On the geometry of certain Fano threefolds

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A Fano manifold is a complex projective manifold $X$ with $\det(T_X)$ ample, i.e., $\det(T_X)$ has a Hermitian metric with positive curvature.

\footnote{Gino Fano (Mantova, 1871 – Verona, 1952).}
A *Fano*\(^1\) manifold is a complex projective manifold \(X\) with \(\text{det}(T_X)\) ample, i.e., \(\text{det}(T_X)\) has a Hermitian metric with positive curvature.

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- Dimension 3: all classified (17 + 88 families).

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A Fano\textsuperscript{1} manifold is a complex projective manifold $X$ with $\det(T_X)$ ample, i.e., $\det(T_X)$ has a Hermitian metric with positive curvature.

- Dimension 1: $\mathbb{P}^1$ (1 family).
- Dimension 2 (del Pezzo\textsuperscript{2} surfaces): $\mathbb{P}^2$ blown-up in at most 8 points, or $\mathbb{P}^1 \times \mathbb{P}^1$ (10 families).
- Dimension 3: all classified ($17 + 88$ families).
- Dimension $n$: finitely many families.

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Examples

The projective space $\mathbb{P}^n$. Submanifolds in $\mathbb{P}^n + s$ defined by (general) homogeneous equations of degrees $d_1, \ldots, d_s > 1$ with $d_1 + \cdots + d_s \leq n + s$, such as a cubic surface $X_3 \subset \mathbb{P}^3$; a cubic threefold $X_3 \subset \mathbb{P}^4$.

The threefold $X_{10}$ intersection in $\mathbb{P}(\wedge^2 C_5) = \mathbb{P}^9$ of the Grassmannian $G(2,5)$, a quadric, and a $\mathbb{P}^7$.
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  - A cubic surface \( X_3 \subset \mathbb{P}^3 \);
  - A cubic threefold \( X_3 \subset \mathbb{P}^4 \);
  - The threefold \( X_{10} \) intersection in \( \mathbb{P}(\bigwedge^2 \mathbb{C}^5) = \mathbb{P}^9 \) of the Grassmannian \( G(2,5) \), a quadric, and a \( \mathbb{P}^7 \).
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Rationality

X is rational if a (Zariski) open subset of X is isomorphic to an open subset of a \( \mathbb{P}^n \), i.e., there exists a birational isomorphism \( \mathbb{P}^n \cong X \).

Equivalently, the field \( \mathbb{C}(X) \) of rational functions on X is a purely transcendental extension \( \mathbb{C}(t_1, \ldots, t_n) \) of \( \mathbb{C} \).

Or, (a dense open subset of) X can be parametrized in a one-to-one way by rational functions in \( t_1, \ldots, t_n \).
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Rationality of cubic surfaces

A (smooth) cubic surface in $\mathbb{P}^3$ is rational.\(^\text{2}\)

\(^\text{2}\)Rudolf Friedrich Alfred Clebsch (Königsberg, Germany (now Kaliningrad, Russia), 1833 – Göttingen, 1872).
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Proof. A smooth cubic surface $X \subset \mathbb{P}^3$ contains skew lines $\ell$ and $\ell'$. 

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- dominant and $\ell \times \ell' \simrightarrow P^2$;
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is

- dominant and $\ell \times \ell' \sim \mathbb{P}^2$;
- birational: the only preimage of $x'' \in X$ is

$$x = \langle \ell', x'' \rangle \cap \ell, \quad x' = \langle \ell, x'' \rangle \cap \ell'.$$
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Unirationality

Unirationality: $X$ is unirational if there exists a dominant rational map $\mathbb{P}^n \rightarrow X$. Equivalently, the field $\mathbb{C}(X)$ is contained in a purely transcendental extension $\mathbb{C}(t_1,\ldots,t_n)$ of $\mathbb{C}$. Or, (a dense open subset of) $X$ can be parametrized in a finite-to-one way by rational functions in $t_1,\ldots,t_n$. 

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- For curves and surfaces, rationality is equivalent to unirationality.\(^1\)

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- For curves and surfaces, rationality is equivalent to unirationality.\(^1\)
- A (smooth) cubic hypersurface in $\mathbb{P}^{n+1}$ is unirational for $n \geq 2$.\(^2\)

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$$\mathbb{P}(T_x|\ell) \rightarrow X$$

$$\ell_x \subset T_{X,x} \quad \mapsto \quad \text{third point of intersection of } \ell_x \text{ with } X$$

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- dominant and $\mathbb{P}(T_X|_\ell) \sim \mathbb{P}^n$;
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- of degree 2: if $x' \in X$, the intersection of $X$ with the plane $\langle \ell, x' \rangle$ is the union of $\ell$ and a conic $c$. The two preimages of $x'$ are the two points of $\ell \cap c$. 

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Some (smooth) cubic hypersurfaces in $\mathbb{P}^n$, $n \geq 5$, such as

$$(x_0^3 + \cdots + x_{2m+1}^3 = 0) \subset \mathbb{P}^{2m+1}$$

are rational (odd-dimensional examples were constructed by M. Mella)...
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are rational (odd-dimensional examples were constructed by M. Mella)...

...but the rationality of general cubics is unknown!
Jacobians

This is the abelian variety $J(C) = \mathbb{H}^1(C, \mathbb{Z}) = (\text{g-dim'l vector space}) / (\text{lattice})$.

Natural target for integrating:
Jacobian of a curve $C$. 

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\[ C \rightarrow J(C) \quad \text{Abel–Jacobi}^1 \text{ map} \]

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$$x \mapsto \left( \omega \mapsto \int_{x_0}^x \omega \right) \quad (\text{well-defined modulo } H_1(C, \mathbb{Z}))$$

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$$C \longrightarrow J(C)$$

$$x \longmapsto \left(\omega \longmapsto \int_{x_0}^x \omega\right)$$

(well-defined modulo $H_1(C, \Z)$)

($x_0$ fixed point of $C$.)

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Intermediate Jacobians

Intermediate Jacobian of a Fano threefold $X$:

$$J(X) = H^2,1(X) \vee \text{Im} H^3(X, Z)$$

These are principally polarized abelian varieties: they contain a hypersurface $\Theta$ uniquely defined up to translation. The pair $(J(X), \Theta)$ carries information about $X$.

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Intermediate Jacobian of a Fano threefold $X$: $J(X) = H^2,1(X) \cap \text{Im} H^3(X,\mathbb{Z})$. These are principally polarized abelian varieties: they contain a hypersurface $\Theta$ uniquely defined up to translation. The pair $(J(X), \Theta)$ carries information about $X$. 

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If $X$ is rational, we get, by Hironaka's resolution of indeterminacies, $\tilde{P}^3$ composition of blow-ups of points and of smooth curves $C_1, \ldots, C_r$.

\[ \tilde{P}^3 \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow 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If $X$ is rational, we get, by Hironaka’s resolution of indeterminacies,
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$$\widetilde{\mathbb{P}^3} \xrightarrow{\text{composition of blow-ups of points and of smooth curves } C_1, \ldots, C_r} \mathbb{P}^3 \rightarrow X$$
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$J(\tilde{\mathbb{P}}^3) \simeq J(C_1) \times \cdots \times J(C_r)$;
If $X$ is rational, we get, by Hironaka's resolution of indeterminacies,

\[ \widetilde{\mathbb{P}^3} \leftarrow \mathbb{P}^3 \leftarrow X \]

composition of blow-ups of points and of smooth curves $C_1, \ldots, C_r$

- $J(\widetilde{\mathbb{P}^3}) \simeq J(C_1) \times \cdots \times J(C_r)$;
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If $X$ is rational, we get, by Hironaka’s resolution of indeterminacies,

$$
\begin{array}{c}
\widetilde{\mathbb{P}}^3 \\
\downarrow \\
\mathbb{P}^3 \rightarrow X
\end{array}
$$

composition of blow-ups of points and of smooth curves $C_1, \ldots, C_r$

- $J(\widetilde{\mathbb{P}}^3) \simeq J(C_1) \times \cdots \times J(C_r)$;
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- hence

$$
codim_{J(X)} \text{Sing}(\Theta) \leq 4
$$
Back to a cubic threefold $X_3 \subset \mathbb{P}^4$. 
Back to a cubic threefold $X_3 \subset \mathbb{P}^4$. Projection from a line $\ell \subset X_3$ yields
Back to a cubic threefold $X_3 \subset P^4$. Projection from a line $\ell \subset X_3$ yields

\[ \xymatrix{ X_3 \ar@{-->}[r] & P^2 \ar[d] \ar[r] & \tilde{C} \ar[d] \ar[r] & \tilde{X}_3 \ar@{-->}[ld] \ar[d] & \tilde{X}_3 \ar@{-->}[l] \ar@{-->}[d] \ar@{-->}[r] & \tilde{X}_3 \ar@{-->}[ld] } \]

- $\tilde{C}$ double étale cover
- $\tilde{C}$ discriminant curve
- $J(X_3)$ has dimension 5;
Back to a cubic threefold $X_3 \subset \mathbb{P}^4$. Projection from a line $\ell \subset X_3$ yields

- Blow-up of $\ell$
- Conic bundle
- Double étale cover
- Discriminant curve

$J(X_3)$ has dimension 5;
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---

$^1$Friedrich Emil Prym (Düren, Germany, 1841 – Bonn, 1915).
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$$\xymatrix{ X_3 \ar[r] \ar@{-->}[dr] & \mathbb{P}^2 \ar[d] \ar[dl] \ar[r] & \tilde{C} \ar[d] \ar@{-->}[dl] \\
& \text{conic bundle} & \text{double \ étale \ cover} \\
\text{blow-up of } \ell & \tilde{X}_3 & C \text{ discriminant curve}
}$$

- $J(X_3)$ has dimension 5;
- $J(X_3)$ is isomorphic to the Prym\(^1\) variety of the cover $\tilde{C} \rightarrow C$;
- $\Theta$ has a unique singular point $o$, which is a triple point (Beauville);

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Nonrationality of cubic threefolds

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- $J(X_3)$ has dimension 5;
- $J(X_3)$ is isomorphic to the Prym\(^1\) variety of the cover $\tilde{C} \to C$;
- $\Theta$ has a unique singular point $o$, which is a triple point (Beauville);
- $X_3$ is \underline{not} rational (because $\text{codim} J(X_3) \text{Sing}(\Theta) = 5$).

\(^1\)Friedrich Emil Prym (Düren, Germany, 1841 – Bonn, 1915).
The Abel–Jacobi map

$X$ Fano threefold.
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Geometric information about $(J(X), \Theta)$ is in general hard to get, in particular when $X$ is *not* a conic bundle as before.
The Abel–Jacobi map

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Some information can be obtained from families of curves on $X$:

- $(C_t)_{t \in T}$ (connected) family of curves on $X$, base-point $0 \in T$;
The Abel–Jacobi map

\( \mathcal{X} \) Fano threefold.

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Some information can be obtained from families of curves on \( \mathcal{X} \):

- \( (C_t)_{t \in T} \) (connected) family of curves on \( \mathcal{X} \), base-point \( 0 \in T \);
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- $(C_t)_{t \in T}$ (connected) family of curves on $X$, base-point $0 \in T$;
- $C_t$ is algebraically, hence homologically, equivalent to $C_0$, so $C_t - C_0$ is the boundary of a (real) 3-chain $\Gamma_t$, defined modulo $H_3(X, \mathbb{Z})$;
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- define the *Abel–Jacobi map*

\[
T \longrightarrow J(X) = H^{2,1}(X)^{\vee} / H_3(X, \mathbb{Z})
\]

\[
t \longmapsto \left( \omega \longmapsto \int_{\Gamma_t} \omega \right)
\]
The Torelli problem

Lines on a cubic threefold $X^3 \subset P^4$ are parametrized by a smooth surface $F$. The Abel–Jacobi map $F \to J(X^3)$ is an embedding, and $\Theta = F - F$. We recover Beauville's result: $o$ is a singular point of $\Theta$. Moreover, the projectified tangent cone to $\Theta$ at $o$ is isomorphic to $X^3$: $X^3 \cong P(TC_\Theta, o) \subset P(TJ(X^3), o) \cong P^4$. In particular, $X^3$ can be reconstructed from its intermediate Jacobian. This is the Torelli theorem for cubic threefolds.
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The Torelli problem

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We recover Beauville’s result: $o$ is a singular point of $\Theta$. Moreover, the projectified tangent cone to $\Theta$ at $o$ is isomorphic to $X_3$:

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This is the Torelli theorem for cubic threefolds.\textsuperscript{1}

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\textsuperscript{1}Ruggiero Torelli (Napoli 1884 – Isonzo, 1915) showed that a curve $C$ is uniquely determined by its Jacobian $(J(C), \Theta)$. 

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Olivier DEBARRE | On the geometry of certain Fano threefolds
Joint work in progress with A. Iliev (Sofia) and L. Manivel (Grenoble).
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Most Fano threefolds of the first kind of degree $c_1^3 = 10$ are obtained as follows:
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- $V$ 5-dimensional complex vector space;
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Most Fano threefolds of the first kind of degree $c_1^3 = 10$ are obtained as follows:

- $\mathcal{V}$ 5-dimensional complex vector space;
- $G(2, \mathcal{V}) \subset \mathbb{P}(\wedge^2 \mathcal{V}) = \mathbb{P}^9$ the Plücker embedding;

\footnote{Julius Plücker (Elberfeld, 1801 – Bonn, 1868).}
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Most Fano threefolds of the first kind of degree $c_1^3 = 10$ are obtained as follows:

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- $G(2, V) \subset \mathbb{P}(\wedge^2 V) = \mathbb{P}^9$ the Plücker\(^3\) embedding;
- $\mathbb{P}^7 \subset \mathbb{P}(\wedge^2 V)$ general codimension-2 linear space;

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- $G(2, V) \cap P^7$ is a smooth 4-fold of degree 5 (independent, modulo the action of $PGL(V)$, of the choice of $P^7$);

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- $\mathbb{P}^7 \subset \mathbb{P}(\wedge^2 V)$ general codimension-2 linear space;
- $G(2, V) \cap \mathbb{P}^7$ is a smooth 4-fold of degree 5 (independent, modulo the action of PGL($V$), of the choice of $\mathbb{P}^7$);
- $X_{10} = G(2, V) \cap \mathbb{P}^7 \cap \Omega$, where $\Omega$ is a (general) quadric.

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All $X_{10}$ are *unirational*. 
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Some (smooth) degenerations $X'$ of $X_{10}$ are conic bundles for which (by Prym theory) $\text{codim}_{J(X')} \text{Sing}(\Theta) > 4$. 
All $X_{10}$ are unirational.

Some (smooth) degenerations $X'$ of $X_{10}$ are conic bundles for which (by Prym theory) $\dim J(X') \text{Sing}(\Theta) > 4$.

This implies the same for a general $X_{10}$, which is therefore not rational.
For a cubic threefold $X_3 \subset \mathbb{P}^4$, we used lines to parametrize (via the Abel–Jacobi map) the theta divisor of $J(X_3)$. Conics on $X_{10}$ are parametrized by a smooth connected surface $F(X)$. The surface $F(X)$ is the blow-up at one point of a minimal surface of general type $F_m(X)$. 
For a cubic threefold $X_3 \subset \mathbb{P}^4$, we used lines to parametrize (via the Abel–Jacobi map) the theta divisor of $J(X_3)$.

For $X_{10}$, we will use conics to disprove the Torelli theorem.
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The surface $F(X)$ is the blow-up at one point of a minimal surface of general type $F_m(X)$. 
Elementary transformation along a conic

Let $c$ be a general conic contained in $X$. 
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\[ Y \xrightarrow{\varphi} \bar{Y} \quad \text{quadric in } \mathbb{P}^4 \]

\[ c \leftrightarrow \text{projection from } \langle c \rangle \]

\[ \text{blow-up of } c \quad \varepsilon \]

\[ X \]
Let $c$ be a general conic contained in $X$. Consider

The only curves contracted by $\varphi$ are:

- the (strict transforms of the 20) lines in $X$ that meet $c$;
- (the strict transform of) a conic $\iota(c)$ that meets $c$ in 2 points.

It is a small contraction that can be flopped: $\chi: Y \xrightarrow{\varphi} \bar{Y}$ quartic in $\mathbb{P}^4$, projection from $\langle c \rangle$.
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Elementary transformation along a conic

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\[
\begin{array}{c}
\text{Y} \\
\downarrow \\
\text{X}
\end{array}
\xrightarrow{\varepsilon} \begin{array}{c}
\text{blow-up of } c \\
\downarrow \\
\text{X}
\end{array}
\xrightarrow{\varphi} \begin{array}{c}
\bar{Y} \\
\downarrow \\
\text{quartic in } \mathbb{P}^4
\end{array}
\xrightarrow{\pi_c} \begin{array}{c}
\text{projection from } \langle c \rangle
\end{array}
\]

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The only curves contracted by $\varphi$ are:
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It is a *small contraction* that can be *flopped*:

$$\chi : Y \xrightarrow{\varphi} \bar{Y} \xleftarrow{\varphi'} Y'$$ smooth projective threefold

(isomorphism outside curves contracted by $\varphi$ or $\varphi'$).
Elementary transformation along a conic

We obtain:

\[ Y \xrightarrow{\varphi} \bar{Y} \xrightarrow{\psi_c} X \xrightarrow{\pi_c} X' = X_c \]

where

\[ Y' \xrightarrow{\varphi'} \bar{Y} \xrightarrow{\psi_c} X \xrightarrow{\pi_c} X' = X_c \]

\[ \varepsilon \xrightarrow{} \varepsilon' \]

\[ Y' \xrightarrow{\varphi'} \bar{Y} \xrightarrow{\psi_c} X \xrightarrow{\pi_c} X' = X_c \]

where

\[ X' \text{ is again a smooth Fano threefold of degree } 10 \text{ in } P^7; \]

\[ \varepsilon' \text{ is the blow-up of a smooth conic } c' \text{ in } X'; \]

the picture is symmetric:

\[ \psi_c' = \psi_c - 1 \]

\[ X' = X_c \]

Are \( X \) and \( X' \) isomorphic?
Elementary transformation along a conic

We obtain:

\[
\begin{array}{c}
Y \\ \downarrow \varphi \\ \downarrow \varepsilon \\
\rightarrow \bar{Y} \\
X \\
\end{array} \quad \chi \quad \begin{array}{c}
Y' \\ \downarrow \varphi' \\ \downarrow \varepsilon' \\
\rightarrow \bar{Y} \\
X' = X_c \\
\end{array}
\]

where

- \( X' \) is again a smooth Fano threefold of degree 10 in \( \mathbb{P}^7 \);
Elementary transformation along a conic

We obtain:

\[
\begin{array}{ccc}
Y & \xrightarrow{\chi} & Y' \\
\downarrow^{\varepsilon} & & \downarrow^{\varepsilon'} \\
\bar{Y} & \xleftarrow{\psi_c} & X' = X_c \\
\uparrow^{\varphi} & & \uparrow^{\varphi'} \\
X & \xrightarrow{\pi_c} & \bar{Y} & \xleftarrow{\pi_{c'}} \downarrow & X' = X_c \\
\end{array}
\]

where

- \(X'\) is again a smooth Fano threefold of degree 10 in \(\mathbb{P}^7\);
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Elementary transformation along a conic

We obtain:

\[
\begin{array}{c}
Y \xrightarrow{\varphi} \bar{Y} \xleftarrow{\pi_c} X \\
\epsilon \downarrow \quad \epsilon' \downarrow \\
Y' \xrightarrow{\varphi'} \bar{Y} \xleftarrow{\pi_{c'}} X' = X_c
\end{array}
\]

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- \(X'\) is again a smooth Fano threefold of degree 10 in \(\mathbb{P}^7\);
- \(\epsilon'\) is the blow-up of a smooth conic \(c'\) in \(X'\);
- the picture is symmetric: \(\psi_{c'} = \psi_c^{-1} : X' \to X\);
Elementary transformation along a conic

We obtain:

\[
\begin{array}{ccccccccc}
Y & \xrightarrow{\chi} & Y' \\
\downarrow & & \uparrow \\
\varepsilon & & \phi & & \psi & & \chi & & \varepsilon' \\
\downarrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow \\
X & \xrightarrow{\pi_c} & \bar{Y} & \xleftarrow{\pi_{c'}} & X' = X_c
\end{array}
\]

where

- \(X'\) is again a smooth Fano threefold of degree 10 in \(\mathbf{P}^7\);
- \(\varepsilon'\) is the blow-up of a smooth conic \(c'\) in \(X'\);
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- the intermediate Jacobians of \(X\) and \(X'\) are isomorphic.
Elementary transformation along a conic

We obtain:

\[
\begin{array}{c}
Y \xrightarrow{\varphi} \tilde{Y} \xrightarrow{\pi_{c'}} \bar{X} \xrightarrow{\psi_c} X' = X_c \\
\downarrow \varepsilon \quad \downarrow \varepsilon' \\
\end{array}
\]

where

- \(X'\) is again a smooth Fano threefold of degree 10 in \(\mathbb{P}^7\);
- \(\varepsilon'\) is the blow-up of a smooth conic \(c'\) in \(X'\);
- the picture is symmetric: \(\psi_{c'} = \psi_{c}^{-1} : X' \rightarrow X\);
- the intermediate Jacobians of \(X\) and \(X'\) are isomorphic.

Are \(X\) and \(X'\) isomorphic?
To answer this question (negatively), we study $F(X')$. 

Since the automorphism group of a minimal surface of general type is finite, we have a 2-dim'l family of $X_{10}$ with isomorphic intermediate Jacobians.
To answer this question (negatively), we study $F(X')$. We construct

$$\varphi_c : F(X) \simrightarrow F(X')$$
The Torelli problem for $X_{10}$

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- for $\bar{c} \in F(X)$ general, $\langle c, \bar{c} \rangle$ is a 5-plane in $\mathbb{P}^7$;
- $X \cap \langle c, \bar{c} \rangle$ is a canonically embedded genus-6 curve $c + \bar{c} + \Gamma_{c,\bar{c}}$;
The Torelli problem for $X_{10}$

To answer this question (negatively), we study $F(X')$. We construct

$$\varphi_c : F(X) \sim F(X')$$

- for $\bar{c} \in F(X)$ general, $\langle c, \bar{c} \rangle$ is a 5-plane in $\mathbb{P}^7$;
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The Torelli problem for $X_{10}$

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Olivier DEBARRE | On the geometry of certain Fano threefolds
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and $F(X')$ is isomorphic to the surface $F_m(X)$ blown up at the point $c$. Since the automorphism group of a minimal surface of general type is finite, we have a 2-dim’l family of $X_{10}$ with isomorphic intermediate Jacobians.
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The transforms $(X_\ell)_c$, for $c \in F(X_\ell)$, again form a 2-dimensional family of Fano threefolds with same intermediate Jacobian as $X$. 

The Torelli problem for $X_{10}$

We expect this family to be distinct from the first family obtained earlier, and that these two families should yield all Fano threefolds with the same intermediate Jacobians as $X_{10}$, thereby describing the (general) fibers of the period map $\{22\text{-dim'l family of Fano threefolds } X_{10}\} \to \{55\text{-dim'l family of p.p.a.v. of dimension 10}\}$. (General) fibers should be the union of two surfaces.
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\begin{array}{c}
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Related problems

Hopeless to classify all Fano varieties of dimension $\geq 4$. For Mori's program, need Fano varieties with mild singularities. Full classification is harder, but many interesting examples occur and other problems such as smoothing are also interesting.

Relax condition $\det(T_X)$ ample to $\det(T_X)$ big and nef (not all are degenerations of Fano varieties).

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