HOW TO CLASSIFY FANO VARIETIES?

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Abstract. We review some of the methods used in the classification of Fano varieties and the description of their birational geometry. Mori theory brought important simplifications to this classical theory which we will illustrate with a few examples.

1. Introduction

For us, a *Fano variety* will be a *smooth* complex projective algebraic variety whose anticanonical bundle (i.e., the determinant of the tangent bundle) is ample. In other words, the first Chern class can be represented by a positive definite differential form. Here are some examples: all projective spaces are Fano varieties (more generally, all smooth complete intersections in $\mathbf{P}^n$ of hypersurfaces of degrees $d_1, \dots, d_c$ are Fano varieties, provided $d_1 + \cdots + d_c \le n$) and, more generally, so are all homogeneous projective varieties under a connected linear algebraic group (and their linear sections of small enough degrees).

From the differential geometry viewpoint, Fano varieties are (by the Calabi-Yau theory) compact Kähler manifolds with positive definite Ricci curvature.

In this talk, I will present some aspects of the classification of Fano varieties of low dimensions.

We begin with some notation and a couple of remarks. If X is a Fano variety, the classical approach to classification is via the anticanonical linear system $|-K_X|$. If it is very ample, it defines a closed embedding $X \hookrightarrow \mathbf{P}^{h^0(X,-K_X)-1}$ whose image has degree d.

It is customary to call the greatest integer r such that one can write $K_X \underset{\text{num}}{\equiv} -rH$ (with H ample, called a *fundamental divisor*) the *index* of X. For all $i > 0$

$$H^i(X, sH) = H^i(X, K_X + (r+s)H) = 0$$

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by Kodaira vanishing for $s > -r$. In particular, $h^0(X,H) = \chi(X,H)$ can (sometimes) be computed by Riemann-Roch or other tricks (see §4).

As usual, we denote by $\rho(X)$ the Picard number of X (since $h^2X, \mathscr{O}_X) = 0$ by Kodaira vanishing, this is also $b_2(X)$). Fano varieties with $\rho = 1$ are called *prime.*

2. Curves and surfaces

If X is a smooth projective curve of genus g, the degree of $-K_X$ is $2-2g$. For $-K_X$ to be ample, we need this number to be positive, hence $g = 0$. Conversely, smooth projective curves of genus 0 are isomorphic to $\mathbf{P}^1$, which is a Fano curve. Hence the only Fano curve is $\mathbf{P}^1$.

The situation becomes more complicated for surfaces (Fano surface are actually called *del Pezzo* surfaces). We study the linear system $|-K_X|$. As explained above, one has by Riemann-Roch

$$h^0(X, -K_X) = \chi(X, -K_X) = K_X^2 + 1.$$

Here are two classical results (see §III.3 in [K]).

Theorem 1. *Let X be a del Pezzo surface of degree $d := (-K_X)^2$. Then,*

- *we have $1 \le d \le 9$;*
- *if $d \ge 3$, the linear system $|-K_X|$ is very ample hence induces an embedding $X \subset \mathbf{P}^d$ whose image is a smooth surface of degree d.*

This theorem implies in particular that del Pezzo surfaces of degree 3 are (smooth) cubic surfaces in $\mathbf{P}^3$ (easy). Del Pezzo surfaces of degree 4 are (smooth) complete intersections of two quadrics in $\mathbf{P}^4$ (why?).

Let us also mention the following result, which shows that prime del Pezzo surfaces are not very interesting.

Theorem 2. *Any del Pezzo surface with $\rho = 1$ is isomorphic to $\mathbf{P}^2$.*

3. Fano threefolds: the classical method

Let X be a Fano threefold of degree $d := (-K_X)^3$. One has (see §4)

$$h^0(X, -K_X) = \chi(X, -K_X) = \frac{1}{2}d + 3.$$

It is customary to call the integer $g := \frac{1}{2}d + 1$ the *genus* of X (it is the genus of any smooth curve obtained as the complete intersection of two elements of $|-K_X|$). The analog of Theorem 1 above is the following (see [IP], Proposition 4.1.11 and Corollary 4.1.13).

Theorem 3 (Iskovskikh)**.** *Let X be a* prime *Fano threefold of degree d. If $g \geq 4$, the linear system $|-K_X|$ is very ample hence induces an embedding $X \subset \mathbf{P}^{g+1}$ whose image is a smooth threefold of degree $d = 2g-2$. If moreover $g \geq 5$, the image is an intersection of quadrics.*

There is also a bound $d \leq 64$ which is obtained from the classification.[1]

From now on, we will consider prime Fano threefolds $X \subset \mathbf{P}^{g+1}$, anticanonically embedded of degree $d = 2g - 2$ (these are actually the only threefolds that Fano himself considered).

3.1. **Lines.** Assume then that $X \subset \mathbf{P}^{g+1}$ is an anticanonically embedded Fano threefold (so that $g \geq 3$) such that $\mathrm{Pic}(X) = \mathbf{Z}[H]$. Note that the degree of any surface contained in X is a multiple of $H^3 = d \geq 4$. In particular, X contains no planes.

Assume that X contains a line ℓ. The exact sequence

$$0 \to T_\ell \to T_X|_\ell \to N_{\ell/X} \to 0$$

implies

$$\deg(N_{\ell/X}) = \deg(T_X|_\ell) - \deg(T_\ell) = -K_X \cdot \ell - 2 = -1.$$

In particular, by Riemann-Roch, $\chi(\ell, N_{\ell/X}) = 1$, hence the scheme $L(X)$ of lines contained in X has everywhere positive dimension.

Lemma 4. *There are only two possibilities:*

$$\begin{array}{lcll} N_{\ell/X} & \simeq & \mathscr{O}_\ell(-1) \oplus \mathscr{O}_\ell & \textit{(lines of the first type);} \\ N_{\ell/X} & \simeq & \mathscr{O}_\ell(-2) \oplus \mathscr{O}_\ell(1) & \textit{(lines of the second type).} \end{array}$$

In the first case, $L(X)$ is smooth of dimension 1 at $[\ell]$; in the second case, either $L(X)$ is smooth of dimension 2 at $[\ell]$, or it is singular of dimension 1.

Proof. By Bertini, a general hyperplane section of X containing ℓ is a smooth K3 surface S. We have $K_S \cdot \ell = 0$, hence, by adjunction, $\ell^2 = -2$, so that $N_{\ell/S} \simeq \mathscr{O}_\ell(-2)$. The normal exact sequence

$$\begin{array}{ccccccccc} 0 & \to & N_{\ell/S} & \to & N_{\ell/X} & \to & N_{S/X}|_\ell & \to & 0 \\ & & \| & & & & \| & & \\ & & \mathscr{O}_\ell(-2) & & & & \mathscr{O}_\ell(1) & & \end{array}$$

gives what we need. ∎

Lemma 5. *The scheme $L(X)$ has pure dimension 1.*

[1] It should be said here that *all* Fano threefolds have been classified! There are 17 families of prime Fano threefolds (all but one known to Fano himself; see §4.2) and 88 families of nonprime Fano threefolds.

Proof. If not, it has a smooth component of dimension 2 whose points correspond to lines of the second type. These lines are not free hence cannot cover X, but only a surface in X. But only one surface contains a 2-dimensional family of lines: $\mathbf{P}^2$. This is absurd since X contains no planes. $\square$

Elementary considerations ([IP], Proposition 4.3.1) show that for $g \geq 4$, given any line $\ell \subset X$,

only finitely many lines contained in X meet ℓ.

3.2. **Elementary transformations.** Let again $X \subset \mathbf{P}^{g+1}$ be an anti-canonically embedded prime Fano threefold that contains a line ℓ. The projection

$$\pi_\ell : X \dashrightarrow \mathbf{P}^{g-1}$$

can be resolved by the blow-up $\varepsilon : \widetilde{X} \to X$ of ℓ. The composition

$$\widetilde{X} \xrightarrow{\varepsilon} X \overset{\pi_\ell}{\dashrightarrow} \mathbf{P}^{g-1}$$

is the morphism associated with the base-point-free linear system $|-K_{\widetilde{X}}| = |\varepsilon^* H - E|$. We let $\bar{X}$ be the normalization of its image and denote by $\varphi : \widetilde{X} \to \bar{X}$ the induced morphism.

What are the fibers? Outside of E, they are the intersections with X of planes containing ℓ (minus ℓ). Since X contains no planes, they are the curves residual to ℓ in these intersections. Assume $g \geq 5$, so that X is an intersection of quadrics. Fibers can then only be (outside of E)

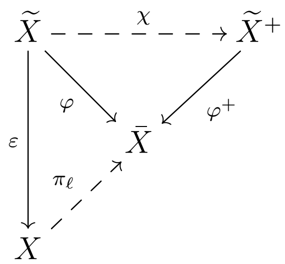
- one point;
- or a line meeting ℓ.

This (almost) proves that

the morphism $\varphi : \widetilde{X} \to \bar{X}$ is a small contraction.

Assume that φ is not an isomorphism (since $-K_{\widetilde{X}} = \varepsilon^*(-K_X) - E = \varphi^* \bar{H}$, this is exactly saying that $\widetilde{X}$ is *not* a Fano threefold). Since $\rho(\widetilde{X}) = 2$, we have the two extremal rays of the Mori cone of curves of $\widetilde{X}$, with contractions ε and φ (with relative Picard number 1).

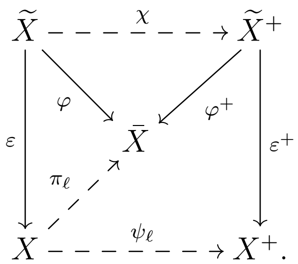
In that situation, we can perform a flop:[2]



where

- χ is an isomorphism in codimension 1;
- the projective threefold $\widetilde{X}^+$ is smooth;
- we have $-K_{\widetilde{X}^+} \underset{\text{lin}}{\equiv} \varphi^{+*}\bar{H}$.

We have $\rho(\widetilde{X}^+) = \rho(\widetilde{X}) = 2$. Since the extremal ray defined by φ^+ has $K_{\widetilde{X}^+}$-degree 0 and $K_{\widetilde{X}^+}$ is not nef, the other extremal ray is $K_{\widetilde{X}^+}$-negative and defines a Mori contraction $\varepsilon^+ : \widetilde{X}^+ \to X^+$ (where X^+ has dimension 1, 2 or 3). So we may complete the diagram above as follows:



The rational map ψ_ℓ is called the *elementary transformation with center ℓ.*

3.3. **Classification.** Extremal contractions in dimension 3 have been classified by Mori. Their types are rather restricted ([IP], Theorem 1.4.3) and, together with computations of intersection numbers on $\widetilde{X}$ and $\widetilde{X}^+$, this leads to a complete description of all possible cases (assuming X contains a line!). In particular, we have the following boundedness result.

[2]If ℓ' is a line contracted by φ, we have $\varphi^*\bar{H} \cdot \ell' = 0$ and $E \cdot \ell' = 1$. Since $\varphi(E) \subsetneq \varphi(\widetilde{X})$, we have $|m\varphi^*\bar{H} - E| \neq \varnothing$ for $m \gg 0$, and for D in that linear system, $D \cdot \ell' < 0$. Since the relative Picard number of φ is 1, this means that D is φ-antiample. In that situation, we can perform a D-flop χ, which is nothing else than a flip for the klt pair $(\widetilde{X}, tD)$ for $t > 0$ small. The divisor χ_*D is now φ^+-ample.

Theorem 6. *Let $X \subset \mathbf{P}^{g+1}$ be an anticanonically embedded prime Fano threefold that contains a line. Then, $g \leq 12$ and $g \neq 11$.*

But how to prove that X always contains a line? This is unfortunately not easy.

The general classification of anticanonically embedded prime Fano threefolds is actually obtained with a slightly different method (Takeguchi) that bypasses completely this problem: given a general point $x \in X$, starting from the projection $X \dashrightarrow \mathbf{P}^{g-3}$ from the projective tangent space to X at x (this is classically called a "double projection"), one constructs as above an *elementary transformation* $\psi_x : X \dashrightarrow X_x$ *with center* x whose analysis is similar, but more difficult than for ψ_ℓ (there are more possibilities for curves contracted by φ). One then gets directly Theorem 6 and a complete classification ([IP], Theorem 4.5.8; one then needs to prove that they actually occur). A case-by-case check shows that any X from the list does contain a line!

4. The vector bundle method

Gushel realized around 1982 that some Fano threefolds (from the list above) are contained in a Grassmannian of the type $G(2, m)$. This gave him the idea to classify them (differently) by producing directly the corresponding vector bundle of rank 2 using the Serre construction and a suitable elliptic curve.

Mukai vastly generalized this idea around 1988 (see [M]) and completely classified prime Fano varieties X of dimension $n \geq 3$ and index $n - 2$.[3]

Before explaining Mukai's method, let us make some remarks on the index r of a Fano variety X of dimension n. There is a very neat trick of Shokurov's that is very easy to explain.

We write as before $-K_X \underset{\text{lin}}{\equiv} rH$, with H ample. By Riemman-Roch, $P(m) := \chi(X, mH)$ is a polynomial of degree n and leading coefficient $\alpha := H^n/n!$. As explained in the introduction, it is equal to $h^0(X, mH)$ for $-r < m$, hence it vanishes for $-r < m < 0$. This implies already $r \leq n + 1$[4] and gives strong restrictions on P when r is large.

Let me also mention the following facts:

- when $r = n + 1$, the variety X is isomorphic to $\mathbf{P}^n$;

[3]Actually, Mukai needed the extra assumption that the intersection of $n - 2$ general members of $|H|$ be a smooth surface; this was proved in all dimensions by Mella in 1996.

[4]This can also be obtained, in all characteristics, using Mori theory.

- when $r = n$, the variety X is isomorphic to a smooth quadric in $\mathbf{P}^{n+1}$;
- when $r = n - 1$, one says that X is a *del Pezzo varieties*; they have been classified.

Going back to our case ($r = n - 2$) and to Shokurov's trick, we can write

$$P(m) = (m + 1) \cdots (m + n - 3)(\alpha m^3 + \beta m^2 + \gamma m + \delta).$$

Then $P(0) = 1$ and $P(m) = (-1)^n P(-m - n + 2)$ by Serre duality, and all this implies

$$h^0(X, H) = P(1) = g + n - 1,$$

where $g := \frac{1}{2}H^n + 1$ is an integer.

It is known that $|H|$ is base-point-free, hence defines a morphism $X \to \mathbf{P}^{g+n-2}$ which is an embedding for $g \geq 4$. Note that any smooth linear section of X (if of dimension ≥ 3) is still a Fano variety of the same type.

Theorem 7. *Let $X \subset \mathbf{P}^{g+n-2}$ be a prime Fano n-fold of index $n - 2$. Then, $g \leq 12$ and $g \neq 11$.*

Moreover, there is a complete description of all possible X. When $g \geq 6$, they are all linear sections of various Grassmannians (some quite exotic) and $n \leq 10$.

Proof. A general surface section of X is a (smooth) K3 surface S of degree d and Mukai's idea is to construct a vector bundle on S and then extend it to X.

For the bound on g, we may assume $n = 3$. Consider the (19-dimensional) moduli space $\mathscr{F}_g$ of polarized K3 surfaces of genus g, the $(g + 19)$-dimensional moduli space $\mathscr{P}_g$ of pairs (S, C) where $S \subset \mathbf{P}^g$ is a K3 surface of genus g and $C \subset S$ a hyperplane section, and the $(3g - 3)$-dimensional moduli space $\mathscr{M}_g$ of curves of genus g. There are maps

$$\begin{array}{ccc} \mathscr{P}_g & \overset{\varphi_g}{\dashrightarrow} & \mathscr{M}_g \\ \downarrow & & \\ \mathscr{F}_g & & \end{array}$$

Consider a general pencil $P = \{S_t \mid t \in \mathbf{P}^1\}$ of hyperplane sections of X. All the K3 surfaces S_t contain the (smooth) base curve C hence P lifts to a curve in $\mathscr{P}_g$ which is contained in the fiber of φ_g over $[C]$ (the curve $P \dashrightarrow \mathscr{F}_g$ is not constant, because P contains both singular and smooth members). One checks that $[C]$ is a general member of the image of φ_g, hence φ_g is not generically finite onto its image. But Mukai computed the differential of the map φ_g at a general point and proved that it is injective for $g = 11$ or $g \geq 13$. This proves the theorem. $\square$

Let us now turn to the description of X, assuming $g \in \{6, 7, 8, 9, 10, 12\}$. We want to construct a vector bundle $\mathscr{E}$ on X. Then (for $g \neq 7$), for each decomposition $g = rs$, Mukai constructs a stable rank-r vector bundle $\mathscr{E}_S$ on S with $h^0(S, \mathscr{E}_S) = r + s$ and $c_1(\mathscr{E}_S) = H|_S$, which is unique with these properties. It is generated by its global sections.

Assume $r, s > 1$. A theorem of Fujita then says that one can extend $\mathscr{E}_S$ to a stable rank-r vector bundle $\mathscr{E}$ on X with $h^0(X, \mathscr{E}) = r + s$ and $c_1(\mathscr{E}) = H$, which is again unique with these properties. The vector bundle $\mathscr{E}$ is still generated by its global sections hence defines a morphism

$$\varphi_{\mathscr{E}} : X \to G(r, r + s)$$

such that $\varphi_{\mathscr{E}}^* \mathscr{S}_r^\vee \simeq \mathscr{E}$ (in particular, $\varphi_{\mathscr{E}}^* \mathscr{O}_G(1) \simeq \mathscr{O}_X(H)$, hence $\varphi_{\mathscr{E}}$ is finite and $n \leq rs = g$).

4.1. **Case** $g = 6$. We have $X \subset \mathbf{P}^{n+4})$ and we take $r = 2$ and $s = 3$. We obtain $n \leq 6$ and a morphism $\varphi : X \to G(2, V_5)$, where $V_5 := H^0(X, \mathscr{E})^\vee$.

We have a linear map

$$\eta : \wedge^2 H^0(X, \mathscr{E}) \to H^0(X, \wedge^2 \mathscr{E}) \simeq H^0(X, H)$$

and a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi_{\mathscr{E}}\ } & G(2, V_5) \\ {\scriptstyle \varphi_H} \big\downarrow & & \big\downarrow {\scriptstyle \text{Plücker}} \\ \mathbf{P}(H^0(X, H)^\vee) & \overset{\eta^\vee}{\dashrightarrow} & \mathbf{P}(\wedge^2 V_5). \end{array}$$

There are two cases:

- *either η is surjective;* the map $\eta^\vee$ is injective with image a $\mathbf{P}^{n+4} \subset \mathbf{P}(\wedge^2 V_5)$. The intersection $W := G(2, V_5) \cap \mathbf{P}^{n+4}$ is smooth of dimension $n + 1$ and degree 5, and $\varphi_H(X) \subset W$ is a divisor of degree 10, hence is the intersection of W with a quadric.
- *or the corank of η is 1;* consider the cone $CG \subset \mathbf{P}(\mathbf{C} \oplus \wedge^2 V_5)$, with vertex $v = \mathbf{P}(\mathbf{C})$, over $G(2, V_5)$. In the above diagram, we may view $H^0(X, H)^\vee$ as embedded in $\mathbf{C} \oplus \wedge^2 V_5$ by mapping $(\mathrm{Im}(\eta))^\perp$ to $\mathbf{C}$, and $\eta^\vee$ as the projection from v, with image a $\mathbf{P}^{n+3} \subset \mathbf{P}(\wedge^2 V_5)$. Again, $W := G(2, V_5) \cap \mathbf{P}^{n+3}$ is smooth and $\varphi_H(X)$ is the intersection of the cone CW with a quadric. The

map $\varphi : X \to W$ is a double cover branched along the (smooth) intersection of W with a quadric.

For example, when $n = 6$ (the maximal possible dimension when $g = 6$), we are in the second case and $\varphi_{\mathscr{E}} : X \to G(2, V_5)$ is a double cover branched along the (smooth) intersection of $G(2, V_5)$ with a quadric (a "quadratic line complex," a Fano fivefold of index 3).

4.2. **Case $g = 12$.** This case was overlooked by Fano and discovered by Iskovskikh. We take $r = 3$ and $s = 4$. Mukai shows $n = 3$ and he proves that $X \subset G(3, 7)$ is the zero locus of a section of the rank-9 vector bundle $(\wedge^2 \mathscr{S}_3^\vee)^{\oplus 3}$. Together with Umemura, he also proved that there is a unique such threefold with automorphism group $\mathrm{PGL}(2, \mathbf{C})$; it is a compactification of $\mathbf{C}^3$.

5. What's next?

5.1. **Prime Fano n-folds of index $n - 2$.** Now that we have a good description of these Fano n-folds, one can ask about

- their birational properties (and in particular, whether they are rational);
- their moduli spaces;
- their period maps.

These are very difficult questions that are only partially answered. The elementary transformations introduced earlier are very useful because they often give nontrivial rational maps between Fano threefolds with sometimes different numerical invariants. They can also be generalized in higher dimensions (but more care is needed, because images of flops are no longer smooth in general).

5.2. **Positive characteristics.** Shepherd-Barron completed the classification of *prime* Fano threefolds in positive characteristic. There are no new families.

5.3. **Singular Fano varieties.** For Mori's minimal model program, one need to consider singular Fano varieties. For example, one may only require that some multiple of the Weil divisor $-K_X$ be a Cartier divisor. Even for surfaces, the classification in this case is (to my knowledge) not complete.

5.4. **Nonalgebraically closed fields.** There is a nice discussion of del Pezzo surfaces over any field in [K], §III.3.

5.5. **Higher dimensions.** Corti and his collaborators have embarked on a program to classify Fano fourfolds using mirror symmetry. The first step would be to recover the known classification of Fano threefolds in a more systematic way. Very heavy computer calculations are involved.

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