CONES OF HIGHER-CODIMENSIONAL CYCLES

OLIVIER DEBARRE

July 14, 2010

This is joint work with L. Ein, R. Lazarsfeld, and C. Voisin (eprint arXiv:1003.3183, to appear in Compos. Math.).

1. Introduction

Let X be a smooth complex projective variety of dimension n. Divisors on X have long been studied by looking at their classes in the finite-dimensional real vector space

$$N^1(X) := \{\mathbf{R}\text{-divisors on } X\}/\text{numerical equivalence}$$

and by studying the (closed convex) cones

$$\mathrm{Psef}^1(X) \subset N^1(X),$$

generated by classes of hypersurfaces in X, and

$$\mathrm{Nef}^1(X) \subset N^1(X),$$

generated by classes of nef divisors on X. Classes in the interior of the nef cone are ample (Kleiman's criterion); some multiple is therefore effective, hence

$$\mathrm{Nef}^1(X) \subset \mathrm{Psef}^1(X) \subset N^1(X).$$

We can make similar definitions in higher codimensions. Set, for each $k \in \{0, \dots, n\}$,

$$N^k(X) := \{\text{codimension-}k \text{ algebraic } \mathbf{R}\text{-cycles on } X\}/\text{numerical equivalence}.$$

In this finite-dimensional real vector space, we consider the closed convex cone

$$\mathrm{Psef}^k(X) \subset N^k(X)$$

generated by classes of subvarieties of codimension k. There is a perfect pairing

$$N^k(X) \times N^{n-k}(X) \to \mathbf{R}$$

given by intersection, and we define

$$\mathrm{Nef}^k(X) \subset N^k(X)$$

to be the closed convex cone dual to $\mathrm{Psef}^{n-k}(X)$.

Our main result is that the nef cone is in general no longer contained in the pseudo-effective cone. We given an example with $k = 2$ and X is an abelian variety (in which case $\mathrm{Psef}^k(X) \subset \mathrm{Nef}^k(X)$), answering negatively a question of Grothendieck (1964). This shows that nefness, as defined here, is not a good notion of positivity.

2. Positive classes on abelian varieties

Let $X = V/\text{lattice}$ be an abelian variety, where V is a complex vector space of dimension n. Since numerical and homological equivalence coincide on X, we have

$$N^1(X) \subset H^{1,1}(X) \cap H^2(B, \mathbf{R}) = \{\text{real } (1,1)\text{-forms on } V\} = \{\text{Hermitian forms on } V\} := N^1(V).$$

If

$$\mathrm{Semi}^1(V) \subset N^1(V)$$

is the cone of semipositive Hermitian forms, it is well-known that

$$\mathrm{Psef}^1(X) = \mathrm{Nef}^1(X) = \mathrm{Semi}^1(V) \cap N^1(X) := \mathrm{Semi}^1(X).$$

Similarly, we have

$$N^k(X) \subset H^{k,k}(X) \cap H^{2k}(B, \mathbf{R}) = \{\text{real } (k,k)\text{-forms on } V\} = \{\text{Hermitian forms on } \wedge^k V\} := N^k(V).$$

But for real (k,k)-forms, there are several notions of positivity:

$$\begin{array}{rcl} \mathrm{Strong}^k(V) & = & \text{convex cone generated by } i^{k^2}\alpha \wedge \overline{\alpha}, \text{ where } \alpha \text{ is a decomposable } k\text{-form on } V \\ \cap & & \\ \mathrm{Semi}^k(V) & = & \{\text{semipositive Hermitian forms}\} \\ & = & \text{convex cone generated by } i^{k^2}\alpha \wedge \overline{\alpha}, \text{ where } \alpha \text{ is a } k\text{-form on } V \\ \cap & & \\ \mathrm{Weak}^k(V) & = & \text{cone dual to } \mathrm{Strong}^{n-k}(V) \\ & = & \{\text{forms that restrict to a nonnegative volume form on each } W_k \subset V\}. \end{array}$$

The inclusions are strict for $2 \leq k \leq n-2$.

By taking intersections with $N^k(X)$, we obtain a series of cones:

$$\mathrm{Psef}^k(X) \subset \mathrm{Strong}^k(X) \subset \mathrm{Semi}^k(X) \subset \mathrm{Weak}^k(X) \subset \mathrm{Nef}^k(X).$$

Note that these cones have no a priori resaons to be distinct anymore, nor is duality preserved.

Proof. We prove the rightmost inclusion. If ω is a weakly positive form and $Y \subset X$ a subvariety of dimension k, the restriction of ω to each tangent space $T_{Y,y}$ is a nonnegative volume form, hence $\int_Y \omega \geq 0$. The other inclusion $\mathrm{Psef}^k(X) \subset \mathrm{Strong}^k(X)$ is proved in a similar fashion. □

Remark. These cones are invariant under isogenies, so we may restrict ourselves to the principally polarized case.

3. Self-products of CM elliptic curves

Let E be a CM elliptic curve and let $X = E^n = V/\text{lattice}$.

Theorem 3.1. *For all k,*

$$\mathrm{Psef}^k(X) = \mathrm{Strong}^k(X) = \mathrm{Strong}^k(V) \subset \mathrm{Weak}^k(V) = \mathrm{Weak}^k(X) = \mathrm{Nef}^k(X).$$

For $2 \leq k \leq n-2$, the middle inclusion is strict.

Proof. Since $\operatorname{Strong}^k(V) = \mathbf{S}^k \operatorname{Strong}^1(V)$, and $\mathbf{S}^k \operatorname{Psef}^1(X) \subset \operatorname{Psef}^k(X)$, it suffices to show

$$\operatorname{Strong}^1(V) \subset \operatorname{Psef}^1(X).$$

But this follows from the well-known fact that $N^1(X) = N^1(V)$ (this indeed characterizes abelian varieties that are isogeneous to the self-product of a CM elliptic curve). □

4. Self-product of an abelian variety

Let (A, θ) be a principally polarized abelian variety of dimension g. In $N^1(A \times A)$, we consider the classes

$$\theta_1 = p_1^* \theta \quad , \quad \theta_2 = p_2^* \theta \quad , \quad \lambda = c_1(\text{Poincaré line bundle}).$$

We denote by $N^\bullet_{\text{can}}(A \times A)$ the subalgebra of $N^\bullet(A \times A)$ generated by these classes.

Theorem 4.1 (Tankeev, Ribet)**.** *When (A, θ) is a very general principally polarized abelian variety,*

$$N^k_{\text{can}}(A \times A) = N^k(A \times A).$$

Proposition 4.2. *For every $k \in \{0, \dots, g\}$,*

$$\mathbf{S}^k N^1_{\text{can}}(A \times A) \simeq N^k_{\text{can}}(A \times A).$$

We define cones $\operatorname{Psef}^k_{\text{can}}(A \times A)$, etc. by taking intersections with the smaller vector space $N^k_{\text{can}}(A \times A)$.

Proposition 4.3. *The cones*

$$\operatorname{Strong}^k_{\text{can}}(A \times A) \subset \operatorname{Semi}^k_{\text{can}}(A \times A) \subset \operatorname{Weak}^k_{\text{can}}(A \times A)$$

are independent of (A, θ).

Proof. This is because in a suitable basis, the matrices of the associated Hermitian forms are

$$\theta_1 \to \begin{pmatrix} I_g & 0 \\ 0 & 0 \end{pmatrix}, \; \theta_2 \to \begin{pmatrix} 0 & 0 \\ 0 & I_g \end{pmatrix}, \; \lambda \to \begin{pmatrix} 0 & I_g \\ I_g & 0 \end{pmatrix},$$

hence are independent of (A, θ). □

In this basis, we have therefore

$$\operatorname{Psef}^1_{\text{can}}(A \times A) = \operatorname{Nef}^1_{\text{can}}(A \times A) = \{a_1\theta_1 + a_2\theta_2 + a_3\lambda \mid a_1 \geq 0, a_2 \geq 0, a_1 a_2 \geq a_3^2\}.$$

Remark. The inclusion $\mathrm{GL}_2(\mathbf{Z}) \subset \operatorname{End}(A \times A)$ gives rise to an action of $\mathrm{GL}_2(\mathbf{R})$ on $N^\bullet_{\text{can}}(A \times A)$. The closed convex cone $\operatorname{Psef}^1_{\text{can}}(A \times A)$ is generated by the orbit of θ_1.

We now restrict to the case $g = 2$. Our main result is the following.

Theorem 4.4. *Let A be a very general principally polarized abelian surface. Then*

$$\operatorname{Psef}^2(A \times A) = \operatorname{Strong}^2(A \times A) = \operatorname{Semi}^2(A \times A) \subsetneq \operatorname{Weak}^2(A \times A) \subsetneq \operatorname{Nef}^2(A \times A).$$

Furthermore,

$$\operatorname{Psef}^2(A \times A) = \mathbf{S}^2 \operatorname{Psef}^1(A \times A).$$

Everything there takes place in the 6-dimensional real vector space $\langle \theta_1^2, \theta_1\theta_2, \theta_2^2, \theta_1\lambda, \theta_2\lambda, \lambda^2 \rangle$.

Corollary 4.5. *Let A be a principally polarized abelian variety of dimension g. Then*

$$\mathrm{Semi}^2_{\mathrm{can}}(A \times A) = \mathrm{Psef}^2_{\mathrm{can}}(A \times A) = \mathbf{S}^2 \,\mathrm{Psef}^1_{\mathrm{can}}(A \times A).$$

Proof. The equalities are a statement in convexity theory: a class

$$\alpha = a_1\theta_1^2 + a_2\theta_1\theta_2 + a_3\theta_2^2 + a_4\theta_1\lambda + a_5\theta_2\lambda + a_6\lambda^2$$

is associated with the following Hermitian form on $\wedge^2 V$ (in a suitable basis):

$$\begin{pmatrix} a_2 - 2a_6 & & & & & \\ & a_2 - 2a_6 & & & & \\ & & a_2 - 2a_6 & & & \\ & & & a_1 & a_4 & a_6 \\ & & & a_4 & a_2 + 2a_6 & a_5 \\ & & & a_6 & a_5 & a_3 \end{pmatrix}$$

we must show that the ‘spectrahedron” defined by the condition that this matrix be semipositive is generated by

$$(1, a^2 + b^2, a^2b^2, a + b, ab(a + b), ab),$$

for a, $b \in \mathbf{R}$.

One proves the two strict inclusions also by direct calculations. □

We obtain for example, for $t \in \mathbf{R}$, and A a very general abelian surface,

$$4\theta_1\theta_2 + t\lambda^2 \quad \text{is} \quad \begin{cases} \text{semipositive} & \text{iff } t = 0; \\ \text{weakly positive} & \text{iff } |t| \leq 1; \\ \text{nef} & \text{iff } -1 \leq t \leq \frac{3}{2}. \end{cases}$$

The classes $4\theta_1\theta_2 + t\lambda^2$ and $2\theta_1^2 + 2\theta_2^2 - \lambda^2$ are weakly positive, hence nef, but their product is -8.

5. Questions and conjectures

In this section we propose some questions and conjectures concerning positivity for cycles of codimension > 1.

Abelian varieties. Let (A, θ) be a principally polarized abelian variety.

Problem 5.1. *Are the canonical cones $\mathrm{Nef}^k_{\mathrm{can}}(A \times A)$ and $\mathrm{Psef}^k_{\mathrm{can}}(A \times A)$ independent of (A, θ)?*

Problem 5.2. *Is $\mathrm{Psef}^k(X) = \mathrm{Strong}^k(X)$ for an arbitrary abelian variety X?*

Although this holds in all the examples considered above, we suspect that this is not the case in general, but it would be nice to find an actual example where it fails. One direction where to look is the Weil construction of abelian varieties where Hodge classes are not generated by classes of divisors.

Characterization of big classes. Let X be a smooth projective variety of dimension n. By analogy with the case of divisors, one defines a class $\alpha \in N^k(X)$ to be *big* if it lies in the interior of $\mathrm{Psef}^k(X)$. There has been a certain amount of recent interest in the question of trying to recognize big cycles geometrically, the intuition being that they should be those that "sit positively" in X, or that "move a lot."

Peternell's question: if $Y \subset X$ is a smooth subvariety with $N_{Y/X}$ ample, is $[Y]$ big?

Voisin gave a counter-example involving a codimension two subvariety Y that moves in a family covering X.

Voisin's conjecture: if $Y \subset X$ is "very moving", i.e., if, through a general point of X, there is a deformation of Y that passes through x with general tangent space at x, is $[Y]$ big?

Note that if $X = V/\Lambda$ is an abelian variety of dimension n that satisfies

$$\mathrm{Psef}^k(X) = \mathrm{Strong}^k(X),$$

Voisin's conjecture holds for codimension k subvarieties of X.

Proof. If $[Y]$ lies on the boundary of $\mathrm{Psef}^k(X) = \mathrm{Strong}^k(X)$, it is also on the boundary of $\mathrm{Strong}^k(V)$, i.e., $\int_Y \omega = 0$ for some weakly positive nonzero form ω on V. But $\omega|_{T_{Y,y}} > 0$ for some $y \in Y$: contradiction. $\square$

Ein and Lazarsfeld's conjecture: α is big if and only if

$$\exists C > 0 \;\; \exists m \text{ arb. large } \;\; \exists Z \text{ eff.} \in m\alpha \;\; \text{passing through} \geq Cm^{n/k} \text{ v. gen. points of } X.$$

The implication $\Rightarrow$ is elementary, and the exponent $\frac{n}{k}$ appearing here is the largest that can occur. The statement has been verified in one nontrivial case, namely when $k = 2$ and $\mathrm{Pic}(X) = \mathbf{Z}$.

Note that contrary to the case of divisors and curves, it might happen that no multiple of a big class is represented by an irreducible subvariety (hence one needs Z to be a cycle).