

PERIODS AND MODULI

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We work over the field of complex numbers.

It has been known since the nineteenth century that there is a group structure on the points of the plane curve with (Weierstraß) equation

$$y^2 = x^3 + ax + b.$$

This curve is (unaptly) called an *elliptic curve*. Their higher-dimensional analog are *complex tori* V/Γ , where Γ is a lattice in a (finite-dimensional) complex vector space V . The group structure and the analytic structure are obvious, but not all tori are algebraic. For that, we need an additional (Riemann) condition: the existence of a definite positive Hermitian form on V whose (skew-symmetric) imaginary part is integral on Γ . An algebraic complex torus is called an *abelian variety*. When this skew-symmetric form is in addition unimodular on Γ , we say that the abelian variety is *principally polarized*. It contains a hypersurface uniquely determined up to translation (“the” *theta divisor*).

The combination of the algebraic and group structures makes the geometry of abelian varieties very rich. This is one of the reasons why it is useful to associate, whenever possible, an abelian variety (if possible principally polarized) to a given geometric situation.

1. ATTACHING AN ABELIAN VARIETY TO AN ALGEBRAIC OBJECT

1.1. **Curves.** Given a smooth projective curve C of genus g , we have the Hodge decomposition

$$H^1(C, \mathbf{Z}) \subset H^1(C, \mathbf{C}) = H^{0,1}(C) \oplus H^{1,0}(C),$$

where the right-hand-side is a $2g$ -dimensional complex vector space and $H^{1,0}(C) = \overline{H^{0,1}(C)}$. The g -dimensional complex torus

$$J(C) = H^{0,1}(C)/H^1(C, \mathbf{Z})$$

is a principally polarized abelian variety (the polarization corresponds to the unimodular intersection form on $H^1(C, \mathbf{Z})$). We therefore have an additional geometric object: the theta divisor $\Theta \subset J(C)$, uniquely defined up to translation.

The geometry of the theta divisor of the Jacobian of a curve has been intensively studied since Riemann. One may say that it is well-known.

1.2. Prym varieties. Given a double étale cover $\pi : \tilde{C} \rightarrow C$ between smooth projective curves, one can endow the abelian variety $P(\tilde{C}/C) = J(\tilde{C})/\pi^*J(C)$ with a natural principal polarization. The dimension of $P(\tilde{C}/C)$ is

$$g(\tilde{C}) - g(C) = g(C) - 1;$$

it is called the *Prym variety* attached to π . By results of Mumford and Beauville, we have a rather good understanding of the geometry of the theta divisor of $P(\tilde{C}/C)$.

1.3. Threefolds. If X is a smooth projective threefold, the Hodge decomposition is

$$H^3(X, \mathbf{Z}) \subset H^3(X, \mathbf{C}) = (H^{0,3}(X) \oplus H^{1,2}(X)) \oplus (\text{complex conjugate})$$

and we may again define the *intermediate Jacobian* of X as

$$J(X) = (H^{0,3}(X) \oplus H^{1,2}(X))/H^3(X, \mathbf{Z}).$$

This is in general only a complex torus, because the intersection form has opposite signs on $H^{0,3}(X)$ and $H^{1,2}(X)$. In case $H^{0,3}(X)$ vanishes, however, we still have a principally polarized abelian variety.

In some situations, the intermediate Jacobian is a Prym. For example if X is a smooth projective threefold with a conic bundle structure $X \rightarrow \mathbf{P}^2$ (i.e., a morphism with fibers isomorphic to conics), define the discriminant curve $C \subset \mathbf{P}^2$ as the locus of points whose fibers are reducible conics, i.e., unions of two lines. The choice of one of these lines defines a double cover $\tilde{C} \rightarrow C$. Although C may have singular points, they are only nodes, and we can still define a Prym variety $P(\tilde{C}/C)$. We have $H^{0,3}(X) = 0$, and

$$J(X) \simeq P(\tilde{C}/C)$$

as principally polarized abelian varieties. This isomorphism is a powerful tool for proving nonrationality of some threefolds: the intermediate Jacobian of a rational threefold must have a very singular theta divisor and the theory of Prym varieties can sometimes tell that this does not happen.

Example 1.1 (Cubic threefolds). If $X \subset \mathbf{P}^4$ is a smooth cubic hypersurface, we have $h^{0,3}(X) = 0$ and $h^{1,2}(X) = 5$, so that $J(X)$ is a 5-dimensional principally polarized abelian variety.

Any such X contains a line ℓ . Projecting from this line induces a conic bundle structure $\tilde{X} \rightarrow \mathbf{P}^2$ on the blow-up \tilde{X} of ℓ in X . The

discriminant curve $C \subset \mathbf{P}^2$ is a quintic and $J(X) \simeq P(\tilde{C}/C)$ (this agrees with the fact that $J(X)$ has dimension $g(C) - 1 = \binom{4}{2} - 1 = 5$). This isomorphism can be used to prove that the theta divisor $\Theta \subset J(X)$ has a unique singular point, which has multiplicity 3. In particular, it is not “singular enough,” and X is not rational (Clemens-Griffiths).

Example 1.2 (Quartic double solids). If $p : X \rightarrow \mathbf{P}^3$ is a double cover branched along a quartic surface $B \subset \mathbf{P}^3$ (a *quartic double solid*), we have $h^{0,3}(X) = 0$ and $h^{1,2}(X) = 10$, so that $J(X)$ has dimension 10. There is in general no conic bundle structure on X . However, when B acquires a node s , the variety X becomes singular, but there is a (rational) conic bundle structure $X \dashrightarrow \mathbf{P}^2$ obtained by composing p with the projection $\mathbf{P}^3 \dashrightarrow \mathbf{P}^2$ from s . The discriminant curve is a sextic. This degeneration can be used to prove that for X general, the singular locus of the theta divisor $\Theta \subset J(X)$ has dimension 5 (Voisin) and has a unique component of that dimension (Debarre). Again, this implies that X is not rational.

Example 1.3 (Fano threefolds of degree 10). If $X \subset \mathbf{P}^9$ is the smooth complete intersection of the Grassmannian $G(2, 5)$ in its Plücker embedding, two hyperplanes, and a smooth quadric, we have $h^{0,3}(X) = 0$ and $h^{1,2}(X) = 10$, so that $J(X)$ has dimension 10.

1.4. Odd-dimensional varieties. Let X be a smooth projective variety of dimension $2n + 1$ whose Hodge decomposition is of the form

$$H^{2n+1}(X, \mathbf{C}) = H^{n,n+1}(X) \oplus H^{n+1,n}(X).$$

We may again define the *intermediate Jacobian* of X as

$$J(X) = H^{n,n+1}(X)/H^{2n+1}(X, \mathbf{Z}).$$

This is again principally polarized abelian variety.

Example 1.4 (Intersections of two quadrics). If $X \subset \mathbf{P}^{2n+3}$ is the smooth base-locus of a pencil Λ of quadrics, its Hodge decomposition satisfies the conditions above, so we can form the principally polarized abelian variety $J(X)$.

The choice of one of the two components of the family of \mathbf{P}^{n+1} contained in a member of Λ defines a double cover $C \rightarrow \mathbf{P}^1$ ramified exactly over the $2n + 4$ points corresponding to the singular members of the pencil. The curve C is smooth, hyperelliptic, of genus $n + 1$, and its Jacobian is isomorphic to $J(X)$ (Reid, Donagi).¹

¹Although this will not concern us here, let me mention that the set of \mathbf{P}^n contained in X is also isomorphic to $J(X)$ (Reid, Donagi), and the set of \mathbf{P}^{n-1} contained in X is isomorphic to the moduli space of stable vector bundles of rank 2 and fixed determinant of odd degree (Desale-Ramanan).

Example 1.5 (Intersections of three quadrics). If $X \subset \mathbf{P}^{2n+4}$ is the smooth complete intersection of a net of quadrics $\Pi = \langle Q_1, Q_2, Q_3 \rangle$, its Hodge decomposition satisfies the conditions above, so we can form the principally polarized abelian variety $J(X)$.

When $n \geq 1$, the variety X contains a line ℓ . The map $X \dashrightarrow \Pi$ defined by sending a point $x \in X$ to the unique quadric in Π that contains the 2-plane $\langle \ell, x \rangle$, is a (rational) *quadric* bundle structure on X . The discriminant curve $C \subset \Pi$ parametrizes singular quadrics; its equation is

$$\det(\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3) = 0.$$

It has degree $2n + 5$. The choice of a component of the set of \mathbf{P}^{n+1} contained in a singular quadric of the net Π defines a double étale cover $\tilde{C} \rightarrow C$ and (Beauville, Tjurin), $J(X) \simeq P(\tilde{C}/C)$.

2. PERIODS AND PERIOD MAPS

Assume now that we have a family $\mathcal{X} \rightarrow S$ of smooth projective threefolds, whose fibers X_s all satisfy $H^{0,3}(X_s) = 0$. We can construct for each $s \in S$ the intermediate Jacobian $J(X_s)$. Let us look at this from a slightly different point of view. Assume that the base S is simply connected; we can then identify all $H^3(X_s, \mathbf{Z})$ with the fixed lattice $H = H^3(X_0, \mathbf{Z})$ of rank $2g$ and define an algebraic *period map* with values in a Grassmannian:

$$\begin{aligned} \wp : S &\longrightarrow G(g, H_{\mathbf{C}}) \\ s &\longmapsto H^{2,1}(X_s) \end{aligned}$$

Letting Q be the skew-symmetric intersection form on $H_{\mathbf{C}}$, we also have

- $H^{2,1}(X_s)$ is totally isotropic for Q ,
- the Hermitian form $iQ(\cdot, \bar{\cdot})$ is definite on $H^{2,1}(X_s)$,

so that \wp takes its values into a dense open subset of an isotropic Grassmannian which is isomorphic to the Siegel upper half-space $\mathcal{H}_g = \mathrm{Sp}(2g, \mathbf{R})/\mathrm{U}(g)$. If $\wp(s)$ correspond to $\tau(s) \in \mathcal{H}_g$, we have

$$J(X_s) = H_{\mathbf{C}}/(H \oplus \tau(s)H).$$

Back to the case where the base S is general, with universal cover $\tilde{S} \rightarrow S$, we obtain a diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{\wp}} & \mathcal{H}_g \\ \downarrow & & \downarrow \\ S & \xrightarrow{\wp} & \mathcal{H}_g/\mathrm{Sp}(2g, \mathbf{Z}) = \mathcal{A}_g \end{array}$$

where $\tilde{\varphi}$ is holomorphic, φ is algebraic, and

$$\mathcal{A}_g = \{\text{ppav's of dimension } g\}/\text{isomorphism}$$

is the *moduli space* of principally polarized abelian varieties of dimension g . It has a natural structure of a quasi-projective variety of dimension $g(g+1)/2$.

We want to generalize this construction for any smooth projective variety X of dimension n . Even if the Hodge decomposition

$$H^k(X, \mathbf{C}) = \bigoplus_{p=0}^k H^{p, k-p}(X)$$

does not have level one (i.e., only two pieces), we can still use it to define a period map as follows (Griffiths). Choose an ample class $h \in H^2(X, \mathbf{Z}) \cap H^{1,1}(X)$ and define the *primitive cohomology* by

$$H^k(X, \mathbf{C})_{\text{prim}} = \text{Ker} \left(H^k(X, \mathbf{C}) \xrightarrow{\smile h^{n-k+1}} H^{2n-k+2}(X, \mathbf{C}) \right).$$

Set $H^{p,q}(X)_{\text{prim}} = H^{p,q}(X) \cap H^k(X, \mathbf{C})_{\text{prim}}$ and

$$F^r = \bigoplus_{p \geq r} H^{p, k-p}(X)_{\text{prim}}.$$

Define a bilinear form on $H^k(X, \mathbf{C})_{\text{prim}}$ by

$$Q(\alpha, \beta) = \alpha \smile \beta \smile h^{n-k}.$$

The associated *period domain* is then the set \mathcal{D} of flags

$$0 = F^{k+1} \subset F^k \subset \dots \subset F^1 \subset F^0 = H^k(X, \mathbf{C})$$

satisfying

- $F^r \oplus \overline{F^{k-r+1}} = F^0$,
- $F^r = (F^{k-r+1})^{\perp_Q}$,
- the Hermitian form $i^k Q(\cdot, \bar{\cdot})$ is definite on each $H^{p, k-p}(X)_{\text{prim}}$ (with known sign).

These conditions define a period domain \mathcal{D} which is a homogeneous space, quotient of a Lie group by a compact subgroup which may be not maximal, so that \mathcal{D} is not in general Hermitian symmetric.

Given a family $\mathcal{X} \rightarrow S$ of polarized varieties, we obtain as above a *holomorphic* map

$$\tilde{\varphi} : \tilde{S} \longrightarrow \mathcal{D}.$$

Note that the lattice $H^k(X, \mathbf{Z})$ has played no role here yet. It will however if we want to define a period map on S : one needs to quotient by the action of $\pi_1(S, 0)$ and this group acts via the monodromy

representation

$$\pi_1(S, 0) \longrightarrow \text{Aut}(H^k(X_0, \mathbf{Z})).$$

The discrete group $\Gamma = \text{Aut}(H^k(X_0, \mathbf{Z}))$ acts on $H^k(X_0, \mathbf{C})_{\text{prim}}$, and properly on \mathcal{D} , hence we obtain a diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{\varphi}} & \mathcal{D} \\ \downarrow & & \downarrow \\ S & \xrightarrow{\varphi} & \mathcal{D}/\Gamma, \end{array}$$

where \mathcal{D}/Γ is an analytic space and φ is holomorphic.

Example 2.1 (Quartic surfaces). If $B \subset \mathbf{P}^3$ is a quartic (hence K3) surface, we have

$$\begin{array}{rcl} H^2(B, \mathbf{C}) & = & H^{0,2}(B) \oplus H^{1,1}(B) \oplus H^{1,0}(B) \\ \text{dimensions:} & & 1 \qquad 20 \qquad 1 \\ H^2(B, \mathbf{C})_{\text{prim}} & = & H^{0,1}(B) \oplus H^{1,1}(B)_{\text{prim}} \oplus H^{1,0}(B) \\ \text{dimensions:} & & 1 \qquad 19 \qquad 1 \end{array}$$

Because of its properties, this decomposition is completely determined by the point of $\mathbf{P}(H^2(B, \mathbf{C})_{\text{prim}})$ defined by the line $H^{1,0}(B)$. The period map takes values in the 19-dimensional period domain

$$\begin{aligned} \mathcal{D}^{19} &= \{[\omega] \in \mathbf{P}^{20} \mid Q(\omega, \omega) = 0, Q(\omega, \bar{\omega}) > 0\} \\ &\simeq \text{SO}(19, 2)^0 / \text{SO}(19) \times \text{SO}(2), \end{aligned}$$

where Q is a quadratic form, integral on a lattice H , with signature $(19, 2)$ on $H_{\mathbf{R}}$. It is a bounded symmetric domain of type IV and the discrete group Γ_{19} can be explicitly described.

Note that quartic surfaces are in one-to-one correspondance with quartic double solids (Example 1.2), so we may also associate to B the 5-dimensional intermediate Jacobian $J(X)$ of the double solid $X \rightarrow \mathbf{P}^3$ branched along B and get another kind of period map with values in \mathcal{A}_5 .

Example 2.2 (Cubic fourfolds). If $X \subset \mathbf{P}^5$ is a smooth cubic fourfold, the situation is completely analogous: the decomposition

$$\begin{array}{rcl} H^4(X, \mathbf{C})_{\text{prim}} & = & H^{1,3}(X) \oplus H^{2,2}(X)_{\text{prim}} \oplus H^{3,1}(X) \\ \text{dimensions:} & & 1 \qquad 20 \qquad 1 \end{array}$$

is completely determined by the point $[H^{3,1}(X)]$ of $\mathbf{P}(H^4(X, \mathbf{C})_{\text{prim}})$, and

$$\begin{aligned} \mathcal{D}^{20} &= \{[\omega] \in \mathbf{P}^{21} \mid Q(\omega, \omega) = 0, Q(\omega, \bar{\omega}) > 0\} \\ &\simeq \text{SO}(20, 2)^0 / \text{SO}(20) \times \text{SO}(2). \end{aligned}$$

Here the quadratic form Q , integral on a lattice H , has signature $(20, 2)$ on $H_{\mathbf{R}}$. The domain \mathcal{D} is again a bounded symmetric domain of type IV. The discrete group Γ^{20} can be explicitly described.

Example 2.3 (Cubic surfaces). If $X \subset \mathbf{P}^3$ is a smooth cubic surface, with equation $F(x_0, x_1, x_2, x_3) = 0$, we have $H^2(X, \mathbf{C}) = H^{1,1}(X)$, so the period map is trivial.

Proceeding as in Example 2.1, where we associated to a quartic surface in \mathbf{P}^3 the double cover of \mathbf{P}^3 branched along this surface, we may consider the cyclic triple cover $\tilde{X} \rightarrow \mathbf{P}^3$ branched along X . It is isomorphic to the cubic threefold with equation $F(x_0, x_1, x_2, x_3) + x_4^3 = 0$ in \mathbf{P}^4 , so its Hodge structure is as in Example 1.1 and the period domain is \mathcal{H}_5 . On the other hand, the Hodge structure carries an action of μ_3 . The eigenspace $H_{\omega}^3(\tilde{X})$ for the eigenvalue $e^{2i\pi/3}$ splits as

$$\begin{array}{rcl} H_{\omega}^3(\tilde{X}) & = & H_{\omega}^{1,2}(\tilde{X}) \oplus H_{\omega}^{2,1}(\tilde{X}) \\ \text{dimensions:} & & 1 \qquad\qquad 4 \end{array}$$

Following Allcock, Carlson, and Toledo, one can then define a period map with values in the 4-dimensional space

$$\{[\omega] \in \mathbf{P}^4 \mid Q(\omega, \bar{\omega}) < 0\},$$

where the quadratic form Q is integral on a lattice H , with signature $(4, 1)$ on $H_{\mathbf{R}}$. It is isomorphic to the complex hyperbolic space \mathbf{B}^4 , which is much smaller than \mathcal{H}_5 ! The discrete group Γ^4 can be explicitly described.

Example 2.4 (Cubic threefolds, II). Similarly, if $X \subset \mathbf{P}^4$ is a smooth cubic threefold, we consider the cyclic triple cover $\tilde{X} \rightarrow \mathbf{P}^4$ branched along X . It is a cubic fourfold in \mathbf{P}^5 , so its Hodge structure is as in Example 2.2, with an extra symmetry of order three. With analogous notation as above, $H_{\omega}^{3,1}(\tilde{X})$ has dimension 1 and $H_{\omega}^{2,2}(\tilde{X})$ has dimension 10. So we (Laza) can define a period map with values in the 10-dimensional space

$$\{\omega \in \mathbf{P}^{10} \mid Q(\omega, \bar{\omega}) < 0\} \simeq \mathbf{B}^{10},$$

where again the quadratic form Q is integral on a lattice H , with signature $(10, 1)$ on $H_{\mathbf{R}}$.

3. IS THE PERIOD MAP INJECTIVE?

3.1. Curves. This is the famous *Torelli theorem*: a curve C is determined (up to isomorphism) by the pair $(J(C), \theta)$. It is customary to

call *Torelli problem* the question of deciding whether an algebraic object is determined by a polarized abelian variety attached to it. In fancy terms, the period map

$$\begin{array}{c} \mathcal{M}_g = \{\text{curves of genus } g\}/\text{isomorphism} \\ \downarrow \wp_g \\ \mathcal{A}_g = \{\text{ppav's of dimension } g\}/\text{isomorphism} \end{array}$$

is injective.

3.2. Prym varieties. The period map

$$\begin{array}{c} \mathcal{R}_g = \{\text{double étale covers of genus-}g \text{ curves}\}/\text{isom.} \\ \downarrow \wp_g \\ \mathcal{A}_{g-1} = \{\text{ppav's of dimension } g-1\}/\text{isom.} \end{array}$$

cannot be injective in low genera for dimensional reasons. The following table sums up the situation:

g	$\dim(\mathcal{R}_g)$	$\dim(\mathcal{A}_{g-1})$	\wp_g
2	3	1	dominant, not injective
3	6	3	dominant, not injective
4	9	6	dominant, not injective
5	12	10	dominant, not injective
6	15	15	dominant, 27:1
$g \geq 7$	$3g-3$	$g(g-1)/2$	generically injective, <i>not</i> injective

The injectivity defect for $g \geq 7$ is not yet entirely understood.

3.3. Cubic threefolds. The period map

$$\begin{array}{c} \mathcal{M}_{\text{ct}}^{10} = \left\{ \begin{array}{l} \text{smooth cubic} \\ \text{threefolds} \end{array} \right\} / \text{isom.} = \mathbf{P}(H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(3)))^0 / \text{PGL}(5) \\ \downarrow \wp_{\text{ct}} \\ \mathcal{A}_5 \end{array}$$

for cubic threefolds is injective. This can be seen as follows (Beauville): if $X \subset \mathbf{P}^4$ is a cubic, we explained in Example 1.1 that the theta divisor $\Theta \subset J(X)$ has a unique singular point s , which has multiplicity 3. It turns out that the projectified tangent cone

$$\mathbf{P}(TC_{X,s}) \subset \mathbf{P}(T_{J(X),s}) = \mathbf{P}^4$$

is isomorphic to X .

Of course, \wp_{ct} is not dominant, since it maps a 10-dimensional space to a 15-dimensional space. Its image was characterized geometrically by Casalaina-Martin and Friedman: it is essentially the set of elements of \mathcal{A}_5 whose theta divisor has a point of multiplicity 3.

Recall (Example 2.4) that Allcock, Carlson, and Toledo defined another period map

$$\wp'_{\text{ct}} : \mathcal{M}_{\text{ct}} \longrightarrow \mathbf{B}^{10}/\Gamma^{10}.$$

They prove (among other things) that \wp'_{ct} induces an isomorphism onto an open subset whose complement is explicitly described.

3.4. Quartic double solids and quartic surfaces. The period map

$$\begin{array}{c} \mathcal{M}_{\text{qds}}^{19} = \left\{ \begin{array}{l} \text{smooth quartic} \\ \text{double solids} \end{array} \right\} / \text{isom.} = \mathbf{P}(H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(4)))^0 / \text{PGL}(4) \\ \downarrow \wp_{\text{qds}} \\ \mathcal{A}_{10} \end{array}$$

for quartic double solids is injective. This can be seen as follows: as mentioned in Example 1.2, if $X \rightarrow \mathbf{P}^3$ is a smooth quartic double solid, the singular locus of the theta divisor $\Theta \subset J(X)$ has a unique 5-dimensional component S . General points s of S are double points on Θ , and the projectified tangent cones $\mathbf{P}(TC_{\Theta,s})$ are, after translation, quadrics in $\mathbf{P}(T_{J(X),0}) = \mathbf{P}^9$. The intersection of these quadrics is isomorphic to the image of the branch quartic surface $B \subset \mathbf{P}^3$ by the Veronese morphism $v_2 : \mathbf{P}^3 \rightarrow \mathbf{P}^9$ (Clemens).

Again, \wp_{qds} maps a 19-dimensional space to a 45-dimensional space, so it cannot be dominant. However, X is determined by the quartic surface $B \subset \mathbf{P}^3$, and we have another period map (Example 2.1)

$$\wp'_{\text{qds}} : \mathcal{M}_{\text{qds}}^{19} \longrightarrow \mathcal{D}^{19}/\Gamma^{19},$$

which is an isomorphism onto an explicitly described open subset (Piatetski-Shapiro, Shafarevich).

3.5. Intersections of two quadrics. The period map

$$\begin{array}{c} \mathcal{M}_{\text{i2q}}^{2n+1} = \left\{ \begin{array}{l} \text{intersections of 2} \\ \text{quadrics in } \mathbf{P}^{2n+3} \end{array} \right\} / \text{isom.} \\ = G(2, H^0(\mathbf{P}^{2n+3}, \mathcal{O}_{\mathbf{P}^{2n+3}}(2)))^0 / \text{PGL}(2n+4) \\ \downarrow \wp_{\text{i2q}} \\ \mathcal{A}_{n+1} \end{array}$$

for intersections of two quadrics is injective. This is because, by the Torelli theorem for curves, from the intermediate Jacobian $J(X)$, one

can reconstruct the hyperelliptic curve C (see Example 1.4), hence its Weierstrass points, hence the pencil of quadrics that defines X . The image of \wp_{i2q} is the set of hyperelliptic Jacobians, hence it is not dominant for $n \geq 2$.

3.6. Intersections of three quadrics. The period map

$$\begin{aligned} \mathcal{M}_{i3q}^{2n^2+13n+12} &= \left\{ \begin{array}{l} \text{intersections of 3} \\ \text{quadrics in } \mathbf{P}^{2n+4} \end{array} \right\} / \text{isom.} \\ &= G(3, H^0(\mathbf{P}^{2n+4}, \mathcal{O}_{\mathbf{P}^{2n+4}}(2)))^0 / \text{PGL}(2n+5) \\ &\quad \downarrow \wp_{i3q} \\ &\quad \mathcal{A}_{(n+1)(2n+5)} \end{aligned}$$

for intersections of three quadrics is injective: using *ad hoc* geometric constructions, Debarre showed how to recover, from the theta divisor $\Theta \subset J(X)$, the double cover \tilde{C} of the discriminant curve C and the involution that defines the cover, and from there, it was classically known how to reconstruct X .

Again, for dimensional reasons, \wp_{i3q} is not dominant.

3.7. Cubic surfaces, II. Allcock, Carlson, and Toledo proved that the modified period map

$$\begin{aligned} \mathcal{M}_{cs}^4 &= \left\{ \begin{array}{l} \text{smooth cubic} \\ \text{surfaces} \end{array} \right\} / \text{isom.} = \mathbf{P}(H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3)))^0 / \text{PGL}(4) \\ &\quad \downarrow \wp'_{cs} \\ &\quad \mathbf{B}^4 / \Gamma^4 \end{aligned}$$

constructed in Example 2.3 induces an isomorphism with an explicit open subset of \mathbf{B}^4 / Γ^4 .

3.8. Cubic fourfolds, II. The period map (Example 2.2)

$$\begin{aligned} \mathcal{M}_{cf}^{20} &= \left\{ \begin{array}{l} \text{smooth cubic} \\ \text{fourfolds} \end{array} \right\} / \text{isom.} = \mathbf{P}(H^0(\mathbf{P}^5, \mathcal{O}_{\mathbf{P}^5}(3)))^0 / \text{PGL}(6) \\ &\quad \downarrow \wp_{cf} \\ &\quad \mathcal{D}^{20} / \Gamma^{20} \end{aligned}$$

for cubic fourfolds is injective (Voisin) and induces an isomorphism with an explicitly described open subset of $\mathcal{D}^{20} / \Gamma^{20}$ (Laza).

3.9. Fano threefolds of degree 10. I am referring here to the threefolds $X \subset \mathbf{P}^7$ considered in Example 1.3. Their moduli space \mathcal{M}^{22} has dimension 22, so the period map

$$\wp : \mathcal{M}^{22} \longrightarrow \mathcal{A}_{10}$$

can certainly not be dominant. Furthermore, Debarre, Iliev, and Manivel proved that its fibers have everywhere dimension 2.

Here is a sketch of the construction. Conics $c \subset X$ are parametrized by a smooth connected projective surface $F(X)$ which is the blow-up at one point of a smooth minimal surface $F_m(X)$ of general type. Given such a smooth conic c , one can construct another smooth Fano threefold X_c of degree 10 and a birational map $X \dashrightarrow X_c$ which is an isomorphism in codimension 1. In particular, it induces an isomorphism $J(X_c) \simeq J(X)$. However, one shows that the surface $F(X_c)$ is isomorphic to the blow-up of $F_m(X)$ at the point corresponding to c . In particular, it is (in general) *not* isomorphic to $F(X)$, so X_c is also *not* isomorphic to X . We actually prove that this construction (and a variant thereof) produces *two smooth proper 2-dimensional connected components* of each general fiber.

4. MODULI SPACES

Up to now, little care was taken to define the exact structure of the various “moduli spaces” we encountered. There are two main methods for constructing quasi-projective moduli spaces:

- Geometric Invariant Theory; roughly speaking, one “naturally” embeds the objects we want to classify into some fixed projective space, then quotients the corresponding subset of the Hilbert scheme by the action of the special linear group using GIT.
- constructing directly an ample line bundle on the functor; roughly speaking, one needs to construct on the base S of every family $X \rightarrow S$ of objects, a “natural” ample line bundle λ_S .

The advantage of the GIT method is that it also produces automatically a compactification of the moduli space. Its draw-back is that it is difficult to apply in practice. The second method, pioneered by Kollár and Viehweg, is more general, but technically more difficult. It can also produce compactifications, but there, one needs to decide which kind of singular objects one needs to add to make the moduli space compact. This approach now seems to have had complete success for varieties with ample canonical bundle.

Once a compactification is constructed, one may then try to extend the various period maps constructed above to various known compactifications of the period domain \mathcal{D}/Γ (such as the compactifications coming from the Baily-Borel theory). This turns out to be very useful, in some cases, in order to characterize the image of the original period map.

Here are a few examples.

4.1. Hypersurfaces. Hypersurfaces of degree d in \mathbf{P}^n are parametrized by the projective space $|dH| = \mathbf{P}(H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)))$. Let $|dH|^0$ be the dense open subset corresponding to *smooth* hypersurfaces. The complement $|dH| - |dH|^0$ is a hypersurface, because the condition that the equation F and its partial derivatives $\partial F/\partial x_0, \dots, \partial F/\partial x_n$ have a common zero is given by the vanishing of a single (homogeneous) polynomial in the coefficients of F . It follows that $|dH|^0$ is an affine open set, invariant by the action of the reductive group $\mathrm{SL}(n+1)$.

For $d \geq 3$, this action is *regular* in the sense of GIT (the dimensions of the stabilizers are locally constant) hence closed (the orbits are closed). Since $\mathcal{O}_{\mathbf{P}^n}(1)$ admits a $\mathrm{SL}(n+1)$ -linearization, $|dH|^0$ is contained in the set $|dH|^s$ of stable points associated with these data. The GIT theory implies that the quotient $|dH|^0/\mathrm{SL}(n+1)$, which is the moduli space of smooth hypersurfaces of degree d in \mathbf{P}^n , can be realized as an open set in the GIT quotient $|dH|^{ss}/\mathrm{SL}(n+1)$, which is a projective variety.

The precise description of the semistable points, i.e., of the kind of singularities one needs to add to obtain the GIT compactification of the moduli space, is a difficult task, impossible to achieve in general. Some cases are known: plane curves of degree ≤ 6 and cubic surfaces (Hilbert, Mumford, Shah), quartic surfaces (Shah), cubic threefolds (Allcock), cubic fourfolds (Laza)...

Example 4.1 (Cubic surfaces, III). Stable points correspond to cubic surfaces that have at most ordinary double points (“type A_1 ”). Semistable points correspond to cubic surfaces whose singular points are all of type A_1 or A_2 . GIT theory yields a compactification $\overline{\mathcal{M}}_{\mathrm{cs}}^4$ of the moduli space of smooth cubic surfaces (§3.7) and the modified period map extends to an isomorphism

$$\overline{\mathcal{M}}_{\mathrm{cs}}^4 \xrightarrow{\sim} \overline{\mathbf{B}^4/\Gamma^4},$$

where the right-hand-side is the Baily-Borel compactification of \mathbf{B}^4/Γ^4 .

Example 4.2 (Quartic surfaces, II). There is a list of all allowed singularities on quartic surfaces corresponding to semistable points (Shah).

Again, the period map (§3.4) induces an isomorphism (Kulikov)

$$\overline{\mathcal{M}}^{19} \xrightarrow{\sim} \overline{\mathcal{D}}^{19}/\Gamma^{19},$$

where the left-hand-side is the GIT-compactification and the right-hand-side is the Baily-Borel compactification.

Example 4.3 (Cubic threefolds, III). Stable points correspond to cubic threefolds whose singular points are of type A_n , with $1 \leq n \leq 4$. There is also a list of all possible singularities of cubic threefolds that correspond to semistable points (Allcock). The modified period map (§3.3) induces an morphism (which contracts a rational curve)

$$\overline{\mathcal{M}}_{\text{ct}}^{10} \longrightarrow \overline{\mathbf{B}}^{10}/\Gamma^{10},$$

where $\overline{\mathcal{M}}_{\text{ct}}^{10}$ is an explicit blow-up of the GIT-compactification and $\overline{\mathbf{B}}^{10}/\Gamma^{10}$ is the Baily-Borel compactification.

Example 4.4 (Cubic fourfolds, II). There are complete lists of all possible singularities of cubic fourfolds that correspond to stable and semistable points (Laza). The period map (§3.8) induces an isomorphism

$$\overline{\mathcal{M}}_{\text{ct}}^{20} \longrightarrow \overline{\mathcal{D}}^{20}/\Gamma^{20},$$

where $\overline{\mathcal{M}}_{\text{ct}}^{20}$ is an explicit blow-up of the GIT-compactification and $\overline{\mathcal{D}}^{20}/\Gamma^{20}$ is the Looijenga compactification (a modification of the Baily-Borel compactification).

Example 4.5 (Intersections of two quadrics, II). The moduli space of smooth intersections of two quadrics in \mathbf{P}^n can be constructed as the GIT quotient of an affine dense open set of the Grassmannian $G(2, H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(2)))$ (see §3.5) by the reductive group $SL(n+1)$. Using a slightly different presentation, Avritzer and Miranda prove that smooth intersections correspond exactly to stable points. Therefore, the moduli space is a quasi-projective variety.

Example 4.6 (Fano threefolds of degree 10, II). These threefolds $X \subset \mathbf{P}^7$ were considered in Example 1.3 and §3.9: they are obtained as intersections, in \mathbf{P}^9 , of the Grassmannian $G(2, 5)$ in its Plücker embedding, two hyperplanes, and a smooth quadric, and their moduli space \mathcal{M}^{22} can be seen as follows.

Let $G = G(8, \wedge^2 \mathbf{C}^5)$ be the 16-dimensional Grassmannian parametrizing pencils of skew forms on 5, and let \mathcal{S} be the tautological rank-8 vector bundle on G . The composition

$$\wedge^4 V_5^\vee \hookrightarrow \text{Sym}^2(\wedge^2 V_5^\vee) \rightarrow \text{Sym}^2 \mathcal{S}^\vee$$

is everywhere injective and its cokernel \mathcal{E} is a vector bundle of rank 31 on G . To each point λ of $\mathbf{P}(\mathcal{E})$, one can associate a codimension-2 linear subspace of $\mathbf{P}(\wedge^2 V_5)$ and a quadric in that subspace, well-defined up to the space of quadrics that contain $G(2, \mathbf{C}^5) \subset \mathbf{P}(\wedge^2 V_5)$, hence a threefold $X_\lambda \subset \mathbf{P}^7$ of degree 10, which is in general smooth.

The group $\mathrm{SL}(5)$ acts on $\mathbf{P}(\mathcal{E})$ and one checks that the stabilizers, which correspond to the automorphisms group of X_λ , are finite when X_λ is smooth. So we expect that the moduli space should be an open subset of the GIT quotient $\mathbf{P}(\mathcal{E})//\mathrm{SL}(5)$. However, the relationship between the smoothness of X_λ and stability for λ (which of course involves the choice of a polarization, since $\mathbf{P}(\mathcal{E})$ has Picard number 2) is not clear at the moment (work in progress of Debarre, Iliev, and Manivel).

Remark 4.7. These examples show that several GIT-moduli spaces admit, as ball quotients, complex hyperbolic structures. Another large class of examples of moduli spaces as ball quotients is due to Deligne and Mostow, in their exploration of moduli of points on \mathbf{P}^1 and hypergeometric functions. “Our” examples are not directly of Deligne-Mostow type, since the discrete groups do not appear on the various lists of Deligne-Mostow and Thurston. However, Brent Doran found, by taking a view of hypergeometric functions based on intersection cohomology valued in local systems, links between these two types of examples.