

# A NEW FAMILY OF SYMPLECTIC FOURFOLDS

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## 1. IRREDUCIBLE SYMPLECTIC VARIETIES

It follows from work of Beauville and Bogomolov that any smooth complex compact Kähler manifold  $M$  with  $c_1(M) = 0$  has a finite étale cover which is a product of (Kähler) manifolds of one of the following types:

- complex tori (they have  $\chi(\mathcal{O}) = 0$ );
- Calabi-Yau manifolds, i.e., simply connected projective manifolds  $X$  with  $H^0(X, \Omega_X^p) = 0$  for  $0 < p < \dim(X)$  (they have  $\chi(\mathcal{O}) = 1 + (-1)^{\dim}$ );
- irreducible symplectic manifolds, i.e., compact even-dimensional Kähler manifolds  $X$  with an everywhere non-degenerate 2-form  $\omega$  such that  $H^0(X, \Omega_X^p)$  is 0 for  $p$  odd and generated by  $\omega^{p/2}$  for  $p \in \{0, 2, \dots, \dim(X)\}$  (they have  $\chi(\mathcal{O}) = 1 + \frac{1}{2}\dim$ ).

It is very easy to construct Calabi-Yau manifolds, for example by taking complete intersections in Fano manifolds. Irreducible symplectic manifolds are much rarer. Beauville constructed

- the  $n$ -th punctual Hilbert scheme  $S^{[n]}$  for a K3 surface  $S$  (it has  $b_2 = 23$ );
- the inverse image  $K^n(A)$  of the origin by the sum morphism  $A^{[n+1]} \rightarrow A$ , where  $A$  is a 2-dimensional torus (it has  $b_2 = 7$ ).

and O'Grady constructed two other families in dimensions 6 and 10.

## 2. EXISTING DESCRIPTIONS OF A GENERAL DEFORMATION OF $S^{[2]}$

When  $S$  is algebraic, so is  $S^{[n]}$ , and it has Picard number 2 for  $S$  general, whereas a general algebraic deformation has Picard number 1. In fact, the  $S^{[n]}$  form a hypersurface in their polarized deformation space and the problem we want to contribute to is to give a geometric description of the general deformation of  $S^{[n]}$ . This has been done in a few cases, and only for  $n = 2$

**The Beauville-Donagi construction.** Let  $X \subset \mathbf{P}(V_6)$  be a cubic hypersurface and let  $F(X) \subset G(2, V_6)$  be the scheme parametrizing

lines contained in  $X$ . Any equation of  $X$  can be seen as a section of  $\mathrm{Sym}^3 \mathcal{S}_2^*$  (where  $\mathcal{S}_d$  will be the canonical rank- $d$  subsheaf on any Grassmannian  $G(d, V)$ ). Since this rank-4 vector bundle is globally generated,  $F(X)$  will be smooth of the expected dimension  $8 - 4 = 4$  for  $X$  general (infinitesimal computations show that this is actually the case as soon as  $X$  is smooth). By adjunction,  $c_1(F(X)) = 0$ .

The Koszul resolution of  $\mathcal{O}_{F(X)}$  and a computer package such as Macaulay2 gives  $\chi(F(X), \mathcal{O}_{F(X)}) = 3$  from which it follows, together with the classification from §1, that  $F(X)$  is an irreducible symplectic fourfold. Beauville and Donagi have a more explicit proof of this fact: they show that for particular cubics (“Pfaffian cubics”),  $F(X)$  is actually isomorphic to  $S^{[2]}$ , where  $S$  is a general K3 surface of genus 8.

**The Iliev-Ranestad construction.** Let  $X \subset \mathbf{P}(V_6)$  be again a general cubic. The variety of sums of powers of  $X$  also degenerates to the same  $S^{[2]}$ , but with a polarization of a different degree. So this gives a geometric description of general algebraic deformation of  $S^{[2]}$  with a different polarization.

**The O’Grady construction.** O’Grady shows that certain double covers of Eisenbud-Popescu-Walter sextics are general deformations of  $S^{[2]}$ , where  $S$  is a general K3 surface of genus 6.

### 3. OUR CONSTRUCTION

Let  $V_{10}$  be a complex vector space of dimension 10 and let  $\sigma \in \bigwedge^3 V_{10}^*$  be a 3-form (a dimension count shows that the moduli space of such  $\sigma$  is 20-dimensional). We define a subvariety

$$Y_\sigma = \{[W_6] \in G(6, V_{10}) \mid \sigma|_{W_6} \equiv 0\}.$$

As in the Beauville-Donagi construction,  $Y_\sigma$  is the zero-set of a section of the globally generated rank-20 vector bundle  $\bigwedge^3 \mathcal{S}_6^*$ , hence is smooth of dimension  $24 - 20 = 4$  for  $\sigma$  general.

**Theorem 3.1.** *For  $\sigma$  general,  $Y_\sigma$  is an irreducible symplectic fourfold.*

*Proof.* There are several ways to prove that. The quickest (with the computer), but the least enlightening, is to compute the Euler characteristic using the Koszul resolution as in the Beauville-Donagi construction.

Alternatively, as shown to us by Manivel and Han, using the Koszul resolution of  $\mathcal{O}_{Y_\sigma}$ , Bott’s theorem, and properties of the irreducible representations that occur in  $\bigwedge^i(\bigwedge^3 V_6)$  (or, alternatively, the program LiE), one can prove directly  $h^2(Y_\sigma, \mathcal{O}_{Y_\sigma}) = 1$ .

There is also a more geometric proof. Let

$$G_\sigma = \{([W_3], [W_6]) \in G(3, V_{10}) \times G(6, V_{10}) \mid W_3 \subset W_6, \sigma|_{W_6} \equiv 0\},$$

with its two projections

$$Y_\sigma \xleftarrow{p} G_\sigma \xrightarrow{q} F_\sigma,$$

where

$$F_\sigma = \{[W_3] \in G(3, V_{10}) \mid \sigma|_{W_3} \equiv 0\}$$

is a Plücker hyperplane section. It induces a cohomological correspondence

$$p_*q^* : H^{20}(F_\sigma, \mathbf{Q})_{\text{van}} \rightarrow H^2(Y_\sigma, \mathbf{Q}).$$

One can compute, using Griffiths' description of the Hodge structure on the vanishing cohomology of an ample hypersurface, the Hodge structure on the left-hand-side: it has  $h^{9,11} = h^{11,9} = 1$  and  $h^{10,10} = 20$ , and the other Hodge numbers are 0. For  $\sigma$  very general, this Hodge structure is also simple, and one proves that  $p_*q^*$  is non-zero, hence injective. This implies easily the theorem.  $\square$

From the last proof, we get  $b_2(Y_\sigma) \geq 23$  because  $p_*q^*$  takes its values in  $H^2(Y_\sigma, \mathbf{Q})_{\text{van}} \subsetneq H^2(Y_\sigma, \mathbf{Q})$ . There is actually equality by work of Guan, who proved that 23 is the maximal possible second Betti number for an irreducible symplectic fourfold, and that since  $b_2(Y_\sigma) = 23$ , it has the same Hodge numbers as the second punctual Hilbert scheme of a K3 surface.

#### 4. THE MANIFOLDS $Y_\sigma$ ARE DEFORMATIONS OF $S^{[2]}$

Unfortunately, we were unable to identify special  $\sigma$  for which  $Y_\sigma$  is actually isomorphic to an  $S^{[2]}$ . Instead, we will prove the following result.

**Theorem 4.1.** *The polarized manifolds  $(Y_\sigma, \mathcal{O}_{Y_\sigma}(1))$  are the general deformations of  $S^{[2]}$ , where  $S$  is a general K3 surface of genus 12, endowed with an explicit (nonample) line bundle.*

**Work of Gritsenko, Hulek, and Sankaran on moduli spaces of polarized irreducible symplectic fourfolds.** They prove that polarized irreducible symplectic fourfolds which are deformation equivalent to  $(S^{[2]}, h)$ , for some class  $h$ , admit a quasi-projective coarse moduli space  $\mathcal{M}_h$  which is finite over a dense open subset of a locally symmetric modular variety  $\mathcal{S}_h$ . There are two “types” of classes  $h$ ; when  $h$  is “of split type,”  $\mathcal{S}_h$  (hence also every component of  $\mathcal{M}_h$ ) is of general type for  $d := \frac{1}{2}q(h) \geq 12$  and of nonnegative Kodaira dimension for  $d = 9$  or  $11$  ( $q$  is the Beauville-Bogomolov quadratic form). In our

case,  $h$  is of “nonsplit type” and  $d = 11$ , and our construction proves that one component of  $\mathcal{M}_h$  (hence also  $\mathcal{S}_h$ ) is unirational.

*Proof of the theorem.* We will study general  $\sigma$  for which the hyperplane section  $F_\sigma \subset G(3, V_{10})$  is singular. This corresponds to the existence of  $W \subset V_{10}$  of dimension 3 such that

$$\sigma|_{W \times W \times V_{10}} \equiv 0$$

and generically, this is a single such  $W$ . The corresponding variety  $Y_\sigma$  remains irreducible, 4-dimensional, and normal, but becomes singular along the subvariety

$$Y'_\sigma = \{[W_6] \in Y_\sigma \mid W \subset W_6\}.$$

The projection  $p : V_{10} \rightarrow V_{10}/W$  induces a morphism  $Y'_\sigma \rightarrow G(3, V_{10}/W)$  whose image is easily checked to be the zero-set of a (general) section of  $\mathcal{O}(1) \oplus (\wedge^2 \mathcal{S}_3^*)^{\oplus 3}$ . This is, by work of Mukai, a (general) K3 surface  $S$  of genus 12. One also gets a *birational* map

$$\phi : S^{[2]} \dashrightarrow Y_\sigma$$

by sending a pair  $(W', W'')$  to the unique  $[W_6] \in Y_\sigma$  such that  $p(W_6) = W' \oplus W''$ .

To finish the proof, we follow ideas of Huybrechts, who proved that birational equivalence implies deformation equivalence for irreducible symplectic manifolds. However, we are in a situation where only a singular degeneration of  $Y_\sigma$  is birationally equivalent to an  $S^{[2]}$ , to which we cannot apply directly Huybrechts’ theorem. In particular,  $L$  is *not* ample on  $S^{[2]}$ .

Consider a general deformation  $(\mathcal{X}, \mathcal{L}) \rightarrow \Delta$  of the pair  $(S^{[2]}, L)$ , with  $L = \phi^* \mathcal{O}_{Y_\sigma}(1)$ , so that for  $t \in \Delta$  very general, the group  $\text{Pic}(X_t)$  has rank 1. The steps are then the following.

- One checks by explicit Riemann-Roch calculations that for all  $k \in \mathbf{Z}$ ,

$$\chi(S^{[2]}, L^k) = \chi(Y_\sigma, \mathcal{O}_{Y_\sigma}(k)).$$

- Using this formula, and a criterion for projectivity of D. Huybrechts, one proves that  $X_t$  is projective, and that  $L_t$  is ample for  $t$  very general, hence for  $t$  general.
- One then shows that  $\phi$  induces isomorphisms

$$\phi^* : H^0(Y_\sigma, \mathcal{O}_{Y_\sigma}(k)) \simeq H^0(S^{[2]}, L^k)$$

for all  $k \geq 0$ .

- It follows that for  $k \gg 0$  and  $t$  general,

$$\begin{aligned} h^0(S^{[2]}, L^k) &= h^0(Y_\sigma, \mathcal{O}_{Y_\sigma}(k)) = \chi(Y_\sigma, \mathcal{O}_{Y_\sigma}(k)) \\ &= \chi(S^{[2]}, L^k) = \chi(X_t, L^k) = h^0(X_t, L_t^k). \end{aligned}$$

This implies that all  $\pi_*(\mathcal{L}^k)$  are locally free in a neighborhood of 0 in  $\Delta$ . In the flat projective family

$$\mathcal{Y} = \mathcal{P}roj\left(\bigoplus_{k \geq 0} \pi_*(\mathcal{L}^k)\right) \rightarrow \Delta,$$

the central fiber is isomorphic to  $(Y_\sigma, \mathcal{O}_{Y_\sigma}(1))$ , whereas the fiber over  $t \neq 0$  is  $X_t$  endowed with the ample line bundle  $L_t$ . It only remains to check that any small deformation of  $(Y_\sigma, \mathcal{O}_{Y_\sigma}(1))$  is given by a deformation of  $\sigma$ .  $\square$

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