

CONES OF HIGHER-CODIMENSIONAL CYCLES

OLIVIER DEBARRE

July 14, 2010

This is joint work with L. Ein, R. Lazarsfeld, and C. Voisin (eprint arXiv:1003.3183, to appear in Compos. Math.).

1. INTRODUCTION

Let X be a smooth complex projective variety of dimension n . Divisors on X have long been studied by looking at their classes in the finite-dimensional real vector space

$$N^1(X) := \{\mathbf{R}\text{-divisors on } X\}/\text{numerical equivalence}$$

and by studying the (closed convex) cones

$$\text{Psef}^1(X) \subset N^1(X),$$

generated by classes of hypersurfaces in X , and

$$\text{Nef}^1(X) \subset N^1(X),$$

generated by classes of nef divisors on X . Classes in the interior of the nef cone are ample (Kleiman's criterion); some multiple is therefore effective, hence

$$\text{Nef}^1(X) \subset \text{Psef}^1(X) \subset N^1(X).$$

We can make similar definitions in higher codimensions. Set, for each $k \in \{0, \dots, n\}$,

$$N^k(X) := \{\text{codimension-}k \text{ algebraic } \mathbf{R}\text{-cycles on } X\}/\text{numerical equivalence}.$$

In this finite-dimensional real vector space, we consider the closed convex cone

$$\text{Psef}^k(X) \subset N^k(X)$$

generated by classes of subvarieties of codimension k . There is a perfect pairing

$$N^k(X) \times N^{n-k}(X) \rightarrow \mathbf{R}$$

given by intersection, and we define

$$\text{Nef}^k(X) \subset N^k(X)$$

to be the closed convex cone dual to $\text{Psef}^{n-k}(X)$.

Our main result is that the nef cone is in general no longer contained in the pseudo-effective cone. We give an example with $k = 2$ and X is an abelian variety (in which case $\text{Psef}^k(X) \subset \text{Nef}^k(X)$), answering negatively a question of Grothendieck (1964). This shows that nefness, as defined here, is not a good notion of positivity.

2. POSITIVE CLASSES ON ABELIAN VARIETIES

Let $X = V/\text{lattice}$ be an abelian variety, where V is a complex vector space of dimension n . Since numerical and homological equivalence coincide on X , we have $N^1(X) \subset H^{1,1}(X) \cap H^2(B, \mathbf{R}) = \{\text{real } (1, 1)\text{-forms on } V\} = \{\text{Hermitian forms on } V\} := N^1(V)$.
If

$$\text{Semi}^1(V) \subset N^1(V)$$

is the cone of semipositive Hermitian forms, it is well-known that

$$\text{Psef}^1(X) = \text{Nef}^1(X) = \text{Semi}^1(V) \cap N^1(X) := \text{Semi}^1(X).$$

Similarly, we have

$$N^k(X) \subset H^{k,k}(X) \cap H^{2k}(B, \mathbf{R}) = \{\text{real } (k, k)\text{-forms on } V\} = \{\text{Hermitian forms on } \wedge^k V\} := N^k(V).$$

But for real (k, k) -forms, there are several notions of positivity:

$$\begin{aligned} \text{Strong}^k(V) &= \text{convex cone generated by } i^{k^2} \alpha \wedge \bar{\alpha}, \text{ where } \alpha \text{ is a decomposable } k\text{-form on } V \\ &\cap \\ \text{Semi}^k(V) &= \{\text{semipositive Hermitian forms}\} \\ &= \text{convex cone generated by } i^{k^2} \alpha \wedge \bar{\alpha}, \text{ where } \alpha \text{ is a } k\text{-form on } V \\ &\cap \\ \text{Weak}^k(V) &= \text{cone dual to } \text{Strong}^{n-k}(V) \\ &= \{\text{forms that restrict to a nonnegative volume form on each } W_k \subset V\}. \end{aligned}$$

The inclusions are strict for $2 \leq k \leq n - 2$.

By taking intersections with $N^k(X)$, we obtain a series of cones:

$$\text{Psef}^k(X) \subset \text{Strong}^k(X) \subset \text{Semi}^k(X) \subset \text{Weak}^k(X) \subset \text{Nef}^k(X).$$

Note that these cones have no a priori reasons to be distinct anymore, nor is duality preserved.

Proof. We prove the rightmost inclusion. If ω is a weakly positive form and $Y \subset X$ a subvariety of dimension k , the restriction of ω to each tangent space $T_{Y,y}$ is a nonnegative volume form, hence $\int_Y \omega \geq 0$. The other inclusion $\text{Psef}^k(X) \subset \text{Strong}^k(X)$ is proved in a similar fashion. \square

Remark. These cones are invariant under isogenies, so we may restrict ourselves to the principally polarized case.

3. SELF-PRODUCTS OF CM ELLIPTIC CURVES

Let E be a CM elliptic curve and let $X = E^n = V/\text{lattice}$.

Theorem 3.1. *For all k ,*

$$\text{Psef}^k(X) = \text{Strong}^k(X) = \text{Strong}^k(V) \subset \text{Weak}^k(V) = \text{Weak}^k(X) = \text{Nef}^k(X).$$

For $2 \leq k \leq n - 2$, the middle inclusion is strict.

Proof. Since $\text{Strong}^k(V) = \mathbf{S}^k \text{Strong}^1(V)$, and $\mathbf{S}^k \text{Psef}^1(X) \subset \text{Psef}^k(X)$, it suffices to show

$$\text{Strong}^1(V) \subset \text{Psef}^1(X).$$

But this follows from the well-known fact that $N^1(X) = N^1(V)$ (this indeed characterizes abelian varieties that are isogeneous to the self-product of a CM elliptic curve). \square

4. SELF-PRODUCT OF AN ABELIAN VARIETY

Let (A, θ) be a principally polarized abelian variety of dimension g . In $N^1(A \times A)$, we consider the classes

$$\theta_1 = p_1^* \theta \quad , \quad \theta_2 = p_2^* \theta \quad , \quad \lambda = c_1(\text{Poincaré line bundle}).$$

We denote by $N_{\text{can}}^\bullet(A \times A)$ the subalgebra of $N^\bullet(A \times A)$ generated by these classes.

Theorem 4.1 (Tankeev, Ribet). *When (A, θ) is a very general principally polarized abelian variety,*

$$N_{\text{can}}^k(A \times A) = N^k(A \times A).$$

Proposition 4.2. *For every $k \in \{0, \dots, g\}$,*

$$\mathbf{S}^k N_{\text{can}}^1(A \times A) \simeq N_{\text{can}}^k(A \times A).$$

We define cones $\text{Psef}_{\text{can}}^k(A \times A)$, etc. by taking intersections with the smaller vector space $N_{\text{can}}^k(A \times A)$.

Proposition 4.3. *The cones*

$$\text{Strong}_{\text{can}}^k(A \times A) \subset \text{Semi}_{\text{can}}^k(A \times A) \subset \text{Weak}_{\text{can}}^k(A \times A)$$

are independent of (A, θ) .

Proof. This is because in a suitable basis, the matrices of the associated Hermitian forms are

$$\theta_1 \rightarrow \begin{pmatrix} I_g & 0 \\ 0 & 0 \end{pmatrix}, \quad \theta_2 \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & I_g \end{pmatrix}, \quad \lambda \rightarrow \begin{pmatrix} 0 & I_g \\ I_g & 0 \end{pmatrix},$$

hence are independent of (A, θ) . \square

In this basis, we have therefore

$$\text{Psef}_{\text{can}}^1(A \times A) = \text{Nef}_{\text{can}}^1(A \times A) = \{a_1 \theta_1 + a_2 \theta_2 + a_3 \lambda \mid a_1 \geq 0, a_2 \geq 0, a_1 a_2 \geq a_3^2\}.$$

Remark. The inclusion $\text{GL}_2(\mathbf{Z}) \subset \text{End}(A \times A)$ gives rise to an action of $\text{GL}_2(\mathbf{R})$ on $N_{\text{can}}^\bullet(A \times A)$. The closed convex cone $\text{Psef}_{\text{can}}^1(A \times A)$ is generated by the orbit of θ_1 .

We now restrict to the case $g = 2$. Our main result is the following.

Theorem 4.4. *Let A be a very general principally polarized abelian surface. Then*

$$\text{Psef}^2(A \times A) = \text{Strong}^2(A \times A) = \text{Semi}^2(A \times A) \subsetneq \text{Weak}^2(A \times A) \subsetneq \text{Nef}^2(A \times A).$$

Furthermore,

$$\text{Psef}^2(A \times A) = \mathbf{S}^2 \text{Psef}^1(A \times A).$$

Characterization of big classes. Let X be a smooth projective variety of dimension n . By analogy with the case of divisors, one defines a class $\alpha \in N^k(X)$ to be *big* if it lies in the interior of $\text{Psef}^k(X)$. There has been a certain amount of recent interest in the question of trying to recognize big cycles geometrically, the intuition being that they should be those that “sit positively” in X , or that “move a lot.”

Peternell’s question: if $Y \subset X$ is a smooth subvariety with $N_{Y/X}$ ample, is $[Y]$ big?

Voisin gave a counter-example involving a codimension two subvariety Y that moves in a family covering X .

Voisin’s conjecture: if $Y \subset X$ is “very moving”, i.e., if, through a general point of X , there is a deformation of Y that passes through x with general tangent space at x , is $[Y]$ big?

Note that if $X = V/\Lambda$ is an abelian variety of dimension n that satisfies

$$\text{Psef}^k(X) = \text{Strong}^k(X),$$

Voisin’s conjecture holds for codimension k subvarieties of X .

Proof. If $[Y]$ lies on the boundary of $\text{Psef}^k(X) = \text{Strong}^k(X)$, it is also on the boundary of $\text{Strong}^k(V)$, i.e., $\int_Y \omega = 0$ for some weakly positive nonzero form ω on V . But $\omega|_{T_{Y,y}} > 0$ for some $y \in Y$: contradiction. \square

Ein and Lazarsfeld’s conjecture: α is big if and only if

$$\exists C > 0 \quad \exists m \text{ arb. large} \quad \exists Z \text{ eff.} \in m\alpha \text{ passing through } \geq Cm^{n/k} \text{ v. gen. points of } X.$$

The implication \Rightarrow is elementary, and the exponent $\frac{n}{k}$ appearing here is the largest that can occur. The statement has been verified in one nontrivial case, namely when $k = 2$ and $\text{Pic}(X) = \mathbf{Z}$.

Note that contrary to the case of divisors and curves, it might happen that no multiple of a big class is represented by an irreducible subvariety (hence one needs Z to be a cycle).