscr{X}_{\sigma\text{-plane}}$ of fourfolds containing a σ-plane has codimension 2 in $\mathscr{X}$. The period map p induces a dominant map $\mathscr{X}_{\sigma\text{-plane}} \to \mathscr{D}''_{10}$.

The unique irreducible component of $p^{-1}(\mathscr{D}''_{10})$ that contains $\mathscr{X}_{\sigma\text{-plane}}$ is the codimension-1 family $\mathscr{X}_{\text{quintic}}$ of fourfolds containing a quintic del Pezzo surface.

A general member of each of these two families is rational.

3) The codimension-1 family $\mathscr{X}_{\tau\text{-quadric}}$ of fourfolds containing a τ-quadric surface is an irreducible component of $p^{-1}(\mathscr{D}'_{10})$. The period

map p induces a dominant map $\mathscr{X}_{\tau\text{-quadric}} \to \mathscr{D}'_{10}$ whose general fiber is rationally dominated by the symmetric square of a degree-10 K3 surface.

A general member of $\mathscr{X}_{\tau\text{-quadric}}$ is rational.

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