LINES ON SMOOTH HYPERSURFACES

OLIVIER DEBARRE

Let $\mathbf{k}$ be an algebraically closed field of characteristic p. If X is a subvariety of $\mathbf{P}^n_{\mathbf{k}}$ and x a point on X, we denote by $\mathbf{T}_x X$ the projective closure of the embedded Zariski tangent space to X at x.

It is known that the family of lines on a *general* hypersurface of degree d in $\mathbf{P}^n$ is smooth of dimension $2n-3-d$. When n is very large with respect to d, any smooth hypersurface of degree d in $\mathbf{P}^n$ has this property ([HMP]). However, the bound in [HMP] is very large, whereas

- the family of lines on a smooth cubic in $\mathbf{P}^n$, with $n \geq 3$, is smooth of the expected dimension $2n-6$.
- the family of lines on a smooth quartic in $\mathbf{P}^n$, with $n \geq 4$, has the expected dimension $2n-7$ for $p \neq 2, 3$ (Collino). However, it may be reducible and non-reduced.

Conjecture 1. *Assume $p = 0$ or $p \geq d$. The family of lines on a smooth hypersurface of degree d in $\mathbf{P}^n$ has the expected dimension $2n-3-d$ for $n \geq d$.*

These bounds are the best possible: the family of lines contained in a Fermat hypersurface in $\mathbf{P}^n$ has dimension at least $n-3$, which is larger than the expected dimension for $n < d$. Also, when $p > 0$, the family of lines contained in a Fermat hypersurface of degree $p+1$ in $\mathbf{P}^n$ has dimension $2n-4$, which is larger than the expected dimension.

Note that by taking general hyperplane sections,[1] it is enough to prove Conjecture 1 in the case $n = d$. In this case, X cannot be covered by lines (at least in characteristic zero): the normal bundle to a generic line must be generated by its global sections hence be non-negative, but its total degree is $n-1-d < 0$. Conjecture 1 would therefore follow from the following conjecture.

Date: November 2003.

[1]A family of lines meets the set of lines contained in a given hyperplane in codimension 2, except if they all pass through the same point. Lines contained in X and passing through the same point x are contained in $X \cap \mathbf{T}_x X$ hence form a family of dimension at most $n-2$ with equality if and only if $X = \mathbf{T}_x X$. Since $n-3 \leq 2n-d-3$ for $d \leq n$, the conjecture is proved in this case.

Conjecture 2. *An $(n-2)$-dimensional family of lines contained in a smooth hypersurface X of degree $\geq n-2$ in $\mathbf{P}^n$ must cover X.*

In other words, *a subvariety of $\mathbf{P}^n$ of codimension $c \geq 2$ that contains ∞^{n-2} lines (i.e., ∞^{c-1} lines through a general point) is not contained in a* smooth *hypersuface of degree $\geq n-2$.*

The assumption on the degree is necessary: a smooth quadric X in $\mathbf{P}^5$ contains a 2-dimensional family of 2-planes. A subfamily of dimension 1 of these 2-planes sweeps out a hypersurface in X with a 3-dimensional family of lines on it. Also, if X is a smooth hypersurface of degree $n-3$, the family of lines contained in a general hyperplane section has dimension $n-2$. More generally, a general complete intersection of multidegree $(d_0, \dots, d_s)$ in $\mathbf{P}^n$ contains $\infty^{2n-2-\sum(d_j+1)}$ lines; if $\sum(d_j+1) \leq n$, it contains an $(n-2)$-dimensional family of lines and is at the same time contained in smooth hypersurfaces of degrees $d_0, \dots, d_s$.

We prove below Conjecture 2 for $n \leq 5$, hence Conjecture 1 for $d \leq 5$. Note that all present proofs for bounding the dimension of the family of lines contained in a hypersurface are based on an estimate of $h^0(\ell, N_{\ell/P})$ (and even of $h^0(\ell, N_{\ell/P}(-1))$). In our case however, it may well happen that $h^0(\ell, N_{\ell/P}) > 2n-3-d$ everywhere, so that the family is non-reduced.

1. The results

We begin with a preliminary result.

Lemma 3. *Let P be an r-plane contained in a the smooth locus of a hypersurface X of degree d in $\mathbf{P}^n$.*

- *We have $r \leq \dfrac{n-1}{2}$ unless X is a hyperplane.*
- *If $r = \frac{n-1}{2}$, we have*

$$H^0(P, N_{P/X}(d-3)) = 0$$

  In particular, X contains at most finitely many such planes when $d \geq 3$.

Proof. Let F be a homogeneous polynomial of degree d defining X and let $x_{r+1} = \cdots = x_n = 0$ be linear equations defining P. We may write

$$F = x_{r+1}F_{r+1} + \cdots + x_n F_n$$

Any common zero of the $n-r$ polynomials $F_{r+1}, \dots, F_n$ on P is a singular point of X hence $n-r \geq r+1$ if these polynomials are nonconstant.

If $r = \frac{n-1}{2}$, these polynomials form a regular sequence and the Koszul complex

$$0 \to \mathcal{O}_P(-(r+1)(d-1)) \longrightarrow \cdots \longrightarrow \mathcal{O}_P(-2(d-1))^{\binom{n-r}{2}} \longrightarrow \mathcal{O}_P(-(d-1))^{n-r} \xrightarrow{\cdot(F_{r+1},\dots,F_n)} \mathcal{O}_P \to 0$$

is exact. On the other hand, the normal bundle to P in X fits in an exact sequence

$$0 \to N_{P/X} \longrightarrow \mathcal{O}_P(1)^{n-r} \xrightarrow{\cdot(F_{r+1},\dots,F_n)} \mathcal{O}_P(d) \longrightarrow 0$$

hence we get a long exact sequence

$$0 \to \mathcal{O}_P(d-(r+1)(d-1)) \longrightarrow \cdots \longrightarrow \mathcal{O}_P(d-2(d-1))^{\binom{n-r}{2}} \longrightarrow N_{P/X} \to 0$$

Since the sheaves in this free resolution with r terms have only non-zero H^0 and H^r, the associated spaces of sections form an exact sequence and this still holds after tensoring by $\mathcal{O}_P(d-3)$. □

From now on, we assume that the smooth hypersurface X of degree $d \geq 2$ in $\mathbf{P}^n$ contains an $(n-2)$-dimensional irreducible family of lines which covers an integral subvariety S of X of dimension $k = n-1-s$. Varieties with many lines have been studied extensively ([S], [R]). The results are as follows.

Theorem 4. *A k-dimensional irreducible subvariety S of $\mathbf{P}^n$ contains at most ∞^{2k-2} lines.*

- *If S contains ∞^{2k-2} lines, it is a k-plane.*
- *If S contains ∞^{2k-3} lines,*
  a) *either S contains ∞^1 $(k-1)$-planes;*
  b) *or S is a quadric.*
- *If S contains ∞^{2k-4} lines,*
  a) *either S contains ∞^2 $(k-2)$-planes;*
  b) *or S contains ∞^1 $(k-1)$-dimensional quadrics;*
  c) *or S is a section of $G(1,4) \subset \mathbf{P}^9$ by $6-k$ hyperplanes;*
  d) *or the linear span of S has dimension $\leq k+2$.*

Corollary 5. *If $d \geq 3$, we have*

$$s \leq \frac{1}{2}(n-4)$$

Proof. The first item of the theorem yields

$$s \leq \frac{1}{2}(n-2)$$

and if there is equality, X contains an $\frac{n}{2}$-plane, which contradicts Lemma 3.

If $s = \frac{1}{2}(n-3)$, either X contains ∞^1 $\frac{1}{2}(n-1)$-planes, which contradicts Lemma 3, or X contains a $\frac{1}{2}(n+1)$-dimensional quadric, which contradicts the following lemma. □

Lemma 6. *If X contains a q-dimensional hypersurface of degree δ, either $d = \delta$ or $q \leq \frac{n-1}{2}$.*

Proof. Let L be the linear span of the hypersurface Y contained in X. We may assume, by lemma 3, that L is not contained in X. Since the Gauss map of X is finite, the inverse image of the $(n-q-2)$-plane $\{H \in \mathbf{P}^{n*} \mid H \supset L\}$ has dimension at most $n-q-2$. If $n-q-2 \leq q-2$, the hypersurface $X \cap L$ in L is non-singular in codimension 1 hence integral hence equal to Y, and $d = \delta$. □

This proves Conjecture 2 for $n \leq 5$.[2]

References

[C] Collino, A., Lines on quartic threefolds, *J. London Math. Soc.* **19** (1979), 257–267.

[HMP] Harris, J., Mazur, B., Pandharipande, R., Hypersurfaces of low degree, *Duke Math. J.* **95** (1998), 125–160.

[R] Rogora, E., Varieties with many lines, *Manuscr. Math.* **82** (1994), 207–26.

[S] Segre, B., Sulle V_n contenenti più di ∞^{n-k} S_k, I and II. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.* **5** (1948), 193–197 and 275–280.

[2] Added in 2006: in the article "Lines on projective hypersurfaces", by R. Beheshti (*J. Reine Angew. Math.* **592** (2006), 1–21), Conjecture 1 is proved for $d \leq 6$ and $p = 0$.