ON PRIME FANO VARIETIES OF DEGREE 10 AND COINDEX 3

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Abstract. According to Gushel and Mukai, most Fano varieties of degree 10, dimension n, and coindex 3 (index $n-2$) are obtained as linear sections of the intersection of the Grassmannian $G(2,5)$ in its Plcker embedding with a quadric. In particular, n can only be 3, 4, or 5; they depend on 22, 24, and 25 parameters respectively. We will study the geometry of these varieties, their links with the double EPW-sextics studied by O'Grady, and their various period maps. We work over the complex numbers, and this is joint work in progress with A. Iliev and L. Manivel.

1. Prime Fano varieties of degree 10 and coindex 3

A *Fano variety* is a smooth projective variety X whose canonical bundle is anti-ample. It is *prime* if the Picard group of X is isomorphic to $\mathbf{Z}$; if H is a generator, the *coindex* of X is the number r such that $-K_X \underset{\text{lin}}{\equiv} (n+1-r)H$, where $n := \dim(X)$. The *degree* of X is the positive number H^n.

According to Gushel and Mukai, any prime Fano n-fold of coindex 3 and degree 10 is obtained as follows. Consider the Grassmannian $G := G(2, V_5) \subset \mathbf{P}(\wedge^2 V_5) = \mathbf{P}^9$ in its Plücker embedding. Then $n \in \{3,4,5\}$ and X is

- either a smooth linear section by a $\mathbf{P}^{n+4}$ of the intersection of $G \subset \mathbf{P}^9$ with a quadric;
- or a double cover of a smooth linear section by a $\mathbf{P}^{n+3}$ of $G \subset \mathbf{P}^9$, branched along its intersection with a quadric.

Note that the second case (the " Gushel" case) is a degeneration of the first case: consider, in $\mathbf{P}^{10}$, the 7-dimensional cone CG over $G \subset \mathbf{P}^9$, with vertex v, and its intersection with a general quadric and a $\mathbf{P}^{n+4}$. If the $\mathbf{P}^{n+4}$ does not contain v, everything takes place in that linear space and we are in the first case; if the $\mathbf{P}^{n+4}$ does contain v, we are in the second case.

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One checks that when $n = 3$, 4, or 5, the corresponding family $\mathscr{X}_n$ depends on 22, 24, and 25 parameters respectively. Also, the points corresponding to Gushel n-folds form a subfamily which is birational to $\mathscr{X}_{n-1}$.

2. Prime Fano fourfolds of degree 10 and coindex 3

Let X be a fourfold of this type.

2.1. **Local deformations.** By Kodaira-Akizuki-Nakano vanishing, we have $H^i(X, T_X) = 0$ for $i > 1$ (since $T_X \simeq \Omega^3_X(2)$), so local deformations of X are unobstructed. One computes $H^0(X, T_X) = 0$ hence $H^1(X, T_X) = \chi(X, T_X) = 24$. So we have a local universal deformation

$$\mathscr{X} \to T,$$

where T is a 24-dimensional polydisk.

2.2. **The local period map.** One computes the Hodge diamond of X (Iliev-Manivel):

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&0&&0&&& \\ &&0&&1&&0&& \\ &0&&0&&0&&0& \\ 0&&1&&22&&1&&0 \end{array}$$

In particular, the Hodge structure on $H^4(X)$ is of K3 type. The Hodge structure $H^4(G)$ embeds into $H^4(X)$ and we consider as usual the *primitive cohomology*

$$H^4(X)_0 := H^4(G)^\perp \subset H^4(X).$$

The intersection form has signature $(20, 2)$ on that space. Moreover, one checks that the lattice $H^4(X, \mathbf{Z})_0$, with the intersection form, is isomorphic to

$$\Lambda := 2E_8 \oplus 2U \oplus 2\langle 2\rangle.$$

The local universal deformation $\mathscr{X} \to T$ is $\mathscr{C}^\infty$-trivial. This means that we can identify all the lattices $H^4(X_t, \mathbf{Z})_0$ (with their intersection form) with the fixed lattice Λ.

Inside the fixed vector space $\Lambda \otimes_{\mathbf{Z}} \mathbf{C}$, the line $H^{3,1}(X_t)$ moves. This defines a *local period map*

$$T \to \mathbf{P}(\Lambda \otimes_{\mathbf{Z}} \mathbf{C}).$$

Because of Riemann's bilinear relations, the period actually lands into the smaller 20-dimensional subset

$$\Omega := \{\omega \in \mathbf{P}(\Lambda \otimes_{\mathbf{Z}} \mathbf{C}) \mid q(\omega) = 0, q(\omega + \bar\omega) > 0\}$$

called the *local period domain.*

2.3. **The global period map.** If one wants to globalize the construction, one needs to take the quotient of Ω under the action of a suitably defined arithmetic group Γ of automorphisms of Λ. One obtains the global period map

$$\wp : \mathscr{X}_4 \to \mathscr{D} := \Gamma\backslash\Omega.$$

A local calculation shows that the local period map is a submersion, hence $\wp$ is dominant, and its general fibers have dimension 4. Our aim is to describe these fibers.

3. Cubic fourfolds

Since the situation is very similar to the case of cubic fourfolds, I will briefly describe what is known in that case.

3.1. **The period map.** Let $X \subset \mathbf{P}^5$ be a smooth cubic fourfold. Its primitive integral cohomology $H^4(X, \mathbf{Z})_0$ is isomorphic to the lattice

$$2E_8 \oplus 2U \oplus \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The moduli space of smooth cubic fourfolds is, by GIT, a quasi-projective 20-dimensional variety $\mathscr{C}_4$, and we have as above a dominant global period map

$$\wp : \mathscr{C}_4 \to \mathscr{D}'.$$

However, in this case, we have local Torelli ($\wp$ is unramified) and global Torelli (Voisin): $\wp$ is an open embedding.

Moreover, the complement of the image of the period map in the Baily-Borel compactification $\overline{\mathscr{D}}'$ is known by recent work of Laza: it is the union of two irreducible hypersurfaces. Laza also extends the period map to the GIT compactification of $\mathscr{C}_4$.

3.2. **The variety of lines.** Let $F(X) \subset G(1, \mathbf{P}^5)$ be the (smooth projective) variety that parametrizes lines contained in X. It is an irreducible symplectic fourfold. By Beauville and Donagi's work, we know that the primitive cohomology $H^4(X, \mathbf{Z})_0$ is isomorphic to the primitive cohomology $H^4(F(X), \mathbf{Z})_0$.

More precisely, consider the incidence variety

$$I := \{(x, \ell) \in X \times F(X) \mid x \in \ell\}$$

with its projections $p : I \to X$ and $q : I \to F(X)$. Then the *Abel-Jacobi map*

$$a := q_* p^* : H^4(X, \mathbf{Z})_0 \to H^2(F(X), \mathbf{Z})_0$$

is an isomorphism of polarized Hodge structures. The family of $F(X)$, as X describes $\mathscr{C}_4$, is a locally complete family of deformations. The Torelli problem for these varieties was recently solved by Verbitsky. We will come back to that.

4. Double EPW sextics

We will show that one can also attach to a prime Fano fourfold of degree 10 and coindex 3 an irreducible symplectic fourfold.

Assume for simplicity that we are in the non-Gushel case: X is the (smooth) intersection, in $\mathbf{P}^9$, of $G := G(2, V_5)$, a hyperplane $\mathbf{P}^8$, and a quadric Q.

4.1. **The associated sextic.** The vector space $I_G(2)$ of quadrics containing G has dimension 5 and consists of (rank-6) *Plücker quadrics.* More precisely, we have an isomorphism

$$\begin{array}{ccc} V_5 & \xrightarrow{\sim} & I_G(2) \\ \alpha & \longmapsto & (\omega \mapsto \omega \wedge \omega \wedge \alpha). \end{array}$$

Then, $I_X(2) \simeq I_G(2) \oplus \mathbf{C}Q$. Let $V_4 \subset V_5$ be a hyperplane. All Plücker quadrics restrict in $\mathbf{P}(\wedge^2 V_4)$ to the sole (smooth) Plücker quadric defining $G(2, V_4) \subset \mathbf{P}(\wedge^2 V_4)$, hence elements of $I_X(2)$ restrict to a pencil. A general element of that pencil is smooth, and a finite number of elements are singular. These elements define a finite number of hyperplanes in $I_X(2)$, hence points in $\mathbf{P}(I_X(2)^\vee)$. When V_4 varies, we get a subvariety

$$Z_X \subset \mathbf{P}(I_X(2)^\vee) = \mathbf{P}^5.$$

Theorem 4.1 (Iliev-Manivel)**.** *For X general in $\mathscr{X}_4$, the fourfold $Z_X \subset \mathbf{P}^5$ is an EPW sextic hypersurface.*

4.2. **Double EPW sextics.** What is an EPW sextic? Let U_6 be a 6-dimensional vector space. If we choose an isomorphism $\wedge^6 U_6 \simeq \mathbf{C}$, the vector space $\wedge^3 U_6$ inherits a non-degenerate skew-symmetric form given by wedge product. Let $A \subset \wedge^3 U_6$ be a general Lagrangian subspace. Then

$$Z_A := \{U_5 \subset U_6 \mid \wedge^3 U_5 \cap A \neq 0\} \subset \mathbf{P}(U_6^\vee)$$

is a *EPW sextic.* Its singular locus is the smooth surface

$$S_A := \{U_5 \subset U_6 \mid \dim(\wedge^3 U_5 \cap A) \geq 2\}.$$

There is a canonically defined double cover (O'Grady)

$$\pi : Y_A \to Z_A,$$

such that

- the branch locus of π is the surface S_A;
- the fourfold Y_A is a (smooth) irreducible symplectic fourfold.

4.3. **A conjecture.** Iliev and Manivel gave a beautiful geometric construction of the canonical double cover of the EPW sextic Z_X. It goes as follows. They prove that the variety $C(X)$ of (possibly degenerate) conics contained in X is a smooth projective fivefold.

A general conic $c \subset X$ is contained in a unique $G(2, V_4)$, for some hyperplane $V_4 \subset V_5$. The 4-plane $\mathbf{P}(\wedge^2 V_4) \cap \mathbf{P}^8$ contains two quadrics:

- its intersection with $G(2, V_4)$,
- its intersection with Q,

whose intersection is contained is $X \cap \mathbf{P}(\wedge^2 V_4)$. They both contain c, but not both $\langle c \rangle$, because X contains no 2-planes. Thus, there exists a unique quadric in this pencil that contains $\langle c \rangle$, and this quadric must be singular (because a smooth quadric in $\mathbf{P}^4$ contains no 2-planes). This defines a morphism $C(X) \to Z_X$ whose Stein factorisation is

$$C(X) \xrightarrow{\beta} Y_X \xrightarrow{\pi} Z_X,$$

where π is the canonical double cover defined above. On the other hand, O'Grady computed that the primitive cohomology lattice $H^2(Y_X, \mathbf{Z})_0$ is isomorphic to the lattice Λ defined earlier, hence to $H^4(X, \mathbf{Z})_0$. As in the case of cubic fourfolds, we may define an Abel-Jacobi map

$$a : H^4(X, \mathbf{Z}) \to H^2(C(X), \mathbf{Z})$$

which is a morphism of Hodge structures.

Conjecture 4.2. On the primitive cohomology, the Abel-Jacobi map factors as

$$a : H^4(X, \mathbf{Z})_0 \xrightarrow{\sim} H^2(Y_X, \mathbf{Z})_0 \overset{\beta^*}{\hookrightarrow} H^2(C(X), \mathbf{Z}).$$

Moreover,

$$\forall u, v \in H^4(X, \mathbf{Z})_0 \qquad q_{Y_X}(a(u), a(v)) = -u \cdot v.$$

4.4. **Moduli and period maps.** Let $\mathscr{E}\mathscr{P}\mathscr{W}$ be the 20-dimensional moduli space of EPW sextics (O'Grady proved that it is quasi-projective by GIT). By the theorem above, there is a rational map

$$\begin{array}{rccc} \mathrm{epw} : & \mathscr{X}_4 & \dashrightarrow & \mathscr{E}\mathscr{P}\mathscr{W} \\ & [X] & \longmapsto & Z_X. \end{array}$$

Iliev and Manivel proved that it is dominant. Its general fibers therefore have dimension 4.

On the other hand, there is also a global period map

$$p : \mathscr{E}\mathscr{P}\mathscr{W} \to \mathscr{D}$$

with values in the same period domain as for our fourfolds, which takes an EPW sextic to the period of its canonical double cover. By work of Verbitsky on the Torelli problem for irreducible symplectic manifolds, *p is dominant and birational.*

If we believe the conjecture above, we get a commutative diagram

$$\begin{tikzcd} \mathscr{X}_4 \arrow[rr, dashed, "\mathrm{epw}"] \arrow[dr, "\wp"'] & & \mathscr{E}\mathscr{P}\mathscr{W} \arrow[dl, "p", "1:1"'] \\ & \mathscr{D} & \end{tikzcd}$$

and (at least birationally), the general (4-dimensional) fibers of $\wp$ and epw are the same.

4.5. **Gushel fourfolds.** We explained earlier that the Gushel constructions induces a morphism $\mathscr{X}_3 \to \mathscr{X}_4$. In terms of Hodge theory, the Hodge diamond of a threefold $B \in \mathscr{X}_3$ is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 10 & & 10 & & 0 \end{array}$$

so there is a 10-dimensional intermediate Jacobian

$$J(B) := H^{1,2}(B)/H^3(B,\mathbf{Z})$$

and a period map $\mathscr{X}_3 \to \mathscr{A}_{10}$.

However, the relation between the Hodge theory of B and that of the associated Gushel fourfold X is not clear at all (although very similar constructions have been studied in great detail for cubic hypersurfaces by Allcock, Carlson, and Toledo).

Here is what we can do at the moment:

- identify the fibers of $\mathscr{X}_3 \to \mathscr{X}_4 \xrightarrow{\mathrm{epw}} \mathscr{E}\mathscr{P}\mathscr{W}$;
- prove that the period map $\mathscr{X}_3 \to \mathscr{A}_{10}$ factors as

$$\mathscr{X}_3 \longrightarrow \mathscr{E}\mathscr{P}\mathscr{W} \longrightarrow \mathscr{A}_{10}.$$

O'Grady proved that the (projective) dual of an EPW sextic is still of that type, so that duality induces a (non-trivial) involution on $\mathscr{E}\mathscr{P}\mathscr{W}$. We prove a further factorization

$$\mathscr{X}_3 \longrightarrow \mathscr{E}\mathscr{P}\mathscr{W} \longrightarrow \mathscr{E}\mathscr{P}\mathscr{W}/\mathrm{duality} \xrightarrow{\gamma} \mathscr{A}_{10}.$$

Note that since $p : \mathscr{E}\mathscr{P}\mathscr{W} \to \mathscr{D}$ is birational, we may also write this factorization as

$$\mathscr{X}_3 \longrightarrow \mathscr{D} \longrightarrow \mathscr{D}/r \dashrightarrow \mathscr{A}_{10}.$$

So the situation is subtle: there are threefolds with isomorphic polarized Hodge structures, but such that the Hodge structures of the associated Gushel fourfolds differ by the involution r.

We make two final conjectures:

- the map γ is birational onto its image;
- the fiber of $\wp$ through a general X is isomorphic to the dual EPW sextic $Z_X^\vee$.

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