A NEW FAMILY OF SYMPLECTIC FOURFOLDS

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1. Irreducible symplectic varieties

It follows from work of Beauville and Bogomolov that any smooth complex compact Kähler manifold M with $c_1(M) = 0$ has a finite étale cover which is a product of (Kähler) manifolds of one of the following types:

- complex tori (they have $\chi(\mathscr{O}) = 0$);
- Calabi-Yau manifolds, i.e., simply connected projective manifolds X with $H^0(X, \Omega_X^p) = 0$ for $0 < p < \dim(X)$ (they have $\chi(\mathscr{O}) = 1 + (-1)^{\dim}$);
- irreducible symplectic manifolds, i.e., compact even-dimensional Kähler manifolds X with an everywhere non-degenerate 2-form ω such that $H^0(X, \Omega_X^p)$ is 0 for p odd and generated by $\omega^{p/2}$ for $p \in \{0, 2, \dots, \dim(X)\}$ (they have $\chi(\mathscr{O}) = 1 + \frac{1}{2}\dim$).

It is very easy to construct Calabi-Yau manifolds, for example by taking complete intersections in Fano manifolds. Irreducible symplectic manifolds are much rarer. Beauville constructed

- the n-th punctual Hilbert scheme $S^{[n]}$ for a K3 surface S (it has $b_2 = 23$);
- the inverse image $K^n(A)$ of the origin by the sum morphism $A^{[n+1]} \to A$, where A is a 2-dimensional torus (it has $b_2 = 7$).

and O'Grady constructed two other families in dimensions 6 and 10.

2. Existing descriptions of a general deformation of $S^{[2]}$

When S is algebraic, so is $S^{[n]}$, and it has Picard number 2 for S general, whereas a general algebraic deformation has Picard number 1. In fact, the $S^{[n]}$ form a hypersurface in their polarized deformation space and the problem we want to contribute to is to give a geometric description of the general deformation of $S^{[n]}$. This has been done in a few cases, and only for $n = 2$

The Beauville-Donagi construction. Let $X \subset \mathbf{P}(V_6)$ be a cubic hypersurface and let $F(X) \subset G(2, V_6)$ be the scheme parametrizing lines contained in X. Any equation of X can be seen as a section of $\mathrm{Sym}^3 \mathscr{S}_2^*$ (where $\mathscr{S}_d$ will be the canonical rank-d subscheaf on any Grassmannian $G(d, V)$). Since this rank-4 vector bundle is globally generated, $F(X)$ will be smooth of the expected dimension $8 - 4 = 4$ for X general (infinitesimal computations show that this is actually the case as soon as X is smooth). By adjunction, $c_1(F(X)) = 0$.

The Koszul resolution of $\mathscr{O}_{F(X)}$ and a computer package such as Macaulay2 gives $\chi(F(X), \mathscr{O}_{F(X)}) = 3$ from which it follows, together with the classification from §1, that $F(X)$ is an irreducible symplectic fourfold. Beauville and Donagi have a more explicit proof of this fact: they show that for particular cubics ("Pfaffian cubics"), $F(X)$ is actually isomorphic to $S^{[2]}$, where S is a general K3 surface of genus 8.

The Iliev-Ranestad construction. Let $X \subset \mathbf{P}(V_6)$ be again a general cubic. The variety of sums of powers of X also degenerates to the same $S^{[2]}$, but with a polarization of a different degree. So this gives a geometric description of general algebraic deformation of $S^{[2]}$ with a different polarization.

The O'Grady construction. O'Grady shows that certain double covers of Eisenbud-Popescu-Walter sextics are general deformations of $S^{[2]}$, where S is a general K3 surface of genus 6.

3. Our construction

Let V_{10} be a complex vector space of dimension 10 and let $\sigma \in \bigwedge^3 V_{10}^*$ be a 3-form (a dimension count shows that the moduli space of such σ is 20-dimensional). We define a subvariety

$$Y_\sigma = \{[W_6] \in G(6, V_{10}) \mid \sigma|_{W_6} \equiv 0\}.$$

As in the Beauville-Donagi construction, Y_σ is the zero-set of a section of the globally generated rank-20 vector bundle $\bigwedge^3 \mathscr{S}_6^*$, hence is smooth of dimension $24 - 20 = 4$ for σ general.

Theorem 3.1. *For σ general, Y_σ is an irreducible symplectic fourfold.*

Proof. There are several ways to prove that. The quickest (with the computer), but the least enlightening, is the compute the Euler characteristic using the Koszul resolution as in the Beauville-Donagi construction.

Alternatively, as shown to us by Manivel and Han, using the Koszul resolution of $\mathscr{O}_{Y_\sigma}$, Bott's theorem, and properties of the irreducible representations that occur in $\bigwedge^i(\bigwedge^3 V_6)$ (or, alternatively, the program LiE), one can prove directly $h^2(Y_\sigma, \mathscr{O}_{Y_\sigma}) = 1$.

There is also a more geometric proof. Let

$$G_\sigma = \{([W_3],[W_6]) \in G(3,V_{10}) \times G(6,V_{10}) \mid W_3 \subset W_6,\ \sigma|_{W_6} \equiv 0\},$$

with its two projections

$$Y_\sigma \xleftarrow{p} G_\sigma \xrightarrow{q} F_\sigma,$$

where

$$F_\sigma = \{[W_3] \in G(3,V_{10}) \mid \sigma|_{W_3} \equiv 0\}$$

is a Plücker hyperplane section. It induces a cohomological correspondence

$$p_*q^* : H^{20}(F_\sigma, \mathbf{Q})_{\mathrm{van}} \to H^2(Y_\sigma, \mathbf{Q}).$$

One can compute, using Griffiths' description of the Hodge structure on the vanishing cohomology of an ample hypersurface, the Hodge structure on the left-hand-side: it has $h^{9,11} = h^{11,9} = 1$ and $h^{10,10} = 20$, and the other Hodge numbers are 0. For σ very general, this Hodge structure is also simple, and one proves that p_*q^* is non-zero, hence injective. This implies easily the theorem. □

From the last proof, we get $b_2(Y_\sigma) \geq 23$ because p_*q^* takes its values in $H^2(Y_\sigma, \mathbf{Q})_{\mathrm{van}} \subsetneq H^2(Y_\sigma, \mathbf{Q})$. There is actually equality by work of Guan, who proved that 23 is the maximal possible second Betti number for an irreducible symplectic fourfold, and that since $b_2(Y_\sigma) = 23$, it has the same Hodge numbers as the second punctual Hilbert scheme of a K3 surface.

4. The manifolds Y_σ are deformations of $S^{[2]}$

Unfortunately, we were unable to identify special σ for which Y_σ is actually isomorphic to an $S^{[2]}$. Instead, we will prove the following result.

Theorem 4.1. *The polarized manifolds $(Y_\sigma, \mathscr{O}_{Y_\sigma}(1))$ are the general deformations of $S^{[2]}$, where S is a general K3 surface of genus 12, endowed with an explicit (nonample) line bundle.*

Work of Gritsenko, Hulek, and Sankaran on moduli spaces of polarized irreducible symplectic fourfolds. They prove that polarized irreducible symplectic fourfolds which are deformation equivalent to $(S^{[2]}, h)$, for some class h, admit a quasi-projective coarse moduli space $\mathscr{M}_h$ which is finite over a dense open subset of a locally symmetric modular variety $\mathscr{S}_h$. There are two "types" of classes h; when h is "of split type," $\mathscr{S}_h$ (hence also every component of $\mathscr{M}_h$) is of general type for $d := \frac{1}{2}q(h) \geq 12$ and of nonnegative Kodaira dimension for $d = 9$ or 11 (q is the Beauville-Bogomolov quadratic form). In our case, h is of "nonsplit type" and $d = 11$, and our construction proves that one component of $\mathscr{M}_h$ (hence also $\mathscr{S}_h$) is unirational.

Proof of the theorem. We will study general σ for which the hyperplane section $F_\sigma \subset G(3, V_{10})$ is singular. This corresponds to the existence of $W \subset V_{10}$ of dimension 3 such that

$$\sigma|_{W \times W \times V_{10}} \equiv 0$$

and generically, this is a single such W. The corresponding variety Y_σ remains irreducible, 4-dimensional, and normal, but becomes singular along the subvariety

$$Y'_\sigma = \{[W_6] \in Y_\sigma \mid W \subset W_6\}.$$

The projection $p : V_{10} \to V_{10}/W$ induces a morphism $Y'_\sigma \to G(3, V_{10}/W)$ whose image is easily checked to be the zero-set of a (general) section of $\mathscr{O}(1) \oplus (\bigwedge^2 \mathscr{S}_3^*)^{\oplus 3}$. This is, by work of Mukai, a (general) K3 surface S of genus 12. One also gets a *birational* map

$$\phi : S^{[2]} \dashrightarrow Y_\sigma$$

by sending a pair (W', W'') to the unique $[W_6] \in Y_\sigma$ such that $p(W_6) = W' \oplus W''$.

To finish the proof, we follow ideas of Huybrechts, who proved that birational equivalence implies deformation equivalence for irreducible symplectic manifolds. However, we are in a situation where only a singular degeneration of Y_σ is birationally equivalent to an $S^{[2]}$, to which we cannot apply directly Huybrechts' theorem. In particular, L is *not* ample on $S^{[2]}$.

Consider a general deformation $(\mathscr{X}, \mathscr{L}) \to \Delta$ of the pair $(S^{[2]}, L)$, with $L = \phi^* \mathscr{O}_{Y_\sigma}(1)$, so that for $t \in \Delta$ very general, the group $\mathrm{Pic}(X_t)$ has rank 1. The steps are then the following.

- One checks by explicit Riemann-Roch calculations that for all $k \in \mathbf{Z}$,
$$\chi(S^{[2]}, L^k) = \chi(Y_\sigma, \mathscr{O}_{Y_\sigma}(k)).$$
- Using this formula, and a criterion for projectivity of D. Huybrechts, one proves that X_t is projective, and that L_t is ample for t very general, hence for t general.
- One then shows that ϕ induces isomorphisms
$$\phi^* : H^0(Y_\sigma, \mathscr{O}_{Y_\sigma}(k)) \simeq H^0(S^{[2]}, L^k)$$
for all $k \geq 0$.

- It follows that for $k \gg 0$ and t general,

$$\begin{aligned} & h^0(S^{[2]}, L^k) = h^0(Y_\sigma, \mathscr{O}_{Y_\sigma}(k)) = \chi(Y_\sigma, \mathscr{O}_{Y_\sigma}(k)) \\ = \ & \chi(S^{[2]}, L^k) = \chi(X_t, L^k) = h^0(X_t, L_t^k). \end{aligned}$$

The implies that all $\pi_*(\mathscr{L}^k)$ are locally free in a neighborhood of 0 in Δ. In the flat projective family

$$\mathscr{Y} = \mathscr{P}roj\Big(\bigoplus_{k\geq 0} \pi_*(\mathscr{L}^k)\Big) \to \Delta,$$

the central fiber is isomorphic to $(Y_\sigma, \mathscr{O}_{Y_\sigma}(1))$, whereas the fiber over $t \neq 0$ is X_t endowed with the ample line bundle L_t. It only remains to check that any small deformation of $(Y_\sigma, \mathscr{O}_{Y_\sigma}(1))$ is given by a deformation of σ. □

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