On the geometry of abelian varieties

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Introduction

The geometry of varieties embedded in a projective space has been investigated for a long time. In some sense, there are many similarities with the geometry of subvarieties of an abelian variety, while at the same time there are major differences. We try to review in this series of lectures some of this material.

CHAPTER 1

Subvarieties of an abelian variety

All varieties and subvarieties are irreducible and reduced and defined over the complex numbers (only otherwise specified).

Subvarieties of the projective space $\mathbf{P}^n$ have the fundamental property that they meet if the sum of their dimensions is at least n. For subvarieties of an abelian variety, this is no longer true in general and one has to impose nondegeneracy properties on the subvarieties, which can take several forms.

If V and W are subsets of an abelian variety X, we set

$$V \pm W = \{v \pm w \mid v \in V,\ w \in W\}.$$

If V is a variety and v is a point in V, we write $T_{V,v}$ for the Zariski tangent space to V at v. If X is an abelian variety and $x \in X$, we identify $T_{X,x}$ with $T_{X,0}$ by translation. In particular, if V is a subvariety of X and $v \in V$, the tangent space $T_{V,v}$ will be considered as a vector subspace of $T_{X,0}$.

The results of this chapter are taken from the last chapter of [**D1**].

1. Abelian subvariety generated by a subset

Let X be an abelian variety. The abelian subvariety of X generated by a connected subset V of X is the intersection of all abelian subvarieties K of X such that some translate of K contains V; we denote it by $\langle V\rangle$. It is also the intersection of all abelian subvarieties of X that contain $V - V$. We say that V *generates* X if $\langle V\rangle = X$.

If V is not connected, we define $\langle V\rangle$ to be the intersection of all abelian subvarieties of X that contain the connected component of $V - V$ which contains the origin. In particular, $\langle V\rangle = 0$ if and only if V is finite.

If V is a subvariety of X,

$$\langle V\rangle = \overbrace{(V - V) + \cdots + (V - V)}^{m \text{ times}}$$

for all $m \gg 0$.

We will need two lemmas.

LEMMA 1.1. *Let X be an abelian variety, let V be a subvariety of X, and let U be a dense subset of V. The vector space $T_{\langle V\rangle,0}$ is spanned by all the $T_{V,v}$, for $v \in U$.*

PROOF. Let m be an integer such that the map $\sigma : V^{2m} \to \langle V\rangle$ defined by

$$\sigma(v_1,\dots,v_{2m}) = v_1 - v_2 + \cdots + v_{2m-1} - v_{2m}$$

is surjective. The tangent map to σ at a general point of V^{2m}, hence at some point $(v_1,\dots,v_{2m})$ of U^{2m}, is surjective. This map is

$$\begin{array}{ccc} T_{V,v_1}\times\cdots\times T_{V,v_{2m}} & \longrightarrow & T_{\langle V\rangle,0} \\ (t_1,\dots,t_{2m}) & \longmapsto & t_1 - t_2 + \cdots + t_{2m-1} - t_{2m} \end{array}$$

hence the lemma. □

LEMMA 1.2. *Let X be an abelian variety, let V be a proper variety, and let G be a subvariety of $X \times V$. Let $q : G \to V$ be the second projection. The abelian subvariety $\langle q^{-1}(v)\rangle$ of X is independent of v general in $q(G)$.*

PROOF. Let v_0 be a point of $q(G)$ such that the dimension of $K = \langle q^{-1}(v_0)\rangle$ is *minimal*. Let $p : X \to X/K$ be the quotient map, let G' be the image of G by the surjection $(p,\mathrm{Id}) : X\times V \to (X/K)\times V$, and let $q' : G' \to q(G)$ be the map induced by the second projection. The fiber $q'^{-1}(v_0)$ is finite. Hence, for v general in $q(G)$, the fiber $q'^{-1}(v)$ is finite, so that $\langle q^{-1}(v)\rangle$ is contained in K. By choice of v_0, there is equality. □

2. Geometrically nondegenerate subvarieties

DEFINITION 2.1 (Ran). A subvariety V of an abelian variety X is *geometrically nondegenerate* if, for any abelian subvariety K of X, either $V + K = X$ or $\dim(V + K) = \dim(V) + \dim(K)$.

Equivalently,

$$\dim(V + K) = \min(\dim(V) + \dim(K), \dim(X))$$

or, for any abelian quotient $p : X \to Y$, either $p(V) = Y$ or the restriction of p to V is generically finite onto its image.

EXAMPLES 2.2. 1) An irreducible curve in X is geometrically nondegenerate if and only if it generates X. An irreducible hypersurface D in X is geometrically nondegenerate if and only if the divisor D is ample.

2) Any subvariety of a *simple* abelian variety is geometrically nondegenerate.

The following technical result will be our main tool.

THEOREM 2.3. *Let V and W be subvarieties of an abelian variety X. Let K be the largest abelian subvariety of X such that $V+W+K = V+W$ and let $p : X \to X/K$ be the quotient map. We have*

$$\dim(p(V)+p(W)) = \dim(p(V)) + \dim(p(W))$$

PROOF. Replacing X by X/K, we may assume $K = 0$. If $\sigma : V \times W \to X$ is the sum map, the fiber $\sigma^{-1}(x)$ is isomorphic via the first projection to $F_x = V \cap (x - W)$.

Let G be the image of $V \times W$ by the automorphism

$$\begin{array}{ccc} X \times X & \longrightarrow & X \times X \\ (x,y) & \longmapsto & (x, x+y) \end{array}$$

and let $u : G \to V + W$ be the map induced by the second projection, so that $u^{-1}(x) = F_x \times \{x\}$. Lemma 1.2 implies that $K' = \langle F_x \rangle$ is independent of x general in $V+W$. For all v in F_x, we have $F_x - v \subset V + W - x$, hence $T_{F_x,v} \subset T_{V+W,x}$. Lemma 1.1 then implies

$$T_{K',0} = T_{\langle F_x \rangle,0} \subset T_{V+W,x}.$$

By Lemma 2.4 below, this in turn implies $V+W = V+W+K'$, hence $K' = 0$. It follows that F_x is finite, hence σ is generically finite and $V \times W$ and $V + W$ have the same dimension. □

LEMMA 2.4. *Let X be an abelian variety and let V be a subvariety of X. Let K be an abelian subvariety of X such that, for all v general in V, we have $T_{K,0} \subset T_{V,v}$. Then $V + K = V$.*

PROOF. Let $p : X \to X/K$ be the quotient map. The tangent map to the surjection $V \to p(V)$ at a general point v of V is surjective, hence

$$\begin{aligned} \dim(p(V)) &= \dim(V) - \dim(T_{V,v} \cap T_{K,0}) \\ &= \dim(V) - \dim(K). \end{aligned}$$

This proves the lemma. □

COROLLARY 2.5. *Let X be an abelian variety and let V and W be subvarieties of X. Assume that V is geometrically nondegenerate.*

a) *We have either $V + W = X$ or $\dim(V + W) = \dim(V) + \dim(W)$.*
b) *If W is also geometrically nondegenerate, so is $V + W$.*

PROOF. By Theorem 2.3, and with its notation, we have

$$\begin{aligned}\dim(V+W) &= \dim(p(V)+p(W))+\dim(K)\\ &= \dim(p(V))+\dim(p(W))+\dim(K)\\ &= \min(\dim(V),\dim(X/K))+\dim(p(W))+\dim(K)\\ &\geq \min(\dim(V),\dim(X/K))+\max(\dim(W),\dim(K))\\ &\geq \min(\dim(V)+\dim(W),\dim(X/K)+\dim(K))\end{aligned}$$

This shows a).

If W is also geometrically nondegenerate, we have for all abelian subvarieties K' of X, using a),

$$\begin{aligned}\dim(V+W+K') &= \min(\dim(V)+\dim(W+K'),\dim(X))\\ &= \min(\dim(V)+\dim(W)+\dim(K'),\dim(X))\\ &\geq \min(\dim(V+W)+\dim(K'),\dim(X))\end{aligned}$$

This shows b). □

COROLLARY 2.6. *A subvariety V of an abelian variety X is geometrically nondegenerate if and only if it meets any subvariety of X of dimension* $\geq \operatorname{codim}(V)$.

PROOF. Assume V meets any subvariety of X of dimension $\geq \operatorname{codim}(V)$. Let K be an abelian subvariety of X with quotient map $p: X \to X/K$ and $p(V) \neq X/K$. Let W be a subvariety of X/K such that $\dim(p(V))+\dim(W)=\dim(X/K)-1$ and $p(V)\cap W=\varnothing$. Then $p^{-1}(W)$ does not meet V and has dimension $\dim(X)-\dim p(V)-1$, hence $\dim(X)-\dim(p(V))-1<\operatorname{codim}(V)$. This proves that V is geometrically nondegenerate.

Conversely, if V is geometrically nondegenerate and W is a subvariety of X such that $\dim(V)+\dim(W)\geq\dim(X)$, we have $V-W=X$ by Corollary 2.5.a) hence $0\in V-W$. □

3. Ampleness of the normal bundle

To any vector bundle (or more generally, coherent sheaf) E on a variety X, one can associate a bundle $\mathbf{P}E \to X$ whose fibers are the projectifications (in the Grothendieck sense) of the fibers of E and a canonically defined line bundle $\mathscr{O}_{\mathbf{P}E}(1)$ on $\mathbf{P}E$. We say that E is *ample* if $\mathscr{O}_{\mathbf{P}E}(1)$ is ample on $\mathbf{P}E$.

Any quotient vector bundle of an ample vector bundle is ample. Using the Euler exact sequence, this implies that the tangent bundle to $\mathbf{P}^n$ is ample, and so is the normal bundle to any smooth subvariety of $\mathbf{P}^n$. This is not always the case for subvarieties of abelian varieties.

LEMMA 3.1. *Let V be a smooth subvariety of an abelian variety X. The normal bundle to V in X is ample if and only if, for all hyperplanes H in $T_{X,0}$, the set*

$$\{v \in V \mid T_{V,v} \subset H\}$$

is finite.

PROOF. The surjection $T_X|_V \twoheadrightarrow N_{V/X}$ induces a morphism

$$g : \mathbf{P}(N_{V/X}) \longrightarrow \mathbf{P}(T_X|_V) = V \times \mathbf{P}(T_{X,0}) \xrightarrow{p_2} \mathbf{P}(T_{X,0})$$

such that $g^*\mathscr{O}_{\mathbf{P}(T_{X,0})}(1) = \mathscr{O}_{\mathbf{P}(N_{V/X})}(1)$. It follows that $N_{V/X}$ is ample if and only if g is finite. If H is a hyperplane in $T_{X,0}$, the fiber $g^{-1}(H)$ maps isomorphically to $\{v \in V \mid T_{V,v} \subset H\}$ by the projection $\mathbf{P}(N_{V/X}) \to V$. $\square$

PROPOSITION 3.2. *Any smooth subvariety V of a* simple *abelian variety X has ample normal bundle.*

PROOF. Let H be a hyperplane in $T_{X,0}$ and let C be a subvariety of $\{v \in V \mid T_{V,v} \subset H\}$. For all v smooth in C, we have $T_{C,v} \subset H$, hence by Lemma 1.1, $T_{\langle C\rangle,0} \subset H$. Since X is simple, this implies $\langle C\rangle = 0$, hence C is a single point. $\square$

PROPOSITION 3.3. *If the normal bundle to a smooth subvariety V of an abelian variety X is ample, V is geometrically nondegenerate.*

PROOF. Let $p : X \to Y$ be an abelian quotient such that $p(V) \neq Y$. Let y be a smooth point of $p(V)$ and let $H \subset T_{Y,0}$ be a hyperplane that contains $T_{p(V),y}$. For any $v \in p^{-1}(y) \cap V$, we have $T_{V,v} \subset (Tp)^{-1}(H)$, hence $p^{-1}(y) \cap V$ is finite by Lemma 3.1. This implies that $p(V)$ has same dimension as V. $\square$

As a partial converse, if V is geometrically nondegenerate, for any hyperplane H of $T_{X,0}$, one can show that

$$\{v \in V \mid T_{V,v} \subset H\}$$

has dimension $\leq \operatorname{codim}(V) - 1$ (one says that the normal bundle to V in X is $(\operatorname{codim}(V) - 1)$-*ample*).

4. Connectedness theorems

We are now interested in the connectedness of the intersection of two subvarieties of an abelian variety. For applications, it is better to have a more general result, whose statement is unfortunately a bit technical.

We say that a pair (V, W) of subvarieties of an abelian variety X is *nondegenerate* if

- either V and W are nondegenerate and $\dim(V) + \dim(W) > \dim(X)$;
- or $V = X$ and W generates X.

THEOREM 4.1. *Let V and W be normal projective varieties, let X be an abelian variety, and let $f : V \to X$ and $g : W \to X$ be morphisms such that $(f(V), g(W))$ is nondegenerate. There exist an isogeny $p : X' \to X$ and factorizations*

$$f : V \xrightarrow{f'} X' \xrightarrow{p} X$$

and

$$g : W \xrightarrow{g'} X' \xrightarrow{p} X$$

such that

- *$V \times_{X'} W$ is connected;*
- *the sequence*

$$\pi_1(V \times_{X'} W) \longrightarrow \pi_1(V) \times \pi_1(W) \xrightarrow{\pi_1(f') - \pi_1(g')} \pi_1(X') \longrightarrow 0 \tag{1}$$

is exact.

Instead of proving the theorem (the proof can be found in [**D1**], §8.3), we will deduce from it a few geometric consequences.

COROLLARY 4.2. *Let X be an abelian variety and let V be a normal geometrically nondegenerate subvariety of X of dimension $> \frac{1}{2}\dim(X)$. The morphism $\pi_1(V) \to \pi_1(X)$ is bijective.*

PROOF. We first show that this morphism is surjective. Since $\pi_1(X)$ is a free abelian group, it is enough to show that the composition $\pi_1(V) \to \pi_1(X) \to \pi_1(X)/n\pi_1(X)$ is surjective for all integers $n > 0$, i.e., that the inverse image of V by the isogeny $\mathbf{n}_X : X \to X$ is connected. Any two irreducible components of $\mathbf{n}_X^{-1}(V)$ are geometrically nondegenerate, of dimension $> \frac{1}{2}\dim(X)$, hence they meet by Corollary 2.6. It follows that $\mathbf{n}_X^{-1}(V)$ is connected.

Apply now Theorem 4.1, taking for f and g the inclusion ι of V in X, so that $V \times_X V$ is the diagonal of $V \times V$. Since $\pi_1(\iota)$ is surjective, the isogeny p is an isomorphism and we have an exact sequence

$$\begin{array}{ccccccc} \pi_1(V) & \longrightarrow & \pi_1(V) \times \pi_1(V) & \longrightarrow & \pi_1(X) & \longrightarrow & 0 \\ t & \longmapsto & (t,t) & & & & \\ & & (t,t') & \longmapsto & \pi_1(\iota)(t-t') & & \end{array}$$

which shows that $\pi_1(\iota)$ is bijective. □

Sommese extended in [**S**] this result, proving that if V is a *smooth* subvariety of a *simple* abelian variety X,

$$\pi_i(V) \simeq \pi_i(X) \qquad \text{for} \qquad i \leq 2\dim(V) - \dim(X).$$

This should still be valid for geometrically nondegenerate subvarieties which are local complete intersections.

Corollary 4.3. *Let X be a* simple *abelian variety of dimension n, let V be a normal projective variety, and let $f : V \to X$ be a finite surjective morphism of degree d.*

For $v \in V$, let $e_f(v)$ be the local degree of f at v (i.e., the number of sheets that meet in v). For any integer $\ell \leq \min\{d-1, n\}$, the subset

$$R_\ell = \{v \in V \mid e_f(v) > \ell\} \tag{2}$$

has codimension $\leq \ell$.

If $d \leq n$, the induced morphism $\pi_1(f) : \pi_1(V) \to \pi_1(X)$ is injective with finite cokernel.

Proof. Let $X' \to X$ be an isogeny of maximal degree such that f factorizes as $V \xrightarrow{f'} X' \to X$. Replacing f by f', we may assume that f factorizes through no nontrivial isogeny.

We show by induction on ℓ that R_ℓ is nonempty for $\ell \leq \min\{d-1, n\}$. It is then a result of Lazarsfeld (which generalizes the purity theorem) that it has codimension $\leq \ell$ in V.

We have $R_0 = V$. Let $\ell \leq \min\{d-1, n\}$ and let R be the normalization of an irreducible component of $R_{\ell-1}$. It has dimension $\geq n - \ell + 1 > 0$ in V, hence $f(R)$ generates X (because X is simple) and $R \times_X V$ is connected by Theorem 4.1. But $R \times_X V$ contains the diagonal Δ_R of R. If they are equal, $R \subset R_{d-1} \subset R_\ell$. Otherwise, another component of $R \times_X V$ meets Δ_R and if v is a point in the intersection, we have $e_f(v) > \ell$, hence R_ℓ is nonempty.

If $d \leq n$, R_{d-1} is nonempty and contains a curve R, with normalization $R' \to R$, such that the morphism $f^{-1}(R) \to R$ induced by f is bijective. We use Theorem 4.1 again: there is an exact sequence

$$\pi_1(V \times_X R') \longrightarrow \pi_1(V) \times \pi_1(R') \longrightarrow \pi_1(X) \longrightarrow 0.$$

But $V \times_X R'$ is homeomorphic to R', hence $\pi_1(V) \to \pi_1(X)$ is bijective. $\square$

CHAPTER 2

Minimal cohomology classes and Jacobians

The results of this chapter are taken from [**D3**].

1. Nondegenerate subvarieties

Let X be an abelian variety of dimension n. We say that a subvariety W of X, of dimension d, is *nondegenerate* if the restriction map

$$H^0(X,\Omega_X^d) \longrightarrow H^0(W_{\mathrm{reg}},\Omega_{W_{\mathrm{reg}}}^d) \tag{3}$$

is injective. By [**R**], Lemma II.1, this is equivalent to each of the following properties:

- the cup-product map $\cdot[W] : H^{d,0}(X) \longrightarrow H^{n,n-d}(X)$ is injective, where $[W] \in H^{n-d,n-d}(X)$ is the cohomology class of W,
- the contraction map $\cdot\{W\} : H^{n,d}(X) \longrightarrow H^{n-d,0}(X)$ is injective, where $\{W\} \in H_{d,d}(X)$ is the homology class of W.

Example 1.1. If ℓ is an ample class, by the Lefschetz theorem, any subvariety with class a multiple of ℓ^{n-d} is nondegenerate.

Proposition 1.2. *Any nondegenerate subvariety of X is geometrically nondegenerate.*

In fact, one can show that a subvariety W of an abelian variety X is geometrically nondegenerate if and only if the kernel of the restriction (3) contains no nonzero *decomposable* forms.

Proof. Let $W \subset X$ be a nondegenerate subvariety of X, of dimension d, and let $p : X \to Y$ be an abelian quotient of such that $p(W) \neq Y$ and $m = \dim(p(W)) < \dim(W) = d$. For any 1-forms $\eta_1,\dots,\eta_{m+1}$ on Y and $\omega_1,\dots,\omega_{d-m-1}$ on X, we have $\eta_1 \wedge \cdots \wedge \eta_{m+1}|_{p(Y)} = 0$, hence $p^*\eta_1 \wedge \cdots \wedge p^*\eta_{m+1} \wedge \omega_1 \wedge \cdots \wedge \omega_{d-m-1}$ vanishes on W. This contradicts the injectivity of the restriction (3). $\square$

Let V and W be two subvarieties of X, of respective codimensions d and $n-d$. Assume that $s = V\cdot W > 0$. The addition map $V \times W \to X$ is then surjective, hence generically étale. It follows that for x general in X, the varieties V and $x-W$ meet transversally at distinct smooth points $v_1,\dots,v_s$.

Let $P_i : T_{X,0} \to T_{X,0}$ be the projector with image T_{V,v_i} and kernel $T_{W,x-v_i}$. Let $c(V,W)$ be the endomorphism $\sum_{i=1}^s \bigwedge^{n-d} P_i$ of $\bigwedge^{n-d} T_{X,0}$; Ran proves ([**R**], Theorem 2) that the transposed endomorphism ${}^t c(V,W)$ of $\bigwedge^{n-d} T^*_{X,0} \simeq H^{n-d,0}(X)$ is equal to the composition:

$$H^{n-d,0}(X) \xrightarrow{\cdot[V]} H^{n,d}(X) \xrightarrow{\cdot\{W\}} H^{n-d,0}(X).$$

In particular, when V and W are nondegenerate, $c(V,W)$ is an automorphism.

Proposition 1.3 (Ran). *If V and W are nondegenerate subvarieties of a n-dimensional abelian variety, of respective codimensions d and $n-d$, we have*

$$V\cdot W \geq \binom{n}{d}. \tag{4}$$

Proof. Since $c(V,W)$ is the sum of $V\cdot W$ projectors of rank 1, we have

$$\binom{n}{d} = \operatorname{rank}\bigl(c(V,W)\bigr) \leq V\cdot W.$$

□

Let (JC,θ) be the principally polarized Jacobian of a smooth curve C of genus n. For each $d\in\{0,\dots,n\}$, the class of the subvariety

$$W_{n-d}(C) = \overbrace{C+\cdots+C}^{n-d \text{ times}}$$

is $\theta_d = \theta^d/d!$. This class is *minimal,* i.e., nondivisible, in $H^{2d}(A,\mathbf{Z})$. The variety $W_{n-d}(C)$ is nondegenerate (Example 1.1). Since $\theta^n = n!$, we have

$$W_{n-d}(C)\cdot W_d(c) = \binom{n}{d}$$

The main result of Ran is that equality in (4) has strong geometrical consequences. To state it, we need a definition.

Definition 1.4. Let V and W be two subvarieties of an abelian variety X and let $\sigma : V\times W \to X$ be the sum map. We say that *V has property $(\mathscr{P})$ with respect to W* if, for v general in V, the only subvariety of $\sigma^{-1}(v+W)$ which dominates both $v+W$ via σ, and W via the second projection, is $\{v\}\times W$.

THEOREM 1.5 (Ran). *Let V and W be nondegenerate subvarieties of a n-dimensional abelian variety, of respective codimensions d and $n-d$. If $V \cdot W = \binom{n}{d}$, the variety V has property $(\mathscr{P})$ with respect to W.*

PROOF. We keep the same notation as above. If α_i spans the line $\bigwedge^{n-d} T_{V,v_i}$ in $\bigwedge^{n-d} T_{X,0}$, then $\{\alpha_1, \ldots, \alpha_s\}$ (with $s = \binom{n}{d}$) is a basis for $\bigwedge^{n-d} T_{X,0}$. The choice of an identification $\bigwedge^n T_{X,0} \simeq \mathbf{C}$ induces an isomorphism $\bigwedge^d T_{X,0} \simeq \bigwedge^{n-d} T^*_{X,0}$. Let β_i be an element of $\bigwedge^d T_{W,x-v_i}$ such that $\beta_i(\alpha_i) = 1$. Then $c(V,W) = \sum_{i=1}^s \alpha_i \otimes \beta_i$. In particular, for $i \neq j$:

$$\beta_i\big(c(V,W)^{-1}(\alpha_j)\big) = 0.$$

Fix $v = v_1$ in V and let x vary in $v + W$, so that $v \in V \cap (x - W)$. We get:

$$\bigwedge^{d} T_{x-v_i} W \wedge c(V,W)^{-1}\Big(\bigwedge^{n-d} T_v V\Big) = 0$$

for $i > 1$. Since W is nondegenerate, the points $x - v_1, \ldots, x - v_s$ must therefore describe proper subvarieties of W. This proves the last part of the theorem, since $p_2\sigma^{-1}(x) = W \cap (x - V) = \{x - v, x - v_1, \ldots, x - v_s\}$. $\square$

For $1 < d < n$, apart from Jacobians of curves, I know of only one other family of principally polarized abelian varieties of dimension n with a subvariety with class θ_d: in the 5-dimensional intermediate Jacobian JZ of a smooth cubic hypersurface Z in $\mathbf{P}^4$, the image by any Abel-Jacobi map of the Fano surface of lines contained in Z is a surface F in JZ with class θ_3.

For any positive integer n, let $\mathscr{A}_n$ be the moduli space of complex principally polarized abelian varieties of dimension n, let $\mathscr{J}_n$ be the closure in $\mathscr{A}_n$ of the subvariety which corresponds to Jacobians of smooth curves of genus n, and let $\mathscr{C}T_5$ be the closure in $\mathscr{A}_5$ of the subvariety which corresponds to (principally polarized) intermediate Jacobians of smooth cubic threefolds. For $0 < d \leq n$, the subset $\mathscr{C}_{n,d}$ of $\mathscr{A}_n$ which corresponds to principally polarized abelian varieties for which θ_d is the class of an effective algebraic cycle (so that $\mathscr{C}_{n,1} = \mathscr{C}_{n,n} = \mathscr{A}_n$), is *closed* in $\mathscr{A}_n$.

CONJECTURE 1.6. For $1 < d < n$ and $(n,d) \neq (5,3)$, we have $\mathscr{C}_{n,d} = \mathscr{J}_n$. Furthermore, $\mathscr{C}_{5,3} = \mathscr{J}_5 \cup \mathscr{C}T_5$.

It follows from a criterion of Matsusaka that the conjecture holds for $d = n - 1$ and any n. It also holds for $n = 4$ and $d = 2$ by [**R**].

I proved in [**D3**] a weak version of this conjecture: $\mathscr{J}_n$ is an irreducible component of $\mathscr{C}_{n,d}$ for all $1 < d < n$, and $\mathscr{C}T_5$ is an irreducible component of $\mathscr{C}_{5,3}$.

2. Subvarieties with minimal classes in Jacobians

We now prove our main result.

THEOREM 2.1. *Let (JC, θ) be the Jacobian of a smooth curve C of genus n. Any nondegenerate subvariety V of JC of codimension d such that $V \cdot W_d(C) = \binom{n}{d}$ is a translate of either $W_{n-d}(C)$ or $-W_{n-d}(C)$.*

Any subvariety with minimal class θ_d satisfies the hypotheses of the theorem.

SKETCH OF PROOF. We prove by induction on $n-d$ that any $(n-d)$-dimensional irreducible subvariety V of JC which has property $(\mathscr{P})$ with respect to $W_d(C)$ is a translate of some $W_r(C) - W_{n-d-r}(C)$, with $0 \le r \le n-d$. This is obvious for $n-d = 0$, hence we assume $n-d > 0$.

For any positive integer e, we define the sum map

$$\sigma_e : V \times W_e(C) \to JC$$

and the difference map

$$\delta_e : V \times W_e(C) \to JC.$$

Since $V \cdot W_d(C) > 1$, the maps σ_d and δ_d are not birational. Let m be the smallest integer such that either σ_m or δ_m is not birational onto its image.

We will assume that σ_m is not birational onto its image: for v general in V, there exists an irreducible component Γ of $(\sigma_m)^{-1}(v + W_m(C))$, distinct from $\{v\} \times W_m(C)$, which dominates $v + W_m(C)$. Our aim is to prove that V can be written as $U + C$ for some subvariety U of $J(C)$.

By Theorem 1.5, V has property $(\mathscr{P})$ with respect to $W_d(C)$. One checks that V still has property $(\mathscr{P})$ with respect to $W_m(C)$. The projection $p_2(\Gamma) \subset W_m(C)$ therefore has dimension $r < m$. Furthermore, for D general in $p_2(\Gamma)$, there exists a subvariety S_D of V of dimension $m - r > 0$ such that

$$S_D + D \subset v + W_m(C).$$

One checks that for E general in $W_r(C)$, the subvariety $E + W_{m-r}(C)$ of $W_m(C)$ must meet $p_2(\Gamma)$ at some point $D = E + E'$ and that, analogously, the subvariety $S_D + D - v$ of $W_m(C)$ meets $E' + W_r(C)$

at s points $E'+E_1,\dots,E'+E_s$ with $s>0$, i.e., there exist points $v_1,\dots,v_s$ of $S_D\subset V$ such that

$$v_i+(E+E')-v=E'+E_i\ ,\qquad\text{i.e.,}\qquad v_i-E_i=v-E.$$

Since $r<m$, the map δ_r is birational onto its image. Therefore, since v and E are general, $E_i=E$ for all i and $v\in S_D$. In other words,

$$(S_D+D-v)\cap\bigl(E'+W_r(C)\bigr)=\{D\}$$

as subsets of $W_m(C)$. This is actually a scheme-theoretic equality. A simple argument shows that this is only possible if there exists an effective divisor E_D of degree r such that

$$S_D+D-v=W_{m-r}(C)+E_D.$$

Moreover, since $v\in S_D$, the divisor $E'_D=D-E_D$ is effective, and

$$S_D+E'_D=v+W_{m-r}(C).$$

If c is in the support of E'_D and if x is any point of C, the point $v+x-c$ is in S_D hence in V. Since v is general, this implies that some irreducible component T of the scheme

$$\{\ (v,c)\in V\times C\ \big|\ v-c+C\subset V\ \}$$

dominates V by the first projection. Note that $U=\delta_1(T)$ satisfies $U+C\subset V$. In particular, $\dim(U)<\dim(V)\le\dim(T)$, hence $\dim(U)=\dim(T)-1$. This implies $V=U+C$.

Now it is not difficult to check that U still has property $(\mathscr{P})$ with respect to $W_{d+1}(C)$ and the induction hypothesis applies.

The conclusion of the proof is now easy: if C is hyperelliptic, $-C$ is a translate of C, hence $-W_{n-d-r}(C)$ is a translate of $W_{n-d-r}(C)$, and V is a translate of $W_{n-d}(C)$. If C is nonhyperelliptic, one checks

$$\bigl(W_r(C)-W_{n-d-r}(C)\bigr)\cdot W_d(C)=\binom{n-d}{r}\binom{n}{d}.$$

This number is equal to $\binom{n}{d}$ only if $r=n-d$, or $r=0$, hence V is a translate of either $W_{n-d}(C)$ or $-W_{n-d}(C)$. $\square$

Höring has recently proved ([**H2**]) that in the intermediate Jacobian of a *general* cubic hypersurface $Z\subset\mathbf{P}^4$, any surface with class θ_3 is a translate of either the Fano surface F, or of $-F$.

CHAPTER 3

Cohomological methods

Most results of this chapter are taken from [**PP1**] and [**D2**].

1. The vector bundle associated to a covering

Let $f : Y \to X$ be a covering, i.e., a surjective finite morphism, of degree d, between smooth projective (complex) varieties of the same dimension n. The morphism f is flat, hence $f_*\mathscr{O}_Y$ is locally free of rank d and there is a *trace morphism* $\mathrm{Tr}_{Y/X} : f_*\mathscr{O}_Y \to \mathscr{O}_X$ such that the composition

$$\mathscr{O}_X \longrightarrow f_*\mathscr{O}_Y \xrightarrow{\mathrm{Tr}_{Y/X}} \mathscr{O}_X$$

is multiplication by d. Let E_f be the vector bundle, of rank $d-1$, dual to the kernel of $\mathrm{Tr}_{Y/X}$, so that

$$f_*\mathscr{O}_Y = \mathscr{O}_X \oplus E_f^\vee.$$

Also, by duality for a finite flat morphism

$$f_*\omega_{Y/X} = \mathscr{O}_X \oplus E_f.$$

This vector bundle is interesting because of the following proposition.

Proposition 1.1. *There is a factorization*

$$\begin{array}{ccc} Y & \hookrightarrow & E_f \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & X & \end{array}$$

Proof. The inclusion $E_f^\vee \subset f_*\mathscr{O}_Y$ induces a morphism

$$\mathrm{Sym}\, E_f^\vee \longrightarrow f_*\mathscr{O}_Y$$

of $\mathscr{O}_Y$-algebras which is surjective since $\mathrm{Sym}^0 E_f^\vee$ is mapped onto $\mathscr{O}_X$ and $\mathrm{Sym}^1 E_f^\vee$ onto $E_f^\vee$. Taking relative spectra, we get

$$Y = \mathbf{Spec}\, f_*\mathscr{O}_Y \hookrightarrow \mathbf{Spec}\,\mathrm{Sym}\, E_f^\vee = E_f$$

∎

EXAMPLE 1.2. If $f : Y \to X$ is a double cover, E_f is a line bundle on X and the branch locus of f in X is a smooth divisor which is the zero locus of a section $s \in H^0(X, E_f^{\otimes 2})$. The variety Y sits in the total space of E_f as

$$Y = \{(x,t) \in E_f \mid t^2 = s(x)\}.$$

We will be interested in situations where E_f is *ample.* This has several consequences.

Consequence 1. *Assume E_f is* ample. *If S is a proper variety and $g : S \to X$ is a morphism whose image has dimension ≥ 1, $S \times_X Y$ is connected.*

PROOF. Let $T = S \times_X Y$ and let $f' : T \to S$ be the morphism deduced from f. If g is *finite,* $E_{f'} = g^* E_f$ is still ample, hence $H^0(S, E_{f'}^\vee) = 0$ (because S is integral of dimension ≥ 1). This implies

$$h^0(T, \mathscr{O}_T) = h^0(S, f'_* \mathscr{O}_T) = h^0(S, \mathscr{O}_S) + h^0(S, E_{f'}^\vee) = 1$$

hence T is connected. In general, consider the *Stein factorization* $g : S \xrightarrow{p} S' \xrightarrow{g'} X$, where p has connected fibers and g' is finite. The morphism $T \to S' \times_X Y$ induced by p has connected image and connected fibers, hence T is connected. $\square$

Consequence 2. *Assume E_f is* ample. *The induced morphisms*

$$H^i(f, \mathbf{C}) : H^i(X, \mathbf{C}) \longrightarrow H^i(Y, \mathbf{C})$$

are bijective for $i \leq n - d + 1$.

PROOF. Consider the projective completion $\pi : \bar{E}_f = \mathbf{P}(E_f^\vee \oplus \mathscr{O}_X) \to X$ of E_f. Let $\xi \in H^{2e}(\bar{E}_f, \mathbf{C})$ be the class of the divisor at infinity $\bar{E}_f$ – E_f and let $[Y] \in H^{2e}(\bar{E}_f, \mathbf{C})$ be the class of the image of $Y \overset{j}{\hookrightarrow} \bar{E}_f$, with $e = d - 1$. Consider the commutative diagram

$$\begin{array}{ccccc} H^{n-e+i}(Y,\mathbf{C}) & \xrightarrow{j_*} & H^{n+e+i}(\bar{E}_f,\mathbf{C}) = H^{n+e+i}(X,\mathbf{C})[\xi] & \xleftarrow{\pi^*} & H^{n+e+i}(X,\mathbf{C}) \\ & \searrow^{\cdot j^*[Y]} & \downarrow{\scriptstyle j^*} & \swarrow_{f^*} & \\ & & H^{n+e+i}(Y,\mathbf{C}) & & \end{array}$$

with $i \geq 0$. The class of Y in $\bar{E}_f$ is d times that of the zero section; since $j^*\xi = 0$, this implies $j^*[Y] = dc_e(f^*E_f)$. An analogue of the Hard Lefschetz Theorem, due to Sommese, shows that since f^*E_f is ample, the cup-product by its top Chern class is surjective when $e \leq n$. The map j^* is then surjective, and since $j^*\xi = 0$, so is f^*. Since f^* is injective in all degrees, it is here bijective, and the statement follows by Poincaré duality. □

It is expected that the maps $H^i(f, \mathbf{Z})$, as well as the maps $\pi_i(f) : \pi_i(Y) \to \pi_i(X)$, are still bijective in the same range.

When X is a projective space, the ampleness of E_f was proved by Lazarsfeld in [**L**].

THEOREM 1.3. *Let Y be a smooth projective variety and let $f : Y \to \mathbf{P}^n$ be a covering. The bundle $E_f(-1)$ is globally generated. In particular, E_f is ample.*

PROOF. It is enough to show that $E_f(-1)$ is 0-regular, i.e.,

$$H^i(\mathbf{P}^n, E_f(-i-1)) = 0$$

for all $i > 0$. This follows from Kodaira vanishing on Y. □

This was later generalized by Kim and Manivel ([**KM**]) to the case where X is a Grassmannian or Lagrangian Grassmannian. When X is an abelian variety, E_f is not ample in general (if f is an isogeny, E_f is a sum of numerically trivial line bundles), but only nef (i.e., “limit” of amples; [**PS**]).

However, there is a large class of situations in which E_f is ample. We will prove the next theorem in the rest of this chapter.

THEOREM 1.4. *Let X be a* simple *abelian variety, let Y be a smooth projective variety, and let $f : Y \to X$ be a covering. If f does not factor through any nontrivial isogeny $X' \to X$, the bundle E_f is ample.*

In particular, Consequence 1 above applies (the conclusion also follows from Theorem 4.1), and so does Consequence 2.[1]

2. The Mukai transform

Let X be an abelian variety of dimension n defined over an algebraically closed field $\mathbf{k}$ and let $\widehat{X} = \mathrm{Pic}^0(X)$ be its dual abelian variety. For $\xi \in \widehat{X}$, we denote by P_ξ the corresponding numerically trivial line bundle on X.

[1]When $d \leq n$, we showed in (the proof of) Corollary 4.3 that $\pi_1(f) : \pi_1(Y) \to \pi_1(X)$ is bijective. In particular $H^1(f, \mathbf{Z}) : H^1(X, \mathbf{Z}) \to H^1(Y, \mathbf{Z})$ is bijective, hence $H_1(Y, \mathbf{Z})$ is torsion-free, and so is $H^2(Y, \mathbf{Z})$ by the Universal Coefficients Theorem. When $d < n$, the morphism $H^2(f, \mathbf{Z}) : H^2(X, \mathbf{Z}) \to H^2(Y, \mathbf{Z})$ is injective with finite cokernel, and so is $\mathrm{Pic}(f) : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$. It is expected that both morphisms are bijective.

There is on $X \times \widehat{X}$ a unique line bundle $\mathscr{P}$ that satisfies, for all ξ in $\widehat{X}$,

$$\mathscr{P}|_{X\times\{\xi\}} \simeq P_\xi \quad \text{and} \quad \mathscr{P}|_{\{0\}\times\widehat{X}} \simeq \mathscr{O}_{\widehat{X}}.$$

If $\mathscr{F}$ is a coherent sheaf on X, or more generally a complex of coherent sheaves, we set

$$\widehat{\mathscr{S}}(\mathscr{F}) = q_*(p^*\mathscr{F} \otimes \mathscr{P}),$$

where $p : X \times \widehat{X} \to X$ and $q : X \times \widehat{X} \to \widehat{X}$ are the two projections. This defines a functor from the category of coherent $\mathscr{O}_X$-modules to the category of coherent $\mathscr{O}_{\widehat{X}}$-modules. Let

$$\mathbf{R}\widehat{\mathscr{S}} : \mathbf{D}(X) \to \mathbf{D}(\widehat{X})$$

be its derived functor. Since $\widehat{\widehat{X}}$ is canonically isomorphic to X, there is another functor

$$\mathbf{R}\mathscr{S} : \mathbf{D}(\widehat{X}) \to \mathbf{D}(X).$$

THEOREM 2.1 (Mukai). *Both* $\mathbf{R}\widehat{\mathscr{S}}$ *and* $\mathbf{R}\mathscr{S}$ *are equivalences of categories and there is a canonical isomorphism of functors*

$$\mathbf{R}\mathscr{S} \circ \mathbf{R}\widehat{\mathscr{S}} \simeq (-\mathbf{1}_X)^*[-n].$$

EXAMPLE 2.2. If $\xi \in \widehat{X}$, it is easy to check that the complex $\mathbf{R}\mathscr{S}(\mathbf{k}_\xi)$ is the single sheaf P_ξ placed in degree 0. It follows that $\mathbf{R}\widehat{\mathscr{S}}(P_\xi)$ is the single sheaf $\mathbf{k}_{-\xi}$ placed in degree n.

The cohomology sheaves of the complex $\mathbf{R}\widehat{\mathscr{S}}(\mathscr{F})$ are

$$R^i\widehat{\mathscr{S}}(\mathscr{F}) = R^i q_*(p^*\mathscr{F} \otimes \mathscr{P}).$$

This is a sheaf on $\widehat{X}$ whose support is contained in

$$V_i(\mathscr{F}) = \{\xi \in \widehat{X} \mid H^i(X, \mathscr{F} \otimes P_\xi) \neq 0\}. \tag{5}$$

Following Mukai, we will say that a coherent cheaf $\mathscr{F}$

- satisfies property WIT_j if $R^i\widehat{\mathscr{S}}(\mathscr{F}) = 0$ for all $i \neq j$ (we then write $R^j\widehat{\mathscr{S}}(\mathscr{F}) = \widehat{\mathscr{F}}$);
- satisfies (the stronger) property IT_j if $V_i(\mathscr{F}) = \varnothing$ for all $i \neq j$, i.e.,

$$H^i(X, \mathscr{F} \otimes P_\xi) = 0 \text{ for all } \xi \in \widehat{X} \text{ and all } i \neq j.$$

Of course, these definitions also make sense for a *complex* of coherent sheaves on X.

Mukai proves ([**M1**], Corollary 2.4) that if $\mathscr{F}$ satisfies WIT_j, the sheaf $\widehat{\mathscr{F}}$ satisfies WIT_{n-j}, and $\widehat{\widehat{\mathscr{F}}} = (-\mathbf{1}_X)^*\mathscr{F}$.

EXAMPLE 2.3. The sheaf $\mathbf{k}_\xi$ satisfies IT_0, hence P_ξ satisfies WIT_n (Example 2.2). However, P_ξ does not satisfies IT_n since $V_i(P_\xi) = \{-\xi\}$ for all $i \in \{0, \dots, n\}$.

An ample line bundle L on X satisfies IT_0, and $\widehat{L}$ is a vector bundle on $\widehat{X}$ of rank $h^0(X, L)$. In particular, if L defines a principal polarization, $\widehat{L}$ is a line bundle which is $(\varphi_L^{-1})^* L^{-1}$ ([**M1**], Proposition 3.11.(3)), where $\varphi_L : X \to \widehat{X}$ is the isomorphism defined by L.

Similarly, $L^\vee$ satisfies IT_n, and $\widehat{L^\vee} = (-\mathbf{1}_X)^* \widehat{L}^\vee[-n]$.

DEFINITION 2.4. A coherent sheaf $\mathscr{F}$ on X is *M-regular* if

$$\operatorname{codim}\bigl(\operatorname{Supp} R^i \widehat{\mathscr{S}}(\mathscr{F})\bigr) > i$$

for all $i > 0$.

One can show that this property is equivalent to the more concrete condition

$$\operatorname{codim}(V_i(\mathscr{F})) > i$$

for all $i > 0$. Obviously, sheaves that satisfy WIT_0 are M-regular (but the converse is false in general).

3. M-regular sheaves

We now show that M-regular sheaves have remarkable positivity properties.

THEOREM 3.1 (Pareschi–Popa, Debarre). *Let $\mathscr{F}$ be a coherent sheaf on an abelian variety X. Consider the following properties:*

(i) *$\mathscr{F}$ is M-regular;*

(ii) *for any ample line bundle L on X, there exists a positive integer N such that, for $(\xi_1, \dots, \xi_N) \in \widehat{X}^N$ general, the map*

$$\bigoplus_{i=1}^{N} H^0(X, \mathscr{F} \otimes P_{\xi_i}) \otimes H^0(X, L \otimes P_{\xi_i}^\vee) \to H^0(X, \mathscr{F} \otimes L) \tag{6}$$

is surjective;

(iii) *$\mathscr{F}$ is* continuously globally generated*: there exists a positive integer N such that, for $(\xi_1, \dots, \xi_N) \in \widehat{X}^N$ general,*

$$\bigoplus_{i=1}^{N} H^0(X, \mathscr{F} \otimes P_{\xi_i}) \otimes P_{\xi_i}^\vee \to \mathscr{F} \tag{7}$$

is surjective;

(iv) *there exists an isogeny $\pi : X' \to X$ such that $\pi^*(\mathscr{F} \otimes P_\xi)$ is globally generated for all $\xi \in \widehat{X}$;*

(v) $\mathscr{F}$ *is ample.*

Then, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v)*.*

Implication (v) $\Rightarrow$ (iv) does not hold in general.

Proof. Let us show (i) $\Rightarrow$ (ii). Surjectivity of (6) is equivalent to that of

$$\bigoplus_{\xi\in U} H^0(X,\mathscr{F}\otimes P_\xi)\otimes H^0(X,L\otimes P_\xi^\vee)\to H^0(X,\mathscr{F}\otimes L)$$

for any dense open subset U of $\widehat{X}$, hence to the injectivity of its dual

$$H^0(X,\mathscr{F}\otimes L)^\vee\to\bigoplus_{\xi\in U} H^0(X,\mathscr{F}\otimes P_\xi)^\vee\otimes H^0(X,L\otimes P_\xi^\vee)^\vee$$

or, by Serre duality, to the injectivity of

$$\text{(8)}\qquad \operatorname{Ext}^n_X(\mathscr{F},L^\vee)\to\bigoplus_{\xi\in U}\operatorname{Hom}_{\mathbf{C}}(H^0(X,\mathscr{F}\otimes P_\xi),H^n(X,L^\vee\otimes P_\xi)).$$

One checks that there exists a fourth-quadrant spectral sequence ($p\ge 0$, $q\le 0$)

$$E_2^{pq}=\operatorname{Ext}^p_{\widehat{X}}(R^{-q}\widehat{\mathscr{S}}(\mathscr{F}),\widehat{L^\vee})\Rightarrow\operatorname{Ext}^{p+q}_{\mathbf{D}(\widehat{X})}(\mathbf{R}\widehat{\mathscr{S}}(\mathscr{F}),\widehat{L^\vee}).$$

By Serre duality, we have

$$\operatorname{Ext}^p_{\widehat{X}}(R^{-q}\widehat{\mathscr{S}}(\mathscr{F}),\widehat{L^\vee})\simeq H^{n-p}(\widehat{X},R^{-q}\widehat{\mathscr{S}}(\mathscr{F})\otimes\widehat{L^\vee})^\vee$$

which, since $\mathscr{F}$ is M-regular, vanishes for $q<0$ and $n-p\ge n-(-q)$, i.e., $p\le -q$. On the other hand, these groups can only be nonzero for $0\le p\le n$ and $-n\le q\le 0$. Differentials coming into E_r^{00} being always zero, we have a chain of inclusions

$$\operatorname{Hom}_{\mathbf{D}(\widehat{X})}(\mathbf{R}\widehat{\mathscr{S}}(\mathscr{F}),\widehat{L^\vee})=E_\infty^{00}\subset\cdots\subset E_3^{00}\subset E_2^{00}=\operatorname{Hom}_{\widehat{X}}(R^0\widehat{\mathscr{S}}(\mathscr{F}),\widehat{L^\vee})$$

and a result of Mukai implies that the left-hand-side term is isomorphic to $\operatorname{Ext}^n_X(\mathscr{F},L^\vee)$. We obtain an injective map

$$\operatorname{Ext}^n_X(\mathscr{F},L^\vee)\longrightarrow\operatorname{Hom}_{\widehat{X}}(R^0\widehat{\mathscr{S}}(\mathscr{F}),\widehat{L^\vee})$$

which is the morphism on global sections associated with the morphism of sheaves

$$\varphi:\operatorname{Ext}^n_X(\mathscr{F},L^\vee)\otimes\mathscr{O}_{\widehat{X}}\longrightarrow\mathscr{H}om_{\mathscr{O}_{\widehat{X}}}(R^0\widehat{\mathscr{S}}(\mathscr{F}),\widehat{L^\vee}).$$

At a general $\xi\in\widehat{X}$, the morphism between fibers induced by φ is the map

$$\operatorname{Ext}^n_X(\mathscr{F},L^\vee)\to\operatorname{Hom}_{\mathbf{C}}(H^0(X,\mathscr{F}\otimes P_\xi),H^n(X,L^\vee\otimes P_\xi))$$

which appears in (8). Since the sheaf $\mathscr{H}om_{\mathscr{O}_{\widehat{X}}}(R^0\widehat{\mathscr{S}}(\mathscr{F}), \widehat{L^\vee})$ is torsion-free, the map (8) is injective.

Let us show (ii) $\Rightarrow$ (iii). Let L be an ample line bundle on X such that $\mathscr{F}\otimes L$ is globally generated. In the commutative diagram

$$\begin{array}{ccc} \bigoplus_{i=1}^N H^0(X,\mathscr{F}\otimes P_{\xi_i})\otimes H^0(X,L\otimes P_{\xi_i}^\vee)\otimes\mathscr{O}_X & \longrightarrow & H^0(X,\mathscr{F}\otimes L)\otimes\mathscr{O}_X \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\ \bigoplus_{i=1}^N H^0(X,\mathscr{F}\otimes P_{\xi_i})\otimes L\otimes P_{\xi_i}^\vee & \longrightarrow & \mathscr{F}\otimes L \end{array}$$

the top arrow is surjective by (ii), and so is the evaluation. It follows that the bottom map is also surjective, hence (iii).

Let us show (iii) $\Rightarrow$ (iv). Let $\xi_0\in\widehat{X}$. Since torsion points are dense in $\widehat{X}^N$, the open subset of points of $\widehat{X}^N$ for which the map (7) is surjective and all $h^0(X,\mathscr{F}\otimes P_{\xi_i})$ take their minimal value contains a point of the type

$$(\xi_0+\eta_1(\xi_0),\dots,\xi_0+\eta_N(\xi_0)),$$

where $(\eta_1(\xi_0),\dots,\eta_N(\xi_0))$ is torsion, hence also contains

$$U_{\xi_0}+(\eta_1(\xi_0),\dots,\eta_N(\xi_0)),$$

where U_{ξ_0} is a neighborhood of ξ_0 in $\widehat{X}$. Since $\widehat{X}$ is quasi-compact, it is covered by finitely many such neighborhoods, say $U_{\xi_1},\dots,U_{\xi_M}$.

Let $\pi: X'\to X$ be an isogeny such that the kernel of $\widehat{\pi}:\widehat{X}\to\widehat{X}'$ contains $\eta_i(\xi_j)$, for all $i\in\{1,\dots,N\}$ and $j\in\{1,\dots,M\}$. Fix $j\in\{1,\dots,M\}$. The map

$$\bigoplus_{i=1}^N H^0(X,\mathscr{F}\otimes P_\xi\otimes P_{\eta_i(\xi_j)})\otimes\pi^*P_\xi^\vee\otimes\pi^*P_{\eta_i(\xi_j)}^\vee\longrightarrow\pi^*\mathscr{F}$$

is surjective for all $\xi\in U_{\xi_j}$. But this map is

$$\bigoplus_{i=1}^N H^0(X,\mathscr{F}\otimes P_\xi\otimes P_{\eta_i(\xi_j)})\otimes\pi^*P_\xi^\vee\longrightarrow\pi^*\mathscr{F}$$

and since each $H^0(X,\mathscr{F}\otimes P_\xi\otimes P_{\eta_i(\xi_j)})$ is a subspace of $H^0(Y,\pi^*(\mathscr{F}\otimes P_\xi))$, the sheaf $\pi^*(\mathscr{F}\otimes P_\xi)$ is globally generated for all $\xi\in U_{\xi_j}$, hence for all $\xi\in\widehat{X}$.

Let us show (iv) $\Rightarrow$ (v). Let C be a curve in X'. If there is a trivial quotient $\pi^*\mathscr{F}|_C\twoheadrightarrow\mathscr{O}_C$, there are surjections $\pi^*(\mathscr{F}\otimes P_\xi)|_C\twoheadrightarrow\pi^*(P_\xi)|_C$ for each $\xi\in\widehat{X}$. Since $\pi^*(\mathscr{F}\otimes P_\xi)$ is globally generated, so is $\pi^*(P_\xi)|_C$.

This implies that the map $\widehat{X} \to \widehat{X}' \to \mathrm{Pic}^0(C)$ vanishes, hence $\pi(C)$ is a point, which is absurd.

So the restriction of $\pi^*\mathscr{F}$ to any curve in X has no trivial quotient. A lemma of Gieseker implies that $\pi^*\mathscr{F}$ is ample, hence so is $\mathscr{F}$. □

4. Proof of Theorem 1.4

Green and Lazarsfeld have shown that any irreducible component of the locus

$$V_i = \{\xi \in \widehat{X} \mid H^{n-i}(Y, f^*P_\xi^\vee) \neq 0\}$$

is a translated abelian subvariety of codimension $\geq i$ in $\widehat{X}$. Since X is simple, V_i is finite for all $i > 0$. Let $\widehat{Y} = \mathrm{Pic}^0(Y)$. Since Y is connected,

$$\begin{aligned} V_n &= \{\xi \in \widehat{X} \mid H^0(Y, f^*P_\xi^\vee) \neq 0\} \\ &= \{\xi \in \widehat{X} \mid f^*P_\xi^\vee \simeq \mathscr{O}_Y\} \\ &= \ker(\widehat{f} : \widehat{X} \to \widehat{Y}), \end{aligned}$$

hence $V_n = \{0\}$ because $\widehat{f}$ is injective. By Serre duality on Y, we have

$$\begin{aligned} V_i &= \{\xi \in \widehat{X} \mid H^i(Y, \omega_Y \otimes f^*P_\xi) \neq 0\} \\ &= \{\xi \in \widehat{X} \mid H^i(X, f_*\omega_Y \otimes P_\xi) \neq 0\}. \end{aligned}$$

Since $f_*\omega_Y = \mathscr{O}_X \oplus E_f$, we have $V_i(E_f) \subset V_i$ and $V_n(E_f) = \varnothing$. The sheaf E_f is M-regular, hence ample by Theorem 3.1.

CHAPTER 4

Generic vanishing and minimal cohomology classes on abelian varieties

In this section, we show how a cohomological property of the ideal sheaf of a subvariety V of a principally polarized abelian variety implies that V has minimal class in the sense of Chapter 2. Since all known subvarieties with minimal classes (see Chapter 2) have this cohomological property, it is conjectured that this property should actually characterize them (Conjecture 6.1).

Most results in this section are due to Hacon, Pareschi, and Popa.

1. The theta dual of a subvariety of a principally polarized abelian variety

For any subvariety[1] V of a principally polarized abelian variety (X, Θ), we define

$$T(V) = \{x \in X \mid V \subset \Theta + x\}$$

and we call it the *theta dual* of V (we assume here and in the rest of this chapter that Θ is symmetric, i.e., $-\Theta = \Theta$).

We have $V - T(V) \subset \Theta$. The image of $T(V)$ by the isomorphism $X \to \widehat{X}$ provided by the principal polarization is, in the notation of (5) of Chapter 3,

$$\widehat{T}(V) = \{\xi \in \widehat{X} \mid H^0(X, \mathscr{I}_V(\Theta) \otimes P_\xi) \neq 0\} = V_0(\mathscr{I}_V(\Theta)).$$

EXAMPLE 1.1. If C is a smooth projective curve of genus n and $d \in \{0, \dots, n-1\}$, we have $\Theta = W_d(C) + W_{n-d-1}(C)$ and $T(W_d(C))$ is $-W_{n-d-1}(C)$.

If F is the Fano surface of lines in the 5-dimensional intermediate Jacobian (JZ, Θ) of a smooth cubic hypersurface Z in $\mathbf{P}^4$ (see Chapter 2, §1), we have $\Theta = F - F$ and $T(F) = F$.

The point here is that subvarieties V for which $V - T(V) = \Theta$ are very special. Indeed, the examples above are conjectured to be the

[1] In this section only, subvarieties will still be reduced, but will be allowed to have several irreducible components *of the same dimension.*

only ones on an indecomposable principally polarized abelian variety. We will relate this conjecture with Conjecture 1.6 in Chapter 2 and to cohomological properties of the sheaf $\mathscr{I}_V(\Theta)$ analogous to the ones studied in Chapter 3.

2. Weakly M-regular sheaves

Let X be an abelian variety. In §2 of Chapter 3, we defined the Mukai transform

$$\mathbf{R}\widehat{\mathscr{S}} : \mathbf{D}(X) \xrightarrow{\sim} \mathbf{D}(\widehat{X})$$

from the derived category of coherent $\mathscr{O}_X$-modules to the derived category of coherent $\mathscr{O}_{\widehat{X}}$-modules. A coherent sheaf $\mathscr{F}$ on X was said to be M-regular if

$$\operatorname{codim}\bigl(\operatorname{Supp} R^i\widehat{\mathscr{S}}(\mathscr{F})\bigr) > i$$

for all $i > 0$ (Definition 2.4). This property can be shown to be equivalent to

$$\operatorname{codim}(V_i(\mathscr{F})) > i$$

for all $i > 0$, where $V_i(\mathscr{F})$ is defined in (5).

Definition 2.1. A coherent sheaf $\mathscr{F}$ on an abelian variety X is *weakly M-regular* if

$$\operatorname{codim}\bigl(\operatorname{Supp} R^i\widehat{\mathscr{S}}(\mathscr{F})\bigr) \geq i$$

for all $i > 0$.[2]

Again, this property can be shown to be equivalent to

$$\operatorname{codim}(V_i(\mathscr{F})) \geq i$$

for all $i \geq 0$.

These definitions also make sense for *complexes* of coherent sheaves on X.

Remark 2.2. We proved in Theorem 3.1 that M-regular sheaves are ample. This result is used in [**PP3**] to prove that weakly M-regular sheaves are nef (i.e., "limits" of amples).

We will be interested in subvarieties V of X such that the sheaf $\mathscr{I}_V(\Theta)$ is weakly M-regular.[3]

[2]Pareschi and Popa say that $\mathscr{F}$ is *a GV-sheaf.*

[3]This property is equivalent to: $\mathscr{O}_V(\Theta)$ is M-regular and $h^0(V, \mathscr{O}_V(\Theta) \otimes P_\xi) = 1$ for $\xi \in \widehat{X}$ general. Indeed, since $H^i(X, \mathscr{O}_X(\Theta) \otimes P_\xi) = 0$ for all $\xi \in \widehat{X}$ and all $i > 0$, we get $V_i(\mathscr{I}_V(\Theta)) = V_{i-1}(\mathscr{O}_V(\Theta))$ for $i \geq 2$ while $V_1(\mathscr{I}_V(\Theta)) = \{\xi \in \widehat{X} \mid h^0(V, \mathscr{O}_V(\Theta) \otimes P_\xi) > 1\}$.

EXAMPLE 2.3. If C is a smooth projective curve, Pareschi and Popa have shown that $\mathscr{I}_{W_d(C)}(\Theta)$ is weakly M-regular.

If F is the Fano surface of lines in the 5-dimensional intermediate Jacobian (JZ,Θ) of a smooth cubic hypersurface $Z\subset \mathbf{P}^4$ (see Chapter 2, §1), Höring has shown in [**H1**] that $\mathscr{I}_F(\Theta)$ is also weakly M-regular.

3. A criterion for weak M-regularity

Given an object $\mathscr{F}^\bullet$ in $\mathbf{D}(X)$, we set

$$\mathbf{R}\Delta(\mathscr{F}^\bullet)=\mathbf{R}\mathscr{H}om(\mathscr{F}^\bullet,\mathscr{O}_X)$$

(and we define $\mathbf{R}\widehat{\Delta}$ similarly on $\mathbf{D}(\widehat{X})$). We have

$$(\mathbf{R}\Delta\circ\mathbf{R}\Delta)(\mathscr{F}^\bullet)\simeq\mathscr{F}^\bullet$$

and a commutation relation ([**M1**] (3.8))

$$\mathbf{R}\widehat{\Delta}\circ\mathbf{R}\widehat{\mathscr{S}}\simeq((-\mathbf{1}_{\widehat{X}})^*\circ\mathbf{R}\widehat{\mathscr{S}}\circ\mathbf{R}\Delta)[n]. \tag{9}$$

Our main technical tool will be the following result, due to Hacon and Pareschi–Popa, for whose proof we refer to [**PP3**].

THEOREM 3.1. *Let $\mathscr{F}^\bullet$ be a complex of coherent sheaves on an abelian variety of dimension n. The following properties are equivalent:*

(i) *$\mathscr{F}^\bullet$ is weakly M-regular;*
(ii) *$\mathbf{R}\Delta(\mathscr{F}^\bullet)$ satisfies WIT_n.*

4. Mukai transform of twisted ideal sheaves

Let V be a subvariety of a principally polarized abelian variety (X,Θ). We investigate the Mukai transform of $\mathscr{I}_V(\Theta)$, beginning with the following technical result.

PROPOSITION 4.1. *Let V be a subvariety of a principally polarized abelian variety (X,Θ) of dimension n. We have an isomorphism*

$$(-\mathbf{1}_{\widehat{X}})^*R^n\widehat{\mathscr{S}}(\mathbf{R}\Delta(\mathscr{I}_V(\Theta)))\simeq\mathscr{O}_{\widehat{T}(V)}(\widehat{\Theta}).$$

PROOF. We first identify the fibers of the sheaf on the left-hand-side.

LEMMA 4.2. *For any $\xi\in\widehat{X}$, there is an isomorphism*

$$(-\mathbf{1}_{\widehat{X}})^*R^n\widehat{\mathscr{S}}(\mathbf{R}\Delta(\mathscr{I}_V(\Theta)))_\xi\simeq H^0(X,\mathscr{I}_V(\Theta)\otimes P_\xi)^\vee.$$

Proof. Since $\mathbf{H}^{n+1}(X, \mathbf{R}\Delta(\mathscr{I}_V(\Theta)) \otimes P_\xi)$ vanishes, we have by base change for complexes, for any $\xi \in \widehat{X}$, a natural isomorphism

$$(-\mathbf{1}_{\widehat{X}})^* R^n \widehat{\mathscr{S}}(\mathbf{R}\Delta(\mathscr{I}_V(\Theta)))_\xi \simeq \mathbf{H}^n(X, \mathbf{R}\Delta(\mathscr{I}_V(\Theta)) \otimes P_\xi^{-1}).$$

By Grothendieck–Serre duality, this hypercohomology group is isomorphic to $H^0(X, \mathscr{I}_V(\Theta) \otimes P_\xi)^\vee$. □

The fibers therefore have dimension either 0 or 1, and the latter happens if and only if $\xi \in \widehat{T}(V)$.

We have a natural surjective homomorphism

$$R^n \widehat{\mathscr{S}}(\mathbf{R}\Delta(\mathscr{O}_X(\Theta))) \longrightarrow R^n \widehat{\mathscr{S}}(\mathbf{R}\Delta(\mathscr{I}_V(\Theta))).$$

Indeed, as in Lemma 4.2, the morphism betweens fibers over ξ is given by the dual of the injection $H^0(X, \mathscr{I}_V(\Theta) \otimes P_\xi) \hookrightarrow H^0(X, \mathscr{O}_X(\Theta) \otimes P_\xi)$.

Now $\mathbf{R}\Delta(\mathscr{O}_X(\Theta)) \simeq \mathscr{O}_X(-\Theta)$ and $(-\mathbf{1}_{\widehat{X}})^* R^n \widehat{\mathscr{S}}(\mathscr{O}_X(-\Theta)) \simeq \mathscr{O}_{\widehat{X}}(\widehat{\Theta})$ (Example 2.3), which proves the proposition. □

Assume now that $\mathscr{I}_V(\Theta)$ is weakly M-regular. By Theorem 3.1, the complex $\mathbf{R}\Delta(\mathscr{I}_V(\Theta))$ satisfies WIT_n, hence (Proposition 4.1)

$$(-\mathbf{1}_{\widehat{X}})^* \mathbf{R}\widehat{\mathscr{S}}(\mathbf{R}\Delta(\mathscr{I}_V(\Theta))) \simeq \mathscr{O}_{\widehat{T}(V)}(\widehat{\Theta})[-n].$$

Applying the functor $\mathbf{R}\widehat{\Delta}[n]$ to both sides, we obtain on the left-hand-side, using (9),

$$\mathbf{R}\widehat{\Delta}\big((-\mathbf{1}_{\widehat{X}})^* \mathbf{R}\widehat{\mathscr{S}}(\mathbf{R}\Delta(\mathscr{I}_V(\Theta)))[n]\big) \simeq \mathbf{R}\widehat{\mathscr{S}}(\mathscr{I}_V(\Theta)),$$

so that

$$\mathbf{R}\widehat{\mathscr{S}}(\mathscr{I}_V(\Theta)) \simeq \mathbf{R}\widehat{\Delta}(\mathscr{O}_{\widehat{T}(V)}(\widehat{\Theta})). \tag{10}$$

This implies already that $T(V)$ is nonempty, but we have more.

Proposition 4.3. *If V is a subvariety of dimension d of a principally polarized abelian variety (X, Θ) of dimension n such that $\mathscr{I}_V(\Theta)$ is weakly M-regular,* $\dim(T(V)) \geq n - d - 1$.

Proof. Since V has dimension d, we have $R^i \widehat{\mathscr{S}}(\mathscr{O}_V(\Theta)) = 0$ for $i > d$, and so $R^i \widehat{\mathscr{S}}(\mathscr{I}_V(\Theta)) = 0$ for $i > d + 1$, by applying the Mukai functor to the exact sequence

$$0 \to \mathscr{I}_V(\Theta) \to \mathscr{O}_X(\Theta) \to \mathscr{O}_V(\Theta) \to 0. \tag{11}$$

Combining

$$\operatorname{codim}_{\widehat{X}}(\widehat{T}(V)) = \min\{k \mid \mathscr{E}xt^k_{\mathscr{O}_{\widehat{X}}}(\mathscr{O}_{\widehat{T}(V)}, \mathscr{O}_{\widehat{X}}) \neq 0\}$$

with the isomorphism

$$\mathscr{E}xt^i_{\mathscr{O}_{\widehat{X}}}(\mathscr{O}_{\widehat{T}(V)}, \mathscr{O}_{\widehat{X}}) \simeq R^i\widehat{\mathscr{S}}(\mathscr{I}_V(\Theta)) \otimes \mathscr{O}_{\widehat{X}}(\widehat{\Theta}) \tag{12}$$

obtained from (10), we get the desired result. □

THEOREM 4.4. *If V is a subvariety of dimension d of a principally polarized abelian variety (X, Θ) of dimension n such that $\mathscr{I}_V(\Theta)$ is weakly M-regular and $T(V)$ has dimension $n-d-1$, we have*

- *$T(V)$ is Cohen-Macaulay and equidimensional;*
- *$\mathscr{I}_{T(V)}(\Theta)$ is weakly M-regular;*
- *V is Cohen-Macaulay and $T(T(V)) = V$.*

Since $V - T(V) \subset \Theta$, by Proposition 4.3 and Corollary 2.5.a) in Chapter 1, any *geometrically nondegenerate* subvariety V such that $\mathscr{I}_V(\Theta)$ is weakly M-regular satisfies the hypotheses of the theorem.

PROOF. The complex $\mathscr{F}^\bullet = \mathbf{R}\Delta(\mathscr{I}_V(\Theta))$ satifies WIT_n by Theorem 3.1, hence $R^i\widehat{\mathscr{S}\mathscr{F}^\bullet} = 0$ for $i \neq n$, while, by Proposition 4.1,

$$\operatorname{codim}_{\widehat{X}}\big(\operatorname{Supp}(R^n\widehat{\mathscr{S}}(\mathscr{F}^\bullet))\big) = \operatorname{codim}_{\widehat{X}}(\widehat{T}(V)) = d+1.$$

So $\mathscr{F}^\bullet$ is certainly not weakly M-regular, but an extension of Theorem 3.1, for which we refer to [**PP3**], still implies $R^i\widehat{\mathscr{S}}(\mathbf{R}\Delta(\mathscr{F}^\bullet)) = 0$ for $i < d+1$, or equivalently $R^i\widehat{\mathscr{S}}(\mathscr{I}_V(\Theta)) = 0$ for $i < d+1$.

As we have noted above in the proof of Proposition 4.3, we have $R^i\widehat{\mathscr{S}}(\mathscr{I}_V(\Theta)) = 0$ for $i > d+1$. Using the isomorphism (12), we obtain

$$\mathscr{E}xt^i_{\mathscr{O}_{\widehat{X}}}(\mathscr{O}_{\widehat{T}(V)}, \mathscr{O}_{\widehat{X}}) \neq 0 \iff i = d+1, \tag{13}$$

which implies that $T(V)$ is Cohen-Macaulay of pure codimension $d+1$.

This proves the first point. For the other two points, we need the following.

LEMMA 4.5. *If $\mathscr{I}_V(\Theta)$ is weakly M-regular, $\mathbf{R}\widehat{\Delta}(\mathscr{I}_{\widehat{T}(V)}(\widehat{\Theta}))$ satisfies WIT_n, and*

$$R^n\mathscr{S}\big(\mathbf{R}\widehat{\Delta}(\mathscr{I}_{\widehat{T}(V)}(\widehat{\Theta}))\big) \simeq (-\mathbf{1}_X)^*\mathscr{O}_V(\Theta).$$

In other words, we have $\mathbf{R}\mathscr{S}\big(\mathbf{R}\widehat{\Delta}(\mathscr{I}_{\widehat{T}(V)}(\widehat{\Theta}))\big) \simeq (-\mathbf{1}_X)^*\mathscr{O}_V(\Theta)[-n]$.

PROOF. By Mukai's inversion result (Theorem 2.1 of Chapter 3), the lemma is equivalent to

$$\mathbf{R}\widehat{\mathscr{S}}(\mathscr{O}_V(\Theta)) \simeq \mathbf{R}\widehat{\Delta}(\mathscr{I}_{\widehat{T}(V)}(\widehat{\Theta})).$$

Because of the exact sequence (11) and its analogue for $T(V)$, this is in turn equivalent to

$$\mathbf{R}\widehat{\mathscr{S}}(\mathscr{I}_V(\Theta)) \simeq \mathbf{R}\widehat{\Delta}(\mathscr{O}_{\widehat{T}(V)}(\widehat{\Theta})),$$

which is (10). □

By Theorem 3.1, the lemma means that $\mathscr{I}_{\widehat{T}(V)}(\widehat{\Theta})$ is weakly M-regular. Moreover, by Proposition 4.1, the support of the Mukai transform of $\mathbf{R}\widehat{\Delta}(\mathscr{I}_{\widehat{T}(V)}(\widehat{\Theta}))$ is $-\widehat{T}(\widehat{T}(V))$, hence $-\widehat{T}(\widehat{T}(V)) = -V$ by the lemma again.

Since $\widehat{T}(\widehat{T}(V)) = V$ has dimension $n - \dim(\widehat{T}(V)) - 1$, everything we did for V applies to $\widehat{T}(V)$. In particular, $V = \widehat{T}(\widehat{T}(V))$ is Cohen-Macaulay. □

5. Minimal cohomology classes

THEOREM 5.1. *If V is a geometrically nondegenerate subvariety of dimension d of a principally polarized abelian variety (X, Θ) of dimension n such that $\mathscr{I}_V(\Theta)$ is weakly M-regular, V and $T(V)$ both have minimal class.*

PROOF. Since V is Cohen-Macaulay (Theorem 4.4), it has a dualizing sheaf $\omega_V \simeq \mathbf{R}\Delta(\mathscr{O}_V)[n-d]$. Equation (10) for $\widehat{T}(V)$ reads

$$\mathbf{R}\mathscr{S}(\mathscr{I}_{\widehat{T}(V)}(\widehat{\Theta})) \simeq \mathbf{R}\Delta(\mathscr{O}_V(\Theta)) \simeq \omega_V(-\Theta)[d-n]$$

hence by Mukai's inversion theorem,

$$\mathbf{R}\widehat{\mathscr{S}}(\omega_V(-\Theta)) \simeq (-\mathbf{1}_{\widehat{X}})^* \mathscr{I}_{\widehat{T}(V)}(\widehat{\Theta})[d].$$

In other words, $\omega_V(-\Theta)$ satisfies WIT_d and $\widehat{\omega_V(-\Theta)} \simeq (-\mathbf{1}_{\widehat{X}})^* \mathscr{I}_{\widehat{T}(V)}(\widehat{\Theta})$.

We now use the relation

$$\mathrm{ch}_i(\widehat{\mathscr{F}}) = (-1)^{i+j} PD_{2n-2i}(\mathrm{ch}_{n-i}(\mathscr{F}))$$

in cohomology between the Chern character of a sheaf $\mathscr{F}$ on X satisfying WIT_j and that of its Mukai transform, established in [**M2**], Corollary 1.18, where PD denotes the Poincaré duality isomorphism and the components of the Chern character are indexed by the codimension. In our situation, we obtain

$$\mathrm{ch}_{d+1}(\mathscr{I}_{T(V)}(\Theta)) = (-1)^{n+1} PD(\mathrm{ch}_{n-d-1}((-\mathbf{1}_X)^* \omega_V(-\Theta))).$$

We now compute both sides. On the one hand, since the support of $\omega_V(-\Theta)$ has dimension d, the right-hand-side vanishes. On the other hand

$$\mathrm{ch}_{d+1}(\mathscr{I}_{T(V)}(\Theta)) = \mathrm{ch}_{d+1}(\mathscr{O}_X(\Theta)) - \mathrm{ch}_{d+1}(\mathscr{O}_{T(V)}(\Theta)) = \theta_{d+1} - [T(V)],$$

hence the theorem. □

6. A conjecture

As explained in the introduction to this chapter, in view of Example 2.3, it is natural to make the following conjecture.

Conjecture 6.1. Let (X, Θ) be an indecomposable principally polarized abelian variety of dimension n and let V be a subvariety of X of dimension $d \leq n-2$. The following properties are equivalent:

(i) V has minimal cohomology class;
(ii) V is geometrically nondegenerate and $\mathscr{I}_V(\Theta)$ is weakly M-regular.

The implication (ii) $\Rightarrow$ (i) follows from Theorem 5.1.

The conjecture holds for $n = 4$ (because all subvarieties with minimal classes are known in this dimension ([**R**])) and for $d = 1$ by Matsusaka's criterion. It also holds if (X, Θ) is the Jacobian of a curve or the intermediate Jacobian of a general cubic threefold (because again, all subvarieties with minimal classes are known (see §2 of Chapter 2 and Example 2.3)).

If true, Conjecture 6.1 has an amusing consequence relative to Conjecture 1.6 in Chapter 2. If V has minimal class, $\mathscr{I}_V(\Theta)$ is weakly M-regular by Conjecture 6.1 hence, by Theorem 5.1, $T(V)$ also has minimal class. In particular, $\mathscr{C}_{n,d} = \mathscr{C}_{n,n+1-d}$. Since $\mathscr{C}_{n,n-1} = \mathscr{J}_n$ by Matsusaka's criterion, this would imply Conjecture 1.6 for $d = 2$ and all n.

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