Let \( k \) be an algebraically closed field of characteristic \( p \). If \( X \) is a subvariety of \( \mathbb{P}_k^n \) and \( x \) a point on \( X \), we denote by \( T_x X \) the projective closure of the embedded Zariski tangent space to \( X \) at \( x \).

It is known that the family of lines on a general hypersurface of degree \( d \) in \( \mathbb{P}^n \) is smooth of dimension \( 2n - 3 - d \). When \( n \) is very large with respect to \( d \), any smooth hypersurface of degree \( d \) in \( \mathbb{P}^n \) has this property ([HMP]). However, the bound in [HMP] is very large, whereas

- the family of lines on a smooth cubic in \( \mathbb{P}^n \), with \( n \geq 3 \), is smooth of the expected dimension \( 2n - 6 \).
- the family of lines on a smooth quartic in \( \mathbb{P}^n \), with \( n \geq 4 \), has the expected dimension \( 2n - 7 \) for \( p \neq 2, 3 \) (Collino). However, it may be reducible and non-reduced.

**Conjecture 1.** Assume \( p = 0 \) or \( p \geq d \). The family of lines on a smooth hypersurface of degree \( d \) in \( \mathbb{P}^n \) has the expected dimension \( 2n - 3 - d \) for \( n \geq d \).

These bounds are the best possible: the family of lines contained in a Fermat hypersurface in \( \mathbb{P}^n \) has dimension at least \( n - 3 \), which is larger than the expected dimension for \( n < d \). Also, when \( p > 0 \), the family of lines contained in a Fermat hypersurface of degree \( p + 1 \) in \( \mathbb{P}^n \) has dimension \( 2n - 4 \), which is larger than the expected dimension.

Note that by taking general hyperplane sections,\(^1\) it is enough to prove Conjecture 1 in the case \( n = d \). In this case, \( X \) cannot be covered by lines (at least in characteristic zero): the normal bundle to a generic line must be generated by its global sections hence be non-negative, but its total degree is \( n - 1 - d < 0 \). Conjecture 1 would therefore follow from the following conjecture.

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1 A family of lines meets the set of lines contained in a given hyperplane in codimension 2, except if they all pass through the same point. Lines contained in \( X \) and passing through the same point \( x \) are contained in \( X \cap T_x X \) hence form a family of dimension at most \( n - 2 \) with equality if and only if \( X = T_x X \). Since \( n - 3 \leq 2n - d - 3 \) for \( d \leq n \), the conjecture is proved in this case.

Date: November 2003.
Conjecture 2. An \((n - 2)\)-dimensional family of lines contained in a smooth hypersurface \(X\) of degree \(\geq n - 2\) in \(\mathbb{P}^n\) must cover \(X\).

In other words, a subvariety of \(\mathbb{P}^n\) of codimension \(\geq 2\) that contains \(\infty^{n-2}\) lines (i.e., \(\infty^{c-1}\) lines through a general point) is not contained in a smooth hypersurface of degree \(\geq n - 2\).

The assumption on the degree is necessary: a smooth quadric \(X\) in \(\mathbb{P}^5\) contains a 2-dimensional family of 2-planes. A subfamily of dimension 1 of these 2-planes sweeps out a hypersurface in \(X\) with a 3-dimensional family of lines on it. Also, if \(X\) is a smooth hypersurface of degree \(n - 3\), the family of lines contained in a general hyperplane section has dimension \(n - 2\). More generally, a general complete intersection of multidegree \((d_0, \ldots, d_s)\) in \(\mathbb{P}^n\) contains \(\infty^{2n-2-\sum(d_j+1)}\) lines; if \(\sum(d_j + 1) \leq n\), it contains an \((n - 2)\)-dimensional family of lines and is at the same time contained in smooth hypersurfaces of degrees \(d_0, \ldots, d_s\).

We prove below Conjecture 2 for \(n \leq 5\), hence Conjecture 1 for \(d \leq 5\). Note that all present proofs for bounding the dimension of the family of lines contained in a hypersurface are based on an estimate of \(h^0(\ell, N_{\ell/P})\) (and even of \(h^0(\ell, N_{\ell/P}(-1))\)). In our case however, it may well happen that \(h^0(\ell, N_{\ell/P}) > 2n - 3 - d\) everywhere, so that the family is non-reduced.

1. The results

We begin with a preliminary result.

Lemma 3. Let \(P\) be an \(r\)-plane contained in the smooth locus of a hypersurface \(X\) of degree \(d\) in \(\mathbb{P}^n\).

- We have \(r \leq \frac{n - 1}{2}\) unless \(X\) is a hyperplane.
- If \(r = \frac{n - 1}{2}\), we have

\[H^0(P, N_{P/X}(d - 3)) = 0\]

In particular, \(X\) contains at most finitely many such planes when \(d \geq 3\).

Proof. Let \(F\) be a homogeneous polynomial of degree \(d\) defining \(X\) and let \(x_{r+1} = \cdots = x_n = 0\) be linear equations defining \(P\). We may write

\[F = x_{r+1}F_{r+1} + \cdots + x_nF_n\]

Any common zero of the \(n - r\) polynomials \(F_{r+1}, \ldots, F_n\) on \(P\) is a singular point of \(X\) hence \(n - r \geq r + 1\) if these polynomials are non-constant.
If \( r = \frac{n-1}{2} \), these polynomials form a regular sequence and the Koszul complex

\[
0 \to \mathcal{O}_P(-(r+1)(d-1)) \to \cdots \to \mathcal{O}_P(-2(d-1))^{n-r} \to \mathcal{O}_P(-d) \to \cdots
\]

is exact. On the other hand, the normal bundle to \( P \) in \( X \) fits in an exact sequence

\[
0 \to \mathcal{N}_{P/X} \to \mathcal{O}_P(1)^{n-r} \to \mathcal{O}_P(d) \to 0
\]

hence we get a long exact sequence

\[
0 \to \mathcal{O}_P(d-(r+1)(d-1)) \to \cdots \to \mathcal{O}_P(d-2(d-1))^{n-r} \to \mathcal{N}_{P/X} \to 0
\]

Since the sheaves in this free resolution with \( r \) terms have only non-zero \( H^0 \) and \( H^r \), the associated spaces of sections form an exact sequence and this still holds after tensoring by \( \mathcal{O}_P(d-3) \). \( \square \)

From now on, we assume that the smooth hypersurface \( X \) of degree \( d \geq 2 \) in \( \mathbb{P}^n \) contains an \( (n-2) \)-dimensional irreducible family of lines which covers an integral subvariety \( S \) of \( X \) of dimension \( k = n-1-s \). Varieties with many lines have been studied extensively ([S], [R]). The results are as follows.

**Theorem 4.** A \( k \)-dimensional irreducible subvariety \( S \) of \( \mathbb{P}^n \) contains at most \( \infty^{2k-2} \) lines.

- If \( S \) contains \( \infty^{2k-2} \) lines, it is a \( k \)-plane.
- If \( S \) contains \( \infty^{2k-3} \) lines,
  - a) either \( S \) contains \( \infty^1 \) \((k-1)\)-planes;
  - b) or \( S \) is a quadric.
- If \( S \) contains \( \infty^{2k-4} \) lines,
  - a) either \( S \) contains \( \infty^2 \) \((k-2)\)-planes;
  - b) or \( S \) contains \( \infty^1 \) \((k-1)\)-dimensional quadrics;
  - c) or \( S \) is a section of \( G(1,4) \subset \mathbb{P}^9 \) by \( 6-k \) hyperplanes;
  - d) or the linear span of \( S \) has dimension \( \leq k+2 \).

**Corollary 5.** If \( d \geq 3 \), we have

\[
s \leq \frac{1}{2}(n-4)
\]

**Proof.** The first item of the theorem yields

\[
s \leq \frac{1}{2}(n-2)
\]

and if there is equality, \( X \) contains an \( \frac{n}{2} \)-plane, which contradicts Lemma 3.
If $s = \frac{1}{2}(n - 3)$, either $X$ contains $\infty^1 \frac{1}{2}(n - 1)$-planes, which contradicts Lemma 3, or $X$ contains a $\frac{1}{2}(n + 1)$-dimensional quadric, which contradicts the following lemma. \hfill \Box

**Lemma 6.** If $X$ contains a $q$-dimensional hypersurface of degree $\delta$, either $d = \delta$ or $q \leq \frac{n - 1}{2}$.

**Proof.** Let $L$ be the linear span of the hypersurface $Y$ contained in $X$. We may assume, by lemma 3, that $L$ is not contained in $X$. Since the Gauss map of $X$ is finite, the inverse image of the $(n - q - 2)$-plane $\{H \in \mathbb{P}^{n*} \mid H \supset L\}$ has dimension at most $n - q - 2$. If $n - q - 2 \leq q - 2$, the hypersurface $X \cap L$ in $L$ is non-singular in codimension 1 hence integral hence equal to $Y$, and $d = \delta$. \hfill \Box

This proves Conjecture 2 for $n \leq 5$.\footnote{Added in 2006: in the article “Lines on projective hypersurfaces”, by R. Beheshti (J. Reine Angew. Math. 592 (2006), 1–21), Conjecture 1 is proved for $d \leq 6$ and $p = 0$.}

**References**


