HYPERBOLICITY
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Olivier Debarre

There exist several notions of hyperbolicity for *compact* complex varieties and I will try to explain what they are and how they are related.

For a compact Riemann surface C, all these notions coincide. "Hyperbolic" means any of the following equivalent properties:

(H1) C has genus ≥ 2;

(H2) C carries a metric with constant negative curvature;

(H3) any holomorphic map $\mathbf{C} \to C$ is constant;

(H4) if C is defined over a number field K, it has finitely many K-points (Faltings).

The main reason for the equivalence of the first three of these properties is that a compact Riemann surface of genus ≥ 2 is covered by the unit disk $\mathbf{B} \subset \mathbf{C}$. Any holomorphic map $\mathbf{C} \to C$ lifts to a holomorphic map $\mathbf{C} \to \mathbf{B}$, which is constant by Liouville's theorem. Of course, if C has genus 0 or 1, there is a nonconstant holomorphic map $\mathbf{C} \to C$.

The fact that smooth projective curves of genus at least 2 defined over a number field K have finitely many K-points is a difficult theorem of Faltings. Curves of genus 0 or 1 may have infinitely many K-points.

1 Analytic hyperbolicity

Let us first concentrate on property (H3). We say that *a compact complex variety* X (not necessarily smooth) *is analytically hyperbolic if any holomorphic map* $\mathbf{C} \to X$ *is constant.*

1.1 Examples

Let us start with a few examples.

(a) The product of 2 analytically hyperbolic varieties is analytically hyperbolic. In particular, any (finite) product of Riemann surfaces of genus ≥ 2 is analytically hyperbolic.

(b) As we saw above, compact Riemann surfaces of genus ≥ 2 are analytically hyperbolic. Similarly, any compact complex variety covered by a bounded domain in $\mathbf{C}^n$ is analytically hyperbolic.

It is a consequence of the Borel–Hirzebruch proportionality theorem that a smooth compact complex variety covered by the unit ball $\mathbf{B}_n \subset \mathbf{C}^n$ must satisfy $c_1^{n-2}(c_1^2 - 2\frac{n+1}{n}c_2) = 0$. Conversely, Yau proved, as a consequence of the Calabi conjecture, that any smooth compact complex variety of dimension n with ample canonical bundle such that $c_1^{n-2}(c_1^2 - 2\frac{n+1}{n}c_2) = 0$ is covered by the unit ball hence is analytically hyperbolic.

Unfortunately, such examples are difficult to construct. Borel constructed in 1963 compact quotients of $\mathbf{B}_2$ by a discontinuous group of analytic automorphisms. Mumford constructed in 1979 a "fake projective plane" (with $c_1^2 = 3$, $c_2 = 1$, and $b_1 = 0$). Hirzebruch considered in 1983 minimal desingularizations of certain coverings of the projective plane branched along a union of lines. Computing directly their Chern numbers, he found $c_1^2 = 3c_2$ for 3 particular configurations of lines.

1.2 The Lang conjectures

Any map from a complex torus to an analytically hyperbolic variety is constant. It was conjectured by Lang that the converse holds.

Conjecture 1 *A compact complex variety X is analytically hyperbolic if and only if there are no nonconstant holomorphic maps from a complex torus to X.*

There is an even stronger form of this conjecture.

Conjecture 2 *A compact algebraic complex variety is analytically hyperbolic if and only if all its subvarieties are of general type.*[1]

[1] Recall that a smooth projective variety X of dimension n is *of general type* if $h^0(X, (\Omega_X^n)^{\otimes m})$ grows like cm^n when m goes to infinity, for some $c > 0$. If X is singular, one (hence all) desingularizations should have this property.

These conjectures are for the moment out of reach, and most of the recent work has concentrated on constructing examples.

1.3 Analytically hyperbolic subvarieties of complex tori: the "Bloch theorem"

Analytically hyperbolic varieties are in fact very common. This is reflected in a celebrated theorem of Bloch (1926): *a subvariety of a complex torus is analytically hyperbolic if and only if it contains no nonzero (translated) subtorus.* This follows from the more precise result:

Theorem 3 (Bloch, Ochiai, Noguchi) *Let A be a complex torus. The Zariski closure of the image of a holomorphic map $\mathbf{C} \to A$ is a (translated) subtorus of A.*

A very general algebraic complex torus contains no nonzero subtori and many subvarieties. So analytically hyperbolic varieties are indeed very common.

1.4 Analytically hyperbolic hypersurfaces of the projective space: the Kobayashi conjecture

Most of the recent activity around analytical hyperbolicity revolves around the celebrated Kobayashi conjecture: *a very general hypersurface in $\mathbf{P}^n$ of degree $\geq 2n-1$ is analytically hyperbolic.*

This obviously holds for $n = 2$. Hypersurfaces of degree $\leq 2n-3$ always contain lines, and quartics in $\mathbf{P}^3$ always contain elliptic curves, so they are not analytically hyperbolic. Very general hypersurfaces in $\mathbf{P}^n$ of degree $\geq 2n-1$ contain no rational or elliptic curves (Voisin); in fact, all their subvarieties are of general type (Pacienza). So there is no algebraic obstruction to their being analytically hyperbolic.

There are roughly two categories of results.

- *Constructing examples.* Smooth hyperbolic surfaces in $\mathbf{P}^3$ of any degree ≥ 8, smooth hyperbolic hypersurfaces in $\mathbf{P}^n$ of any degree $\geq 4(n-1)^2$, have been constructed.

- *Proving the Kobayashi conjecture.* This is much harder. Demailly and El Goul proved that very general surfaces in $\mathbf{P}^3$ of degree ≥ 15 are hyperbolic. Siu has announced a proof of the Kobayashi conjecture for hypersurfaces in $\mathbf{P}^n$ of sufficiently large degree.

Here is a brief description of a construction of Duval. Consider the surface $S \subset \mathbf{P}^3$ with equation

$$P(z_0, z_1, z_2)^2 - Q(z_2, z_3) = 0$$

where P and Q are general polynomials of respective degrees $d \geq 4$ and $2d$. It is smooth outside the finite set $\Sigma = \{(z_0, z_1, 0, 0) \in \mathbf{P}^3 \mid P(z_0, z_1, 0) = 0\}$. The meromorphic projection map

$$\begin{array}{ccc} S & \dashrightarrow & \mathbf{P}^1 \\ (z_0, z_1, z_2, z_3) & \longmapsto & (z_2, z_3) \end{array}$$

is undefined at the finite subset Σ. This set can be blown up to yield a holomorphic map $\tilde{S} \to \mathbf{P}^1$ which factorizes as $\tilde{S} \xrightarrow{u} C \xrightarrow{p} \mathbf{P}^1$, where C is the hyperelliptic curve with inhomogeneous equation $t^2 = Q(1, z_3)$, the map u is given in inhomogeneous coordinates by

$$u(z_0, z_1, 1, z_3) = (P(z_0, z_1, 1), z_3)$$

and p is the double cover $(t, z_3) \mapsto z_3$. Any holomorphic map $f : \mathbf{C} \to S$ lifts to $\tilde{f} : \mathbf{C} \to \tilde{S}$. Since C has genus $d - 1 \geq 3$, the image of $\tilde{f}$ is contained in a fiber $u^{-1}(t, z_3)$ of u, which is isomorphic to the plane curve with inhomogeneous equation $P(z_0, z_1, 1) = t$. This is a plane curve of degree d with at most one singular point, which is a node; it has therefore genus ≥ 2, hence $\tilde{f}$ is constant. The degree $2d$ surface S is therefore analytically hyperbolic. Any small deformation of S is still analytically hyperbolic: this is because any sequence $f_n : \mathbf{C} \to S_n \subset \mathbf{P}^3$ of nonconstant holomorphic maps can be normalized in such a way that there exists a subsequence that converges to a nonconstant holomorphic map $f : \mathbf{C} \to \mathbf{P}^3$. So we obtain in this way examples of *smooth* analytically hyperbolic surfaces in $\mathbf{P}^3$.

It is harder to give an idea of the Demailly–El Goul proof.

The original idea goes back to Bloch, Cartan, Ahlfors, Ochiai, Green, and Griffiths. Any nonconstant holomorphic map $f : \mathbf{C} \to X$ lifts to the projectified tangent bundle as $f_1 : \mathbf{C} \to \mathbf{P}(T_X)$, defined by $f_1(t) = (f(t), f'(t))$. Assume that for some ample[2] line bundle L on X, the vector bundle $\mathbf{S}^m \Omega_X \otimes L^{-1}$

[2] A line bundle is *ample* if it has a hermitian metric with positive curvature.

has base locus $B \subset \mathbf{P}(T_X)$. If $f_1(\mathbf{C}) \not\subset B$, a hermitian metric with positive curvature on L pulls-back to a hermitian metric on $\mathbf{C}$ that can be compared to the Poincaré metric. A version of the Ahlfors–Poincaré lemma implies that f_1 can only exist on a bounded disk in $\mathbf{C}$, which is absurd. This implies $f_1(\mathbf{C}) \subset B$; in other words, any holomorphic map $f : \mathbf{C} \to X$ must automatically satisfy all algebraic differential equations $P(f, f') = 0$ arising from sections P of $\mathbf{S}^m\Omega_X \otimes L^{-1}$.

We now need to produce sufficiently many sections of $\mathbf{S}^m\Omega_X \otimes L^{-1}$. By Riemann–Roch, this is possible on a surface (of general type) only if $c_1^2 > c_2$, which unfortunately never happens for a surface S in $\mathbf{P}^3$ of degree $d \geq 4$, for which $c_1^2 = d(d-4)^2$ and $c_2 = d(d^2 - 4d + 6)$. The trick is to continue the process by lifting the curve f_1 again as $f_2 : \mathbf{C} \to \mathbf{P}(\mathbf{P}(T_X))$, or rather to a smaller, carefully chosen subbundle $E \subset \mathbf{P}(\mathbf{P}(T_X))$. Riemann-Roch now implies that there are many nonzero sections as soon as $c_1^2 > \frac{9}{13}c_2$, which happens for surfaces in $\mathbf{P}^3$ of degree ≥ 15. This yields nontrivial algebraic differential equations satisfied by f. More work is needed to conclude, of course.

1.5 Weak analytic hyperbolicity

If a smooth projective surface X of general type satisfies $c_1^2 > 2c_2$, then indeed, for $m \gg 0$, the base locus B of $\mathbf{S}^m\Omega_X \otimes L^{-1}$ projects onto a proper subset of X (Schneider–Tancredi), which must be an algebraic curve, rational or elliptic. McQuillan proved a very deep theorem on parabolic leaves of algebraic foliations that together with a finiteness result of Bogomolov's implies the following stronger result.

Theorem 4 (McQuillan, 1997) *A smooth projective surface of general type such that $c_1^2 > c_2$ contains only finitely many rational or elliptic curves and any nonconstant entire curve maps onto one of them.*

This points to another extension of property (H3): we may try to study complex varieties X for which any holomorphic curve $f : \mathbf{C} \to X$ is *algebraically degenerate*, i.e., the image $f(\mathbf{C})$ is contained in a proper algebraic subvariety of X (which may or may not depend on f). The Bloch theorem gives a very large class of varieties that satisfy this property.

Theorem 5 *Let X be a smooth algebraic variety of dimension $< h^0(X, \Omega_X)$. There exists a proper algebraic subvariety $Y \subset X$ such that the image of any holomorphic curve $\mathbf{C} \to X$ is contained in Y.*

PROOF. For any compact Kähler variety X, there is a complex torus A and a holomorphic map (the Albanese map) $\alpha : X \to A$ such that $\alpha(X)$ generates A as a group and any map from X to a complex torus factors through α. The dimension of A is $h^0(X, \Omega_X)$.

Since $\dim(X) < \dim(A)$, the image $\alpha(X)$ is a proper subvariety of A which is not a translated subtorus. One can show that there are only finitely many *maximal* translated subtori in $\alpha(X)$, so that their union is a proper algebraic subvariety Y of $\alpha(X)$. The Bloch theorem implies that the image of any holomorphic curve $\mathbf{C} \to X$ is contained in $\alpha^{-1}(Y)$. □

This is of no use for hypersurfaces of $\mathbf{P}^n$, since they have no nonzero holomorphic forms when $n \geq 3$.

There is another conjectural characterization of these varieties.

Conjecture 6 *A smooth complex algebraic variety X is of general type if and only if the union of the images of all nonconstant holomorphic maps $\mathbf{C} \to X$ is not Zariski dense in X.*

This implies Conjectures 1 and 2. The "first case" to look at is that of surfaces of general type: do they contain only finitely many rational or elliptic curves? This holds for surfaces of general type with $c_1^2 > c_2$ by Theorem 4.

2 Algebraic hyperbolicity

Here are a few possible extensions of property (H1) for a higher-dimensional complex algebraic variety X:

- require that all curves contained in X have genus at least 2;
- require that all holomorphic maps from a complex torus to X are constant;
- require that all subvarieties of X are of general type.

The first property turns out to be too weak and not very interesting.

The other two properties, which are thought to be equivalent to analytic hyperbolicity by Conjectures 1 and 2, are not easy to check, although they are known to hold for very general hypersurfaces in $\mathbf{P}^n$ of degree $\geq 2n-1$ (Pacienza).

A more subtle definition, which is a bit technical, but involves only curves hence is easier to check, was suggested by Demailly: *a complex algebraic variety X is* algebraically hyperbolic *if, given an ample line bundle L on X, there is a positive number ε such that, for every smooth projective curve C of genus g and nonconstant map $f : C \to X$, one has*

$$2g - 2 \geq \varepsilon \deg(f^* L) > 0 \tag{1}$$

(This is independent of the choice of L.) It is known to hold for very general hypersurfaces in $\mathbf{P}^n$ of degree $\geq 2n$ (Voisin).

It is weaker than analytic hyperbolicity, to which it is conjectured to be equivalent, but is stronger than the second property above. Indeed, if we have a nonconstant map $f : A \to X$, consider a curve $C \subset A$ on which f is not constant. If $m_A : A \to A$ is the multiplication by m, we have $\deg((m_A \circ f|_C)^* L) = m^2 \deg(f|_C^* L)$, which eventually violates (1).

For subvarieties of a complex torus, all three properties are equivalent to X being analytically hyperbolic (see Theorem 3).

What is missing here is a direct connection between these various algebraic properties and analytic hyperbolicity.

3 Ampleness of the cotangent bundle

Algebraic geometers know what an ample vector bundle is. For those who don't, a smooth projective variety X has ample cotangent bundle if, given an ample line bundle L on X, there is a positive number ε such that, for every smooth projective curve C, nonconstant map $f : C \to X$, and quotient line bundle $f^* \Omega_X \twoheadrightarrow M$,

$$\deg(M) \geq \varepsilon \deg(f^* L) \tag{2}$$

(Demailly's definition requires this inequality only for the quotient $f^* \Omega_X \twoheadrightarrow \Omega_C$). For mathematicians with a background in differential geometry, a compact complex variety has ample cotangent bundle if it carries a Kähler metric with negative bisectional holomorphic curvature (or negative usual sectional Riemannian curvature). In particular, it can be seen as a generalization of property (H2).

This is a very strong property, strictly stronger than analytic hyperbolicity (the product of two curves of genus ≥ 2 is analytically hyperbolic, but its cotangent bundle is not ample). It is so strong that there were for a long time very few examples, although these varieties are expected to be reasonably abundant.

- *Ball quotients:* smooth complex projective varieties that are uniformized by the ball $\mathbf{B}_n$ inherit from the Bergman metric a metric with positive holomorphic bisectional curvature hence have ample cotangent bundle.[3]

- *Mostow and Siu's construction:* these authors constructed in 1980 a compact Kähler surface not covered by the ball $\mathbf{B}_2$, with negative sectional Riemannian curvature.

- *Bogomolov's construction:* in the product of sufficiently many varieties with big cotangent bundle (for instance, curves of genus at least 2), the intersection of sufficiently many sufficiently ample general hypersurfaces has ample cotangent bundle.

- *Subvarieties of abelian varieties:* in an abelian variety of dimension n, the intersection of at least $n/2$ sufficiently ample general hypersurfaces has ample cotangent bundle.

The cotangent bundle of a smooth subvariety X of dimension $> n/2$ in an abelian variety A of dimension n is never ample. This is because its restriction to a fiber of the Gauss map $\mathbf{P}(T_X) \to \mathbf{P}(T_{A,0})$ has a trivial (hence degree 0) quotient, and this fiber has dimension

$$\geq \dim(\mathbf{P}(T_X)) - \dim(\mathbf{P}(T_{A,0})) = 2\dim(X) - \dim(A)$$

Therefore, if $\dim(X) > \dim(A)/2$, there is a curve C, a nonconstant map $f : C \to X$, and a trivial quotient line bundle $f^*\Omega_X \twoheadrightarrow \mathscr{O}_C$ that violates (2).

Similarly, I conjecture that in $\mathbf{P}^n$, the intersection of at least $n/2$ sufficiently ample general hypersurfaces has ample cotangent bundle.

4 Arithmetic hyperbolicity

How to extend property (H4) to higher-dimensional varieties (defined over a number field K)? In other words, what does it mean to have "few" K-points? Keeping in mind conjectures 1 and 6, we might require of an algebraic variety X defined over a number field K,

[3] By §1.1, this is the case for smooth projective surfaces of general type with $c_1^2 = 3c_2$, or more generally for smooth projective varieties with ample canonical bundle and $c_1^n = 2\frac{n+1}{n}c_1^{n-2}c_2$.

- either that X has only finitely many K-points;
- or that the K-points of X are not Zariski dense, i.e., they are contained in a proper algebraic subvariety of X.

Accordingly, we get the following conjectures.

Conjecture 7 (Lang) *A smooth analytically hyperbolic projective variety defined over a number field K has finitely many K-points.*

This would be a consequence of the following, together with (one direction of) Conjecture 2.

Conjecture 8 (Bombieri, Lang, Vojta) *If a smooth projective variety defined over a number field K has positive dimension and is of general type, its K-points are not Zariski dense.*

A weaker form is the following.

Conjecture 9 *A smooth projective variety with ample cotangent bundle, defined over a number field K, has finitely many K-points.*

An analogous result for varieties defined over function fields of curves was proved by Noguchi and Deschamps.

Moriwaki remarked that, if the cotangent bundle is, in addition to being ample, generated by global sections, Conjecture 9 follows from the following theorem of Faltings, which remains to this day the most powerful (if not the only) tool in the subject: *all K-points of a subvariety X of an abelian variety A defined over a number field K lie in the union of finitely many translated (by K-points) abelian subvarieties of A contained in X.* Indeed, if Ω_X is generated by global sections, the Albanese map $\alpha : X \to A$ (which exists over any field) is unramifed. The image of $\alpha(\overline{X(K)})$ is a union of translated abelian subvarieties of A and a theorem of Lang says that $\overline{X(K)}$ is also a union of abelian varieties.

Apart from that, there are no results in the direction of the conjectures above. For example, the known compact ball quotients (see §3) are quotients by arithmetic groups hence are defined over a number field K, but it is unknown whether they have finitely many K-points.

Although the analytic and algebraic theories may give ideas and inspirations about what could (should?) be true in the arithmetic case, no direct connection between these areas has been found yet.