

Fano varieties and polytopes

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Smooth Fano varieties

A smooth Fano variety (over a fixed algebraically closed field \mathbf{k}) is a smooth projective variety whose anticanonical divisor is ample.

Examples (1) A Fano curve is \mathbf{P}^1 .

(2) Smooth Fano (del Pezzo) surfaces are $\mathbf{P}^1 \times \mathbf{P}^1$ and the blow-up of \mathbf{P}^2 in at most 8 points, with no three colinear.

(3) Smooth Fano threefolds have been classified in characteristic zero.

(4) A finite product of Fano varieties is a Fano variety.

(5) A smooth complete intersection in \mathbf{P}^{n+s} defined by equations of degrees $d_1 \geq \dots \geq d_s > 1$ is a Fano variety if and only if $d_1 + \dots + d_s \leq n + s$. In any given dimension n , there are only finitely many choices for the degrees (because $s \leq \sum(d_i - 1) \leq n$).

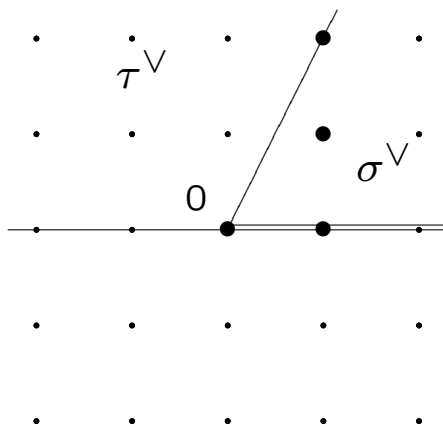
(6) A cyclic covering $X \rightarrow \mathbf{P}^n$ of degree $d > 1$ branched along a smooth hypersurface of degree de is a Fano variety if and only if $(d - 1)e \leq n$ (again, there are only finitely many choices for d and e when n is fixed).

(7) In characteristic zero, a projective variety acted on transitively by a connected linear algebraic group (such as a flag variety) is a smooth Fano variety.

(8) If X is a smooth (Fano) variety and D_1, \dots, D_r are nef divisors on X such that $-K_X - D_1 - \dots - D_r$ is ample, the (Grothendieck) projective bundle $\mathbf{P}(\bigoplus_{i=1}^r \mathcal{O}_X(D_i))$ is a Fano variety.

3) and the dual cone

$$\sigma^\vee = \{u \in M_{\mathbf{R}} \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}$$



the semigroup $\sigma^\vee \cap M$ is finitely generated (by the dots).

4) Define an affine variety by

$$X_\sigma = \text{Spec}(\mathbf{k}[\sigma^\vee \cap M])$$

In our example,

$$\begin{aligned} X_\sigma &= \text{Spec}(\mathbf{k}[U, UV, UV^2]) \\ &\simeq \text{Spec}(\mathbf{k}[X, Y, Z]/(XZ - Y^2)) \end{aligned}$$

5) If τ is a face of σ ,

$$\sigma^\vee \cap M \subset \tau^\vee \cap M$$

The corresponding map $X_\tau \rightarrow X_\sigma$ makes X_τ into a principal open subset of X_σ . (If $u \in M$ is such that $\tau = \sigma \cap u^\perp$, $X_\tau = (X_\sigma)_u$.)

In our example,

$$\begin{aligned} X_\tau &= \text{Spec}(\mathbf{k}[U, U^{-1}, V]) = (X_\sigma)_U \\ X_{\{0\}} &= \text{Spec}(\mathbf{k}[U, U^{-1}, V, V^{-1}]) = (X_\sigma)_{UV} \end{aligned}$$

6) The torus \mathbf{T}^n acts on X_σ and X_τ and the inclusion $X_\tau \rightarrow X_\sigma$ is equivariant.

General toric varieties

A *fan* Δ is a finite set of cones in $N_{\mathbf{R}}$ as above such that:

- every face of a cone in Δ is in Δ ;
- the intersection of any two cones in Δ is in Δ .

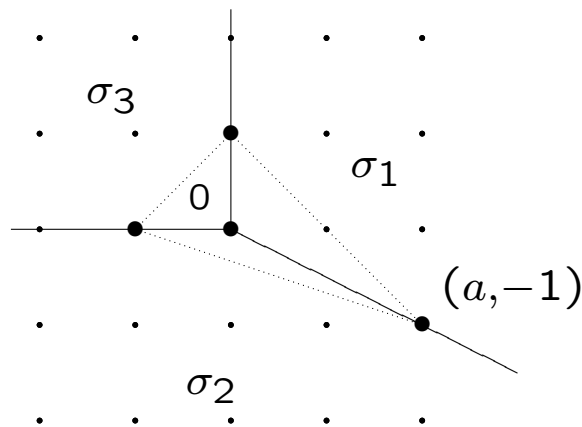
Construct an irreducible n -dimensional \mathbf{k} -scheme X_Δ with an action of \mathbf{T}^n by gluing the affine

varieties X_σ and $X_{\sigma'}$ along the open set $X_{\sigma \cap \sigma'}$, for all σ and σ' in Δ .

The scheme X_Δ is *irreducible, separated and normal*. Each X_σ , hence also X_Δ , contains the open set $X_{\{0\}} = \mathbf{T}^n$.

All toric varieties are obtained via this construction.

Example The toric surface associated with the fan that consists of $\sigma_1, \sigma_2, \sigma_3$ and their faces:



is the projective cone in \mathbf{P}^{a+1} over the rational normal curve in \mathbf{P}^a . For $a = 1$, this is \mathbf{P}^2 .

Projectivity criterion

Δ is the set of cones spanned by the faces of a polytope Q with vertices in N and the origin in its interior $\iff X_{\Delta}$ (or X_Q) projective

Given a polytope Q , we denote by $\mathcal{V}(Q)$ the set of its vertices.

Smooth toric Fano varieties

Let Q be a polytope in $N_{\mathbf{R}}$.

Each facet of Q is
the convex hull of $\Rightarrow X_Q$ smooth Fano
a basis of N

In this case, the anticanonical divisor $-K_{X_Q}$ is *very ample*. We call such a polytope a *smooth Fano polytope*. Its only integral interior point is the origin. Its *dual*

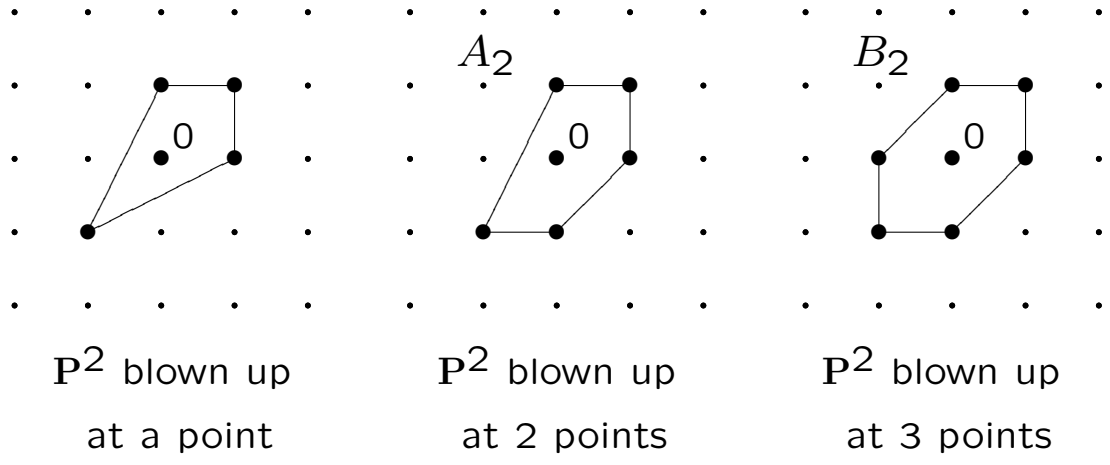
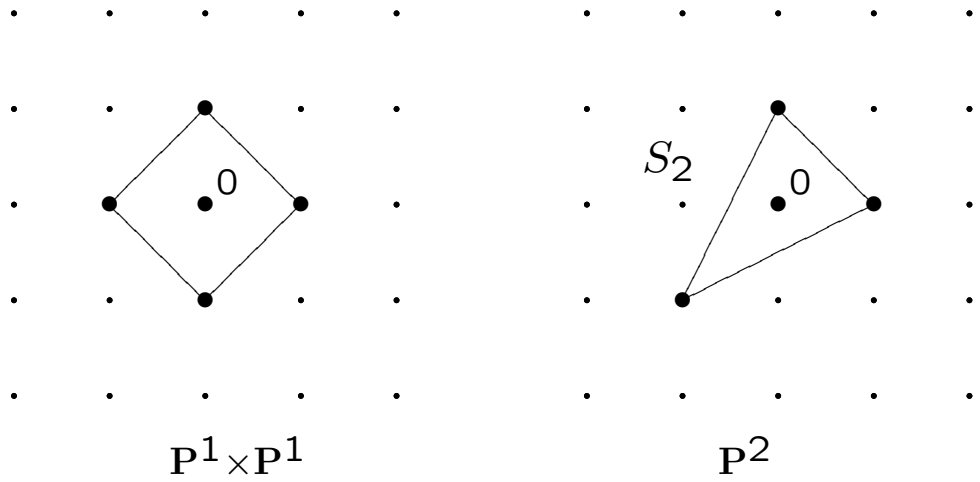
$$P = \{u \in M_{\mathbf{R}} \mid \langle u, v \rangle \leq 1 \text{ for all } v \in Q\}$$

is a polytope with vertices in M . It is usually not a Fano polytope, although its only integral interior point is the origin. We have

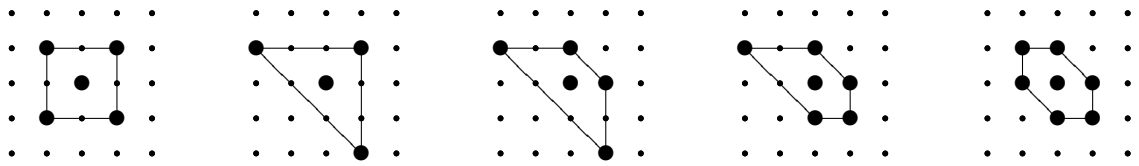
$$\begin{aligned} \text{Pic}(X_Q) &\simeq \mathbf{Z}^{\#\mathcal{V}(Q)-n} \\ h^0(X, -K_X) &= \#(M \cap P) \\ (-K_X)^n &= n! \text{vol}(P) \end{aligned}$$

Examples (1) There is only 1 smooth Fano polytope in \mathbb{R} , the interval $S_1 = [-1, 1]$.

(2) Up to the action of $GL(2, \mathbb{Z})$, there are only 5 smooth Fano polytopes in \mathbb{R}^2 :



The dual polytopes are



and the formulas above are easily checked in these cases.

(3) The product of smooth Fano polytopes Q and Q' is a smooth Fano polytope and $X_{Q \times Q'} = X_Q \times X_{Q'}$.

(4) Let (e_1, \dots, e_n) be a basis for N and set $e_0 = -e_1 \cdots -e_n$. The simplex

$$S_n = \text{Conv}(\{e_0, e_1, \dots, e_n\})$$

is a smooth Fano polytope. When n is even,

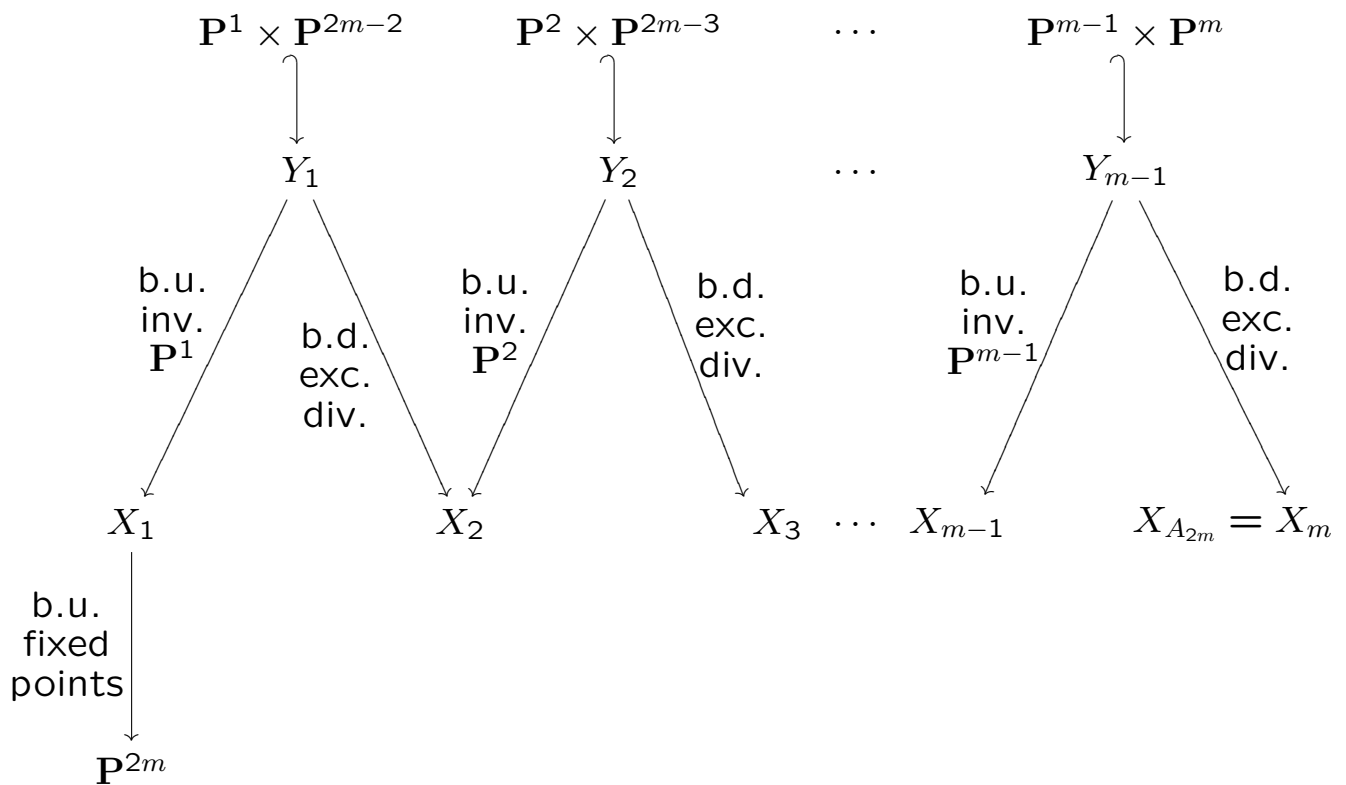
$$A_n = \text{Conv}(\{e_0, \pm e_1, \dots, \pm e_n\})$$

$$B_n = \text{Conv}(\{\pm e_0, \pm e_1, \dots, \pm e_n\})$$

are smooth Fano polytopes.

When $n = 2m + 1$, the $n + 1$ vertices $e_0, -e_1, e_2, \dots, -e_{2m+1}$ are on the same facet $-x_1 + x_2 - x_3 + \cdots - x_{2m+1} = 1$.

One has $X_{S_n} \simeq \mathbf{P}^n$, whereas $X_{A_{2m}}$ and $X_{B_{2m}}$ are obtained from \mathbf{P}^{2m} by a series of blow ups (b.u.) and blow downs (b.d.), such as



The following result, which improves on an earlier result of Ewald, is one of the few classification results that are valid in all dimensions.

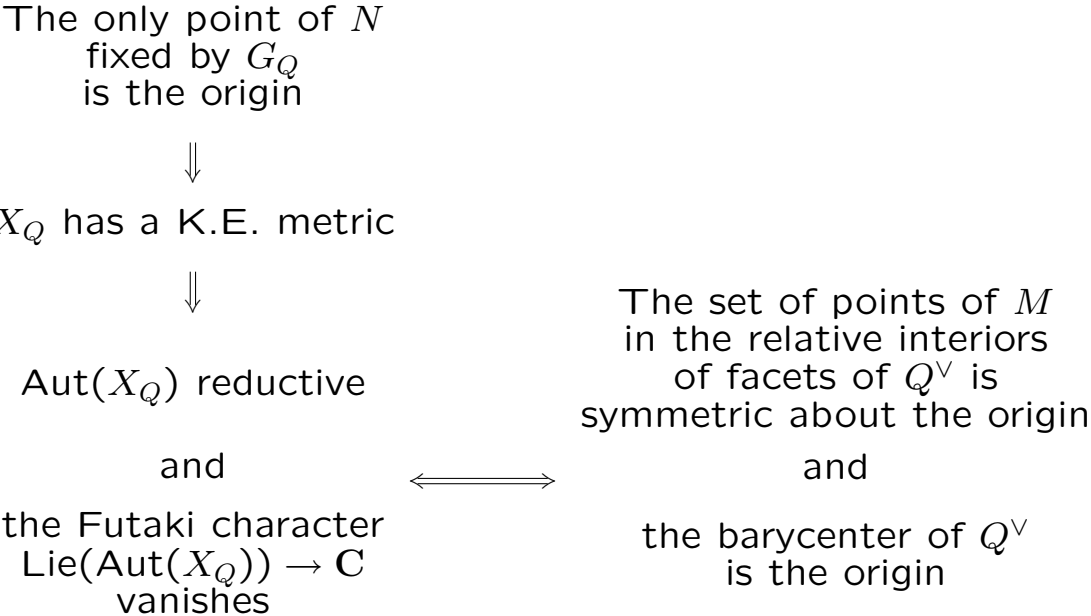
Theorem (Casagrande) *Any smooth Fano polytope in $N_{\mathbf{R}}$ with a set of vertices symmetric about the origin that spans $N_{\mathbf{R}}$ is isomorphic to a product of S_1 , A_{2r} and B_{2s} .*

Kähler–Einstein metrics Compact complex manifolds with a positive Kähler–Einstein metric (i.e., whose Ricci form is a positive (constant) multiple of the Kähler form) are Fano varieties, but it is a difficult problem to find conditions on a complex Fano manifold so that a Kähler–Einstein metric exists.

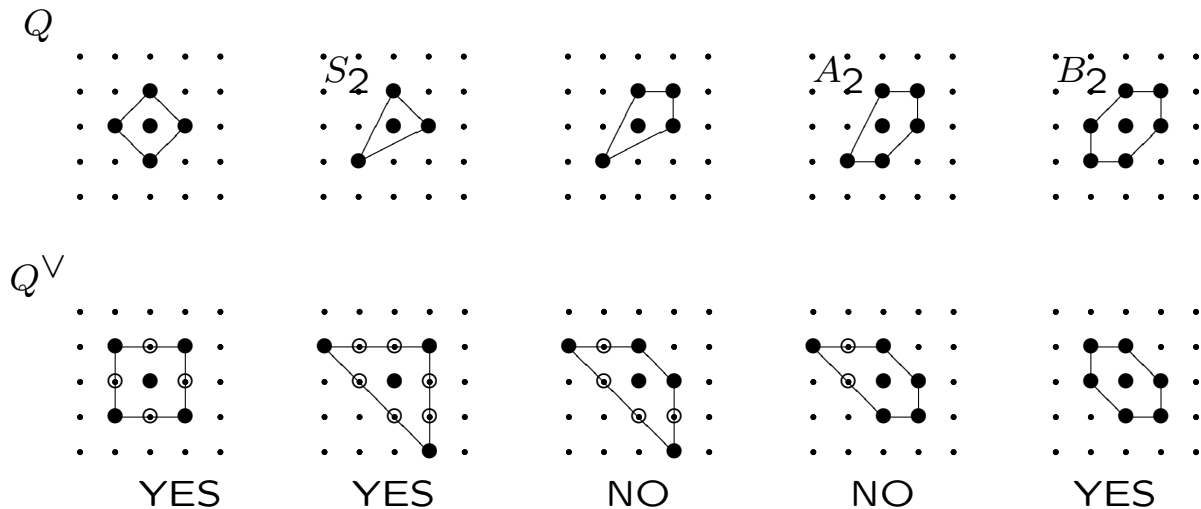
However, there is a criterion for toric Fano manifolds (Tian, Batyrev & Selivanova): if $Q \subset N_{\mathbf{R}}$ is a smooth Fano polytope and *the only point of N fixed by the finite group*

$$G_Q = \{g \in GL(N) \mid g(Q) = Q\}$$

is the origin, the Fano variety X_Q has a Kähler–Einstein metric. This condition implies that the barycenter of Q (resp. of Q^\vee) is the origin. We have the following implications:



Examples (1) In dimension two, the answer to the question “does X_Q have a Kähler–Einstein metric?” is:



The polytopes S_n and B_n all satisfy the Batyrev–Selivanova criterion. Using the criterion above, one checks that the automorphism group of $X_{A_{2m}}$ is not reductive.

(2) Fix positive integers $a_1, \dots, a_m, r_1, \dots, r_m$ with $a_i \leq r_i$. The projective bundle

$$X = \mathbf{P}(\mathcal{O}(a_1, 0, \dots, 0) \oplus \dots \oplus \mathcal{O}(0, \dots, a_m))$$

over $\mathbf{P}^{r_1} \times \dots \times \mathbf{P}^{r_m}$ is a smooth Fano variety (this is a particular case of Example (8)) whose automorphism group is reductive (*by direct computation*). It is the toric variety associated with the smooth Fano polytope Q with

vertices:

$$g_1, \dots, g_m \quad \text{with } g_1 + \dots + g_m = 0$$

$$e_{i1}, \dots, e_{ir_i}, a_i g_i - (e_{i1} + \dots + e_{ir_i})$$

$$\text{for } i = 1, \dots, m$$

If all the r_i are equal and all the a_i are equal, it satisfies the Batyrev–Selivanova criterion, hence X has a Kähler-Einstein metric.

The barycenter of Q^\vee is the origin if and only if the integrals

$$\int_{\substack{g_1, \dots, g_m \leq 1 \\ g_1, \dots, g_m = 0}} g_i \prod_{j=1}^m (r_j + 1 - a_j g_j)^{r_j} d\mu$$

all vanish for $1 \leq i \leq m$. I have checked that for $r_1, \dots, r_m \leq 10$, they do only when all the r_i are equal and all the a_i are equal.

Number of vertices Let $Q \subset N_{\mathbf{R}}$ be a smooth Fano polytope with dual P . There is a one-to-one correspondence

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{vertices of } P \\ \text{(resp. } Q) \end{array} \right\} & \leftrightarrow & \left\{ \begin{array}{l} \text{facets of } Q \\ \text{(resp. } P) \end{array} \right\} \\ u & & F_u \end{array}$$

For any $u \in \mathcal{V}(P)$ and $v \in \mathcal{V}(Q)$, we have

$$\langle u, v \rangle \in \mathbf{Z} \quad , \quad \langle u, v \rangle \leq 1$$

with equality if and only if $v \in F_u$.

Theorem (Voskresenskii, Klyachko) *A smooth Fano polytope of dimension n has most $2n^2$ vertices.*

Proof. Let Q be a smooth Fano polytope, with dual P . The origin is in the interior of P , hence in the interior of the convex hull of at most $2n$ vertices by Steinitz's theorem: we have

$$0 = \lambda_1 u_1 + \cdots + \lambda_d u_d \quad \lambda_1, \dots, \lambda_d > 0$$

where the vertices u_1, \dots, u_d of P generate $M_{\mathbf{R}}$ and $n < d \leq 2n$.

Let v be a vertex of Q . We have $\langle u_i, v \rangle \neq 0$ for at least one i , hence $\langle u_i, v \rangle > 0$ for at least one i . This implies $v \in F_{u_i}$, hence the vertices of Q are all on $F_{u_1} \cup \dots \cup F_{u_d}$. Since each facet has n vertices, Q has at most $nd \leq 2n^2$ vertices. \square

Batyrev conjectures that a smooth Fano polytope of dimension n has at most

$$\begin{cases} 3n & \text{if } n \text{ is even} \\ 3n - 1 & \text{if } n \text{ is odd} \end{cases}$$

vertices. This holds for $n \leq 5$ (Batyrev, Casagrande).

There is equality for n even for $A_2^{n/2}$ and for n odd for $A_2^{(n-1)/2} \times S_1$.

The (asymptotically) best known bound is

$$\begin{aligned} \#\mathcal{V}(Q) &\leq n + 2 + 2\sqrt{(n^2 - 1)(2n - 1)} \\ &\leq 2\sqrt{2n^3} \end{aligned}$$

Corollary *Up to the action of $GL(n, \mathbf{Z})$, there are only finitely many smooth Fano polytopes in \mathbf{R}^n .*

Proof. The number of vertices of such a polytope Q is at most $2n^2$. Since each facet of Q has n vertices, their number is at most $\binom{2n^2}{n}$, hence the volume of Q is at most $\frac{1}{n!} \binom{2n^2}{n}$. Fix the vertices e_1, \dots, e_n of a facet of Q and set $S = \text{Conv}(\{0, e_1, \dots, e_n\})$. For any vertex $v = v_1 e_1 + \dots + v_n e_n$ of Q , the volume of $\text{Conv}(v, S)$ is at most $\frac{1}{n!} \binom{2n^2}{n}$, hence $|v_i| \leq \binom{2n^2}{n}$. The integral vector v may therefore take only finitely many values. \square

Volume of the dual polytope Let Q be a smooth Fano polytope. The integer

$$(-K_{X_Q})^n = n! \operatorname{vol}(Q^\vee)$$

is often called the *degree* of the smooth Fano variety X_Q .

Theorem *Let Q be a smooth Fano polytope of dimension n with $\rho + n$ vertices, $\rho > 1$. We have*

$$\operatorname{vol}(Q^\vee) \leq \left(\frac{(n-1)^{\rho+1} - 1}{n-2} \right)^n \leq n^{\rho n}$$

Equivalently, a smooth toric Fano variety X of dimension n and Picard number $\rho \geq 2$ satisfies

$$(-K_X)^n \leq n! n^{\rho n} \quad (1)$$

Remarks (1) When $\rho = 1$, we have $X \simeq \mathbf{P}^n$, hence

$$(-K_X)^n = (n+1)^n$$

When $\rho = 2$, we have by classification

$$(-K_X)^n \leq n^{2n}$$

For any integers $\rho \geq 2$ and $n \geq 4$ such that $\frac{n}{\log n} \geq 2^{\rho-2}$, there are examples of smooth toric Fano varieties X of dimension n and Picard number ρ with

$$(-K_X)^n \geq \left(\frac{n^\rho}{2^{\rho^2-1}(\log n)^{\rho-1}} \right)^n$$

so the bound (1) is not too far off.

(2) The pseudo-index

$$\iota_X = \min_{C \subset X \text{ rational curve}} (-K_X \cdot C)$$

of X can be computed in a purely combinatorial way from Q . One can get the more precise bound (for $\rho > 1$)

$$\text{vol}(Q^\vee) \leq \left(n(n+1 - \iota_X)^{\rho-1} \right)^n$$

(3) If X_Q is a compact complex manifold of dimension n with a positive Kähler-Einstein metric, methods from differential geometry give

$$\text{vol}(Q^\vee) \leq (2n-1)^n \frac{2^{n+1}n!}{(2n)!} \sim 2(2e)^{n-1/2}$$

(4) Taking $a_1 = r_1$ and $a_2 = \dots = a_m = r_2 = \dots = r_m = 1$ in Example (2) above, one gets

a smooth toric Fano variety X of dimension $n = r_1 + 2m - 2$ and Picard number $m + 1$ with *reductive automorphism group* and

$$(-K_X)^n \geq m^n r_1^{r_1}$$

Taking $m = \left\lfloor \frac{n}{2 \log n} \right\rfloor$, we get $(-K_X)^n \geq \left(\frac{an^2}{\log n} \right)^n$ for some positive constant a . In particular, X does not have a Kähler-Einstein metric.

(5) Ewald conjectures that *given a smooth Fano polytope Q , there is a basis of N in which the vertices of Q have all their coordinates in $\{0, \pm 1\}$* . It holds for all the examples presented here.

Let (e_1, \dots, e_n) be such a basis, let u be a vertex of Q^\vee , and let v_1, \dots, v_n be the vertices of the face F_u of Q . Write

$$v_i = \sum_j \alpha_{ij} e_j, \quad \alpha_{ij} \in \{0, \pm 1\}, \quad \det(\alpha_{ij}) = \pm 1$$

The coordinates $\langle u, e_1 \rangle, \dots, \langle u, e_n \rangle$ of u , solutions of the linear system

$$1 = \langle u, v_i \rangle = \sum_{j=1}^n \alpha_{ij} \langle u, e_j \rangle \quad i = 1, \dots, n$$

are $n \times n$ determinants whose entries are in $\{0, \pm 1\}$. By Hadamard's inequality, their absolute value is bounded by $n^{n/2}$, hence

$$\text{vol}(Q^\vee) \leq 2^n n^{n^2/2}$$

The theorem on the volume of the dual of a smooth Fano polytope is a consequence of the following two lemmas.

Lemma *Let P be a polytope in \mathbf{R}^n whose only integral interior point is the origin. Let b be such that for each vertex u of P and each vertex v of P^\vee , we have $-b \leq \langle u, v \rangle \leq 1$. Then*

$$\text{vol}(P) \leq (b + 1)^n$$

Proof. The hypothesis implies $-\frac{1}{b}P \subset P$. Assume $\text{vol}(P) > (b + 1)^n$ and set

$$P' = \frac{1 - \varepsilon}{b + 1}P$$

We have $\text{vol}(P') > 1$ for ε positive small enough hence, by a theorem of Blichfeldt, P' contains distinct points p and p' such that $p - p' \in \mathbf{Z}^n$. Since P' is convex and $-\frac{1}{b}P' \subset P'$, the point

$$\frac{p - p'}{b + 1} = \frac{p + b \left(-\frac{1}{b}p'\right)}{b + 1}$$

is in P' , hence $p - p'$ is in $(b + 1)P'$, which is contained in the interior of P . This contradicts the fact that the origin is the only integral interior point of P . \square

Lemma *Let Q be a smooth Fano polytope of dimension n with $\rho + n$ vertices, $\rho > 1$. For each vertex u of Q^\vee and each vertex v of Q , we have*

$$-\frac{n-1}{n-2}((n-1)^\rho - 1) \leq \langle u, v \rangle \leq 1$$

Singular Fano varieties

In many cases, such as in the Minimal Model Program, it is necessary to allow some kind of singularities for Fano varieties. We assume the characteristic of \mathbf{k} is 0.

Let X be a normal variety such that K_X is a \mathbf{Q} -Cartier divisor. For any desingularization $f : Y \rightarrow X$, write

$$K_Y \sim f^*K_X + \sum_i a_i E_i$$

where the E_i are f -exceptional divisors, $a_i \in \mathbf{Q}$, and \sim denotes numerical equivalence.

The discrepancy of X is the minimum of the a_i for all possible desingularizations f .

It can be computed from any desingularization f whose exceptional locus is a divisor with normal crossings by the formula

$$\text{discr}(X) = \begin{cases} \min(1, a_i) & \text{if all } a_i \text{ are } \geq -1 \\ -\infty & \text{otherwise} \end{cases}$$

It is therefore either $-\infty$ or a rational number in $[-1, 1]$, and is 1 if X is smooth. We

say that X has *log terminal singularities* if $\text{discr}(X) > -1$.

Definition A *Fano variety* X is a normal projective variety with log terminal singularities such that $-K_X$ is an ample \mathbf{Q} -Cartier divisor.

Example Let Y be a projectively normal smooth subvariety of \mathbf{P}^n , with hyperplane divisor H , such that $K_Y \sim qH$ for some rational number q (e.g., Y a smooth curve).

Set $\pi : \tilde{X} = \mathbf{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(H)) \rightarrow Y$ and let Y_0 be the section such that $Y_0|_{Y_0} \equiv -H$.

The cone X in \mathbf{P}^{n+1} over Y is normal; it is the contraction $f : \tilde{X} \rightarrow X$ of Y_0 , and

$$\begin{aligned} K_{\tilde{X}} &\sim -2Y_0 + \pi^*(K_Y - H) \\ &\sim -2Y_0 + \pi^*((q-1)H) \end{aligned}$$

hence

$$K_X \sim (q-1)H$$

(These two Weil divisors coincide outside of the vertex of X .)

and

| |
|---------------------------------|
| $-K_X \text{ ample} \iff q < 1$ |
|---------------------------------|

Writing $K_{\tilde{X}} \sim f^*K_X + aY_0$, restricting to Y_0 yields $a = -1 - q$, hence

$$X \text{ has log terminal singularities} \iff q < 0$$

It follows that X is a *Fano variety* if and only if Y is a *Fano variety*.

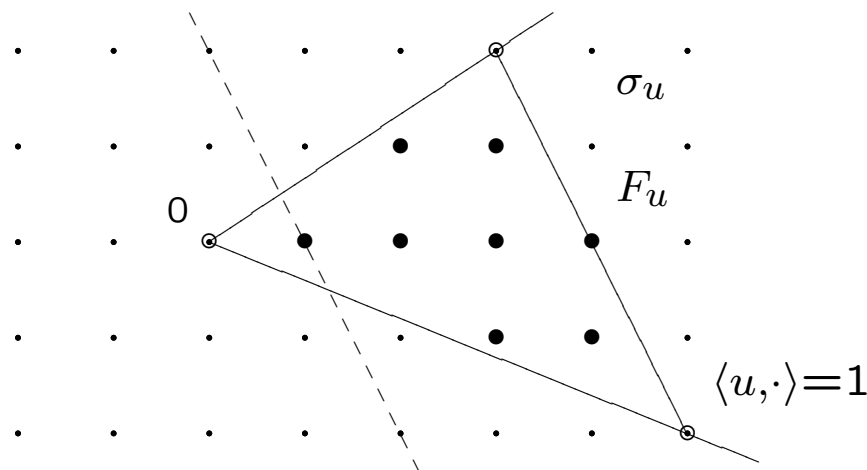
Toric Fano varieties

We have

Vertices of Q are primitive in $N \Rightarrow X_Q$ Fano

Moreover, if $P = Q^\vee$ and if, for each vertex u of P , we let σ_u be the cone in $N_{\mathbf{R}}$ generated by the facet F_u of Q , we have

$$\text{discr}(X_Q) = -1 + \min_{\substack{u \in \mathcal{V}(P) \\ v \in \sigma_u \cap N - \mathcal{V}(Q) - \{0\}}} \langle u, v \rangle$$

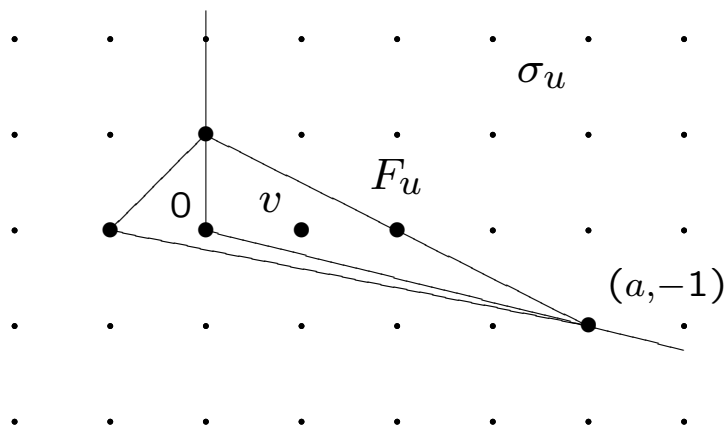


The polytope P has rational vertices and

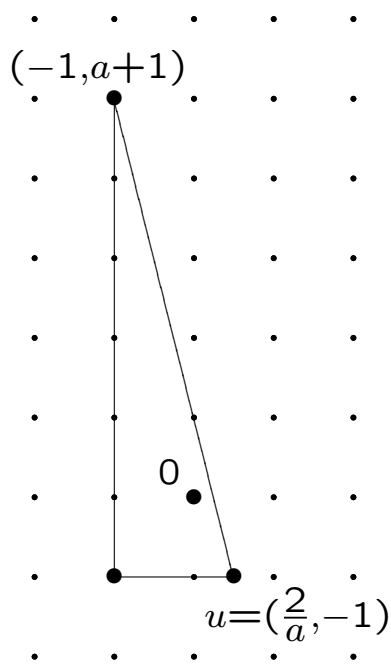
Vertices of mP are in $M \iff mK_X$ Cartier

In particular, X is a Gorenstein variety if and only if P has integral vertices (a polytope Q with this property is called *reflexive*; its dual has the same property)

Example The projective cone in \mathbf{P}^{a+1} over the rational normal curve in \mathbf{P}^a is the toric surface associated with the polytope



with dual



It is a Fano surface (this is a particular case of the example above, with $q = -2/a$). We recover

$$\text{discr}(X) = -1 + \langle u, v \rangle = -1 + \frac{2}{a}$$

Also, the smallest positive integer m such that mK_X is a Cartier divisor is a if a is odd, $a/2$ otherwise.

This shows that there are, in each dimension at least 2, infinitely many isomorphism classes of toric Fano varieties. However, by bounding the discrepancy away from -1 , we get finitely many isomorphism types: since the vertices of a Fano polytope are primitive, we have for any $r > 0$

$$\begin{aligned} \text{discr}(X_Q) \geq -1 + \frac{1}{r} &\iff \\ \langle u, v \rangle \geq \frac{1}{r} &\text{ for all } v \in \sigma_u \cap N \end{aligned}$$

This implies

$$\text{Int}(Q) \cap rN = \{0\}$$

Corollary *Given positive integers n and r , there are only finitely many isomorphism types of toric Fano varieties of dimension n and discrepancy $\geq -1 + \frac{1}{r}$ defined over \mathbf{k} .*

This is due to Borisov & Borisov (1993), who appear to have been unaware of an earlier article of Hensley (1983), which gives the finiteness of the number of isomorphism classes of polytopes $Q \subset \mathbf{R}^n$ such that $\text{Int}(Q) \cap r\mathbf{Z}^n = \{0\}$ under the action of $\text{GL}(\mathbf{Z}^n)$. It also provides explicit (large) bounds in terms of n and r for the smallest integer m such that mK_X is Cartier and for the degree $(-K_X)^n$.

It is conjectured that for each positive integer n and r , there are only finitely many deformation types of Fano varieties of dimension n with discrepancy $\geq -1 + 1/r$. The point is to bound $(-K_X)^n$ and an integer m such that mK_X is a Cartier divisor. McKernan claims in the preprint *Boundedness of log terminal Fano pairs of bounded index* (math.AG/0205214) that given integers n and m , there are only finitely many deformation types of complex Fano varieties X of dimension n such that mK_X is a Cartier divisor.