FANO VARIETIES AND EPW SEXTICS

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ABSTRACT. We explore a connection between smooth projective varieties X of dimension n with an ample divisor H such that $H^n = 10$ and $K_X = -(n-2)H$ and a class of sextic hypersurfaces of dimension 4 considered by Eisenbud, Popescu, and Walter (EPW sextics). This connection makes possible the construction of moduli spaces for these varieties and opens the way to the study of their period maps. This is work in progress in collaboration with Alexander Kuznetsov.

1. INTRODUCTION: COMPLEX CUBIC HYPERSURFACES

1.1. **Smooth projective curves.** Let C be a smooth (complex) projective curve. The Hodge numbers for $H^1(C)$ are

$$g \quad g$$

so the complex torus

$$J(C) := H^{0,1}(C)/H^1(C,\mathbf{Z})$$

is a g-dimensional principally polarized abelian variety. The principal polarization is given by the (unimodular) intersection form on $H^1(C,\mathbf{Z})$ and defines (uniquely up to translations) a *theta divisor* $\Theta_C \subset J(C)$.

The **Torelli property** holds: the curve C is uniquely determined (up to isomorphisms) by the pair $\big(J(C), \Theta_C\big)$.

In higher dimensions, there are a few (much rarer) cases where an analog construction can be made.

1.2. **Smooth cubic threefolds.** Let $X \subset \mathbf{P}^4$ be a smooth (complex) cubic hypersurface. The Hodge numbers for $H^3(X)$ are

$$0 \quad 5 \quad 5 \quad 0$$

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so the complex torus

$$J(X) := H^{1,2}(X)/H^3(X, \mathbf{Z})$$

is a 5-dimensional principally polarized abelian variety (the principal polarization is given by the (unimodular) intersection form on $H^3(X, \mathbf{Z})$).

The following is known:

- **Nonrationality.** A theta divisor Θ of $J(X)$ has a unique singular point, say 0, hence X is not rational by the Clemens–Griffiths criterion.
- **Torelli property.** The cubic X is isomorphic to the projectivized tangent cone to Θ at 0, hence X can be recovered from the data $(J(X), \Theta)$. In fancier terms, the *period map*

$$\begin{array}{ccc} \mathscr{C}_3 & \longrightarrow & \mathscr{A}_5 \\ X & \longmapsto & (J(X), \Theta) \end{array}$$

where $\mathscr{C}_3$ is the GIT, 10-dimensional, affine moduli space for smooth cubic threefolds, and $\mathscr{A}_5$ is the 15-dimensional moduli space for 5-dimensional principally polarized abelian varieties, is *injective.*

There is an alternative period map (Allcock–Carlson–Toledo)

$$\mathscr{C}_3 \longrightarrow \mathscr{D},$$

where $\mathscr{D}$ is a 10-dimensional quasi-projective variety, quotient of a Hermitian symmetric bounded domain by an arithmetic group. It is defined using the Hodge structure of the cyclic triple covering $\widetilde{X} \to \mathbf{P}^4$ branched along X and it is an *open embedding.*

1.3. **Smooth cubic fourfolds.** Let $X \subset \mathbf{P}^5$ be a smooth (complex) cubic hypersurface. The Hodge numbers for $H^4(X)_{\text{prim}}$ are

$$0 \quad 1 \quad 20 \quad 1 \quad 0.$$

We say that the Hodge structure is of K3 type. One can define a period map

$$\mathscr{C}_4 \longrightarrow \mathscr{D},$$

where $\mathscr{C}_4$ is the GIT, 20-dimensional, affine moduli space for smooth cubic fourfolds and $\mathscr{D}$ is a 20-dimensional period domain (quotient of a Hermitian symmetric bounded domain by an arithmetic group), with Baily-Borel compactification $\overline{\mathscr{D}}$.

The following is known:

- **Rationality.** Some smooth cubic fourfolds are rational. No irrational examples are known, although one expects that most cubic fourfolds are irrational.
- **Torelli property.** The above period map is an *open embedding* (Voisin) and its image is the complement in $\overline{\mathscr{D}}$ of 2 (explicit) Heegner divisors (Laza).

One should also mention an interesting construction of Beauville–Donagi: the variety $F(X)$ of lines contained in X is a smooth hyperkähler[1] fourfold and there is an isomorphism of polarized Hodge structures

$$H^4(X)_{\mathrm{prim}} \xrightarrow{\sim} H^2(F(X))_{\mathrm{prim}}(-1).$$

One can then use the injectivity of the period map for $F(X)$ (Verbitsky) to recover Voisin's result.

2. Gushel–Mukai varieties

The above results on cubic hypersurfaces will provide a road map for our study of the following Fano varieties (they were studied by Fano, Gushel, and Mukai, so we will call them *Gushel–Mukai*, or simply *GM varieties*).

Definition 1. Let $\mathbf{k}$ be a field and let $n \le 5$ be an integer. A GM n-fold is a smooth proper intersection

$$X = \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_{n+5}) \cap (\text{quadric}) \subset \mathbf{P}(\textstyle\bigwedge^2 V_5).$$

One computes

$$K_X = (-5 + (10 - (n+5)) + 2)H = -(n-2)H,$$

so that

$$\begin{aligned} \text{when } 3 \le n \le 5, &\qquad X \text{ is a Fano variety of “coindex 3”;} \\ \text{when } n = 2, &\qquad X \text{ is a polarized K3 surface of degree 10;} \\ \text{when } n = 1, &\qquad X \text{ is a curve of genus 6.} \end{aligned}$$

My plan is to discuss the following topics:

- moduli spaces of GM varieties;
- duality;
- birationalities;
- associated hyperkähler fourfold;
- conics on GM variety;

[1] This means that $F(X)$ is simply-connected and the vector space of holomorphic 2-forms on $F(X)$ is generated by a symplectic 2-form.

- periods of GM varieties.

2.1. **Lagrangian geometry.** Let $(\mathbb{V},\omega)$ be a symplectic space and let

$$\mathbb{V} = \mathbb{L}_1 \oplus \mathbb{L}_2$$

be a fixed Lagrangian decomposition. Let $A \subset \mathbb{V}$ be another Lagrangian. We define a quadratic form q^A on A be the formula

$$q^A(x) = \omega(\mathsf{pr}_1(x), \mathsf{pr}_2(x)),$$

hence quadric hypersurfaces $Q_i^A \subset \mathbf{P}(W_i)$, where $W_i = \mathsf{pr}_i(A) \subset \mathbb{L}_i$.

Lemma 2. *The quadrics $Q_1^A \subset \mathbf{P}(\mathbb{L}_1)$ and $Q_2^A \subset \mathbf{P}(\mathbb{L}_2)$ are projectively dual varieties (via the identification $\mathbb{L}_2 \xrightarrow{\sim} \mathbb{L}_1^\vee$ given by ω). The pair of projectively dual quadrics $Q_1^A \subset \mathbf{P}(\mathbb{L}_1)$ and $Q_2^A \subset \mathbf{P}(\mathbb{L}_2)$ uniquely define A and all such pairs are obtained by this construction.*

2.2. **Moduli spaces.** We apply the lemma to the following situation. Let $\mathbb{V} := \bigwedge^3 V_6$, endowed with the symplectic form given by wedge product. Given a direct sum decomposition $V_6 = V_5 \oplus V_1$, we have a Lagrangian decomposition

$$\mathbb{V} = \bigwedge^3 V_5 \oplus (\bigwedge^2 V_5 \wedge V_1).$$

Any Lagrangian $A \subset \mathbb{V}$ then gives rise to a quadric

$$Q_2^A \subset \mathbf{P}(W_2) \subset \mathbf{P}(\bigwedge^2 V_5 \wedge V_1) = \mathbf{P}(\bigwedge^2 V_5)$$

and to a variety

$$X_{A,V_5} := \mathsf{Gr}(2, V_5) \cap Q_2^A,$$

independent of V_1, of expected dimension $n = \dim(W_2) - 5 = 5 - \dim(A \cap \bigwedge^3 V_5)$.

The remarkable fact is that smoothness of X_{A,V_5} only depends on A, not on the choice of the hyperplane $V_5 \subset V_6$.

Theorem 3. *Fix $n \in \{3,4,5\}$. There is an "isomorphism of functors"*

$$\left\{\begin{array}{c} A \subset \bigwedge^3 V_6 \text{ Lagrangian with} \\ \text{no decomposable vectors} \\ \text{and } V_5 \subset V_6 \text{ such that} \\ \dim(A \cap \bigwedge^3 V_5) = 5 - n. \end{array}\right\} /\text{isom.} \xrightarrow{\sim} \{X \text{ GM } n\text{-fold}\}/\text{isom.}$$

The functor on the left was extensively studied by O'Grady:

- there is a 20-dimensional affine GIT moduli space

$$\mathscr{E} := \mathsf{LGr}(\bigwedge^3 V_6)/\!/ \operatorname{PGL}(V_6)$$

for Lagrangians A with no decomposable vectors;

- for such an A, set

$$Y_A^{\geq k} := \{V_5 \in \mathbf{P}(V_6^\vee) \mid \dim(A \cap \bigwedge^3 V_5) \geq k\}.$$

Then $Y_A := Y_A^{\geq 1} \subset Y_A^{\geq 0} = \mathbf{P}(V_6^\vee)$ is an integral sextic hypersurface, singular along $Y_A^{\geq 2}$, itself singular at the finite set $Y_A^{\geq 3}$ (empty for A general).

The sextic fourfolds Y_A were studied by Eisenbud, Popescu, and Walter in a 2001 article and then, in a series of articles, by O'Grady who named them *EPW sextics.*

The right-to-left direction in the theorem is given as follows:

- V_6 is vector the space of quadrics in $\mathbf{P}(W_{n+5})$ containing X;
- the hyperplane $V_5 \subset V_6$ is given by the space of *Plücker quadrics,* i.e., restrictions to $\mathbf{P}(W_{n+5})$ of quadrics that contain $\mathsf{Gr}(2, V_5)$;
- the sextic Y_A can be recovered from the discriminant scheme $\mathrm{Disc}(X)$ of singular quadrics containing X, which splits as

$$\mathrm{Disc}(X) = (n-1)\mathbf{P}(V_5) + Y_A^\vee \subset \mathbf{P}(V_6),$$

and Y_A is the projective dual of $Y_A^\vee$ (which is also the EPW sextic associated with $A^\perp \subset \bigwedge^3 V_6^\vee$).

Corollary 4. *For each $n \in \{3,4,5\}$, there is a coarse quasi-projective moduli space $\mathscr{M}_n$ for GM n-folds and a smooth surjective morphism*

$$\pi_n : \mathscr{M}_n \longrightarrow \mathscr{E}$$

such that $\pi_n^{-1}(A) \simeq Y_A^{5-n} / \operatorname{Aut}(Y_A^{5-n})$.

The dimensions of $\mathscr{M}_5$, $\mathscr{M}_4$, and $\mathscr{M}_3$ are, respectively, 25, 24, and 22.

2.3. **Duality.** I already mentioned that when $A \subset \bigwedge^3 V_6$ is a Lagrangian with no decomposable vectors, so is $A^\perp \subset \bigwedge^3 V_6^\vee$, and the projective dual of $Y_A \subset \mathbf{P}(V_6^\vee)$ is the EPW sextic $Y_{A^\perp} \subset \mathbf{P}(V_6)$. This duality operation defines an involution

$$\delta : \mathscr{E} \to \mathscr{E}.$$

What about GM varieties? Let $X \subset \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_{n+5})$ be a GM n-fold and *choose* a nonPlücker quadric $Q \subset \mathbf{P}(W_{n+5}) \subset \mathbf{P}(\bigwedge^2 V_5)$ containing X; we view Q as a point $[Q]$ of $\mathbf{P}(V_6) \smallsetminus \mathbf{P}(V_5)$. We consider the projective dual $Q^\vee \subset \mathbf{P}(\bigwedge^2 V_5^\vee)$. Then

$$X_Q^\vee := \mathsf{Gr}(2, V_5^\vee) \cap Q^\vee \subset \mathbf{P}(\bigwedge^2 V_5^\vee)$$

is a GM variety and

- its dimension is the integer m such that $[Q] \in Y_X^{\vee(5-m)}$;
- the EPW sextic associated with $X_Q^\vee$ is $Y_X^\vee$.

Smoothness of $X_Q^\vee$ comes from the fact that A has no decomposable vectors if and only if $A^\perp$ has the same property.

Note that $X_Q^\vee$ depends not only on X but also on the choice of Q (since its dimension does!). We will say that $X_Q^\vee$ is a dual of X.

2.4. **Birationalities.** It has been known since the 40s (L. Roth) that all GM fivefolds are rational and that a general GM threefold is not rational (this is obtained using the Clemens–Griffiths criterion on a degeneration). The most interesting case is the case of GM fourfolds. As for cubic fourfolds, some GM fourfolds are rational, but no irrational examples are known, although one expects that most GM fourfolds are irrational.

The following result shows that rationality only depends on the associated EPW sextic (modulo duality, even).

Theorem 5. *All GM varieties associated with isomorphic EPW sextics, or with dual EPW sextics, (but with different choices of hyperplanes) are birationally isomorphic.*

2.5. **Associated hyperkähler fourfold.** Let $Y_A \subset \mathbf{P}(V_6^\vee)$ be an EPW sextic constructed from a Lagrangian A with no decomposable vectors. O'Grady constructed a canonical double cover

$$f_A : \widetilde{Y}_A \to Y_A$$

branched exactly along the surface $\mathrm{Sing}(Y_A) = Y_A^{\geq 2}$. When this surface is smooth (i.e., when $Y_A^{\geq 3} = \varnothing$; this holds for A general), $\widetilde{Y}_A$ is a smooth hyperkähler fourfold called a *double EPW sextic.* It is a deformation of the symmetric square of a K3 surface of degree 10.

2.6. **Conics on Mukai manifolds.** In order to compare the Hodge structures of a Mukai fourfold X and of its associated double dual EPW sextic, we will study conics contained in X.

Let X be a GM n-fold, let $F(X)$ be the (projective) Hilbert scheme of conics contained in X, and let $F(X)^0 \subset F(X)$ be the open (not necessarily dense) subscheme of conics $c \subset X$ such that $\langle c\rangle \not\subset X$. We define a morphism

$$\varphi_X : F(X)^0 \longrightarrow \mathbf{P}(V_6^\vee)$$

by sending a point $[c] \in F(X)^0$ to the hyperplane in V_6 of quadrics containing $\langle c\rangle$.

Theorem 6. *Let X be a GM n-fold.*

- *When $n = 3$, $F(X)^0 = F(X)$ is an irreducible surface with isolated singularities, and φ_X is a double cover of $Y_X^{\vee\geq 2}$ branched along the finite set $Y_X^{\vee\geq 3}$.*

- *When $n = 4$, the scheme $F(X)$ has pure dimension 5. It is the union of a component $F(X)_{\mathrm{main}}$ and of the 5-planes $F(P)$, where P runs through the (finite) set of 2-planes contained in X, and there is a factorization*

$$\varphi_{X,\mathrm{main}} : F(X)_{\mathrm{main}} \xrightarrow[\mathbf{P}^1\text{-fibers}]{\text{generically}} \widetilde{Y}_X \xrightarrow{2:1} Y_X \subset \mathbf{P}(V_6^\vee).$$

- *When $n = 5$, the map φ_X is dominant.*

2.7. **Periods of GM varieties.** Let X be a GM threefold. The Hodge numbers of $H^3(X)$ are

$$0 \quad 10 \quad 10 \quad 0,$$

so we can define the intermediate Jacobian $J(X)$, a 10-dimensional principally polarized abelian variety, and a period map

$$\wp_3 : \mathscr{M}_3 \longrightarrow \mathscr{A}_{10}.$$

Theorem 7. *The period map for GM threefolds factors as*

$$\wp_3 : \mathscr{M}_3 \xrightarrow{\pi_3} \mathscr{E} \longrightarrow \mathscr{E}/\delta \xrightarrow{\varpi} \mathscr{A}_{10},$$

where ϖ is unramified, hence generically finite onto its image.

We expect ϖ to be an embedding.

Let now X be a GM fourfold. The Hodge numbers of $H^4(X)_{\mathrm{prim}}$ are

$$0 \quad 1 \quad 22 \quad 1 \quad 0.$$

The Hodge structure on $H^4(X)$ is therefore of K3 type, and we get a period map

$$\wp_4 : \mathscr{M}_4 \longrightarrow \mathscr{D},$$

where $\mathscr{D}$ is a 20-dimensional quasi-projective variety, quotient of a Hermitian symmetric bounded domain by an arithmetic group. On the other hand, using the so-called Beauville–Bogomolov intersection form on the second cohomology group of a hyperkähler manifold, one can also define a period map

$$\wp^0 : \mathscr{E}^0 \longrightarrow \mathscr{D},$$

where $\mathscr{D}$ is the same as above and $\mathscr{E}^0 \subset \mathscr{E}$ is the dense open subset corresponding to smooth double EPW sextics (i.e., those EPW sextics for which $Y_X^{\geq 3} = \varnothing$).

Theorem 8. a) (Verbitsky) *The period map $\wp^0 : \mathscr{E}^0 \to \mathscr{D}$ is an open embedding.*

b) (O'Grady) *It extends to an open embedding $\wp : \mathscr{E} \to \mathscr{D}$ whose image is contained in the complement of 5 irreducible Heegner divisors.*

The duality involution δ on $\mathscr{E}$ is induced by a lattice-theoretically defined involution $\delta_{\mathscr{D}}$ on $\mathscr{D}$.

We prove that the period map $\wp_4 : \mathscr{M}_4 \longrightarrow \mathscr{D}$ factors through $\pi_4 : \mathscr{M}_4 \to \mathscr{E}$: the period of a Mukai fourfold is the same as the period of its associated double EPW sextic. This theorem is the analog of the Beauville–Donagi isomorphism for smooth cubic fourfolds mentioned in the introduction.

Theorem 9. *The period map for GM fourfolds factors as*

$$\wp_4 : \mathscr{M}_4 \xrightarrow{\pi_4} \mathscr{E} \xrightarrow{\wp} \mathscr{D}.$$

In other words, given a GM fourfold X, there is an isomorphism of polarized Hodge structures

$$H^4(X, \mathbf{Z})_{\mathrm{prim}} \xrightarrow{\sim} H^2(\widetilde{Y}_X, \mathbf{Z})_{\mathrm{prim}}(-1).$$

Let us explain at least where the morphism between these two Hodge structures comes from.

Recall that there is a dominant map

$$\varphi : F(X)_{\mathrm{main}} \dashrightarrow \widetilde{Y}_X.$$

Introducing a smooth resolution $\eta : F \to F(X)_{\mathrm{main}}$ such that $\varphi \circ \eta$ is a morphism and the corresponding incidence correspondence

$$I := \{(x, c) \in X \times F \mid x \in \eta(c)\},$$

with projections $p : I \twoheadrightarrow X$ and $q : I \twoheadrightarrow F$, we obtain a morphism of Hodge structures

$$\alpha := q_* p^* : H^4(X) \to H^2(F)(-1) \qquad \text{and} \qquad \varphi^* : H^2(\widetilde{Y}_X) \to H^2(F).$$

We prove that both φ^* and $q_* p^*$ are injective, and that their images coincide.

Remarks 10. 1) There is also a period map $\wp_5 : \mathscr{M}_5 \to \mathscr{A}_{10}$. We expect that it factors as

$$\wp_5 : \mathscr{M}_5 \xrightarrow{\pi_5} \mathscr{E} \longrightarrow \mathscr{E}/\delta \xrightarrow{\varpi} \mathscr{A}_{10},$$

where ϖ is the map defined in Theorem 7.

2) Using a trick analogous to the cyclic cover trick explained for cubics, one can define alternative period maps

$$\wp_3' : \mathscr{M}_3 \to \mathscr{D}, \quad \wp_4' : \mathscr{M}_4 \to \mathscr{A}_{10}, \quad \wp_5' : \mathscr{M}_5 \to \mathscr{D}.$$

We expect that there should be a generically injective rational map $\mathscr{D}/\delta_{\mathscr{D}} \dashrightarrow \mathscr{A}_{10}$ which would relate the maps $\wp_i'$ and $\wp_i$.

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