MODULI SPACES FOR CERTAIN FANO VARIETIES

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Abstract. It is yet unknown whether quasiprojective moduli spaces exist for all Fano varieties with finite automorphism groups. In this talk, we will describe a class of Fano varieties for which one can give a very concrete description of their moduli spaces, although they do not seem to be constructible by standard techniques, such as Mumford's GIT or Kollár-Viehweg theories. The techniques are very specific to this particular example, but the resulting picture provides a lot of information on the geometry of these varieties.

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1. Moduli spaces

We consider the moduli problem for Fano complex manifolds, i.e., those smooth projective varieties X for which $-K_X$ is ample. The corresponding functor is open and bounded (Matsusaka's big theorem), hence there is an algebraic stack $\mathscr{M}$, which is a Deligne-Mumford stack if the automorphism group of X is finite.[1] If this is the case, there are two more issues:

- the separatedness of the functor;[2] if it holds, there is a separated algebraic moduli space $\mathbf{M}$ (Keel-Mori theorem);
- the quasiprojectivity of $\mathbf{M}$; here very little is known in general.

Viehweg solved in 1995 the analogous problem for varieties X for which K_X is ample by proving that there is always a quasi-projective coarse

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[1]The automorphism group needs to be proper. Being affine, it is finite.

[2]This is not the same as the moduli space being Hausdorff, which seems to be the case for Fano manifolds with a Kähler-Einstein metric. Here we require uniqueness of the total space of a degeneration, as opposed to uniqueness of the limit manifold only (think of smooth quadrics in a projective space!). The precise statement is that if T is the spectrum of a DVR, and $X \to T$ and $X' \to T$ are families of Fano manifolds of the type considered, every isomorphism between the generic fibers should extend to an automorphism between X and X'.

moduli space. One could ask whether the same holds for Fano varieties with finite automorphism groups.

Examples 1. 1) There is a quasiprojective coarse moduli space for del Pezzo surfaces (Fano surfaces) X of degree $d := (-K_X)^2 \in \{1, \dots, 9\}$. It was constructed by Ishii in 1982 as a GIT quotient $\mathrm{Sym}^{9-d}\,\mathbf{P}^2/\!/\,\mathrm{PGL}(2)$. The stack is separated only for $1 \leq d \leq 5$.

2) Geometric Invariant Theory (GIT) constructs an affine coarse moduli space for smooth hypersurfaces of fixed degree $d \geq 3$ of $\mathbf{P}^n$ (they are Fano when $d \leq n$). The argument is classical (Mumford): inside the projective space $\mathbf{P}$ parametrizing all such hypersurfaces, singular hypersurfaces correspond to a divisor, hence smooth hypersurfaces form an open $\mathrm{SL}(n+1)$-invariant affine subset $\mathbf{P}^0$. The action of the reductive group $\mathrm{SL}(n+1)$ on $\mathbf{P}^0$ is linearizable and since the group of automorphisms (preserving the polarization) of any smooth hypersurface of degree $d \geq 3$ is finite, the stabilizers are finite at points of $\mathbf{P}^0$, which is therefore contained in the stable locus.

The moduli space is therefore a dense affine open subset of the projective irreducible GIT quotient $\mathbf{P}/\!/\,\mathrm{SL}(n+1)$.

2. Quadratic line complexes

Mumford's trick can be used to construct a quasiprojective moduli space for smooth sections X of any $Y \subset \mathbf{P}^n$ by degree-d hypersurfaces whenever $\mathrm{Aut}(Y)$ is reductive, the embedding $X \hookrightarrow Y$ is canonical, the set of hypersurfaces whose intersection with Y is singular form a divisor,[3] and $\mathrm{Aut}(X)$ is finite.

One problem (which was already present in Example 1.2) above) is that some smooth deformations of X might not remain of the same type: for example, smooth quartic curves can degenerate to smooth hyperelliptic curves of genus 3 (which are not plane curves), or smooth quintic surfaces can degenerate to (smooth) double covers of ruled surfaces (Horikawa).

There are however a few cases where this is known not to happen.

Theorem 2 (Mukai)**.** *Let X be a smooth Fano 6-fold of index 4 and degree 10.*[4] *Then X is a double cover of $G(2, V_5)$ branched along its (smooth) intersection with a quadric.*[5]

[3] This is equivalent to saying that the projective dual of the image of the embedding of Y by $|\mathscr{O}_{\mathbf{P}^n}(d)|$ is a hypersurface.

[4] This means that there is an ample divisor H such that $\mathrm{Pic}(X) = \mathbf{Z}[H]$, $K_X \underset{\mathrm{lin}}{\equiv} 4H$, and $H^6 = 10$.

[5] Our notation is that V_n stands for a complex vector space of dimension n.

Moreover, this double cover is canonically attached to X, so that isomorphism classes of smooth Fano 6-folds of index 4 and degree 10 are in one-to-one correspondence with smooth quadratic sections X of $G(2, V_5) \subset \mathbf{P}^9$ (classically called quadratic line complexes). One checks that

- singular quadratic sections of $G(2, V_5)$ form a divisor, so that we can apply Mumford's trick;
- $H^0(X, T_X) = 0$ so that $\mathrm{Aut}(X)$ is a discrete subgroup of a linear group, hence is finite.

Theorem 3. *There exists an irreducible affine 25-dimensional coarse moduli space* $\mathbf{X}_6$ *for smooth Fano 6-folds of index 4 and degree 10.*

One way to obtain more information about $\mathbf{X}_6$ is to consider the *period map.* The Hodge numbers of X are as follows

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 0 & & 1 & & 0 & & & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 2 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & 1 & & 22 & & 1 & & 0 & & 0
\end{array}
$$

The Hodge structure on $H^6(X)$ is therefore of K3 type.

If we let $H^6(X)_{\mathrm{van}}$ be the orthogonal for the intersection form $(\cdot, \cdot)$ of the pull-back of $H^6(G(2, V_5))$, the rank-22 lattice $H^6(X, \mathbf{Z})_{\mathrm{van}}$ is isomorphic to $\Lambda := 2E_8(-1) \oplus 2U \oplus 2A_1(-1)$. If we set

$$\mathscr{Q} := \{\omega \in \mathbf{P}(\Lambda \otimes \mathbf{C}) \mid (\omega \cdot \omega) = 0, (\omega \cdot \bar{\omega}) > 0\},$$

a Hermitian symmetric bounded domain, we have a well-defined regular *period map*

$$p_6 : \mathbf{X}_6 \longrightarrow \mathscr{D} := \widetilde{O}(\Lambda) \backslash \mathscr{Q},$$

where $\widetilde{O}(\Lambda)$ is a subgroup of index 2 of the group of isometries of Λ and $\mathscr{D}$ is quasiprojective of dimension 20, with Baily-Borel projective compactification $\overline{\mathscr{D}}^{\mathrm{BB}}$.

Roughly, it sends the isomorphism class of a Fano 6-fold X to the image in $\mathscr{D}$ of the point $[H^{4,2}(X)]$ of $\mathscr{Q}$.

Theorem 4. *The map* $p_6 : \mathbf{X}_6 \to \mathscr{D}$ *is dominant with smooth 5-dimensional fibers.*

Proof. Smoothness is to be understood in the orbifold sense (there might be singularities, but they are all quotient of a smooth germ by a finite group). One computes the differential of the period map; it has same kernel at the morphism

$$\begin{aligned} H^1(X,T_X) &\to \operatorname{Hom}(H^{4,2}(X), H^{4,2}(X)^\perp/H^{4,2}(X)) \\ &\simeq \operatorname{Hom}(H^2(X,\Omega_X^4), H^3(X,\Omega_X^3)) \end{aligned}$$

defined by the natural pairing $H^1(X,T_X)\otimes H^2(X,\Omega_X^4)\to H^3(X,\Omega_X^3)$ (recall that $H^2(X,\Omega_X^4)$ is one-dimensional). Using Bott's theorem and various exact sequences, one shows that this kernel is isomorphic to V_5. □

We will come back to this map later.

Mukai actually classified all smooth prime Fano n-folds X of index $n-2$ and degree 10. He shows in particular $3\le n\le 6$.

For $n=5$, the manifold X is in general a (smooth) quadratic section of $G(2,V_5)$, but there are also (smooth) degenerations, called hyperelliptic, which are not of this type. So $\mathbf{X}_6$ is a coarse moduli space for nonhyperelliptic prime Fano 5-folds of index 3 and degree 10, but we are missing the hyperelliptic ones.

For $n=4$ or 3, X is in general a (smooth) linear section of a nonhyperelliptic 5-fold as above, but there are also hyperelliptic degenerations. We have one-to-one correspondences

$$\{\text{hyperelliptic } n\text{-folds}\}\leftrightarrow\{\text{hyperelliptic } (n-1)\text{-folds}\}.$$

Even if one sticks to nonhyperelliptic X, who sit in a fixed smooth W of dimension one more (a linear section of $G(2,V_5)$), there is a problem because $\operatorname{Aut}(W)$ is *not* reductive for $n\le 4$. Separatedness is therefore already a problem, even in the nonhyperelliptic case.

3. Irreducible holomorphic symplectic (IHS) manifolds

An irreducible holomorphic symplectic manifold is a projective simply-connected manifold Y such that the vector space of holomorphic differential 2-forms on Y is generated by a symplectic (i.e., non-degenerate at each point of Y) 2-form. The dimension of Y is necessarily even and when Y is a surface, it is simply a K3 surface. Examples were constructed by Beauville and O'Grady in every even dimension (but there are not many).

We will show how to associate an IHS manifold with a Fano n-fold of index $n-2$ and degree 10.

Assume $n=5$ and X is nonhyperelliptic so that X is the (smooth) intersection of $G:=G(2,V_5)$ with a quadric Q. The vector space of affine quadrics containing X decomposes as

$$I_X(2) = I_G(2) \oplus \mathbf{C}Q \simeq V_5 \oplus \mathbf{C}Q.$$

Since Plücker quadrics have corank 4, the scheme of singular quadrics in the 5-plane $\mathbf{P}(I_X(2))$ splits as

$$4\mathbf{P}(V_5) + Y_X,$$

where Y_X is a sextic hypersurface (one checks that X is always contained in a smooth quadric). Furthermore,

- Y_X is integral;
- $\mathrm{Sing}(Y_X)$ is an integral surface with finitely many singular points.

In fact, these sextic fourfolds Y_X were studied by Eisenbud, Popescu, and Walter in a 2001 article and then, in a series of article, by O'Grady who, among other things, constructs a canonical double cover

$$f_X : \widetilde{Y}_X \to Y_X$$

branched exactly along the surface $\mathrm{Sing}(Y_X)$. When this surface is smooth (this holds for X general), $\widetilde{Y}_X$ is an IHS fourfold called a double EPW sextic.

O'Grady also constructs, using GIT, an irreducible quasiprojective 20-dimensional coarse moduli space $\mathbf{EPW}$ for smooth double EPW sextics. It is the (affine) complement in its GIT compactification $\overline{\mathbf{EPW}}$ of two irreducible hypersurfaces Σ and Δ.

Theorem 5. *There is a smooth surjective morphism*

$$\pi_6 : \mathbf{X}_6 \to \overline{\mathbf{EPW}} - \Sigma.$$

This is proved by analyzing the relation between X and Y_X. Moreover, the fiber of the point $[Y]$ corresponding to an EPW sextic $Y \subset \mathbf{P}(V_6)$ can be explicitly described: it is isomorphic to the complement in $\mathbf{P}(V_6^\vee)$ of the projective dual $Y^\vee \subset \mathbf{P}(V_6^\vee)$ (which is also an EPW sextic!). The point of the fiber of $[Y_X]$ corresponding to a 6-fold X is the point of $\mathbf{P}(V_6^\vee) = \mathbf{P}(I_X(2)^\vee)$ corresponding to the hyperplane $I_G(2) \subset I_X(2)$.

Periods of IHS manifolds have been studied intensively recently. Let Y be such a manifold, with a polarization h_Y. Contrary to what we did earlier, where we looked at the middle cohomology, we will work with $H^2(Y, \mathbf{C})$. Beauville constructed a nondegenerate quadratic form Q_{BB} on this vector space. We let $H^2(Y, \mathbf{C})_{\mathrm{prim}}$ be the orthogonal of h_Y. One then has

$H^2(Y, \mathbf{C})_{\mathrm{prim}}$	$=$	$H^{0,2}(Y)$	$\oplus$	$H^{1,1}(Y)_{\mathrm{prim}}$	$\oplus$	$H^{2,0}(Y)$.
dimensions:		1		$b_2 - 3$		1

The quadratic form Q_{BB} has signature $(2, b_2-3)$ on $H^2(Y, \mathbf{C})_{\text{prim}}$. The period map takes values in a quotient $\mathscr{D}$ of

$$\{[\alpha] \in \mathbf{P}(H^2(Y, \mathbf{C})_{\text{prim}}) \mid Q_{BB}([\alpha]) = 0\ ,\ Q_{BB}([\alpha + \overline{\alpha}]) > 0\}.$$

Theorem 6 (Verbitsky). *The period map $\mathfrak{p} : \mathbf{EPW} \to \mathscr{D}$ is an open embedding.*

The period domain $\mathscr{D}$ is the same as the one which appears in Theorem 4. Of course, it would be nice to know the image of $\mathfrak{p}$, but we only have a partial result.

Theorem 7 (O'Grady). *The map $\mathfrak{p}$ extends to an open embedding*

$$\mathfrak{p}' : \overline{\mathbf{EPW}} - \Sigma \hookrightarrow \mathscr{D}$$

whose image is contained in the complement of 5 irreducible hypersurfaces $\mathscr{D}_2'$, $\mathscr{D}_2''$, $\mathscr{D}_4$, $\mathscr{D}_8$, and $\mathscr{D}_{10}''$.

O'Grady and I conjecture that the image of $\mathfrak{p}'$ is precisely the complement of these 5 hypersurfaces.

It is likely that the composition

$$\mathbf{X}_6 \xrightarrow{\pi_6} \overline{\mathbf{EPW}} - \Sigma \xrightarrow{\mathfrak{p}'} \mathscr{D}$$

is the period map p_6, but we do not have a proof yet.

In dimensions $n = 4$ or 3, the situation is similar: the decomposition $I_X(2) = V_5 \oplus \mathbf{C}Q$ is still valid and the scheme of singular quadrics splits as

$$(n-1)\mathbf{P}(V_5) + Y_X,$$

where Y_X is again a sextic hypersurface. By analyzing the relation between X and Y_X, we can construct a moduli space for X.

Theorem 8. *Assume $3 \le n \le 6$. There exists a quasiprojective coarse moduli space $\mathbf{X}_n$ for smooth prime Fano n-folds of index $n-2$ and degree 10 and a smooth surjective morphism*

$$\pi_n : \mathbf{X}_n \longrightarrow \overline{\mathbf{EPW}} - \Sigma.$$

One can describe the fibers of π_n as follows. First recall that $\mathbf{X}_n$ contains an open set $\mathbf{X}_n^{\text{nhyp}}$ corresponding to nonhyperelliptic n-folds, with complement $\mathbf{X}_n^{\text{hyp}}$.

Moreover, a hyperelliptic n-fold X is a double cover of an n-dimensional linear section W of G, whose branch locus is a smooth quadratic section of W, i.e., a smooth prime Fano of degree 10 and dimension one less. In other words, we have

$$\mathbf{X}_n = \mathbf{X}_n^{\text{nhyp}} \sqcup \mathbf{X}_{n-1},$$

with $\mathbf{X}_6^{\mathrm{nhyp}} = \varnothing$.

On the other hand, one can construct a $\mathbf{P}^5$-bundle $\pi : \mathbf{P}^\vee \to \overline{\mathbf{EPW}} - \Sigma$ whose fiber over a point $[Y]$, where $Y \subset \mathbf{P}(V_6)$ is an EPW sextic, is the dual $\mathbf{P}(V_6^\vee)$.

Then, all $\mathbf{X}_i$ can be seen as subschemes of $\mathbf{P}^\vee$ and, setting $\pi_n^{\mathrm{nhyp}} := \pi_n|_{\mathbf{X}_n^{\mathrm{nhyp}}}$, we have

- the fiber of $\pi_6 = \pi_5^{\mathrm{nhyp}}$ over a point $[Y]$ is $\mathbf{P}(V_6^\vee) - Y^\vee$;
- the fiber of π_4^{nhyp} over a point $[Y]$ is $Y^\vee - \mathrm{Sing}(Y^\vee)$;
- the fiber of π_3^{nhyp} over a point $[Y]$ is $\mathrm{Sing}(Y^\vee) - \mathrm{Sing}(\mathrm{Sing}(Y^\vee))$

hence

- the fiber of π_5 over a point $[Y]$ is $\mathbf{P}(V_6^\vee) - \mathrm{Sing}(Y^\vee)$;
- the fiber of π_4 over a point $[Y]$ is $Y^\vee - \mathrm{Sing}(\mathrm{Sing}(Y^\vee))$ (which is a projective fourfold for $[Y] \in \overline{\mathbf{EPW}} - \Sigma - \Delta$);
- the fiber of π_3 over a point $[Y]$ is the projective surface $\mathrm{Sing}(Y^\vee)$.

4. Period maps

We already introduced the period map $p_6 : \mathbf{X}_6 \longrightarrow \mathscr{D}$ and asked about the commutativity of the diagram

$$\begin{array}{ccc} \mathbf{X}_6 & \xrightarrow{\ \pi_6\ } & \overline{\mathbf{EPW}} - \Sigma \\ & {\scriptstyle p_6}\searrow \quad \swarrow{\scriptstyle \mathfrak{p}'} & \\ & \mathscr{D}. & \end{array}$$

It turns out that the period domain for fourfolds in $\mathbf{X}_4$ is also $\mathscr{D}$, hence we have a period map $p_4 : \mathbf{X}_4 \longrightarrow \mathscr{D}$ and a similar diagram.

Theorem 9. *The diagram*

$$\begin{array}{ccc} \mathbf{X}_4 & \xrightarrow{\ \pi_4\ } & \overline{\mathbf{EPW}} - \Sigma \\ & {\scriptstyle p_4}\searrow \quad \swarrow{\scriptstyle \mathfrak{p}'} & \\ & \mathscr{D} & \end{array}$$

is commutative.

In other words, if X is such a fourfold, there is an isomorphism of polarized Hodge structures

$$H^4(X, \mathbf{Z})_{\mathrm{van}} \xrightarrow{\sim} H^2(\widetilde{Y}_X, \mathbf{Z})_{\mathrm{prim}}(-1).$$

This is reminiscent of a similar statement proved by Beauville and Donagi: the polarized Hodge structures of a smooth subic fourfold and of its variety of lines (which is also an IHS fourfold) are isomorphic. Our proof actually uses this fact and is done in a very roundabout (and not very satisfactory) way.

There are also period maps for $\mathbf{X}_5$ and $\mathbf{X}_3$: if X is a smooth prime Fano 5-fold of index 3 (resp. Fano 3-fold of index 1) and degree 10, the Hodge structure on $H^5(X)$ (resp. on $H^3(X)$) has level one and defines a principally polarized intermediate Jacobian $J(X)$ which has dimension 10. Therefore, we have period maps

$$q_5 : \mathbf{X}_5 \to \mathscr{A}_{10} \qquad \text{and} \qquad q_3 : \mathbf{X}_3 \to \mathscr{A}_{10}.$$

I studied the map q_3 with Iliev and Manivel and our results on its general fibers imply that it factors as

$$q_3 : \mathbf{X}_3 \xrightarrow{\pi_3} \overline{\mathbf{EPW}} - \Sigma \xrightarrow{\rho} (\overline{\mathbf{EPW}} - \Sigma)/\iota \xrightarrow{\varphi} \mathscr{A}_{10},$$

where ι is the duality involution $[Y] \mapsto [Y^\vee]$. We expect the morphism φ to be injective.

We also expect the composition

$$\mathbf{X}_5 \xrightarrow{\pi_5} \overline{\mathbf{EPW}} - \Sigma \xrightarrow{\varphi\circ\rho} \mathscr{A}_{10}$$

to be the period map q_5. All in all, the morphisms

$$\mathbf{P}^\vee \xrightarrow{\pi^\vee} \overline{\mathbf{EPW}} - \Sigma \overset{\mathfrak{p}'}{\hookrightarrow} \mathscr{D} \dashrightarrow \mathscr{A}_{10}$$

should yield all period maps when restricted to the various $\mathbf{X}_i \subset \mathbf{P}^\vee$.

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