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Zhi JIANG

Sur l'application d'Albanese des variétés algébriques et le cône nef des produits symétriques de courbes

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dirigé par Olivier DEBARRE

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M. Frédéric Campana	Université Nancy I	Rapporteur
M. Olivier Debarre	Université Paris VII, ENS Paris	Directeur de thèse
M. Stéphane Druel	Université Grenoble I	Examinateur
Yujiro Kawamata	Université de Tokyo	Examinateur
Mihnea Popa	Université d'Illinois	Examinateur
Claire Voisin	Université Paris VI	Examinateur
Rapporteur non présent à la soutenance:		
Robert Lazarsfeld	Université de Michigan	

Résumé

Cette thèse se compose de deux parties indépendant.

Dans la première partie, j'étudie les variétés irrégulières et en particulier les variétés de dimension d'Albanese maximale. Pour une variété X générale irrégulière, je donne une condition optimale sur les plurigenres $P_m(X)$ pour que le morphisme d'Albanese soit surjectif et j'obtiens aussi une condition (plus restrictive) toujours optimale sur $P_m(X)$ pour que le morphisme d'Albanese soit un espace fibré algébrique. Pour une variété de dimension d'Albanese maximale, avec quelques hypothèses supplémentaires sur $P_m(X)$ et $q(X)$, je décris (birationnellement) sa structure géométrique. Puis j'étudie les morphismes entre les variétés de dimension d'Albanese maximale. Je fais aussi une remarque sur un travail de Chen et Hacon (Pareschi et Popa) pour montrer que, pour une variété de dimension d'Albanese maximale, $|6K_X|$ induit un modèle de sa fibration d'Iitaka.

Dans la seconde partie, j'étudie un problème très concret: la structure du cône nef du produit symétrique d'une courbe générique. Il y a un théorème intéressant de Kouvidakis sur ce problème. J'utilise une approche par dégénérescence pour étudier ce problème. L'ingrédient principal est une idée de Ein et Lazarsfeld qu'ils ont utilisée pour étudier les constantes de Seshadri. J'améliore le théorème de Kouvidakis.

Abstract

This thesis consists of two independent parts.

In the first part, I study irregular varieties and in particular, varieties with maximal Albanese dimension. For a general irregular variety X, I give an optimal condition on the plurigenera $P_m(X)$ such that the Albanese map should be surjective and I also obtain a (more restrictive) still optimal condition on $P_m(X)$ such that the Albanese map should be an algebraic fiber space. For a variety X of maximal Albanese dimension with some additional assumptions on $P_m(X)$ and $q(X)$, I describe (birationally) its geometry structure. Then I study morphisms between varieties of maxiaml Albanese dimension. I also make a remark about a work of Chen and Hacon (Pareschi and Popa) to show that for a varieties of maximal Albanese dimension, $|6K_X|$ induces a model of its Iitaka fibration.

In the second part, I study a very concrete problem: the structure of the

nef cone of the symmetric product of a generic curve. There is an interesting theorem of Kouvidakis about this problem. I use a degeneration approach to study this problem. The ingredient is an idea due to Ein and Lazarsfeld which they used to study the Seshadri constants of surfaces. I can improve Kouvidakis'result.

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Contents

1 Around the Albanese maps

1.1 Introduction

1.1.1 Notation and conventions

We work throughout over the complex numbers $\mathbb{C}$.

Definition 1.1 *Let X be a normal irreducible projective variety and L a line bundle on X. The Iitaka dimension of L is defined to be*

$$\kappa(L) = \kappa(X, L) = \limsup_{m\to\infty} \frac{\log h^0(X, L^m)}{\log m},$$

provided there is an $m > 0$ such that $H^0(X, L^m) \neq 0$. If $H^0(X, L^m) = 0$ for all $m > 0$, we set $\kappa(X, L) = -\infty$.

Definition 1.2 *Let X be a smooth projective variety. Then,*

1) *we denote by $P_m(X) = h^0(X, mK_X)$ the m-th plurigenera of X, $q(X) = h^1(X, \mathcal{O}_X)$ the irregularity of X, and $\kappa(X) = \kappa(X, K_X)$ the Kodaira dimension of X.*

2) *there exists a morphism $a_X : X \to \mathrm{Alb}(X)$ from X to an abelian variety called the Albanese morphism of X satisfying the following universal property: for any morphism $f : X \to A$ from X to an abelian variety A, there exists a morphism $g : \mathrm{Alb}(X) \to A$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "a_X"] \arrow[dr, "f"'] & \mathrm{Alb}(X) \arrow[d, "g"] \\ & A. \end{tikzcd}$$

3) *we call a morphism $f : X \to Y$ from X to a smooth projective variety Y an algebraic fiber space, if f is surjective and $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

Remark 1.3 It is well known that $P_m(X)$, $\kappa(X)$, and $q(X)$ are all birational invariants.

Let X be a normal projective variety and L a line bundle on X. For each $m > 0$ such that $H^0(X, L^m) \neq 0$, we can define a rational map

$$\phi_m : X \dashrightarrow Y_m \subset \mathbb{P}H^0(X, L^m).$$

The following theorem can be found in [La, section 2.1.c]:

Theorem 1.4 *Under the above assumptions, for sufficiently large m such that $H^0(X, L^m) \neq 0$, the rational mappings $\phi_m : X \dashrightarrow Y_m$ are birationally equivalent to a fixed algebraic fiber space*

$$\phi_\infty : X_\infty \to Y_\infty,$$

and the restriction of L to a very general fiber of ϕ_∞ has Iitaka dimension $= 0$. We call $\phi_\infty : X_\infty \to Y_\infty$ the Iitaka fibration of L. It is unique up to birational equivalence.

Remark 1.5 We can define the Iitaka fibration of a $\mathbb{Q}$-divisor D on a smooth projective variety X as the Iitaka fibration of ND on X where N is any integer > 0 such that ND is a divisor on X.

For a surjective morphism $f : X \to Y$ between smooth projective varieties and a $\mathbb{Q}$-divisor D on X, I will say that the Iitaka fibration of (X, D) dominates Y if there exists an integer $N > 0$ and an ample divisor H on Y such that $ND - f^*H$ is an effective divisor.

1.1.2 Main results

In [Ka], Kawamata proved a fundamental theorem:

Theorem 1.6 (Kawamata) *Let X be a smooth projective variety of Kodaira dimension 0. Then the Albanese morphism $a_X : X \to \mathrm{Alb}(X)$ is an algebraic fiber space.*

However, $\kappa(X) = 0$ is not an effective condition and it is natural to try to prove the same result with weaker and effective assumptions. There are already several generalizations. For instance, Ein and Lazarsfeld in [EL1] (Proposition 2.1 and Proposition 2.2) prove that if $P_1(X) = P_2(X) = 1$, the Albanese map is surjective. In the same paper they also give a very simple proof of Theorem 1.6. I briefly review their beautiful proof.

If $\kappa(X) = 0$, by a standard covering argument, there exists a generically finite cover $\pi : Y \to X$ such that $\kappa(Y) = 0$ and $P_1(Y) = 1$. Since $P_1(Y) = P_2(Y) = 1$, the map a_Y is surjective. Hence a_X is also surjective.

Let $X \xrightarrow{g_X} V \xrightarrow{b} \mathrm{Alb}(X)$ be the Stein factorization of the Albanese map. After resolution of singularities we can suppose that V is smooth. If the Albanese map a_X has nonconnected fibers, since it does not factor through a nontrivial isogeny, b is not étale and by a result of Kollár, we have $P_4(V) \geq 2$ ([Kol1] 11.1). Let $g = g_X \circ \pi$. The main ingredient of Ein and Lazarsfeld's proof is the following:

Lemma 1.7 (Ein-Lazarsfeld) *In the above situation, there exists $m \geq 0$ such that $mK_Y \succeq g^*K_V$.*

Then $P_{4m}(Y) \geq 2$, which contradicts the equality $\kappa(Y) = 0$. This finishes the proof of Kawamata's Theorem.

If we can estimate m in Lemma 1.7, we will get an effective version of Kawamata's theorem. Following Ein and Lazarsfeld's proof, with the help of a proposition of Chen and Hacon, I prove the following:

Theorem 1.8 (=Theorem 1.42) *Let X be a smooth projective variety. If $P_1(X) = P_2(X) = 1$, the Albanese morphism $a_X : X \to \mathrm{Alb}(X)$ is an algebraic fiber space.*

On the other hand, Kollár in [Kol1] studied the m^{th} plurigenera of algebraic varieties for $m \geq 2$. As a by-product, he proved that the Albanese map is surjective if $P_3 = 1$ or $0 < P_m(X) \leq 2m - 6$ for some $m \geq 4$. Certainly this is an effective criterion. Hacon and Pardini in [HP] strengthened Kollár's result:

Theorem 1.9 (Hacon-Pardini) *Let X be a smooth projective variety. If*

$$0 < P_m(X) \leq 2m - 3$$

for some $m \geq 2$, $a_X : X \to \mathrm{Alb}(X)$ is surjective.

In [CH5], Chen and Hacon prove that the Albanese morphism is surjective if $P_3(X) = 4$, which suggests that the above theorem may not be optimal. In section 1.3, I give an optimal bound for $m \geq 2$.

Theorem 1.10 (=Theorem 1.39) *Let X be a smooth projective variety. If*

$$0 < P_m(X) \leq 2m - 2,$$

for some $m \geq 2$, the Albanese map $a_X : X \to \mathrm{Alb}(X)$ is surjective.

Remark 1.11 Let C be a smooth projective curve of genus 2, then $P_m(C) = 2m-1$, for $m \geq 2$. However $a_C : C \to \mathrm{Alb}(C)$ is not surjective. This example shows that without other assumptions, the above bound is optimal.

The results of Kollár, Chen, and Hacon make one half of Kawamata's theorem effective. We then show in Section 1.4 that there exists a similar criterion for the Albanese morphism to be an algebraic fiber space.

Theorem 1.12 (=Theorem 1.44) *Let X be a smooth projective variety and*

$$0 < P_m(X) \leq m - 2,$$

for some $m \geq 3$, the Albanese map $a_X : X \to \mathrm{Alb}(X)$ is an algebraic fiber space.

Hacon and Pardini in [HP] classify varieties with $P_3(X) = 2$ and $q(X) = \dim(X)$. For such a variety, the Albanese map $a_X : X \to \mathrm{Alb}(X)$ is a double covering. Hence a_X is surjective but does not have connected fibers. Furthermore, $P_m(X) = m-1$ for any odd number $m \geq 3$. From this example, the above criterion is optimal to a large extent.

When we take the Stein factorization of an Albanese morphism, we will get a variety of maximal Albanese dimension. Such varieties have been studied a lot since the classical paper of Ein and Lazarsfeld [EL1]. In particular, several classification results of varieties with $q(X) = \dim X$ and small $P_3(X)$ have been proved in [CH1], [CH5], and [HP]. I will first apply Theorem 1.10 to study varieties with $P_2(X) = 2$ and $q(X) = \dim X$. A similar classification theorem is the following:

Theorem 1.13 (=Theorem 1.66) *Let X be a smooth projective variety with $P_2(X) = 2$ and $q(X) = \dim X$, then $\kappa(X) = 1$ and X is birational to a quotient $(K \times C)/G$, where K is an abelian variety and C is a curve, G is a finite group that acts diagonally and freely on $K \times C$, and $C \to C/G$ is branched at 2 points.*

The main ingredient of the proofs of these classification theorems above is to exhibit the possible Kodaira dimensions of X assuming that X is of maximal Albanese dimension and $P_m(X)$ is small. For instance, if X is of maximal Albanese dimension and $P_3(X) = 4$, Chen and Hacon ([CH5, Theorem 1.2]) proved that $\kappa(X)$ is either 1 or 2. It is a bit surprising that a more general result holds:

Theorem 1.14 (=Theorem 1.61) *Let X be a smooth projective variety with $q(X) = \dim X$ and $0 < P_m(X) \leq 2m - 2$, for some $m \geq 4$. Then $\kappa(X) \leq 1$.*

Another interesting property of varieties of maximal Albanese dimension is that their pluricanonical maps behave like that of surfaces. In particular, Chen and Hacon proved in [CH3] that for a variety of general type and of maximal Albanese dimension, $|6K_X|$ induces a birational map. Pareschi and Popa [PP3] gave an elegant proof of the same result using their theory of GV-sheaves. In section 1.6, we borrow Pareschi and Popa's argument to show that a more general theorem holds true: for a variety X of maximal Albanese dimension, $|6K_X|$ always induces a model of its Iitaka fibration.

It is also interesting to study morphisms between varieties with maximal Albanese dimension. It was first noticed by Hacon and Pardini in [HP, Theorem 3.2] that:

Theorem 1.15 *Let $f : X \to Y$ be a dominant morphism of smooth n-dimensional projective varieties, with Y of maximal Albanese dimension, and let $X \to V$ and $Y \to W$ be the Iitaka fibrations of X, respectively Y. Taking appropriate models, we may assume that f induces a morphism $g : V \to W$. If $P_j(X) = P_j(Y)$ for some $j \geq 2$, the induced morphism g has connected fibers. In particular, if Y is of general type, f is birational.*

Remark 1.16 This theorem shows that the birational geometry for varieties of maximal Albanese dimension is very special. We have the following example. Let C_1 and C_2 be smooth projective curves of genus 2 and let i_1 and i_2 be respectively the hyperelliptic involutions of C_1 and C_2. Define Y to be the minimal resolution of singularities of $(C_1 \times C_2)/(i_1, i_2)$. Let X be the blow-up of $C_1 \times C_2$ at the 36 fixed points of (i_1, i_2). There is a 2 to 1 morphism $f : X \to Y$. We have $K_Y^2 = \frac{1}{2}K_{C\times C}^2 = 4$ and $c_2(Y) = \frac{1}{2}(c_2(C_1 \times C_2) - 36) + 72 = 56$. Since Y is a minimal surface, we have $P_2(Y) = K_Y^2 + \frac{1}{12}(K_Y^2 + c_2(Y)) = 9$. We also have $P_2(X) = P_2(C_1 \times C_2) = 3 \times 3 = 9$. Hence we have a non-birational morphism $f : X \to Y$ between smooth projective surfaces of general type with $q(Y) = 0$ and $P_2(X) = P_2(Y) = 9$.

In their proof, Hacon and Pardini actually did not use the assumption that Y is of maximal Albanese dimension; all they needed was that $P_j(X) = P_j(Y) > 0$ and W is of maximal Albanese dimension. However with their assumption, we have a stronger conclusion:

Theorem 1.17 *Under the assumption of Theorem 1.15, the induced map $g : V \to W$ is birational.*

Finally, by a theorem of Kawamata, we can prove that under the assumptions in Theorem 1.15, the structure of $f : X \to Y$ is actually much more restricted: f is birationally equivalent to a quotient by a finite abelian group (see Theorem 1.68 for more details).

In section 1.7.4, I classify all the non-birational morphisms $f : X \to Y$ between surfaces of general type such that $P_m(X) = P_m(Y)$ for some $m \geq 2$ (see Section 1.7.4).

1.2 Preliminaries

In this section we will recall the main tools (multiplier ideal sheaves, Fourier-Mukai transform, etc.).

1.2.1 Multiplier ideals

The standard reference for multiplier ideals is [La, Part 3]. I will draw the definition and some properties of multiplier ideals from that book.

Definition 1.18
- Multiplier ideal of an effective $\mathbb{Q}$-divisor:
  Let D be an effective $\mathbb{Q}$-divisor on a smooth complex variety X, and fix a log resolution $\mu : X^{'} \to X$ of D. Then the multiplier ideal sheaf is defined to be
$$\mathscr{J}(D) = \mu_*(K_{X'/X} - [\mu^* D]).$$
- Multiplier ideal associated to an ideal sheaf:
  Let $\mathfrak{a} \subset \mathscr{O}_X$ be a non-zero ideal sheaf, and $c > 0$ a rational number. Fix a log resolution $\mu : X^{'} \to X$ of $\mathfrak{a}$ with $\mathfrak{a}\mathscr{O}_{X'} = \mathscr{O}_{X'}(-F)$. The multiplier ideal $\mathscr{J}(c \cdot \mathfrak{a})$ associated to c and $\mathfrak{a}$ is defined as
$$\mathscr{J}(c \cdot \mathfrak{a}) = \mu_*(K_{X'/X} - [c \cdot F]).$$

- Multiplier ideal of a linear series:
  Let $|V| \subseteq |L|$ be a non-empty linear series on X, and let $\mu : X' \to X$ be a log resolution of $|V|$, with $\mu^*|V| = |W| + F$. Given a rational number $c > 0$, the multiplier ideal $\mathscr{J}(c \cdot |V|)$ corresponding to c and $|V|$ is
  $$\mathscr{J}(c \cdot |V|) = \mu_*(K_{X'/X} - [c \cdot F]).$$

Proposition 1.19 *Let L be a divisor on X with $\kappa(X, L) \geq 0$ and fix a rational number $c > 0$. For every integer $k \geq 1$ one has the inclusion*

$$\mathscr{J}(\frac{c}{p} \cdot |pL|) \subseteq \mathscr{J}(\frac{c}{pk} \cdot |pkL|).$$

The ascending chain condition on ideals implies that there is a unique maximal element among the family of ideals $\mathscr{J}(\frac{c}{p} \cdot |pL|)_{p \geq 0}$. We define the asymptotic multiplier ideal sheaf $\mathscr{J}(c \cdot ||L||)$ to be this unique maximal element.

We have the following useful properties for asymptotic multiplier ideal sheaves (see Theorem 11.1.8, Proposition 11.2.10, Theorem 11.2.16 of [La]):

Theorem 1.20 *Let X be a smooth projective variety and let L be an integral divisor of non-negative Iitaka dimension.*

(1) The ideals $\mathscr{J}(||mL||)$ form a decreasing sequence in m, i.e.

$$\mathscr{J}(||mL||) \supset \mathscr{J}(||(m+1)L||).$$

(2) Let $\mathfrak{b}_m \subset \mathscr{O}_X$ be the base ideal of $|mL|$, where by convention we set $\mathfrak{b}_m = (0)$ if $|mL| = \emptyset$. Then

$$\mathfrak{b}_m \cdot \mathscr{J}(||lL||) \subseteq \mathscr{J}(||(m+l)L||).$$

(3) The natural inclusion

$$H^0(X, \mathscr{O}_X(mL) \otimes \mathscr{J}(||mL||)) \to H^0(X, \mathscr{O}_X(mL))$$

is an isomorphism for every $m \geq 1$.

(4) Let $f : Y \to X$ be a finite étale cover of smooth projective varieties. Then

$$\mathscr{J}(Y, ||f^*L||) = f^* \mathscr{J}(X, ||L||).$$

1.2.2 Fourier-Mukai transform

Let X be an algebraic variety. The symbol $\mathbf{D}(X)$ will denote the derived category of complexes of quasi-coherent $\mathscr{O}_X$-modules with coherent cohomology sheaves. Mukai [Mu] found a remarkable integral functor between the derived categories of an abelian variety and that of its dual variety. This functor and its generalizations have also been called Fourier-Mukai transform. There are several references on this theory, for instance [Huy]. We will only need Fourier-Mukai transform on abelian varieties so I will review Mukai's duality following [Mu].

Let X be an abelian variety of dimension g, and denote by $\widehat{X}$ its dual abelian variety. Let $\mathscr{P}$ be the normalized Poincare line bundle on $X \times \widehat{X}$. Here normalized means that both $\mathscr{P}|_{0\times\widehat{X}}$ and $\mathscr{P}|_{X\times 0}$ are trivial line bundles. Let $p_1 : X \times \widehat{X} \to X$ and $p_2 : X \times \widehat{X} \to \widehat{X}$ be the natural projections. We denote by $\mathbf{D}(X)$ the derived category of $\mathscr{O}_X$ modules. Mukai defined the functors

$$R\Phi_{\mathscr{P}}(?) = Rp_{2*}(p_1^*(?) \otimes \mathscr{P}) : \mathbf{D}(X) \to \mathbf{D}(\widehat{X})$$

and

$$R\Psi_{\mathscr{P}} = Rp_{1*}(\mathscr{P} \otimes p_2^*(?)) : \mathbf{D}(\widehat{X}) \to \mathbf{D}(X).$$

The following theorem is fundamental:

Theorem 1.21 *There are isomorphisms of functors:*

$$R\Psi_{\mathscr{P}} \circ R\Phi_{\mathscr{P}} \simeq (-1_X)^*[-g],$$

and

$$R\Phi_{\mathscr{P}} \circ R\Psi_{\mathscr{P}} \simeq (-1_{\widehat{X}})^*[-g].$$

We will call a coherent sheaf $\mathscr{F}$ on X an I.T. sheaf of index i if $H^k(X, \mathscr{F} \otimes P) = 0$ for all $P \in \mathrm{Pic}^0(X)$ and $k \neq i$. We will call a coherent sheaf $\mathscr{F}$ on X a W.I.T. sheaf of index i if $R^k\Phi_{\mathscr{P}}(\mathscr{F}) = 0$ for all $k \neq i$. We see that an I.T. sheaf of index i is always a W.I.T. sheaf of index i.

1.2.3 Cohomological support loci and GV-objects

The cohomological support loci of the canonical bundle were first studied by Green and Lazarsfeld in [GL1] and [GL2], through their generic vanishing theorems and then Simpson (see [S]) also contributed to the subject.

Let X be a smooth projective variety and let $\mathscr{F}$ be a coherent sheaf on X. The cohomological support loci of $\mathscr{F}$ are defined as

$$V_i(X, \mathscr{F}) = \{P \in \mathrm{Pic}^0(X) \mid H^i(X, \mathscr{F} \otimes P) \neq 0\},$$

which we often write $V_i(\mathscr{F})$.

GV-objects were first considered by Hacon in [H] and systematically studied by Pareschi and Popa in [PP2]. In this paper, we just need to consider GV-sheaves with respect to the universal Poincaré line bundle.

Definition 1.22 *A sheaf $\mathscr{F}$ on X is called a GV-sheaf if*

$$\mathrm{codim}_{\mathrm{Pic}^0(X)} V_i(\mathscr{F}) \geq i,$$

for all $i \geq 0$.

We see that an I.T. sheaf of index 0 is always a GV-sheaf.

Let $a_X : X \to A$ be the Albanese map of X; then $\mathrm{Pic}^0(X) = \widehat{A}$. Let M be an ample line bundle on $\widehat{A}$. We denote by $\widehat{M}$ the Fourier-Mukai transform of M, which is a locally free sheaf on A. Let $\phi_M : \widehat{A} \to A$ be the standard isogeny induced by M; then $\phi_M^* \widehat{M}^\vee \simeq H^0(M) \otimes M$. Consider the cartesian diagram:

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\varphi_M} & X \\ {\scriptstyle a_{\widehat{X}}}\downarrow & & {\scriptstyle a_X}\downarrow \\ \widehat{A} & \xrightarrow{\phi_M} & A \end{array} \tag{1}$$

Hacon proved the following theorem in [H], which was generalized by Pareschi and Popa (Theorem A in [PP2]).

Theorem 1.23 *Let $\mathscr{F}$ be a coherent sheaf on a smooth projective variety X. If $H^i(\widehat{X}, \varphi_M^* \mathscr{F} \otimes a_{\widehat{X}}^* M) = 0$, for all $i > 0$ and any sufficiently ample M, then $\mathscr{F}$ is a GV-sheaf.*

A related object to GV-sheaf is the M-regular sheaf on an abelian variety:

Definition 1.24 *A coherent sheaf $\mathscr{F}$ on an abelian variety A is called M-regular if*

$$\mathrm{codim}_{\mathrm{Pic}^0(A)} V_i(\mathscr{F}) > i$$

for any $i \geq 1$.

We can find many interesting properties of M-regular sheaf in [PP1].

1.2.4 Positivity of direct images of dualizing sheaves

We have a result of Kollár about the positivity properties of higher direct images of the canonical bundle, which was generalized later by Esnault and Viehweg.

Theorem 1.25 (Kollár, Esnault-Viehweg, [Kol1] 10.15) *Let $f : X \to Y$ be a surjective morphism from a smooth projective variety X to a normal variety Y. Let L be a line bundle on X such that $L \equiv f^*M + \Delta$, where M is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y and (X, Δ) is klt. Then,*

a) $R^j f_*(\omega_X \otimes L)$ is torsion free for $j \geq 0$;

b) if in addition that M is big and nef, $H^i(Y, R^j f_*(\omega_X \otimes L)) = 0$ for all $i > 0$ and all $j \geq 0$.

The following corollary is suggested to me by Prof. Campana which will be used to prove Lemma 1.32.

Corollary 1.26 *Let $h : X \to Z$ and $k : Z \to Y$ be surjective morphisms between smooth projective varieties. Let A be a $\mathbb{Q}$-ample divisor on Z and let E be a $\mathbb{Q}$-divisor on X such that (X, E) is a klt and $L \equiv f^*A + E$ is a divisor. Then*

$$H^i(Y, R^j(kh)_*(\omega_X \otimes L)) = 0$$

for all $i > 0$ and all $j \geq 0$.

Proof. Let H be an ample divisor on Y. Choose $m > 0$ large enough such that $h^*(mA)$ and $h^*(mA - k^*H)$ have respectively reduced smooth members B_1 and B_2 whose union is a SNC divisor and $(X, E^{'})$ is again klt for the divisor $E^{'} = E + \frac{1}{2m}(B_1 + B_2)$. We then have

$$\begin{aligned} h^*A + E &\equiv \frac{1}{2m}B_1 + \frac{1}{2m}B_2 + \frac{1}{2m}(kh)^*H + E \\ &= \frac{1}{2m}(kh)^*H + E^{'}. \end{aligned}$$

Hence we apply Kollár's theorem 1.25 to give the conclusion. □

Another very useful result of Kollár is about the splitting behavior of direct images of dualizing sheaves (see [Kol4]).

Theorem 1.27 *Let $f : X \to Y$ be a surjective map between projective varieties, X smooth, Y arbitrary. Then*

$$R^{\cdot} f_* \omega_X = \sum R^i f_* \omega_X[-i],$$

and as a corollary, we have

$$h^p(X, \omega_X) = \sum h^i(Y, R^{p-i} f_* \omega_X).$$

Let F be a torsion-free coherent sheaf on a nonsingular quasiprojective variety V. There exists an open subset V^0 of V such that $\operatorname{codim}_V V^0 \geq 2$ and i^*F is locally free where $i : V^0 \to V$ is the inclusion. We will denote $\widehat{S}^a F = i_*(S^a(i^*F))$ for $a \in \mathbb{N}$.

We define a weakly positive sheaf following Viehweg (see [V2]).

Definition 1.28 *Let F be a torsion-free sheaf on a nonsingular quasi-projective variety. We say that F is weakly positive if, for every $a > 0$ and every ample invertible sheaf H, there exists $b > 0$ such that $\widehat{S}^{ab} F \otimes H^b$ is generically globally generated.*

The notion of weakly positivity comes from the following important theorem of Viehweg (see for instance [V2]):

Theorem 1.29 *Let $f : X \to Y$ be a surjective projective morphism of quasi-projective varieties. Then $f_*(\omega^v_{X/Y})$ is weakly positive for every $v > 0$.*

1.2.5 Some useful theorems

A numerical characterization of abelian varieties is first conjectured by Kollár in [Kol2, Conjecture 17.9.1] and later verified by Chen and Hacon in [CH1]:

Theorem 1.30 *Let X be a smooth projective variety with $P_1(X) = P_2(X) = 1$ and $q(X) = \dim X$. Then X is birational to an abelian variety.*

This theorem will be frequently used to estimate Kodaira dimensions of certain varieties with maximal Albanese dimension and we will give another proof of this theorem in section 1.5.

The following elementary lemma is frequently used (see [HP]).

Lemma 1.31 *Let X be a smooth projective variety, let L and M be line bundles on X, and let $T \subset \operatorname{Pic}^0(X)$ be an irreducible subvariety of dimension t. If for some positive integers a and b and all $P \in T$, we have $h^0(X, L \otimes P) \geq a$ and $h^0(X, M \otimes P^{-1}) \geq b$, then $h^0(X, L \otimes M) \geq a + b + t - 1$.*

1.3 When is the Albanese map surjective?

In this section I use the language of asymptotic multiplier ideal sheaves. However many of the ideas come from [Kol1], [HP], and [H].

Lemma 1.32 *Suppose that $f : X \to Y$ is a surjective morphism between smooth projective varieties, L is a $\mathbb{Q}$-divisor on X, and the Iitaka model of (X, L) dominates Y. Assume that D is a nef $\mathbb{Q}$-divisor on Y such that $L + f^*D$ is a divisor on X. Then we have*

$$H^i(Y, R^j f_*(\mathscr{O}_X(K_X + L + f^*D) \otimes \mathscr{J}(||L||) \otimes Q)) = 0,$$

for all $i \geq 1$, $j \geq 0$, and all $Q \in \mathrm{Pic}^0(X)$.

Proof. Let $m > 0$ be sufficient large and divisible. We can assume that mL is a divisor and $\mathscr{J}(||L||) = \mathscr{J}(\frac{1}{m}|mL|)$ ([La], §11.2). Let $\mu : X' \to X$ be a log resolution such that

$$\mu^*|mL| = |A| + E,$$

where $|A|$ is base-point-free and E is an effective divisor with SNC support. Since the Iitaka fibration of (X, L) dominates Y, we may assume that we have the following commutative diagram:

$$\begin{tikzcd} X' \arrow[r, "\mu"] \arrow[d, "h"] & X \arrow[d, "f"] \\ Z \arrow[r, "k"] & Y \end{tikzcd}$$

where all arrays are surjective morphisms between smooth projective varieties and $A = h^*A_1$ for a very ample divisor A_1 on Z and

$$\mathscr{J}(||L||) = \mu_*(K_{X'/X} - \left\lfloor \frac{1}{m}E \right\rfloor)$$

Hence by local vanishing theorem ([La], Theorem 9.4.1),

$$\begin{aligned} & R^j f_*(\mathscr{O}_X(K_X + L + f^*D) \otimes \mathscr{J}(||L||) \otimes Q) \\ = \ & R^j (f \circ \mu)_* \big(\mathscr{O}_{X'}\big(K_{X'} + \mu^*L + \mu^*f^*D - \left\lfloor \frac{1}{m}E \right\rfloor + \mu^*Q\big)\big), \end{aligned} \qquad (2)$$

for all $j \geq 0$. Moreover,

$$\mu^* L + \mu^* f^* D - \left\lfloor \frac{1}{m} E \right\rfloor + \mu^* Q \equiv \frac{1}{m} h^*(A_1 + mk^* D) + \left\{ \frac{1}{m} E \right\}.$$

We then apply corollary 1.26 to conclude Lemma 1.32. $\square$

The following lemma is essentially Proposition 2.12 in [HP]. I use Lemma 1.32 to make the proof a little bit simpler.

Lemma 1.33 *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties and assume that the Iitaka model of X dominates Y. Fix a torsion element $Q \in \mathrm{Pic}^0(X)$ and an integer $m \geq 2$. Then $h^0(X, \omega_X^m \otimes Q \otimes f^* P)$ is constant for all $P \in \mathrm{Pic}^0(Y)$.*

Proof. We consider $h^0(X, \omega_X^m \otimes Q \otimes f^* P)$ as a function of $P \in \mathrm{Pic}^0(X)$. Let $P_0 \in \mathrm{Pic}^0(X)$ be such that $h^0(X, \omega_X^m \otimes Q \otimes f^* P_0) = h$ is maximal. We are going to prove that

$$h^0(X, \omega_X^m \otimes Q \otimes f^* P_0 \otimes f^* P) = h,$$

for any torsion $P \in \mathrm{Pic}^0(X)$. Since $P_0 + \{\text{torsion points}\}$ is dense in $\mathrm{Pic}^0(X)$, we then deduce the lemma from semicontinuity.

Let P_1, P_2, Q_1 be such that $P_1^m = P_0$, $P_2^m = P$ and $Q_1^m = Q$. From the properties of asymptotic multiplier ideal sheaves ([La], Theorem 11.1.8), we know that

$$\begin{aligned} & H^0(X, \omega_X^m \otimes Q \otimes f^* P_0 \otimes f^* P) \\ = \; & H^0\big(X, \omega_X^m \otimes Q \otimes f^* P_0 \otimes f^* P \otimes \mathscr{J}(||\omega_X^m \otimes Q_1^m \otimes f^* P_1^m \otimes f^* P_2^m||)\big) \\ = \; & H^0\big(X, \omega_X^m \otimes Q \otimes f^* P_0 \otimes f^* P \otimes \mathscr{J}(||\omega_X^{m-1} \otimes Q_1^{m-1} \otimes f^* P_1^{m-1} \otimes f^* P_2^{m-1}||)\big). \end{aligned}$$

Since P is a torsion point, there exists $N > 0$ such that $P^N = \mathscr{O}_Y$. For $k > 0$ large enough and divisible, we have

$$\begin{aligned} & \mathscr{J}(||\omega_X^{m-1} \otimes Q_1^{m-1} \otimes f^* P_1^{m-1} \otimes f^* P_2^i||) \\ = \; & \mathscr{J}(\frac{1}{kN}|(\omega_X^{m-1} \otimes Q_1^{m-1} \otimes f^* P_1^{m-1} \otimes f^* P_2^i)^{kN}|) \\ = \; & \mathscr{J}(||\omega_X^{m-1} \otimes Q_1^{m-1} \otimes f^* P_1^{m-1}||), \end{aligned}$$

for all $i \geq 0$. Hence we have

$$
\begin{aligned}
& H^0(X, \omega_X^m \otimes Q \otimes f^*P_0 \otimes f^*P) \\
= \ & H^0\big(X, \omega_X^m \otimes Q \otimes f^*P_0 \otimes f^*P \otimes \mathscr{J}(||\omega_X^{m-1} \otimes Q_1^{m-1} \otimes f^*P_1^{m-1}||)\big) \\
= \ & H^0\big(Y, f_*\big(\omega_X^m \otimes Q_1^{m-1} \otimes f^*P_1^{m-1} \otimes \mathscr{J}(||\omega_X^{m-1} \otimes Q_1^{m-1} \otimes f^*P_1^{m-1}||) \otimes Q_1 \otimes f^*P_1 \otimes f^*P\big)\big).
\end{aligned}
$$

We then apply Lemma 1.32 (the Iitaka model of $(X, \omega_X^{m-1} \otimes Q_1^{m-1} \otimes f^*P_1^{m-1}$ dominates Y by assumption) to get that

$$h^0(X, \omega_X^m \otimes Q \otimes f^*P_0 \otimes f^*P) = \chi\big(Y, f_*\big(\omega_X^m \otimes Q \otimes \mathscr{J}(||\omega_X^{m-1} \otimes Q_1^{m-1} \otimes f^*P_1^{m-1}||)\big)\big)$$

is the constant h. □

Lemma 1.34 *Suppose that $f : X \to Z$ is an algebraic fiber space between smooth projective varieties. Assume that $P_m(X) \neq 0$, for some $m \geq 2$, that H is a big $\mathbb{Q}$-divisor on Z, and that K is a nef $\mathbb{Q}$-divisor on Z such that $H_1 \equiv H + K$ is a big and nef divisor. Then,*

1) we have

$$H^i\big(Z, R^j f_*\big(\mathscr{O}_X(K_X + (m-1)K_{X/Z} + f^*H_1) \otimes \mathscr{J}(||(m-1)K_{X/Z} + f^*H||)\big) \otimes P\big) = 0,$$

for all $i \geq 1$, $j \geq 0$ and all $P \in \mathrm{Pic}^0(Z)$.

2) the sheaf

$$f_*\big(\mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes \mathscr{J}(||(m-1)K_{X/Z} + f^*H||)\big)$$

has rank $P_m(X_z)$, where X_z is a general fiber of f.

Proof. The point here is the weak positivity of $f_*(\omega_{X/Z}^{m-1})$, due to Viehweg ([V2] Theorem 4.1 and Corollary 7.1, or [Kol1] Proposition 10.2). There are two conclusions:

1. the Iitaka model of $(X, (m-1)K_{X/Z} + f^*H)$ dominates Z and
2. there exists $k > 0$ sufficient big and divisible such that the restriction:

   $$H^0(X, \mathscr{O}_X(km(m-1)K_{X/Z} + kmf^*H)) \to H^0(X_z, \mathscr{O}_{X_z}(km(m-1)K_{X_z}))$$

   is surjective, where $z \in Z$ is a general point.

By the first conclusion, we can directly apply Lemma 1.32 to deduce 1).
We take a log resolution $\tau : X' \to X$ such that the restriction $\tau_z : X'_z \to X_z$ is also a log resolution for sufficiently general $z \in Z$ (see [La] Theorem 9.5.35) and fix such a point $z \in Z$. Set

- $\tau^*|km(m-1)K_{X/Z} + kmf^*H| = |L_1| + E_1$,
- $\tau_z^*|mK_{X_z}| = |L_2| + E_2$,

where $|L_1|$ and $|L_2|$ are base-point-free, E_1 and E_2 are the fixed divisors, and $E_1 + \mathrm{Exc}(\tau)$ has SNC support. We have

$$E_1|_{X'_z} \preceq k(m-1)E_2, \tag{3}$$

by the second conclusion. Let $f' : X' \xrightarrow{\tau} X \xrightarrow{f} Z$ be the composition of morphisms. Then f' is flat over a dense Zariski open subset of Z. Hence the sheaf

$$f'_*\big(\mathscr{O}_{X'}(K_{X'} + (m-1)\tau^*K_{X/Z} - \left\lfloor \frac{E_1}{km} \right\rfloor)\big)$$

has rank

$$h^0\big(X'_z, \mathscr{O}_{X'_z}\big(mK_{X'_z} - \left\lfloor \frac{E_1}{km} \right\rfloor |_{X'_z}\big)\big) = P_m(X_z).$$

We have the following inclusions

$$\begin{aligned}
& f_*\tau_*\mathscr{O}_{X'}\big(K_{X'} + (m-1)\tau^*K_{X/Z} - \left\lfloor \frac{E_1}{km} \right\rfloor\big) \\
\subset\ & f_*\big(\mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes \mathscr{J}(||(m-1)K_{X/Z} + f^*H||)\big) \\
\subset\ & f_*(\mathscr{O}_X(mK_X)) \otimes \mathscr{O}_Z(-(m-1)K_Z).
\end{aligned}$$

Since the latter sheaf has rank $P_m(X_z)$, the middle sheaf $f_*\big(\mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes \mathscr{J}(||(m-1)K_{X/Z} + f^*H||)\big)$ also has rank $P_m(X_z)$. $\square$

Under the assumptions of Lemma 1.34, we fix a big and base-point-free divisor H. For $n > 0$, we set

$$\mathscr{J}_{m-1,n} = \mathscr{J}(||(m-1)K_{X/Z} + \frac{1}{n}f^*H||)$$
$$\mathscr{F}_{m-1,n} = f_*\big(\mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes \mathscr{J}_{m-1,n}\big).$$

By Lemma 1.34, $\mathscr{F}_{m-1,n}$ has rank $P_m(X_z) > 0$. These sheaves were first considered by Hacon in [H].

Lemma 1.35 *We have $\mathscr{J}_{m-1,n} \supset \mathscr{J}_{m-1,n+1}$ and there exists $N > 0$ such that for any $n \geq N$, one has $\mathscr{F}_{m-1,n} = \mathscr{F}_{m-1,N}$. We will denote by $\mathscr{F}_{m-1,H}$ the fixed sheaf $\mathscr{F}_{m-1,N}$.*

Proof. We may suppose that $k > 0$ is such that the linear series $|k(n+1)n((m-1)K_{X/Z}+\frac{1}{n}f^*H)|$ and $|k(n+1)n((m-1)K_{X/Z}+\frac{1}{n+1}f^*H)|$ compute $\mathscr{J}_{m-1,n}$ and $\mathscr{J}_{m-1,n+1}$, respectively. Let $\tau : X' \to X$ be a log resolution for both linear series. We can write

$$\tau^*|k(n+1)n(m-1)K_{X/Z} + k(n+1)f^*H| = |L_1| + E_1,$$
$$\tau^*|k(n+1)n(m-1)K_{X/Z} + knf^*H| = |L_2| + E_2,$$

where L_1 and L_2 are base-point-free and E_1 and E_2 are fixed divisors. Since H is base-point-free, we have $E_2 \succeq E_1$. By the definition of asymptotic multiplier ideal sheaves, $\mathscr{J}_{m-1,n} \supset \mathscr{J}_{m-1,n+1}$.
Take H_1 very ample on Z such that $H_1 - H$ is a nef divisor. Then by Lemma 1.34, we have

$$H^i(Z, f_*\big(\mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes \mathscr{J}_{m-1,n}\big) \otimes \mathscr{O}_Z(H_1)) = 0,$$

for $i \geq 1$. Using Hacon's argument in the proof of Proposition 5.1 in [H], there exists $N > 0$ such that for $n \geq N$, the inclusion

$$f_*(\mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes \mathscr{J}_{m-1,N}) \otimes \mathscr{O}_Z(H_1)$$
$$\supset f_*(\mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes \mathscr{J}_{m-1,n}) \otimes \mathscr{O}_Z(H_1)$$

is an equality. This implies that the inclusion

$$f_*(\mathscr{O}_X(K_X+(m-1)K_{X/Z})\otimes\mathscr{J}_{m-1,N}) \supset f_*(\mathscr{O}_X(K_X+(m-1)K_{X/Z})\otimes\mathscr{J}_{m-1,n})$$

is again an equality. □

Lemma 1.36 *Under the above assumptions, namely $f : X \to Z$ is an algebraic fiber space between projective smooth varieties and $P_m(X) \neq 0$ with $m \geq 2$, we suppose moreover that Z is of maximal Albanese dimension and that H is a big and base-point-free divisor on Z pulled back from $\mathrm{Alb}(Z)$. Then $\mathscr{F}_{m-1,H}$ is a nonzero GV-sheaf.*

Proof. We apply Theorem 1.23. Let M be any ample divisor on $\mathrm{Pic}^0(Z)$. We have cartesian diagrams as (1):

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\upsilon_M} & X \\ {\scriptstyle \widehat{f}}\downarrow & & {\scriptstyle f}\downarrow \\ \widehat{Z} & \xrightarrow{\varphi_M} & Z \\ {\scriptstyle a_{\widehat{Z}}}\downarrow & & {\scriptstyle a_Z}\downarrow \\ \mathrm{Pic}^0(Z) & \xrightarrow{\phi_M} & \mathrm{Alb}(Z) \end{array}$$

where horizontal maps are étale. By Theorem 11.2.16 in [La], for any $n > 0$,

$$\upsilon_M^* \mathscr{J}(||(m-1)K_{X/Z} + \frac{1}{n} f^* H||) = \mathscr{J}(||(m-1)K_{\widehat{X}/\widehat{Z}} + \frac{1}{n}\widehat{f}^* \varphi_M^* H||),$$

hence

$$\begin{aligned} &\varphi_M^* f_* \big(\mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes \mathscr{J}(||(m-1)K_{X/Z} + \frac{1}{n} f^* H||)\big) \\ = &\widehat{f}_* \big(\mathscr{O}_{\widehat{X}}(K_{\widehat{X}} + (m-1)K_{\widehat{X}/\widehat{Z}}) \otimes \mathscr{J}(||(m-1)K_{\widehat{X}/\widehat{Z}} + \frac{1}{n}\widehat{f}^* \varphi_M^* H||)\big). \end{aligned}$$

Hence $\varphi_M^* \mathscr{F}_{m-1,H} = \widehat{f}_* \big(\mathscr{O}_{\widehat{X}}(K_{\widehat{X}} + (m-1)K_{\widehat{X}/\widehat{Z}}) \otimes \mathscr{J}(||(m-1)K_{\widehat{X}/\widehat{Z}} + \frac{1}{n}\widehat{f}^* \varphi_M^* H||)\big)$ for any $n \gg 0$. Since H is a divisor pulled back by a_Z, we can take n such that $n a_{\widehat{Z}}^* M - \varphi_M^* H$ is nef. Then Lemma 1.34 gives us the vanishing of

$$H^i(\widehat{Z}, \varphi_M^* \mathscr{F}_{m-1,H} \otimes a_{\widehat{Z}}^* M),$$

for all $i > 0$ and we are done. $\square$

Lemma 1.37 *In the situation of Lemma 1.36, denoting by $a_Z : Z \to A$ the Albanese morphism of Z, we have $R^j a_{Z*}(\mathscr{F}_{m-1,H}) = 0$, for all $j > 0$. Hence*

$$V_i(\mathscr{F}_{m-1,H}) = V_i(a_{Z*}(\mathscr{F}_{m-1,H})),$$

for all $i \geq 0$.

Proof. Suppose that $R^t a_{Z*}(\mathscr{F}_{m-1,H}) \neq 0$ for some $t > 0$. Let H_1 be a ample divisor on A such that

$$H^k(A, R^j a_{Z*}(\mathscr{F}_{m-1,H}) \otimes \mathscr{O}_A(H_1)) = 0$$

for all $k \geq 1$ and $j \geq 0$ and

$$H^0(A, R^t a_{Z*}(\mathscr{F}_{m-1,H}) \otimes \mathscr{O}_A(H_1)) \neq 0.$$

By the Leray spectral sequence, we have

$$H^t(Z, \mathscr{F}_{m-1,H} \otimes \mathscr{O}_Z(a_Z^* H_1)) \neq 0.$$

Since H is pulled back from A, we may take H_1 such that $a_Z^* H_1 - H$ is big and nef, then by Lemma 1.34, we have $H^t(Z, \mathscr{F}_{m-1,H} \otimes \mathscr{O}_Z(a_Z^* H_1)) = 0$, which is a contradiction. Thus $R^j a_{Z*}(\mathscr{F}_{m-1,H}) = 0$ for all $j > 0$. For any $P \in \mathrm{Pic}^0(Z)$, we have $H^i(Z, \mathscr{F}_{m-1,H} \otimes a_Z^* P) \simeq H^i(A, a_{Z*}(\mathscr{F}_{m-1,H}) \otimes P)$, hence $V_i(\mathscr{F}_{m-1,H}) = V_i(a_{Z*}(\mathscr{F}_{m-1,H}))$ for all $i \geq 0$. $\square$

Corollary 1.38 *The cohomological support $V_0(\mathscr{F}_{m-1,H})$ is not empty.*

Proof. By Lemma 1.36, $\mathscr{F}_{m-1,H}$ is a GV-sheaf, hence ([H], Corollary 3.2)

$$V_0(\mathscr{F}_{m-1,H}) \supset V_1(\mathscr{F}_{m-1,H}) \supset \cdots \supset V_d(\mathscr{F}_{m-1,H}).$$

If $V_0(\mathscr{F}_{m-1,H})$ is empty, $V_i(\mathscr{F}_{m-1,H})$ is empty for all $i \geq 0$, hence

$$H^i(Z, \mathscr{F}_{m-1,H} \otimes a_Z^* P) = H^i(A, a_{Z*}\mathscr{F}_{m-1,H} \otimes P) = 0,$$

for all $i \geq 0$. Then by Fourier-Mukai transform on an abelian variety (see [Mu]), $a_{Z*}\mathscr{F}_{m-1,H} = 0$. However this is impossible since a_Z is generically finite and $\mathscr{F}_{m-1,H}$ is a sheaf with positive rank. $\square$

Theorem 1.39 *Let X be a smooth projective variety. If*

$$0 < P_m(X) \leq 2m - 2,$$

for some $m \geq 2$, the Albanese map $a_X : X \to \mathrm{Alb}(X)$ is surjective.

PROOF. If a_X is not surjective, then by Ueno's theorem ([Mo], Theorem (3.7)), upon replacing X by a birational model, there exists a surjective morphism $f_1 : X \to Z_1$ onto a smooth variety Z_1 of general type of dimension $d > 0$ such that $Z_1 \to \mathrm{Alb}(Z_1)$ is a birational map onto its image and $Z_1 \to \mathbb{P}(H^0(Z_1, K_{Z_1}))$ is a map generically finite onto its image. Obviously, $P_k(Z_1) \geq \binom{d+k}{d}$ for all $k \geq 1$. Taking the Stein factorization and making birational modifications, we may suppose that there is an algebraic fiber space $f : X \to Z$ such that Z is a smooth variety of general type and of maximal Albanese dimension d, and $P_k(Z) \geq \binom{d+k}{k}$ for all $k \geq 1$.

We let H be a big and base-point-free divisor pulled back by the Albanese morphism $a_Z : Z \to \mathrm{Alb}(Z)$. By Corollary 1.38, $V_0(\mathscr{F}_{m-1,H})$ is not empty thus there exists $P \in \mathrm{Pic}^0(Z)$ such that $h^0(Z, \mathscr{F}_{m-1,H} \otimes P) \geq 1$. Hence

(*) $h^0(X, \mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes f^*P) \geq 1.$

On the other hand, we have $h^0(X, \mathscr{O}_X((m-1)f^*K_Z)) \geq \binom{d+m-1}{m-1}$. We get

$$h^0(X, \mathscr{O}_X(mK_X) \otimes f^*P) \geq \binom{d+m-1}{m-1}. \tag{4}$$

Since Z is of general type, the Iitaka model of (X, K_X) dominates Z because of $(*)$, hence we apply Lemma 1.33 to get $h^0(X, \mathscr{O}_X(mK_X)) \geq \binom{d+m-1}{m-1}$.

If $\dim Z = d \geq 2$, then $P_m(X) \geq \binom{m+1}{2} \geq 2m - 1$, which is a contradiction.

If $\dim Z = 1$, $P_m(X) = h^0(Z, f_*(\omega^m_{X/Z}) \otimes \omega^m_Z)$. As in Corollary 3.6 in [V1], $f_*(\omega^m_{X/Z})$ is a nonzero nef vector bundle on Z hence has nonnegative degree. By the Riemann-Roch theorem, we again have $P_m(X) \geq 2m - 1$, again a contradiction. $\square$

Remark 1.40 The proof follows ideas of Kollár's ([Kol1]), later improved by Hacon and Pardini. Briefly speaking, Kollár proved that $P_m(X) \geq P_{m-2}(Z)$ and Hacon and Pardini used the finite map

$$|(m-2)K_Z + P| \times |K_X + (m-1)K_{X/Z} + K_Z - f^*P| \to |mK_X|,$$

where $P \in \mathrm{Pic}^0(Z)$, to give a better estimate of $P_m(X)$. However $h^0(Z, \mathscr{O}_Z(kK_Z))$ grows very fast with k, so my starting point is to prove $P_m(X) \geq P_{m-1}(Z)$ by applying the theory of GV-sheaves.

Corollary 1.41 *Suppose that* $0 < P_m(X) < \binom{d+m}{m-1}$ *for some* $m \geq 2$ *and* $d \geq 1$, *then* $\kappa(a_X(X)) \leq d$.

PROOF. It is just (5) in the proof of Theorem 1.39, where by Ueno's theorem d is the Kodaira dimension of $a_X(X)$. $\square$

1.4 When does the Albanese map have connected fibers?

In section 1, we saw that Ein and Lazarsfeld's proof of Kawamata's theorem is very close to an effective criterion for the Albanese morphism to be an algebraic fiber space. With the help of a proposition of Chen and Hacon, we have the following:

Theorem 1.42 *Let* X *be a smooth projective variety with* $P_1(X) = P_2(X) = 1$. *The Albanese map* $a_X : X \to \mathrm{Alb}(X)$ *is an algebraic fiber space.*

PROOF. Let A be the Albanese variety of X. The Albanese morphism is already surjective by Theorem 1.9. Suppose that it has non-connected fibers. We start with the Stein factorization of a_X and, resolving singularities and indeterminacies, we can assume that a_X admits a factorization:

$$X \xrightarrow{g} V \xrightarrow{b} A,$$

where b is a generically finite non birational morphism, g is surjective with connected fibers, V is smooth and projective. Since a_X is the Albanese morphism of X, V is not birational to an abelian variety. Thus V is of maximal Albanese dimension and by Chen and Hacon's characterization of abelian varieties ([CH1], Theorem 3.2), we have $P_2(V) \geq 2$. We set $\dim X = n$ and $\dim V = \dim A = d$.

Since $P_1(X) = P_2(X) = 1$, $0 \in V_0(X, \omega_X)$ is an isolated point ([EL1], Proposition 2.1). Hence $0 \in V_0(V, g_*\omega_X)$ is also an isolated point. By Proposition 2.5 in [CH4], for any $v \neq 0$ in $H^1(V, \mathscr{O}_V)$, the sequence

$$0 \to H^0(V, g_*\omega_X) \xrightarrow{\cup v} H^1(V, g_*\omega_X) \to \cdots \xrightarrow{\cup v} H^d(V, g_*\omega_X) \to 0$$

is exact. Since b is surjective, through the map b^* we may consider $H^1(A, \mathscr{O}_A)$ as a subspace of $H^1(V, \mathscr{O}_V)$. Then, as in the proof of Theorem 3 in [EL1], we have an exact complex of vector bundles on $\mathbf{P} = \mathbf{P}(H^1(A, \mathscr{O}_A)) = \mathbf{P}^{d-1}$:

$$0 \to H^0(V, g_*\omega_X) \otimes \mathscr{O}_{\mathbf{P}}(-d) \to H^1(V, g_*\omega_X) \otimes \mathscr{O}_{\mathbf{P}}(-d+1) \to \cdots \to H^d(V, g_*\omega_X) \otimes \mathscr{O}_{\mathbf{P}} \to 0.$$

Take $(v_1, \dots, v_d)$ a basis for $H^1(A, \mathscr{O}_A)$. By chasing through the diagram, we obtain that $H^0(V, g_*\omega_X) \xrightarrow{\wedge v_1 \wedge \cdots \wedge v_d} H^d(V, g_*\omega_X)$ is an isomorphism.

By Kollár's splitting Theorem 1.27 (see also [Kol4, Theorem 3.4]),

$$H^d(X, \omega_X) \simeq \bigoplus_i H^i(V, R^{d-i} g_*\omega_X).$$

Hence we have

$$\begin{tikzcd}
H^0(V, g_*\omega_X) \arrow[r, "\wedge v_1 \wedge \cdots \wedge v_d", "\simeq"'] \arrow[d, "\simeq"] & H^d(V, g_*\omega_X) \arrow[d, hook] \\
H^0(X, \omega_X) \arrow[r, "\wedge g^*(v_1 \wedge \cdots \wedge v_d)"] & H^d(X, \omega_X)
\end{tikzcd}$$

By Hodge conjugation and Serre duality $H^d(X, \omega_X) \simeq H^0(X, \Omega_X^{n-d})$. We will denote by $E \subset H^0(X, \Omega_X^{n-d})$ the nonzero subspace corresponding to $H^d(V, g_*\omega_X) \subset H^d(X, \omega_X)$. Let $(\eta_1, \dots, \eta_d)$ in $H^0(A, \Omega_A)$ be the conjugate basis of $(v_1, \dots, v_d)$. By Serre duality and Hodge conjugation, we get from the above diagram that

$$E \xrightarrow{\wedge g^*(\eta_1 \wedge \cdots \wedge \eta_d)} H^0(X, \omega_X)$$

is an isomorphism. Since $\eta_1 \wedge \cdots \wedge \eta_d$ is a nonzero section of K_V, we have $K_X \succeq g^*K_V$. We deduce $P_2(X) \geq P_2(V) \geq 2$, which is a contradiction. $\square$

The proof of Theorem 1.42 is closely related to Green and Lazarsfeld's generic vanishing theorem, which is Hodge theoretic. Meanwhile Theorem 1.39 relies heavily on the weak positivity theorem of Viehweg. It is natural to ask whether we can use the ideas in section 3 to prove other critera to tell when the Albanese map is an algebraic fiber space.

We again let A be $\mathrm{Alb}(X)$. Suppose that $a_X : X \to A$ is surjective but has non-connected fibers. We take the Stein factorization and obtain that a_X factors as $X \xrightarrow{g} V \xrightarrow{b} A$ where V is normal and finite over A with, again $P_2(V) \geq 2$. The problem here is that we cannot expect V to be of general type.

Fortunately, a structure theorem for varieties of maximal Albanese dimension due to Kawamata (Theorem 13 in [Ka]) tells us that the situation is still manageable.

Theorem 1.43 (Kawamata) *Let $b : V \to A$ be a finite morphism from a projective normal algebraic variety to an abelian variety. Then $\kappa(V) \geq 0$ and there are an abelian subvariety K of A, étale covers $\widetilde{V}$ and $\widetilde{K}$ of V and K respectively, a projective normal variety $\widehat{W}$, and a finite abelian group G, which acts on $\widetilde{K}$ and faithfully on $\widehat{W}$, such that:*

(1) $\widehat{W}$ is finite over A/K, of general type and of dimension $\kappa(V)$,

(2) $\widetilde{V}$ is isomorphic to $\widetilde{K} \times \widehat{W}$,

(3) $V = \widetilde{V}/G = (\widetilde{K} \times \widehat{W})/G$, where G acts diagonally and freely on $\widetilde{V}$.

The construction of $\widehat{W}$ and $\widetilde{V}$ are crucial for our purpose so I will recall the proof of this theorem following Kawamata.

Let $\delta : V^{'} \to V$ be a birational modification of V such that $V^{'}$ is smooth and there exists a morphism $h^{'} : V^{'} \to W^{'}$ such that $W^{'}$ is also smooth and $h^{'}$ is a model of the Iitaka fibration of V. Then a general fiber $V^{'}_{w^{'}}$ of $h^{'}$ is smooth, of Kodaira dimension 0 and generically finite over an abelian variety, hence by Kawamata's Theorem 1.6, $V^{'}_{w^{'}}$ is birational to an abelian variety and $(b \circ \delta)(V^{'}_{w^{'}})$ is then an abelian subvariety of A, denoted by $K_{w^{'}}$. Since $w^{'}$ moves continuously, $K_{w^{'}}$ is a translate of a fixed abelian subvariety $K \subset A$ for every $w^{'} \in W^{'}$. Let $\pi : A \to A/K$ be the quotient map.

Consider the Stein factorization

$$\pi \circ b : V \xrightarrow{h} W \xrightarrow{b_W} A/K.$$

Since general fibers of $h^{'}$ are contracted by $\pi \circ b \circ \delta$, hence by $h \circ \delta$, the map $h \circ \delta$ factors through $h^{'}$ by rigidity, and we get the following commutative diagram:

$$\begin{tikzcd}
V^{'} \arrow[r, "\delta"] \arrow[d, "h^{'}"] & V \arrow[r, "b", "\text{finite}"'] \arrow[d, "h"] & V_0 \arrow[r, hook] \arrow[d] & A \arrow[d, "\pi"] \\
W^{'} \arrow[r, "\delta^{'}"] & W \arrow[r, "b_W", "\text{finite}"'] & W_0 \arrow[r, hook] & A/K
\end{tikzcd} \tag{5}$$

where W is normal, b_W is finite, $h : V \to W$ has connected fibers, δ and $\delta^{'}$ are birational and V_0 and W_0 are the images of V and W in A and A/K respectively.

By Poincaré reducibility, there exists an isogeny $\widetilde{A/K} \to A/K$ such that $A \times_{A/K} \widetilde{A/K} \simeq K \times \widetilde{A/K}$. We then apply the étale base change $(\cdot) \times_{A/K} \widetilde{A/K} \to \cdot$ in the diagram (5) and get the following commutative diagram:

$$\begin{tikzcd}
& \widetilde{V} \arrow[rr, "\widetilde{b}", "\text{finite}"'] \arrow[dl] \arrow[dd, "\widetilde{h}"] & & K \times \widetilde{W_0} = \widetilde{V_0} \arrow[rr, hook] \arrow[dl, hook] \arrow[dd] & & K \times \widetilde{A/K} \arrow[dl] \arrow[dd] \\
V \arrow[rr, "b"] \arrow[dd, "h", "\text{f.s.}"'] & & V_0 \arrow[rr, hook] \arrow[dd] & & A \arrow[dd] & \\
& \widetilde{W} \arrow[rr, "\widetilde{b}_W", "\text{finite}"'] \arrow[dl] & & \widetilde{W_0} \arrow[rr, hook] \arrow[dl] & & \widetilde{A/K} \arrow[dl] \\
W \arrow[rr, "b_W", "\text{finite}"'] & & W_0 \arrow[rr, hook] & & A/K &
\end{tikzcd}$$

where $\widetilde{W_0}$ is some connected component of the inverse image of W_0 in $\widetilde{A/K}$, $\widetilde{V}$ is some connected component of $V \times_{V_0} \widetilde{V_0}$, and $\widetilde{W}$ is some connected component of $W \times_{W_0} \widetilde{W_0}$, and all slanted arrows are étale.

Let us look at

$$\begin{tikzcd}
\widetilde{V} \arrow[r, "\widetilde{b}", "\text{finite}"'] \arrow[d, "\widetilde{h}"] & K \times \widetilde{W_0} \arrow[d] \\
\widetilde{W} \arrow[r, "\widetilde{b}_W", "\text{finite}"'] & \widetilde{W_0}
\end{tikzcd}$$

A general fiber of $\widetilde{h}$ is an étale cover of a general fiber of h hence an étale cover of K, thus isomorphic to an abelian variety $\widetilde{K}$.

The morphism $\widetilde{b}$ is étale over a product $K \times U_0$ for U_0 a dense Zariski open subset of $\widetilde{W_0}$:

$$\begin{tikzcd}
\widetilde{h}^{-1}(U) \arrow[r, "\widetilde{b}"] \arrow[d, "\text{smooth}"'] & K \times U_0 \arrow[d] \\
U \arrow[r] & U_0
\end{tikzcd}$$

The group K acts on $\widetilde{V_0} = K \times \widetilde{W_0}$, and on $K \times U_0$. The infinitesimal action corresponds to vector fields, which lift to $\widetilde{b}^{-1}(K \times U_0)$ because $\widetilde{b}$ is étale there.

This induces an action of $\widetilde{K}$ on $\widetilde{h}^{-1}(U) = \widetilde{b}^{-1}(K \times U_0)$ hence a rational action on $\widetilde{V}$. Let $\widetilde{k} \in \widetilde{K}$ and let $k \in K$ be its image. Let $\widetilde{\Gamma} \subset \widetilde{V} \times \widetilde{V}$ and $\Gamma \subset \widetilde{V_0} \times \widetilde{V_0}$ be the graphs of the actions of $\widetilde{k}$ and k respectively. We have

$$\begin{tikzcd}
\widetilde{V} \times \widetilde{V} \arrow[d, "(\widetilde{b},\widetilde{b})"] & \widetilde{\Gamma} \arrow[l, hook'] \arrow[r, "\widetilde{pr_1}"] \arrow[d] & \widetilde{V} \arrow[d, "\widetilde{b}"] \\
\widetilde{V_0} \times \widetilde{V_0} & \Gamma \arrow[l, hook'] \arrow[r, "pr_1"] & \widetilde{V_0}
\end{tikzcd} \tag{6}$$

where $(\widetilde{b}, \widetilde{b})$ is finite and pr_1 is an isomorphism. We see that $\widetilde{pr_1}$ is finite and birational hence an isomorphism because $\widetilde{V}$ is normal. Thus the action of $\widetilde{k}$ is an isomorphism. So $\widetilde{K}$ acts on $\widetilde{V}$ and $\widetilde{b}$ is equivariant for the $\widetilde{K}$-action on $\widetilde{V}$ and the K-action on $\widetilde{V_0}$.

Set $G_1 = \widetilde{K}/K$.

For $y \in \widetilde{W_0}$ general, we have

$$\widetilde{h}^{-1}\widetilde{b_W}^{-1}(y) = \widetilde{b}_W^{-1}(y) \times \widetilde{K} = \widetilde{b}^{-1}(K \times \{y\}),$$

hence

$$\deg \widetilde{b} = \sharp G_1 \cdot \deg \widetilde{b}_W.$$

Set $\widehat{W_0} = \widetilde{b}^{-1}(k \times \widetilde{W_0})$ for $k \in K$ general. Then $\widehat{W_0}$ is normal and G_1 acts on $\widehat{W_0}$ ($\widehat{W_0}$ may be not connected).

We have

$$\begin{tikzcd}
\widehat{W_0} \arrow[r, "\deg \widetilde{b}:1"] \arrow[d, "\sharp G_1:1"] & k \times \widetilde{W_0} \arrow[d, "="] \\
\widetilde{W} \arrow[r, "\deg \widetilde{b}_W:1"] & \widetilde{W_0},
\end{tikzcd}$$

hence $\widehat{W_0}/G_1 = \widetilde{W}$.

Note that G_1 acts on $\widetilde{K} \times \widehat{W_0}$ diagonally and freely (because the action is free on $\widetilde{K}$). By the $\widetilde{K}$-action, we have a morphism $\varphi : \widetilde{K} \times \widehat{W_0} \to \widetilde{V}$ and there is a commutative diagram:

$$\begin{tikzcd}
\widetilde{K} \times \widehat{W_0} \arrow[r, "\varphi"] \arrow[d] & \widetilde{V} \arrow[d, "\widetilde{h}"] \\
\widehat{W_0} \arrow[r, "\text{finite}"] & \widetilde{W}.
\end{tikzcd}$$

Thus φ is finite because any contracted curve is in some $\widetilde{K}\times\widetilde{w}$ but because of the $\widetilde{K}$-action, this is impossible.

¿From the diagram, we have a finite morphism $\widetilde{K}\times\widehat{W}_0\to\widetilde{V}\times_{\widetilde{W}}\widehat{W}_0$. Since it is birational over U, it is an isomorphism. Hence

$$\widetilde{V}=(\widetilde{V}\times_{\widetilde{W}}\widehat{W}_0)/G_1=(\widetilde{K}\times\widehat{W}_0)/G_1.$$

We then let $\widehat{W}$ be a connected component of $\widehat{W}_0$ and let $\widetilde{\widetilde{V}}=\widetilde{K}\times\widehat{W}$. Then $\widetilde{\widetilde{V}}$ is still a Galois étale cover of $\widetilde{V}$. There exists a commutative diagram

$$\begin{tikzcd}
\widetilde{\widetilde{V}} \arrow[r] \arrow[d] & \widetilde{K}\times\widetilde{A/K} \arrow[d] \\
\widetilde{V} \arrow[r] \arrow[d] & K\times\widetilde{A/K} \arrow[d] \\
V \arrow[r] & A
\end{tikzcd}$$

We then conclude that $\widetilde{\widetilde{V}}$ is a connected component of $V\times_A(\widetilde{K}\times\widetilde{A/K})$. Let G_2 be the finite abelian group $(\widetilde{K}\times\widetilde{A/K})/A$. Then $V=\widetilde{\widetilde{V}}/G=(\widetilde{K}\times\widehat{W})/G$, for some quotient group G of G_2, where G acts diagonally. Since any quotient of $\widetilde{K}$ by a subgroup of G is still an abelian variety, we may assume that G acts faithfully on $\widehat{W}$.

A crucial fact is that $\widehat{W}$ is of general type because

$$\kappa(\widehat{W})=\kappa(\widetilde{\widetilde{V}})=\kappa(V)=\dim W=\dim\widehat{W}.$$

We put everything in a commutative diagram:

$$\begin{tikzcd}
\widetilde{\widetilde{V}}=\widetilde{K}\times\widehat{W} \arrow[r, "\pi_{\widetilde{V}}"] \arrow[rr, bend left, "\text{Galois étale}"] \arrow[d, "\widehat{h}=pr_2"] & \widetilde{V} \arrow[r, "\pi_V"] \arrow[d, "\widetilde{h}"] & V \arrow[r, "b", "\text{finite}"'] \arrow[d, "h"] & A \arrow[d, "\pi"] \\
\widehat{W} \arrow[r, "b_{\widetilde{W}}", "\text{Galois}"'] \arrow[rr, bend right, "b_{\widehat{W}}"'] & \widetilde{W} \arrow[r, "\pi_W", "\text{finite}"'] & W \arrow[r, "b_W", "\text{finite}"'] & A/K.
\end{tikzcd} \tag{7}$$

Now we are ready to prove the main theorem.

Theorem 1.44 *Let X be a smooth projective variety. If*

$$0 < P_m(X) \leq m - 2,$$

for some $m \geq 3$, the Albanese map $a_X : X \to A$ is an algebraic fiber space.

Proof. By Theorem 1.39, the Albanese map is already surjective. Suppose that it has non-connected fibers. Again we have $a_X : X \xrightarrow{g} V \xrightarrow{b} A$, where g has connected fibers, V is normal and b is finite not birational. Applying the above description of the structure of V in (5) and (7), we get the following commutative diagram:

$$\begin{tikzcd}
X \times_V \widetilde{\widetilde{\widetilde{V}}} \arrow[r, "\pi_X"] \arrow[d, "\widehat{g}"] & X \arrow[d, "g"] \arrow[dr, "a_X"] & \\
\widetilde{\widetilde{\widetilde{V}}} \arrow[r, "\text{Galois}", "\text{étale}"'] \arrow[d, "\widehat{h}"] & V \arrow[r, "b"] \arrow[d, "h"] & A \arrow[d, "\pi"] \\
\widehat{W} \arrow[r, "b_{\widehat{W}}"] & W \arrow[r, "b_W"] & A/K
\end{tikzcd} \tag{8}$$

where π_X is étale Galois with galois group G, $\widetilde{\widetilde{\widetilde{V}}} = \widehat{W} \times \widetilde{K}$, and $\widehat{W}$ is of general type.

There exists a dense Zariski open subset U of W such that U and $b_{\widehat{W}}^{-1}(U)$ are smooth and $h \circ g$ and $\widehat{h} \circ \widehat{g}$ are smooth over U and $b_{\widehat{W}}^{-1}(U)$ respectively. Through Hironaka's resolution of singularity, we can blow up W and X along smooth subvarieties of $W - U$ and $X - (h \circ g)^{-1}(U)$ respectively and assume that W is smooth. Similarly, let W_1 and X_1 be the smooth projective varieties obtained by blowing-up $\widehat{W}$ and $X \times_V \widetilde{\widetilde{\widetilde{V}}}$ along subvarieties of $\widehat{W} - b_{\widehat{W}}^{-1}(U)$ and $X \times_V \widetilde{\widetilde{\widetilde{V}}} - (b_{\widehat{W}} \circ \widehat{h} \circ \widehat{g})^{-1}(U)$ respectively such that we have the following commutative diagram:

$$
\begin{tikzcd}
X_1 \arrow[rr, "\pi_{X_1}"] \arrow[dr, "\epsilon"] \arrow[dd, "f_1"'] & & X \arrow[dd, "f"] \\
& X \times_V \widetilde{\widetilde{V}} \arrow[ur, "\pi_X"] & \\
W_1 \arrow[rr, "b_{W_1}"] & & W
\end{tikzcd}
\tag{9}
$$

where W_1 is of general type, b_{W_1} is generically finite and ϵ is the blowing-up of $X \times_V \widetilde{\widetilde{V}}$. We write

$$K_{X_1} = \pi_{X_1}^* K_X + E,$$

where E is an exceptional divisor for π_{X_1} and $f_1(E)$ is a subvariety of $W_1 - b_{W_1}^{-1}(U)$ and

$$\pi_{X_1 *}\mathscr{O}_{X_1} = \pi_{X*}\epsilon_*\mathscr{O}_{X_1} = \pi_{X*}\mathscr{O}_{X\times_V \widetilde{V}} = \bigoplus_{\chi\in G^*} P_\chi,$$

where $P_\chi \in \mathrm{Pic}^0(X)$ is the torsion line bundle corresponding to $\chi \in G^*$.

In order to prove the Theorem, we will need to treat two different cases, $\kappa(W) > 0$ or $\kappa(W) = 0$. The strategy of the proofs are the same so I will treat the first case in details and point out that very similar arguments work for the second case.

Lemma 1.45 *Let X be a smooth projective variety with $P_m(X) > 0$ for some $m \geq 2$. Let $f : X \to W$ be as above. The Iitaka model of $(X, (m-1)K_{X/W} + f^*K_W)$ dominates W.*

Proof. We use the same notation as above. In (9), we already know that W_1 is of general type so by Viehweg's result (see the proof of Lemma 1.34), the Iitaka model of $(X_1, (m-1)K_{X_1/W_1} + f_1^*K_{W_1})$ dominates W_1. On the other hand, we can write

$$\begin{aligned} & (m-1)K_{X_1/W_1} + f_1^*K_{W_1} \\ = \;& \pi_{X_1}^*((m-1)K_{X/W} + f^*K_W) - (m-2)f_1^*K_{W_1/W} + (m-1)E. \end{aligned} \tag{10}$$

So the Iitaka model of $(X_1, \pi_{X_1}^*((m-1)K_{X/W} + f^*K_W) + (m-1)E)$ dominates W_1. Hence for any ample divisor H on W, there exists $N > 0$ such that

$\pi_{X_1}^*\mathscr{O}_X(N((m-1)K_{X/W}+f^*K_W)-f^*H)+N(m-1)E$ is effective. Since $\pi_{X_1*}\mathscr{O}_{X_1}(N(m-1)E)=\pi_{X_1*}\mathscr{O}_{X_1}$ is a direct sum of torsion line bundles, there exists $k>0$ such that $kN((m-1)K_{X/W}+f^*K_W)-kf^*H$ is effective. Therefore the Iitaka model of $(X,(m-1)K_{X/W}+f^*K_W)$ dominates W. $\square$

Since K_W is not necessarily big, we cannot directly apply Lemma 1.34. But we still have:

Lemma 1.46 *Under the assumption of Lemma 1.45, the sheaf*

$$f_*(\mathscr{O}_X(K_X+(m-1)K_{X/W}+f^*K_W)\otimes\mathscr{J}(||(m-1)K_{X/W}+f^*K_W||))$$

is nonzero, of rank $P_m(X_w)$, where X_w is a general fiber of f.

Proof. We use the diagram (9). Since W_1 is of general type, as in Lemma 1.34, by Viehweg's result, there exists $k>0$ such that for w_1 a general point of W_1 and $X_{w_1}\subset X_1$ the fiber of f_1, the restriction:

$$H^0(X_1,\mathscr{O}_{X_1}(km(m-1)K_{X_1/W_1}+kmf_1^*K_{W_1}))\to H^0(X_{w_1},\mathscr{O}_{X_{w_1}}(km(m-1)K_{X_{w_1}}))$$

is surjective. Since $K_{W_1/W}\succeq 0$, by (10), we have

$$\begin{aligned}&H^0(X_1,\mathscr{O}_{X_1}(km(m-1)K_{X_1/W_1}+kmf_1^*K_{W_1}))\\ \subseteq\ &H^0(X_1,\mathscr{O}_{X_1}(km(m-1)\pi_{X_1}^*K_{X/W}+km\pi_{X_1}^*f^*K_W+km(m-1)E)).\end{aligned}$$

Since E is π_{X_1}-exceptional, we conclude that

$$\begin{aligned}&|km(m-1)\pi_{X_1}^*K_{X/W}+km\pi_{X_1}^*f^*K_W+km(m-1)E|\\ =\ &|km(m-1)\pi_{X_1}^*K_{X/W}+km\pi_{X_1}^*f^*K_W|+km(m-1)E.\end{aligned}$$

We also know that $f_1(E)$ is a proper subvariety of W_1. These imply that the restriction:

$$\begin{aligned}&H^0(X_1,\mathscr{O}_{X_1}(km(m-1)\pi_{X_1}^*K_{X/W}+km\pi_{X_1}^*f^*K_W))\\ &\qquad\to H^0(X_{w_1},\mathscr{O}_{X_{w_1}}(km(m-1)K_{X_{w_1}}))\qquad(11)\end{aligned}$$

is surjective.

Set $w = b_{W_1}(w_1)$, and let X_w be the fiber of f. In the following diagram

$$\begin{tikzcd} \pi_{X_1}^{-1} f^{-1}(U) \arrow[r] \arrow[d] & f^{-1}(U) \arrow[d] \\ b_{W_1}^{-1}(U) \arrow[r] & U, \end{tikzcd}$$

all the morphisms are smooth. Hence $\pi_{X_{w_1}} = \pi_{X_1}|_{X_{w_1}} : X_{w_1} \to X_w$ is étale and the pull-back of $H^0(X_w, \mathscr{O}_{X_w}(km(m-1)K_{X_w}))$ is a subspace of $H^0(X_{w_1}, \mathscr{O}_{X_{w_1}}(km(m-1)K_{X_{w_1}}))$.

On the other side, we have

$$\begin{aligned} & H^0(X_1, \mathscr{O}_{X_1}(k(m-1)\pi_{X_1}^* K_{X/W} + k\pi_{X_1}^* f^* K_W)) \\ = & \bigoplus_{\chi \in G^*} \pi_{X_1}^* H^0(X, \mathscr{O}_X(k(m-1)K_{X/W} + kf^*K_W) \otimes P_\chi), \end{aligned} \tag{12}$$

Let M be the order of G. Take a resolution $\tau : X' \to X$ such that $\tau : X'_w \to X_w$ is also a resolution and

- $\tau^*|Mkm(m-1)K_{X/W} + Mkmf^*K_W| = |H| + E_M$,
- $\tau^*|\mathscr{O}_X(km(m-1)K_{X/W} + kmf^*K_W) \otimes P_\chi| = |H_\chi| + E_\chi$, for each $\chi \in G^*$,
- $\tau^*|km(m-1)K_{X_w}| = |H_w| + E_w$,
- $\tau^*|mK_{X_w}| = |H'_w| + E'_w$,

such that H, H_χ, H_w, and H'_w are base-point-free and E_M, E_χ, E_w, E'_w are the fixed divisors, with SNC supports.

Let X'_1 be a smooth model of the main component of $X_1 \times_X X'$ (the irreducible component that dominates X_1). We have the following commutative diagram:

$$\begin{tikzcd} X'_1 \arrow[r, "\pi_{X'_1}"] \arrow[d, "\tau_1"] & X' \arrow[d, "\tau"] \\ X_1 \arrow[r, "\pi_{X_1}"] \arrow[d, "f_1"] & X \arrow[d, "f"] \\ W_1 \arrow[r, "b_{W_1}"] & W \end{tikzcd}.$$

Let $U_1 = X_1 - E$. Then π_{X_1} is étale on U_1, hence $U_1 \times_X X'$ is irreducible and smooth. Since $f_1(E)$ is a proper subvariety of W_1, we can assume that there exists a divisor E' of X_1' such that $X_1' - E'$ is just $U_1 \times_X X'$ and $f_1\tau_1(E')$ is a proper subvariety of W_1. Let X_{w_1}' be the fiber of $f_1\tau_1$. Then $\pi_{X_{w_1}'} = \pi_{X_1'}|_{X_{w_1}'} : X_{w_1}' \to X_w'$ is Galois étale. We have another commutative diagram involving the morphisms of the fibers:

$$
\begin{tikzcd}
X_{w_1}' \arrow[r, "\text{étale}", "\pi_{X_{w_1}'}"'] \arrow[d, "\tau_1", "1:1"'] & X_w' \arrow[d, "1:1", "\tau"'] \\
X_{w_1} \arrow[r, "\pi_{X_{w_1}}", "\text{étale}"'] & X_w
\end{tikzcd} .
$$

We then write

$$
\begin{aligned}
& \tau_1^*|km(m-1)\pi_{X_1}^* K_{X/W} + km\pi_{X_1}^* f^* K_W| \\
= \ & |\pi_{X_1}'^{*}\tau^*(km(m-1)K_{X/W} + kmf^*K_W)| \\
= \ & |H'| + E_1',
\end{aligned}
$$

where E_1' is the fixed divisor. Let F be the maximal divisor $\preceq E_\chi$ for all $\chi \in G^*$. By (12), $\pi_{X_1'}^* F \preceq E_1'$. Hence by (11), we conclude that $\pi_{X_1'}^* F|_{X_{w_1}'}$ is fixed in $\tau_1^*|km(m-1)K_{X_{w_1}}|$ and in particular is fixed in $\pi_{X_{w_1}'}^* \tau^*|km(m-1)K_{X_w}|$ so $\pi_{X_1'}^* F|_{X_{w_1}'} \preceq \pi_{X_{w_1}'}^* E_w$. Since $\pi_{X_{w_1}'}$ is étale, we have

$$
\pi_{X_{w_1}'}^*(F|_{X_w'}) \preceq \pi_{X_1'}^* F|_{X_{w_1}'} \preceq \pi_{X_{w_1}'}^* E_w.
$$

We conclude that $F|_{X_w'} \preceq E_w$.

Since for any $\chi \in G^*$, we have the map:

$$
\begin{aligned}
H^0(X, \mathscr{O}_X(k(m-1)K_{X/W} + kf^*K_{W_1}) \otimes P_\chi)^{\otimes M} & \to H^0(X, \mathscr{O}_X(Mk(m-1)K_{X/W} + Mkf^*K_W)) \\
s_1 \otimes \cdots \otimes s_M & \mapsto s_1 \cdots s_M,
\end{aligned}
$$

we obtain $E_M \preceq MF$, hence $E_M|_{X_w'} \preceq ME_w \preceq Mk(m-1)E_w'$. This is just (4) in the proof of 2) of Lemma 1.34, and we can then finish the proof as there. $\square$

Remark 1.47 We may write Lemma 1.46 in a more general form:

Proposition 1.48 *Assume that we have the following commutative diagram between smooth projective varieties:*

$$\begin{tikzcd} X_1 \arrow[r, "\pi_{X_1}"] \arrow[d, "f_1"] & X \arrow[d, "f"] \\ W_1 \arrow[r, "b_{W_1}"] & W, \end{tikzcd}$$

where $P_m(X) > 0$, π_{X_1} is birationally equivalent to an étale morphism with an exceptional divisor E such that $f_1(E)$ is a proper subvariety of W_1, $\pi_{X_1 *}\mathscr{O}_{X_1} = \bigoplus_\alpha P_\alpha$ is a direct sum of torsion line bundles on X, W_1 is of general type, and b_{W_1} is generically finite and surjective. Then the sheaf

$$f_*(\mathscr{O}_X(K_X + (m-1)K_{X/W} + f^*K_W) \otimes \mathscr{J}(||(m-1)K_{X/W} + f^*K_W||))$$

is nonzero, of rank $P_m(X_w)$, where X_w is a general fiber of f.

According to Lemma 1.46,

$$\mathscr{F}_X = b_{W*}f_*(\mathscr{O}_X(K_X+(m-1)K_{X/W}+f^*K_W)\otimes\mathscr{J}(||(m-1)K_{X/W}+f^*K_W||)))$$

is a nonzero sheaf on A/K. By Lemma 1.32 and Lemma 1.45, it is an IT-sheaf of index 0.

Let $\widehat{\mathscr{F}_X}$ be the Fourier-Mukai transform of $\mathscr{F}_X$. By Fourier-Mukai transform on an abelian variety ([Mu], Theorem 2.2), we know that $\widehat{\mathscr{F}_X}$ is a W.I.T-sheaf of index $\dim A/K$ and its Fourier-Mukai transform $\widehat{\widehat{\mathscr{F}_X}}$ is isomorphic to $(-1_{A/K})^*\mathscr{F}_X$. In particular, $\widehat{\mathscr{F}_X} \neq 0$. Therefore by the Base Change Theorem of coherent sheaves and the definition of the Fourier-Mukai transform, there exists $P_0 \in \mathrm{Pic}^0(A/K)$ such that $h^0(A/K, \mathscr{F}_X \otimes P_0) \neq 0$. Thus for any $P \in \mathrm{Pic}^0(A/K)$,

$$h^0(A/K, \mathscr{F}_X \otimes P) = \chi(\mathscr{F}_X \otimes P) = \chi(\mathscr{F}_X \otimes P_0) = h^0(A/K, \mathscr{F}_X \otimes P_0) \geq 1.$$

Hence for any $P \in \mathrm{Pic}^0(A/K)$, we have

$$\begin{aligned} & h^0(X, \mathscr{O}_X(K_X + (m-1)K_{X/W} + f^*K_W) \otimes f^*b_W^*P) \\ \geq\ & h^0\big(X, \mathscr{O}_X(K_X + (m-1)K_{X/W} + f^*K_W) \otimes \mathscr{J}(||(m-1)K_{X/W} + f^*K_W||) \otimes f^*b_W^*P\big) \\ =\ & h^0\big(A/K, b_{W*}f_*(\mathscr{O}_X(K_X + (m-1)K_{X/W} + f^*K_W) \otimes \mathscr{J}(||(m-1)K_{X/W} + f^*K_W||)) \otimes P\big) \\ =\ & h^0(A/K, \mathscr{F}_X \otimes P) \\ \geq\ & 1. \end{aligned} \tag{13}$$

Lemma 1.49 *Let X, W be as in Lemma 1.45. Suppose that $\kappa(W) > 0$. Then for any $r \geq 3$, there exists a translate $T \subset \mathrm{Pic}^0(A/K)$ of a positive-dimensional torus, such that*

$$h^0(W, \mathscr{O}_W((r-2)K_W) \otimes b_W^* P) \geq r - 2,$$

for all $P \in T$.

Proof. Since $\kappa(W) > 0$, there exist a positive-dimensional abelian subvariety $T_0 \subset \mathrm{Pic}^0(A/K)$ and a torsion point $P_0 \in \mathrm{Pic}^0(A/K)$ such that $b_W^*(P_0 + T_0) \subset V_0(\omega_W)$ ([CH2], Corollary 2.4). Then we iterate Lemma 1.31 to get $h^0(W, \mathscr{O}_W((r-2)K_W) \otimes b_W^* P) \geq r-2$, for all $P \in (r-2)P_0 + T_0$. $\square$

If $\kappa(W) > 0$, since $mK_X = K_X + (m-1)K_{X/W} + f^*K_W + (m-2)f^*K_W$, again by (13), Lemma 1.49 and Lemma 1.31,

$$P_m(X) \geq 1 + m - 2 + \dim T - 1 \geq m - 1,$$

which contradicts our assumption. Hence we have finished the proof in the case $\kappa(W) > 0$.

If $\kappa(W) = 0$, in the diagram (8), b_W is surjective and finite and $\kappa(W) = 0$, hence W is an abelian variety by Kawamata's Theorem 1.43. We still have (13), however K_W is trivial, hence it is not enough for us to deduce a contradiction. We will need new versions of Lemma 1.45 and Lemma 1.46.

First we go back to diagrams (8) and (9):

$$\begin{tikzcd}
X_1 \arrow[r, "\pi_{X_1}"] \arrow[d, "g_1"] \arrow[dd, bend right=60, "f_1"'] & X \arrow[d, "g"] \arrow[dd, bend left=60, "f"] \\
V_1 \arrow[r, "\pi_{V_1}"] \arrow[d, "h_1"] & V \arrow[d, "h"] \\
W_1 \arrow[r, "b_{W_1}"] & W,
\end{tikzcd}$$

where V_1 is birational to $\widetilde{K} \times W_1$.

Since $\pi_{V_1} : V_1 \to V$ is birationally equivalent to the étale cover $\widetilde{V} \to V$, we have $\pi_{V_1 *}\omega_{V_1} = \bigoplus_{\chi \in G^*}(\omega_V \otimes P_\chi)$. On the other hand, V_1 is birational to $\widetilde{K} \times W_1$, hence $h_{1*}\omega_{V_1} = \omega_{W_1}$. Therefore, we have

$$b_{W_1 *}\omega_{W_1} = \bigoplus_{\chi \in G^*} h_*(\omega_V \otimes P_\chi).$$

Since b_{W_1} generically finite and W_1 is of general type, by Theorem 2.3 in [CH2], we know that the irreducible components of $V_0(b_{W_1*}\omega_{W_1})$ generate $\mathrm{Pic}^0(W)$. Hence there exists a $\chi \in G^*$ such that $V_0(h_*(\omega_V \otimes P_\chi))$ is a translated positive-dimensional abelian subvariety of $\mathrm{Pic}^0(W)$. We denote $h_*(\omega_V \otimes P_\chi)$ by $\mathscr{F}_\chi$. Since a general fiber of h is an abelian variety, $\mathscr{F}_\chi$ is a rank-1 torsion-free sheaf.

We can again birationally modify X so that $f^*\mathscr{F}_\chi$ is a line bundle on X. We then have the following lemma similar to Lemma 1.45.

Lemma 4.4* *Under the assumptions of Lemma 1.45, assume moreover that $\kappa(W) = 0$ and let $\mathscr{F}_\chi$ be as above. Then the Iitaka model of $(X, (m-1)K_X - (m-2)f^*\mathscr{F}_\chi)$ dominates W.* □

PROOF. The proof is analogue to that of Lemma 1.45. We have

$$(11^*)\quad \begin{aligned}&\pi_{X_1}^*((m-1)K_X - (m-2)f^*\mathscr{F}_\chi) + (m-1)E\\ &= (m-1)K_{X_1/W_1} + f_1^*K_{W_1} + (m-2)f_1^*K_{W_1} - (m-2)\pi_{X_1}^*f^*\mathscr{F}_\chi\ .\end{aligned}$$

Since $\mathscr{F}_\chi \subset b_{W_1*}\omega_{W_1}$, we have an inclusion $b_{W_1}^*\mathscr{F}_\chi \hookrightarrow \omega_{W_1}$, hence an inclusion

$$(m-2)f_1^*b_{W_1}^*\mathscr{F}_\chi = (m-2)\pi_{X_1}^*f^*\mathscr{F}_\chi \hookrightarrow (m-2)f_1^*\omega_{W_1}.$$

Using Viehweg's result as in the proof of Lemma 1.45, we obtain that the Iitaka model of $\pi_{X_1}^*((m-1)K_X - (m-2)f^*\mathscr{F}_\chi) + (m-1)E$ dominates W_1. Then by the same argument in Lemma 1.45, we finish the proof of Lemma 4.4*. □

We also need an analogue of Lemma 1.46.

Lemma 4.5* *Under the same assumption as Lemma 4.4*, the sheaf*

$$f_*(\mathscr{O}_X(mK_X - (m-2)f^*\mathscr{F}_\chi) \otimes \mathscr{J}(||(m-1)K_X - (m-2)f^*\mathscr{F}_\chi||))$$

is nonzero of rank $P_m(X_w)$, where X_w is a general fiber of f. □

PROOF. It is also parallel to the proof of Lemma 1.46. First, by Viehweg's result again, we have the surjectivity of the restriction map:

$$H^0(X_1, \mathscr{O}_{X_1}(km(m-1)K_{X_1/W_1} + kmf_1^*K_{W_1})) \\ \to H^0(X_{w_1}, \mathscr{O}_{X_{w_1}}(km(m-1)K_{X_{w_1}})).$$

Since E is π_{X_1}-exceptional and $(m-2)f_1^*K_{W_1} \succeq (m-2)\pi_{X_1}^* f^*\mathscr{F}_\chi$, by (10*), we have the surjectivity of the restriction map:

$$H^0(X_1, \pi_{X_1}^*(km(m-1)K_X - km(m-2)f^*\mathscr{F}_\chi)) \\ \to H^0(X_{w_1}, \mathscr{O}_{X_{w_1}}(km(m-1)K_{X_{w_1}})).$$

Then the rest of the proof is the same as the proof of Lemma 1.46. $\square$

By Lemma 4.4* and Lemma 4.5*, we again conclude as in (13) that

$$h^0(X, \mathscr{O}_X(mK_X - (m-2)f^*\mathscr{F}_\chi) \otimes f^*P) \geq 1,$$

for any $P \in \mathrm{Pic}^0(W)$.

As in the proof of Lemma 4.8, there exists a translate $T \subset \mathrm{Pic}^0(W)$ of a positive-dimensional abelian variety, such that $h^0(X, \mathscr{O}_X((m-2)f^*\mathscr{F}_\chi) \otimes f^*P) \geq m-2$, for any $P \in T$. We again have $P_m(X) \geq m-1$, which is a contradiction. This finishes the proof of Theorem 1.44 in the case $\kappa(W) = 0$.

In all, we have finished the proof of Theorem 1.44. $\square$

1.5 Varieties with $P_2(X) = 2$ and $q(X) = \dim X$

In this section we classify varieties with $P_2(X) = 2$ and $q(X) = \dim X$ (Theorem 5.9). The strategy follows from that of Chen and Hacon, who classify in [CH4] varieties with $P_3(X) = 4$ and $q(X) = \dim(X)$.

In this section, we will always assume that X is a smooth projective variety of maximal Albanese dimension. We consider an algebraic fiber space $f : X \to Y$ to a smooth projective variety such that f is a model of the Iitaka fibration of X. We have the commutative diagram:

$$\begin{tikzcd} X \arrow[r, "a_X"] \arrow[d, "f"] & \mathrm{Alb}(X) \arrow[d, "f_*"] \\ Y \arrow[r, "a_Y"] & \mathrm{Alb}(Y). \end{tikzcd} \tag{14}$$

By Proposition 2.1 of [HP], a_Y is generically finite, f_* is an algebraic fiber space, $\ker f_*$ is an abelian variety denoted by K, and a general fiber of f is birational to an abelian variety $\widetilde{K}$ isogenous to K. Define G to be the kernel of

$$\operatorname{Pic}^0(X) = \operatorname{Pic}^0(\operatorname{Alb}(X)) \to \operatorname{Pic}^0(K) \to \operatorname{Pic}^0(\widetilde{K}).$$

Then $f^* \operatorname{Pic}^0(Y)$ is contained in G and $G/f^* \operatorname{Pic}^0(Y)$, denoted by $\overline{G}$, is a finite group consisting of elements $\chi_1, \ldots, \chi_r$. Let $P_{\chi_1}, \ldots, P_{\chi_r} \in G$ be torsion line bundles representing lifts of the elements of $\overline{G}$, so that

$$G = \bigsqcup_{i=1}^{r} (P_{\chi_i} + f^* \operatorname{Pic}^0(Y)).$$

There is an easy observation:

Lemma 1.50 *Let X be a smooth projective variety of maximal Albanese dimension and $P \in \operatorname{Pic}^0(X)$. If $H^0(X, \omega_X^m \otimes P) \neq 0$ for some $m > 0$, we have $P \in G$.*

Proof. If $P \notin G$, since a general fiber F of f is birational to the abelian variety $\widetilde{K}$ and $P|_{\widetilde{K}}$ is non-trivial, any section of $\omega_X^m \otimes P$ vanishes on F. Hence $H^0(X, \omega_X^m \otimes P) = 0$, which is a contradiction. □

We then need a corollary of Lemma 1.33:

Corollary 1.51 *Let $f : X \to C$ be a surjective morphism between a smooth projective variety X and an smooth projective curve C of genus $g(C) \geq 1$ such that the Iitaka model of X dominates C. If for some torsion element $Q \in \operatorname{Pic}^0(X)$ and some integer $m \geq 2$, we have $h^0(X, \omega_X^m \otimes Q) \neq 0$, then $f_*(\omega_X^m \otimes Q)$ is an ample vector bundle on C.*

Proof. By Corollary 3.6 in [V1], $f_*(\omega_{X/C}^m \otimes Q)$ is nef.

If $g(C) \geq 2$, ω_C is ample. Hence $f_*(\omega_X^m \otimes Q)$ is an ample vector bundle on C.

If $g(C) = 1$, we first claim that

♣ For any nef vector bundle F on an elliptic curve C, $V_1(F)$ consists of only finitely many points.

We prove the claim by induction on the rank of F. The rank 1 case is trivial. Let $r > 0$ be an integer. Assuming that ♣ has been proved for any nef vector bundle of rank $\leq r$, we just need to prove ♣ for any nef vector bundle of rank $r + 1$.

Let F be a nef vector bundle of rank $r + 1$. We consider the Harder-Narasimhan filtration of F (see Proposition 6.4.7 in [La]):

$$0 = F_n \subset F_{n-1} \subset \ldots \subset F_1 \subset F_0 = F,$$

where F_i are sub-bundles of F with the properties that F_i/F_{i+1} is a semistable bundle for each i and

$$\mu(F_{n-1}/F_n) > \ldots > \mu(F_1/F_2) > \mu(F_0/F_1).$$

Since $F = F_0$ is nef, so is F_0/F_1, hence $\mu(F_0/F_1) \geq 0$. So F_i/F_{i+1} is a semistable vector bundle with positive slope, for each $i \geq 1$. Hence, for each $i \geq 1$, F_i/F_{i+1} is an ample vector bundle (see Main Claim in the proof of Theorem 6.4.15 in [La]). Thus F_1 is also ample, and $V_1(F_1)$ is empty. We just need to prove that $V_1(F_0/F_1)$ consists of only finitely many points.

If $\mu(F_0/F_1) > 0$, we again have that F_0/F_1 is ample and $V_1(F)$ is empty. We are done. If $\mu(F_0/F_1) = 0$, take $P \in V_1(F_0/F_1)$. Then $h^1(C, F_0/F_1 \otimes P) \neq 0$. Hence, by Serre duality, there exists a non-trivial homomorphism of bundles

$$\pi_P : F_0/F_1 \to P^\vee.$$

Since F_0/F_1 is semistable and $\mu(F_0/F_1) = 0$, π_P is surjective. We have an exact sequence of vector bundles:

$$0 \to G \to F_0/F_1 \to P^\vee \to 0.$$

The rank of G is $\leq r$. Since F_0/F_1 is semistable and $\mu(G) = \mu(F_0/F_1) = 0$, G is also semistable. Hence G is a nef vector bundle (by the Main Claim quoted above) of rank $\leq r$ and, by induction, $V_1(G)$ consists of finitely many points. We conclude that $V_1(F) = V_1(G) \cup \{P\}$ consists of only finitely many points. We have finished the proof of the Claim.

Let the line bundle Q and the integer m be as in the assumption. By Lemma 1.33, for $m \geq 2$, $h^0(X, \omega_X^m \otimes Q \otimes f^*P) = h^0(C, f_*(\omega_X^m \otimes Q) \otimes P)$ is constant for all $P \in \mathrm{Pic}^0(C)$, hence $h^1(C, f_*(\omega_X^m \otimes Q) \otimes P)$ is also constant for all $P \in \mathrm{Pic}^0(C)$. By the claim ♣, $h^1(C, f_*(\omega_X^m \otimes Q) \otimes P) = 0$ for all

$P \in \operatorname{Pic}^0(C)$ and $m \geq 2$. Hence $f_*(\omega_X^m \otimes Q)$ is an I.T. vector bundle of index 0 on C. Hence $f_*(\omega_X^m \otimes Q)$ is ample on C by Corollary 3.2 in [D4]. $\square$

We know that $V_0(\omega_X)$ is a torsion translated subtorus of $\operatorname{Pic}^0(X)$ ([GL2], [S]). Furthermore, Chen and Hacon made several useful observations about $V_0(\omega_X)$ ([CH2], Lemma 2.2), which we summarize in the following lemma.

Lemma 1.52 (Chen-Hacon) *Under the above assumptions, we have*

(1) $V_0(\omega_X) \subset G$.

(2) Denote by $V_0(\omega_X, \chi_i)$ *the union of irreducible components of* $V_0(\omega_X)$ *contained in* $P_{\chi_i} + f^* \operatorname{Pic}^0(Y)$. *Then for every* i, $V_0(\omega_X, \chi_i)$ *is not empty.*

(3) If $P_{\chi_i} \notin f^* \operatorname{Pic}^0(Y)$, *there exists a positive-dimensional component of* $V_0(\omega_X, \chi_i)$.

The following lemma is an analogue of Corollary 3.2 in [CH5].

Lemma 1.53 *Let* X *be a variety of maximal Albanese dimension with* $\kappa(X) \geq 2$. *Suppose that there is a surjective morphism* $g : X \to C$ *onto an elliptic curve* C, *and suppose that there is an ample line bundle* L *on* C *with an inclusion* $g^*L \hookrightarrow \omega_X \otimes P_2$ *for some torsion line bundle* $P_2 \in \operatorname{Pic}^0(X)$. *Then we have*

$$h^0(X, \omega_X^m \otimes P_1 \otimes P_2) \geq h^0(X, \omega_X^{m-1} \otimes P_1) + 2,$$

for all torsion line bundles $P_1 \in V_0(\omega_X)$ *and all* $m \geq 3$.

Proof. From the inclusion, we obtain $H^0(X, \omega_X \otimes P_2) \neq 0$, and by (1) and (2) in Lemma 1.52, we conclude that $P_2 \in V_0(\omega_X, \chi_i)$ for some i and we get an exact sequence of sheaves on C:

$$0 \to g_*(\omega_X^{m-1} \otimes P_1) \otimes L \hookrightarrow g_*(\omega_X^m \otimes P_1 \otimes P_2) \to \mathscr{Q} \to 0. \tag{15}$$

By (2) in Lemma 5.2, we have $h^0(X, \omega_X \otimes P_2^\vee \otimes P) \neq 0$ for some $P \in f^* \operatorname{Pic}^0(Y)$ such that $P_2^\vee \otimes P \in V_0(\omega_X, -\chi_i)$. Hence we have an inclusion $g^*L \hookrightarrow \omega_X^2 \otimes P$. Moreover, since P_2 is a torsion line bundle and $V_0(\omega_X)$ is a torsion translated torus of $\operatorname{Pic}^0(X)$, we may assume that $P \in f^* \operatorname{Pic}^0(Y)$ is also a torsion line bundle. Therefore the Iitaka model of X dominates C.

Thus, by Corollary 1.51, both $g_*(\omega_X^{m-1}\otimes P_1)$ and $g_*(\omega_X^m\otimes P_1\otimes P_2)$ are ample, and so is $\mathcal{Q}$. By Serre duality, for any ample vector bundle V on C, we have $H^1(C,V)=0$. Hence, Riemann-Roch gives

$$h^0(X,\omega_X^{m-1}\otimes P_1)=h^0(C,g_*(\omega_X^{m-1}\otimes P_1))=\deg(g_*(\omega_X^{m-1}\otimes P_1)),$$

and

$$h^0(\omega_X^m\otimes P_1\otimes P_2) \;=\; h^0(C,g_*(\omega_X^{m-1}\otimes P_1)\otimes L)+h^0(C,\mathcal{Q}).$$

Let F be a connected component of a general fiber of g. Since $\kappa(X)\geq 2$, we have $\kappa(F)\geq 1$ by the easy addition formula (Corollary 1.7 in [Mo]). Hence we have $P_2(F)\geq 2$ by Theorem 3.2 in [CH1] (see also Remark 1.58). Since $P_1\in V_0(\omega_X)$, we have $h^0(X,\omega_X^m\otimes P_1\otimes P_2)\geq h^0(X,\omega_X^{m-1}\otimes P_2)>0$. Hence we have $h^0(F,\omega_F^m\otimes P_1\otimes P_2)\neq 0$. Then, by Lemma 1.50 and Lemma 1.52, we have $(P_1\otimes P_2)|_F\otimes P^{'}\in V_0(\omega_F)$, where $P^{'}\in \mathrm{Pic}^0(F)$ is pulled back from the Iitaka fibration of F. On the other hand, since $P_1\otimes P_2$ is torsion, we have $h^0(F,\omega_F^m\otimes P_1\otimes P_2)=h^0(F,\omega_F^m\otimes P_1\otimes P_2\otimes P^{'})$ by Lemma 3.2. Therefore, we conclude that

$$\begin{aligned} h^0(F,\omega_F^m\otimes P_1\otimes P_2) &= h^0(F,\omega_F^m\otimes P_1\otimes P_2\otimes P^{'})\\ &\geq P_{m-1}(F)\\ &\geq 2, \end{aligned}$$

where the last inequality holds for $m\geq 3$. Hence,

$$\mathrm{rank}(g_*(\omega_X^m\otimes P_1\otimes P_2))=h^0(F,\omega_F^m\otimes P_1\otimes P_2)\geq 2.$$

Since $P_1\in V_0(\omega_X)$ by assumption, we have $\mathrm{rank}(g_*(\omega_X^{m-1}\otimes P_1))\geq 1$.

If $\mathrm{rank}(g_*(\omega_X^{m-1}\otimes P_1))\geq 2$, we have

$$\begin{aligned} h^0(C,g_*(\omega_X^m\otimes P_1\otimes P_2)) &\geq h^0(C,g_*(\omega_X^{m-1}\otimes P_1)\otimes L)\\ &\geq \deg(g_*(\omega_X^{m-1}\otimes P_1))+\mathrm{rank}(g_*(\omega_X^{m-1}\otimes P_1))\\ &\geq h^0(X,\omega_X^{m-1}\otimes P_1))+2. \end{aligned}$$

If $\mathrm{rank}(g_*(\omega_X^{m-1}\otimes P_1))=1$, $\mathcal{Q}$ has rank ≥ 1. Since $\mathcal{Q}$ is ample, $h^0(C,\mathcal{Q})\geq 1$. We also have

$$\begin{aligned} h^0(X,\omega_X^m\otimes P_1\otimes P_2) &= h^0(C,g_*(\omega_X^{m-1}\otimes P_1)\otimes L)+h^0(C,\mathcal{Q})\\ &\geq h^0(X,\omega_X^{m-1}\otimes P_1)+\mathrm{rank}(\omega_X^{m-1}\otimes P_1))+1\\ &= h^0(X,\omega_X^{m-1}\otimes P_1)+2. \end{aligned}$$

Hence the lemma is proved. $\square$

Proposition 1.54 *Let X be a smooth projective variety with $0 < P_m(X) \leq 2m - 2$ for some $m \geq 2$ and $q(X) = \dim X$. Let $f : X \to Y$ be an algebraic fiber space between smooth projective varieties which is birationally equivalent to the Iitaka fibration of X. Then Y is birational to an abelian variety.*

Proof. This proposition is a corollary of the proof of Proposition 3.6 and Proposition 3.7 in [CH5], although not explicitly stated there.

Since we have $0 < P_m(X) \leq 2m - 2$ for some $m \geq 2$, a_X is surjective by Theorem 1.39. Since $q(X) = \dim X$, we saw above that a_X and a_Y are both surjective and generically finite. We then use diagram (14) and the notation above.

If $\kappa(X) = 1$, then Y is an elliptic curve, for a_Y is surjective.

If $\kappa(X) \geq 2$, we use the same argument as in the proof of Proposition 3.6 in [CH5]. We claim that

(†) $V_0(\omega_X) \cap f^* \operatorname{Pic}^0(Y) = \{\mathscr{O}_X\}$.

Let δ be the maximal dimension of a component of $V_0(\omega_X) \cap f^* \operatorname{Pic}^0(Y)$.

If $\delta = 0$, $V_0(\omega_X) \cap f^* \operatorname{Pic}^0(Y) = \{\mathscr{O}_X\}$ by Proposition 1.3.3 in [CH2].

If $\delta \geq 2$, by Lemma 1.31, there exists $P_0 \in f^* \operatorname{Pic}^0(Y)$ such that

$$h^0(X, \omega_X^2 \otimes P_0) \geq 1 + 1 + 2 - 1 = 3.$$

By Lemma 1.33, $h^0(X, \omega_X^2 \otimes P) = h^0(X, \omega_X^2 \otimes P_0) \geq 3$, for any $P \in f^* \operatorname{Pic}^0(Y)$. We iterate this process and get $P_m(X) \geq 2m - 1$, which is a contradiction.

If $\delta = 1$, there is a 1-dimensional component T of $V_0(\omega_X) \cap f^* \operatorname{Pic}^0(Y)$. Let $E = \operatorname{Pic}^0(T)$ and let $g : X \to E$ be the induced surjective morphism. By Corollary 2.11 and Lemma 2.13 in [CH5], for some torsion element $P \in T$, there exist a line bundle L of degree 1 on E and an inclusion $g^*L \to \omega_X \otimes P$, and $P|_F = \mathscr{O}_F$, where F is a general fiber of g. Since $\kappa(X) \geq 2$, we have $\kappa(F) \geq 1$. Again by Theorem 3.2 in [CH1], $\operatorname{rank}(g_*(\omega_X^2 \otimes P^2)) = P_2(F) \geq 2$. Consider the exact sequence of sheaves on E,

$$0 \to L^2 \to g_*(\omega_X^2 \otimes P^2) \to \mathscr{Q} \to 0,$$

where $\operatorname{rank}(\mathscr{Q}) \geq 1$. Since $g : X \to C$ is dominated by $f : X \to Y$, the Iitaka model of X (i.e. Y) dominates E, hence $g_*(\omega_X^2 \otimes P^2)$ is ample by Corollary 1.51, so is $\mathscr{Q}$ and $h^0(X, \mathscr{Q}) \geq 1$. Hence $h^0(X, \omega_X^2 \otimes P^2) \geq 3$.

For any $k \geq 3$, we apply Lemma 1.53 to get that

$$h^0(X, \omega_X^k \otimes P^k) \geq h^0(X, \omega_X^{k-1} \otimes P^{k-1}) + 2.$$

By induction, we have $h^0(X, \omega_X^m \otimes P^m) \geq 2m - 1$, for all $m \geq 2$. Since $P \in T \subset f^* \operatorname{Pic}^0(Y)$, we have, by Lemma 1.33, $P_m(X) = h^0(X, \omega_X^m \otimes P^m) \geq 2m - 1$, which is a contradiction.

We have proved the claim (†).

Since X and Y are of maximal Albanese dimension, $K_{X/Y}$ is effective (see the proof of Lemma 3.1 in [CH1]). This implies

$$f^* V_0(\omega_Y) \subset V_0(\omega_X) \cap f^* \operatorname{Pic}^0(Y) = \{\mathscr{O}_X\},$$

and hence $\kappa(Y) = 0$ by Theorem 1 in [CH2]. By Kawamata's Theorem 1.6, a_Y is birational. $\square$

We go back to the commutative diagram (14) and suppose that X is a smooth projective variety with maximal Albanese dimension and $\kappa(X) > 0$. Since every component $V_0(\omega_X)$ is a torsion translate of an abelian subvariety of $\operatorname{Pic}^0(X)$ (see [GL2] and [S]), we can write by (1):

$$V_0(\omega_X) = \bigcup_{1 \leq i \leq r} \bigcup_s (P_{\chi_{i,s}} + T_{\chi_{i,s}}) \subset G,$$

where $P_{\chi_{i,s}} \in P_{\chi_i} + f^* \operatorname{Pic}^0(Y)$ is a torsion point and $T_{\chi_{i,s}}$ is an abelian subvariety of $f^* \operatorname{Pic}^0(Y)$.

Definition 1.55 Keeping the notation as above, we call $T_{\chi_{i,s}}$ a maximal component of $V_0(\omega_X)$, if $T_{\chi_{i,s}}$ is maximal for the inclusion among all $T_{\chi_{j,t}}$. By Theorem 2.3 in [CH2], necessarily, $\dim T_{\chi_{i,s}} \geq 1$.

Proposition 1.56 *Assume that X is a smooth projective variety with maximal Albanese dimension and $\kappa(X) > 0$. Let $T_{\chi_{i,s}}$ be a maximal component of $V_0(\omega_X)$. Then, for any (j,t) such that $\dim T_{\chi_{j,t}} \geq 1$, we have $\dim(T_{\chi_{i,s}} \cap T_{\chi_{j,t}}) \geq 1$.*

PROOF. Let $\widehat{T}_{\chi_{i,s}}$ and $\widehat{T}_{\chi_{j,t}}$ be the dual of $T_{\chi_{i,s}}$ and $T_{\chi_{j,t}}$ respectively. Let π_1 and π_2 be the natural morphisms of abelian varieties $\operatorname{Alb}(X) \to \widehat{T}_{\chi_{i,s}}$ and $\operatorname{Alb}(X) \to \widehat{T}_{\chi_{j,t}}$. Take an étale cover $t : \widetilde{X} \to X$ which is induced by an étale cover of $\mathrm{Alb}(X)$ such that $t^*P_{\chi_{i,s}}$ and $t^*P_{\chi_{j,t}}$ are trivial. Let f_1 and f_2 be the compositions of morphisms $\pi_1 \circ a_X \circ t$ and $\pi_2 \circ a_X \circ t$, respectively. We then take the Stein factorizations of f_1 and f_2:

$$\begin{tikzcd} \widetilde{X} \arrow[d, "g_1"] \arrow[dr, "f_1"] & \\ X_1 \arrow[r, "h_1"] & \widehat{T}_{\chi_{i,s}} \end{tikzcd}, \qquad \begin{tikzcd} \widetilde{X} \arrow[d, "g_2"] \arrow[dr, "f_2"] & \\ X_2 \arrow[r, "h_2"] & \widehat{T}_{\chi_{j,t}} \end{tikzcd}$$

After modifications, we can assume that X_1 and X_2 are smooth.

We claim that:

- $h^0(X_1, \omega_{X_1} \otimes h_1^*P) > 0$, for all $P \in T_{\chi_{i,s}}$, and similarly $h^0(X_2, \omega_{X_2} \otimes h_2^*Q) > 0$, for all $Q \in T_{\chi_{j,t}}$.

The argument to prove the claim is due to Chen and Debarre in [CD]. Let c be the codimension of $T_{\chi_{i,s}}$ in $\mathrm{Pic}^0(X)$. By the proof of Theorem 3 of [EL1], $P_{\chi_{i,s}} + T_{\chi_{i,s}}$ is an irreducible component of $V_c(\omega_X)$. Hence

$$h^c(\widetilde{X}, \omega_{\widetilde{X}} \otimes t^*P) \geq h^c(X, \omega_X \otimes P \otimes P_{\chi_{i,s}}) > 0$$

for any $P \in T_{\chi_{i,s}}$.

Again by the proof of Theorem 3 in [EL1], the dimension of a general fiber of g_1 is also c. Since g_1 is an algebraic fiber space, we have $R^c g_{1*}\omega_{\widetilde{X}} = \omega_{X_1}$ ([Kol3], Proposition 7.6), and

$$R^c f_{1*}\omega_{\widetilde{X}} = h_{1*}(R^c g_{1*}\omega_{\widetilde{X}}) = h_{1*}\omega_{X_1},$$

([Kol4], Theorem 3.4). Moreover, the sheaves $R^k f_{1*}\omega_{\widetilde{X}}$, satisfy the generic vanishing theorem ([H], Corollary 4.2), hence $V_j(R^k f_{1*}\omega_{\widetilde{X}}) \neq T_{\chi_{i,s}}$ for any $j > 0$. Pick $P \in T_{\chi_{i,s}} - \bigcup_{j>0,k} V_j(R^k f_{1*}\omega_{\widetilde{X}})$, so that

$$H^j(\widehat{T}_{\chi_i,s}, R^k f_{1*}\omega_{\widetilde{X}} \otimes P) = 0$$

for all $j > 0$ and all k. By the Leray spectral sequence, we have

$$\begin{aligned} 0 \neq h^c(\widetilde{X}, \omega_{\widetilde{X}} \otimes f_1^*P) &= h^0(\widehat{T}_{\chi_{i,s}}, R^c f_{1*}\omega_{\widetilde{X}} \otimes P) \\ &= h^0(\widehat{T}_{\chi_{i,s}}, h_{1*}\omega_{\widetilde{X}} \otimes P). \end{aligned}$$

Hence we conclude the claim by semi-continuity.

If $\dim(T_{\chi_{i,s}} \cap T_{\chi_{j,t}}) = 0$, the morphism

$$\mathrm{Alb}(X) \xrightarrow{(\pi_1,\pi_2)} \widehat{T}_{\chi_{i,s}} \times \widehat{T}_{\chi_{j,t}}$$

is surjective. Now we consider the following diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "t"] \arrow[d, "g_1"] & X \arrow[r, "a_X"] \arrow[dr] & \mathrm{Alb}(X) \arrow[d, "\pi_1"] \\
X_1 \arrow[rr, "h_1"] & & \widehat{T}_{\chi_{i,s}}.
\end{tikzcd}$$

¿From the proof of Theorem 3 in [EL1], we know that the fibers of g_1 fill up the fibers of π_1. Hence we have a surjective morphism $\widetilde{X} \xrightarrow{(g_1,g_2)} X_1 \times X_2$. Since $a_X \circ t : \widetilde{X} \to \mathrm{Alb}(X)$ and $(h_1, h_2) : X_1 \times X_2 \to \widehat{T}_{\chi_{i,s}} \times \widehat{T}_{\chi_{j,t}}$ are generically finite and surjective, by the proof of Lemma 3.1 in [CH1], $K_{\widetilde{X}/X_1 \times X_2}$ is effective. Therefore, it follows from the claim that

$$h^0(\widetilde{X}, \omega_{\widetilde{X}} \otimes t^*P \otimes t^*Q) > 0 \tag{16}$$

for all $P \in T_{\chi_{i,s}}$ and $Q \in T_{\chi_{j,t}}$. Since $t : \widetilde{X} \to X$ is birationally equivalent to an étale cover of X induced by an étale cover of $\mathrm{Alb}(X)$, $t_*\mathscr{O}_{\widetilde{X}} = \bigoplus_\alpha P_\alpha$, where P_α is a torsion line bundle on X for every α. Let

$$T = T_{\chi_{i,s}} + T_{\chi_{j,t}}$$

be the abelian variety generated by $T_{\chi_{i,s}}$ and $T_{\chi_{j,t}}$. Then (15) implies that there exists an α such that

$$P_\alpha + T \subset V_0(\omega_X).$$

Since $\dim T_{\chi_{j,t}} \geq 1$ and $\dim(T_{\chi_{i,s}} \cap T_{\chi_{j,t}}) = 0$, $T_{\chi_{i,s}} \subset T$ is a proper subvariety, contradicting the assumption that $T_{\chi_{i,s}}$ is a maximal component of $V_0(\omega_X)$. This finishes the proof of the proposition. $\square$

Proposition 1.57 *Let X be a smooth projective variety with $P_2(X) = 2$ and $q(X) = \dim X$. Assume that $f : X \to Y$ is a birational model of the Iitaka fibration of X and Y is a smooth projective variety. Then Y is an elliptic curve.*

Proof. We use diagram (14). By Theorem 1.39, a_X is surjective and hence generically finite. By Proposition 1.54, we may assume that $a_Y : Y \to \mathrm{Alb}(Y)$ is an isomorphism.

If $\dim Y = 1$, Y is an elliptic curve. We are done. We now assume that $\dim Y \geq 2$ and deduce a contradiction.

Let $T_{\chi_{i,s}}$ be a maximal component of $V_0(\omega_X)$. By the claim (†) in the proof of Proposition 1.54, we know that $P_{\chi_i} \notin f^* \mathrm{Pic}^0(Y)$. By (3) of Lemma 1.52, there exist a torsion line bundle $P_{-\chi_{i,t}} \in P_{\chi_i}^{\vee} + f^* \mathrm{Pic}^0(Y)$ and a positive-dimensional abelian subvariety $T_{-\chi_{i,t}} \subset f^* \mathrm{Pic}^0(Y)$ such that $P_{-\chi_{i,t}} + T_{-\chi_{i,t}}$ is a connected component of $V_0(\omega_X)$. Take T the neutral component of $T_{\chi_{i,s}} \cap T_{-\chi_{i,t}}$. By Proposition 1.56, $\dim T \geq 1$.

If $\dim T \geq 2$, then by Lemma 1.31,

$$h^0(X, \omega_X^2 \otimes P \otimes Q) \geq 1 + 1 + 2 - 1 = 3,$$

for all $P \in P_{\chi_{i,s}} + T$ and all $Q \in P_{-\chi_{i,t}} + T$. Since $P_{\chi_{i,s}} \otimes P_{-\chi_{i,t}} \in f^* \mathrm{Pic}^0(Y)$, we conclude, by Lemma 1.33, $P_2(X) \geq 3$, which is a contradiction.

Hence T is an elliptic curve and we denote by $\widehat{T}$ its dual.

There exists a projection $\pi : Y \to \widehat{T}$. We then consider the commutative diagram:

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "\bar{f}"] & \\ Y \arrow[r, "\pi"] & \widehat{T}. \end{tikzcd}$$

We define

$$\begin{aligned} F_1 &= \bar{f}_*(\omega_X \otimes P_{\chi_{i,s}}), \\ F_2 &= \bar{f}_*(\omega_X \otimes P_{-\chi_{i,t}}), \\ F_3 &= \bar{f}_*(\omega_X^2 \otimes P_{\chi_{i,s}} \otimes P_{-\chi_{i,t}}). \end{aligned}$$

These are vector bundles on the elliptic curve $\widehat{T}$ and by Corollary 3.6 in [V1] and Corollary 1.51, F_1 and F_2 are nef and F_3 is ample.

Since $P_{\chi_{i,s}} \otimes P_{-\chi_{i,t}} \in f^* \mathrm{Pic}^0(Y)$ and $f : X \to Y$ is a model of the Iitaka fibration of X, we have

$$2 = P_2(X) = h^0(X, \omega_X^2 \otimes P_{\chi_{i,s}} \otimes P_{-\chi_{i,t}}) = h^0(\widehat{T}, F_3).$$

There exists a natural morphism:

$$F_1 \otimes F_2 \xrightarrow{v} F_3,$$

corresponding to the multiplication of sections. Let R_1, R_2, and R_3 be the respective ranks of F_1, F_2, and F_3.

I claim that

♠ $R_3 > \min\{R_1, R_2\}$.

Indeed if $R_1 \geq 2$ and $R_2 \geq 2$, then by Lemma 1.31, $R_3 \geq R_1 + R_2 - 1$. If either R_1 or R_2 is 1, we just need to prove that $R_3 \geq 2$. Let $f|_{X_t} : X_t \to Y_t$ be the restriction of f to a general fiber of $\bar{f}$. Since $f : X \to Y$ is a model of the Iitaka fibration of X, fixing an ample divisor H on Y, there exists a positive integer $N > 0$ such that $NK_X - H$ is effective. Hence $(NK_X - H)|_{X_t} \succeq 0$, therefore, the Iitaka model of X_t dominates Y_t. Indeed, $f|_{X_t} : X_t \to Y_t$ is a birational model of the Iitaka fibration of X_t since a general fiber of $f|_{X_t}$ is isomorphic to a general fiber of f which is birational to an abelian variety. As we have assumed that $\dim Y \geq 2$, we have $\dim Y_t \geq 1$. Thus X_t is of maximal Albanese dimension and $\kappa(X_t) \geq 1$, hence $P_2(X_t) \geq 2$ ([CH1], Theorem 3.2). Since $(P_{\chi_{i,s}} \otimes P_{-\chi_{i,t}})|_{X_t}$ is pulled back from Y_t, we have

$$R_3 = h^0(X_t, (\omega_X^2 \otimes P_{\chi_{i,s}} \otimes P_{-\chi_{i,t}})|_{X_t}) = P_2(X_t) \geq 2,$$

where the second equality holds because of Lemma 1.33.

Consider the Harder-Narasimhan filtrations of F_1 and F_2 and let G_1 (resp. G_2) be the unique maximal sub-bundle of F_1 (resp. F_2) of largest slope. By definition, G_1 and G_2 are semistable.

I claim that

- $\deg(G_1) > 0$ and $\deg(G_2) > 0$ and therefore G_1 and G_2 are ample.

If $\deg(G_1) = 0$, we conclude from the definition of G_1 that $0 = \mu(G_1) \geq \mu(F_1)$. Since F_1 is nef, we conclude that $\deg(F) = \mu(F) = 0$. Then $V_0(F_1) = V_1(F_1)$. By the claim ♣ in the proof of Corollary 1.51, $V_0(F_1) = V_1(F_1)$ consists of only finitely many points. However, since $T \subset T_{\chi_{i,s}}$, we have $h^0(F_1 \otimes P) > 0$ for all $P \in T$, which is a contradiction. So $\deg(G_1) > 0$. Similarly, we prove that $\deg(G_2) > 0$. Since G_1 and G_2 are semistable, they are ample (see the Main Claim in the proof of Theorem 6.4.15 in [La]).

Let $r_1 > 0$ and $r_2 > 0$ be the ranks of G_1 and G_2. Set $G_3 = \upsilon(G_1 \otimes G_2)$ and let r_3 be its rank. Again by Lemma 1.31, we have

$$r_3 \geq r_1 + r_2 - 1 \geq \max\{r_1, r_2\}.$$

Since G_1 and G_2 are semistable and ample, so is $G_1 \otimes G_2$ (see Corollary 6.4.14 in [La]). Therefore the slopes satisfy

$$\mu(G_3) \geq \mu(G_1 \otimes G_2) = \mu(G_1) + \mu(G_2),$$

and G_3 is also ample.

We then apply Riemann-Roch,

$$\begin{aligned} h^0(\widehat{T}, G_3) &\geq r_3(\mu(G_1) + \mu(G_2)) \\ &\geq \deg(G_1) + \deg(G_2) \\ &\geq 2, \end{aligned}$$

where the second inequality holds because $r_3 \geq \max\{r_1, r_2\}$ and the third inequality holds because $\deg(G_1) > 0$ and $\deg(G_2) > 0$.

Since G_3 is a sub-bundle of F_3 and $h^0(\widehat{T}, F_3) = 2$, we have $h^0(\widehat{T}, G_3) = 2$, hence all the inequalities above should be equalities. In particular, $r_3 = r_1 = r_2$. Hence by the claim ♠, $r_3 \leq \min\{R_1, R_2\} < R_3$. Therefore, F_3/G_3 is a sheaf of rank ≥ 1. Since F_3 is ample, so is F_3/G_3, hence $h^0(\widehat{T}, F_3/G_3) \geq 1$. Since G_3 is ample, $h^1(\widehat{T}, G_3) = 0$. Hence, by Riemann-Roch,

$$h^0(\widehat{T}, F_3) = h^0(\widehat{T}, G_3) + h^0(\widehat{T}, F_3/G_3) \geq 3,$$

which is a contradiction. Thus $\dim Y = 1$. This finishes the proof of Proposition 1.57. □

Remark 1.58 It is easy to see that combining Proposition 1.56 and the proof of Proposition 1.57, one can give another proof of Chen and Hacon's characterization of abelian varieties:

> If X is a smooth projective variety with maximal Albanese dimension. If $P_1(X) = P_2(X) = 1$, X is birational to an abelian variety.

Theorem 1.59 *Let X be a smooth projective variety with $P_2(X) = 2$ and $q(X) = \dim X$. Then $\kappa(X) = 1$ and X is birational to a quotient $(K \times C)/G$, where K is an abelian variety and C is a curve, G is a finite group that acts diagonally and freely on $K \times C$, and $C \to C/G$ is branched at 2 points.*

Proof. Since we know by Proposition 1.57 that Y is an elliptic curve, the proof is parallel to the proof of Theorem 5.1 in [CH5]. By Theorem 1.43, there exists a curve C of genus $g \geq 2$, an abelian variety $\widetilde{K}$, and a finite abelian group G, which acts faithfully on C and $\widetilde{K}$, such that X is birational to $(\widetilde{K} \times C)/G$, where G acts diagonally and freely on $\widetilde{K} \times C$.

We then consider the induced morphism $\phi : C \to C/G = Y$. Following [Be1] §VI.12, we have

$$2 = P_2(X) = \dim H^0(C, \omega_C^2)^G = h^0(Y, \mathscr{O}_Y(\sum_{P \in Y} \left\lfloor 2(1 - \frac{1}{e_P}) \right\rfloor P)),$$

where P is a branch point of ϕ, and e_P is the ramification index of a ramification point lying over P.

Since $\left\lfloor 2(1 - \frac{1}{e_P}) \right\rfloor = 1$ for any branch point P, we have only two branch points. $\square$

Example 1.60 Let C be a bi-elliptic curve of genus 2, $\phi : C \to E$ the morphism such that ϕ is branched at two points, and τ the induced involution. Take K an abelian variety and $G = \mathbb{Z}_2$. Let G act on C by τ and on K by translation by a point of order 2. Set $X = (C \times K)/G$, where G acts diagonally. Then $P_2(X) = h^0(C, \omega_C^2)^\tau = 2$. This construction indeed gives all varieties with $P_3(X) = 2$ and $q(X) = \dim X$ (see [HP]).

Finally, as a complement to Theorem 1.66 and Theorem 1.1 and Theorem 1.2 in [CH5], we have:

Theorem 1.61 *Let X be a smooth projective variety with $q(X) = \dim X$ and $0 < P_m(X) \leq 2m - 2$, for some $m \geq 4$. Then $\kappa(X) \leq 1$.*

Proof. We already know, by Theorem 1.39, that the Albanese morphism $a_X : X \to \mathrm{Alb}(X)$ is surjective and hence generically finite. We then use diagram (14). By Proposition 1.54, we may assume that Y, the image of the Iitaka fibration of X, is an abelian variety.

We first assume that

$$\kappa(X) \geq 2, \tag{17}$$

and under this assumption we will deduce a contradiction. This will finish the proof of Theorem 1.61.

Let $T_{\chi_{1,s}}$ be a maximal component of $V_0(\omega_X)$ in the sense of Definition 1.55. If $\dim T_{\chi_{1,s}} = 1$, by Proposition 1.56, we conclude that $T_{\chi_{i,t}} \subset T_{\chi_{1,s}}$ for any (i,t) such that $\dim T_{\chi_{i,t}} > 0$. By Theorem 2.3 in [CH2], $\mathrm{Pic}^0(Y) = T_{\chi_{1,s}}$. Then $\dim Y = \dim \mathrm{Pic}^0(Y) = 1$, which contradicts our assumption (17) that $\kappa(X) \geq 2$. Hence we get that $\dim T_{\chi_{1,s}} \geq 2$.

We then iterate Lemma 1.31 to get

$$h^0(X, \omega_X^{m-i} \otimes P_{\chi_{1,s}}^{m-i}) \geq (m-i-1)\dim T_{\chi_{1,s}} + 1,$$

and by Lemma 1.33, we have

$$h^0(X, \omega_X^{m-i} \otimes P_{\chi_{1,s}}^{m-i} \otimes f^*P) \geq (m-i-1)\dim T_{\chi_{1,s}} + 1, \tag{18}$$

for all $0 \leq i \leq m-2$ and all $P \in \mathrm{Pic}^0(Y)$. According to (2) in Lemma 1.52, $V_0(\omega_X, -(m-1)\chi_1)$ is not empty, namely there exists $P_0 \in \mathrm{Pic}^0(Y)$ such that $h^0(X, \omega_X \otimes P_{\chi_{1,s}}^{-(m-1)} \otimes P_0) > 0$. Thus $h^0(X, \omega_X^m \otimes P_0) \geq h^0(X, \omega_X^{m-1} \otimes P_{\chi_{1,s}}^{m-1})$. Again by Lemma 1.33, we have

$$P_m(X) = h^0(X, \omega_X^m \otimes P_0) \geq h^0(X, \omega_X^{m-1} \otimes P_{\chi_{1,s}}^{m-1}).$$

We have

$$\begin{aligned} 2m-2 \geq P_m(X) &\geq h^0(X, \omega_X^{m-1} \otimes P_{\chi_{1,s}}^{m-1}) \\ &\geq (m-2)\dim T_{\chi_{1,s}} + 1, \end{aligned}$$

where the last inequality holds by taking $i = 1$ in (18). Hence we deduce that $\dim T_{\chi_{1,s}} = 2$.

Claim 1: $(m-1)\chi_1 = 0$.

If $(m-1)\chi_1 \neq 0$, by (3) in Lemma 1.52, there exists a torsion point $P_{-(m-1)\chi_{1,t}} \in \mathrm{Pic}^0(X)$ such that $P_{-(m-1)\chi_{1,t}} + T_{-(m-1)\chi_{1,t}} \subset V_0(\omega_X)$ with $\dim T_{-(m-1)\chi_{1,t}} \geq 1$.

If $\dim T_{-(m-1)\chi_{1,t}} \geq 2$, by (18) (let $i = 1$) and Lemma 2.4, we get $P_m(X) \geq 2m-3+1+2-1 = 2m-1$, which is a contradiction.

Hence $\dim T_{-(m-1)\chi_{1,t}} = 1$. Let $C = \widehat{T}_{-(m-1)\chi_{1,t}}$ and let $\pi : \mathrm{Alb}(X) \to C$ be the dual of the inclusion $T_{-(m-1)\chi_{1,t}} \hookrightarrow \mathrm{Pic}^0(X)$. Then we take $f = \pi \circ a_X$ as in the following commutative diagram:

$$\begin{tikzcd} X \arrow[d, "a_X"'] \arrow[dr, "f"] & \\ \mathrm{Alb}(X) \arrow[r, "\pi"] & C. \end{tikzcd}$$

Since we assume that $\kappa(X) \geq 2$ and $\dim T_{-(m-1)\chi_{1,t}} = 1$, $V_0(\omega_X) \neq \mathrm{Pic}^0(X)$, therefore $\chi(\omega_X) = 0$. By Lemma 2.10 and Corollary 2.11 in [CH5], there exists an ample line bundle L on C such that $f^*L \hookrightarrow \omega_X \otimes P_{-(m-1)\chi_{1,t}}$. We then apply Lemma 1.53 to conclude that

$$\begin{aligned} P_m(X) &= h^0(X, \omega_X^m \otimes P_{\chi_{1,s}}^{m-1} \otimes P_{-(m-1)\chi_{1,t}}) \\ &\geq h^0(X, \omega_X^{(m-1)} \otimes P_{\chi_{1,s}}^{m-1}) + 2 \\ &\geq 2m-1, \end{aligned}$$

where the last inequality holds by (18). This is a contradiction. We have proved the Claim.

Let $\overline{G}$ be defined as in the beginning of Section 5. We have

Claim 2: $\overline{G} \simeq \mathbb{Z}/2$, namely $\overline{G}$ contains only one nonzero element χ_1. In particular, by Claim 1, m is an odd number.

Assuming the claim is not true, there exists $0 \neq \chi_2 \in \overline{G}$ such that $(m-2)\chi_1 + \chi_2 \neq 0$. According to (3) in Lemma 1.52, there exists $P_{\chi_{2,t}} + T_{\chi_{2,t}} \subset V_0(\omega_X, \chi_2)$ with $\dim T_{\chi_{2,t}} \geq 1$. Then as in the proof of Claim 1, by Lemma 1.31 and Lemma 1.53, we conclude that

$$\begin{aligned} h^0(X, \omega_X^{m-1} \otimes P_{\chi_{1,s}}^{m-2} \otimes P_{\chi_{2,t}}) &\geq h^0(X, \omega_X^{m-2} \otimes P_{\chi_{1,s}}^{m-2}) + 2 \\ &\geq 2m-3, \end{aligned}$$

where the last inequality holds because of (18).

Since $(m-2)\chi_1 + \chi_2 \neq 0$, we may repeat the above process to get

$$\begin{aligned} P_m(X) &\geq h^0(X, \omega_X^{m-1} \otimes P_{\chi_{1,s}}^{m-2} \otimes P_{\chi_{2,t}}) + 2 \\ &\geq 2m-1, \end{aligned}$$

which is a contradiction. Hence we have proved Claim 2.

As $m \geq 4$ is odd, $m-2 \geq 3$ and $(m-3)\chi_1 = 0$. By (18) (with $i = 3$), $P_{m-3}(X) \geq 2m-7$. Since $\kappa(X) \geq 2$, by Lemma 1.33 and Lemma 1.31, we have

$$\begin{aligned} 2m-2 \geq P_m(X) &\geq P_{m-3}(X) + P_3(X) + \kappa(X) - 1 \\ &\geq 2m-6 + P_3(X). \end{aligned}$$

Hence $P_3(X) \leq 4$. According to Chen and Hacon's classification of these varieties (see Theorem 1.1 and Theorem 1.2 in [CH5]) and Claim (2), the only possibility is that X is a double cover of its Albanese variety and $\kappa(X) = 2$, as described in Example 2 in [CH5]. Namely, there exists an algebraic fiber space

$$q : \mathrm{Alb}(X) \to S$$

from an abelian variety of dimension ≥ 3 to an abelian surface, and $a_X : X \to \mathrm{Alb}(X)$ is birational to a double cover of $\mathrm{Alb}(X)$ such that $a_{X*}\mathscr{O}_X = \mathscr{O}_{\mathrm{Alb}(X)} \oplus (q^*L \otimes P)^\vee$, where L is an ample divisor of S with $h^0(S, L) = 1$ and $P \in \mathrm{Pic}^0(A)$ with $P \notin \mathrm{Pic}^0(S)$ and $2P \in \mathrm{Pic}^0(S)$. However, for such a variety, we have the inclusion of sheaves $a_X^*(q^*L \otimes P) \hookrightarrow \omega_X$ (see the proof of Claim 4.6 in [CH5]). Thus, as $m \geq 4$ is odd,

$$\begin{array}{rcl} P_m(X) & = & h^0(X, \omega_X^m) \\ & \geq & h^0(\mathrm{Alb}(X), q^*L^m \otimes P^m \otimes a_{X*}\mathscr{O}_X) \\ & \geq & h^0(\mathrm{Alb}(X), q^*L^{m-1} \otimes P^{m-1}) \\ & = & (m-1)^2 \\ & > & 2m - 2, \end{array}$$

which is a contradiction. This concludes the proof of Theorem 1.61. □

1.6 Effective Iitaka fibrations

Chen and Hacon [CH3, Corollary 4.4] proved that for every smooth complex variety X with maximal Albanese dimension and of general type, $|6K_X|$ gives a birational map. Moreover, they also show that if $\chi(\omega_X) > 0$ the tricanonical linear section $|3K_X|$ is enough to give a birational map. Pareschi and Popa [PP3, section 6] provided a conceptually approach to these theorems based on the regularity and vanishing theorems.

We observe that similar statements hold for the pluricanonical maps of varieties with maximal Albanese dimension which are not necessarily of general type. The proof is quite simple and is parallel to that of Pareschi and Popa but does not seem to be known. The starting point is Lemma 1.32.

The basic setting in this section is as follows. Let X be a smooth projective variety with maximal Albanese dimension of dimension n. We may assume that a smooth model of the Iitaka fibration of X is given by an algebraic fiber space $f : X \to Y$. We will use diagram (14):

$$\begin{array}{ccc} X & \xrightarrow{a_X} & \mathrm{Alb}(X) \\ \Big\downarrow f & & \Big\downarrow f_* \\ Y & \xrightarrow{a_Y} & \mathrm{Alb}(Y). \end{array} \tag{19}$$

Theorem 1.62 *Let $f : X \to Y$ be as above. Then $|6K_X|$ gives a rational map which is birationally equivalent to f. In other words, if X is a smooth projective variety with maximal Albanese dimension, $|6K_X|$ gives a model of the Iitaka fibration of X.*

Proof. Since f is a model of the Iitaka fibration of X, $f_*(\omega_X^2 \otimes \mathscr{J}(||\omega_X||))$ is a torsion-free rank 1 sheaf on Y. By Lemma 1.32, we have

$$H^i(Y, f_*(\omega_X^2 \otimes \mathscr{J}(||\omega_X||)) \otimes P) = 0$$

for all $i \geq 1$ and $P \in \mathrm{Pic}^0(Y)$. As in Lemma 1.37, $R^j a_{Y*}(f_*(\omega_X^2 \otimes \mathscr{J}(||\omega_X||))) = 0$ for all $j \geq 1$. Hence

$$\begin{aligned} & H^i(\mathrm{Alb}(Y), a_{Y*}f_*(\omega_X^2 \otimes \mathscr{J}(||\omega_X||)) \otimes P) \\ = \ & H^i(Y, f_*(\omega_X^2 \otimes \mathscr{J}(||\omega_X||)) \otimes P) \\ = \ & 0, \end{aligned}$$

for all $i \geq 1$ and $P \in \mathrm{Pic}^0(Y)$. Thus $a_{Y*}f_*(\omega_X^2 \otimes \mathscr{J}(||\omega_X||))$ is an IT-sheaf of index 0 and in particular, it is M-regular. By [PP3, Corollary 5.3], $a_{Y*}f_*(\omega_X^2 \otimes \mathscr{J}(||\omega_X||))$ is continuously globally generated. Since a_Y is generically finite, we define Z_1 to be the exceptional locus of a_Y, i.e. the inverse image of a_Y of the locus of points of $\mathrm{Alb}(Y)$ has positive dimension. Then $f_*(\omega_X^2 \otimes \mathscr{J}(||\omega_X||))$ is continuously globally generated away from Z_1. By definition, this means that there exist an open subset $U \subset \mathrm{Pic}^0(Y)$ such that the evaluation map

$$\bigoplus_{P \in U} H^0(f_*(\omega_X^2 \otimes \mathscr{J}(||\omega_X||)) \otimes P) \to f_*(\omega_X^2 \otimes \mathscr{J}(||\omega_X||)$$

is surjective away from Z_1.

Define Z_2 to be the maximal subvariety of Y such that $f_*(\omega_X^2 \otimes \mathscr{J}(||\omega_X||)$ is locally free of rank 1 away from Z_2. Let $\mathscr{L}$ be $f_*(\omega_X^2 \otimes \mathscr{J}(||\omega_X||))$. By the same diagram in [PP1, Proposition 2.12]:

$$\begin{tikzcd}
\bigoplus_{i=1}^N H^0(Y, \mathscr{L} \otimes P_i) \otimes H^0(Y, \mathscr{L} \otimes P_i^\vee) \otimes \mathscr{O}_Y \arrow[r] \arrow[d] & H^0(Y, \mathscr{L}^2) \arrow[d] , \\
\oplus_{i=1}^N H^0(Y, \mathscr{L} \otimes P_i) \otimes \mathscr{L} \otimes P_i^\vee \arrow[r] & \mathscr{L}^2
\end{tikzcd}$$

we conclude that $\mathscr{L}^2$ is globally generated away from $Z_1 \cup Z_2$.

Since we have $\mathscr{L} \otimes \mathscr{L} \hookrightarrow f_*(\omega_X^4)$. There exists an open subset $U \hookrightarrow Y - (Z_1 \cup Z_2)$ such that $f_*(\omega_X^4)$ is globally generated. On the other hand, the inclusion

$$H^0(X, \omega_X^4 \otimes \mathscr{J}(||\omega_X^3||)) \hookrightarrow H^0(X, \omega_X^4)$$

is an isomorphism. Therefore, $f_*(\omega_X^4 \otimes \mathscr{J}(||\omega_X^3||))$ is also globally generated on U. By Lemma 1.32, we have the vanishing $H^i(Y, f_*(\omega_X^4 \otimes \mathscr{J}(||\omega_X^3||)) \otimes P) = 0$ for all $i \geq 1$ and all $P \in \mathrm{Pic}^0(Y)$. Let $y \in U$ be a point and denote by $\mathscr{L}_1$ the rank 1 sheaf $f_*(\omega_X^4 \otimes \mathscr{J}(||\omega_X^3||))$. We have the exact sequence

$$0 \to \mathscr{I}_y \otimes \mathscr{L}_1 \to \mathscr{L}_1 \to \mathscr{L}_1|_y \to 0.$$

We thus conclude easily that $a_{Y*}(\mathscr{I}_y \otimes \mathscr{L}_1)$ is an IT-sheaf of index 0 and thus M-regular. Hence $\mathscr{I}_y \otimes \mathscr{L}_1$ is continuous globally generated away from Z_1 for any $y \in U$. Then by [PP1, Proposition 2.12], $\mathscr{I}_y \otimes \mathscr{L}_1 \otimes \mathscr{L}$ is globally generated away form $Z_1 \cup Z_2$. Since $U \hookrightarrow Y - (Z_1 \cup Z_2)$, $\mathscr{L}_1 \otimes \mathscr{L}$ is very ample on U. We have $\mathscr{L}_1 \otimes \mathscr{L} \hookrightarrow f_*(\omega_X^6)$, thus there exists an open subset U_2 of Y such that $f_*(\omega_X^6)$ is very ample on U_2. We conclude the theorem. $\square$

Theorem 1.62 is just an analogue of Theorem 6.7 in [PP3]. The main point is just that $a_{Y*}f_*(\omega_X^2 \otimes \mathscr{J}(||\omega_X||))$ is M-regular. On the other hand, if X is of general type, with maximal Albanese dimension, and that moreover $a_X(X)$ is not ruled by tori, Pareschi and Popa proved that $a_{X*}\omega_X$ is M-regular, which is the main ingredient of the proof of Theorem 6.1 in [PP3]. If X is not of general type and of Kodaira dimension $\kappa(X)$, the image $a_X(X)$ of the Albanese dimension is always ruled by tori of dimension $n - \kappa(X)$. But we still have:

Theorem 1.63 *Let X be as in above. If $a_X(X)$ is not ruled by tori of dimension $> n - \kappa(X)$, then $a_{Y*}f_*(\omega_X)$ is M-regular. Thus $|3K_X|$ gives a model of the Iitaka fibration of X.*

Proof. We just need to show that under the assumption in Theorem 1.63, $a_{Y*}f_*(\omega_X)$ is M-regular. The rest is the same as the proof of Theorem 1.62. Again by Kawamata's theorem 1.43, We have:

$$\begin{tikzcd} \widehat{Y} \times \widetilde{K} \arrow[r, "\pi_X"] \arrow[d, "pr_1"] & X \arrow[r, "a_X"] \arrow[d, "f"] & \mathrm{Alb}(X) \arrow[d, "f_*"] \\ \widehat{Y} \arrow[r, "b_Y"] & Y \arrow[r, "a_Y"] & \mathrm{Alb}(Y), \end{tikzcd}$$

where $\widehat{Y} \times \widetilde{K}$ is a birationally equivalent to a finite étale cover induced by isogeny of $\mathrm{Alb}(X)$, $\widetilde{K}$ is an abelian variety isogenous to $\ker f_*$, $\widehat{Y}$ is a smooth projective variety of general type and b_Y is generically finite. We will write $g_Y = a_Y \circ b_Y$.

Since $a_X(X)$ is not ruled by tori of dimension $> k$, we conclude that $g_Y(\widehat{Y}) = a_Y(Y)$ is not ruled by tori. We have the following claim:

Claim: $g_{Y*}\omega_{\widehat{Y}}$ is M-regular.

We first see how the Claim implies Theorem 1.63. Since $\widetilde{K}$ is an abelian variety, it is obviously $pr_{1*}\omega_{\widehat{Y}\times\widetilde{K}} = \omega_{\widehat{Y}}$. Hence

$$g_{Y*}pr_{1*}\omega_{\widehat{Y}\times\widetilde{K}} = a_{Y*}f_*\pi_{X*}\omega_{\widehat{Y}\times\widetilde{K}}$$

is M-regular on $\mathrm{Alb}(Y)$. On the other hand, ω_X is a direct summand of $\pi_{X*}\omega_{\widehat{Y}\times\widetilde{K}}$ since π_X is biratonally equivalent to an étale cover. Therefore, $a_{Y*}f_*\omega_X$ is a direct summand of $g_{Y*}pr_{1*}\omega_{\widehat{Y}\times\widetilde{K}}$ and hence is M-regular.

We now prove the Claim.

We have $g_Y : \widehat{Y} \to \mathrm{Alb}(Y)$ and the image $g_Y(\widehat{Y})$ is not ruled by tori. The proof of the Claim goes the same way as in the proof of Theorem 3 in [EL1]. We only need to write it carefully. In order to prove that $b_{Y*}\omega_{\widehat{Y}}$ is M-regular, we need to show that $\mathrm{codim}_{\mathrm{Pic}^0(Y)} V_i(b_{Y*}\omega_{\widehat{Y}}) > i$ for $1 \le i \le n - k$.

First we note that since $g_Y(\widehat{Y})$ generates $\mathrm{Alb}(Y)$, $g_Y^* : \mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(\widehat{Y})$ has a finite kernel. Let S be a component of $V_i(b_{Y*}\omega_{\widehat{Y}})$. Let $y \in S$ be a general point such that $H^i(\widehat{Y}, \omega_{\widehat{Y}} \otimes P_y)$ take the minimal value on S. Take $0 \neq v \in H^1(Y, \mathscr{O}_Y)$. Let $\Delta_v(y) \subset \mathrm{Pic}^0(Y)$ be a neighborhood of y in the straight line in $\mathrm{Pic}^0(Y)$ through y in the direction v. We may also regard $\Delta_v(y)$ as a neighborhood of $b_Y^*y \in \mathrm{Pic}^0(\widehat{Y})$ in the straight line through b_Y^*y in the direction b_Y^*v. By [GL2, Corollary 3.3], for $t \in \Delta_v(y)$ in a punctual neighborhood of y, $h^{n-k-i}(\widehat{Y}, P_t^*)$ is the dimension of the homology of the complex:

$$H^{n-k-i-1}(\widehat{Y}, P_y^*) \xrightarrow{\cup b_Y^* v} H^{n-k-i}(\widehat{Y}, P_y^*) \xrightarrow{\cup b_Y^* v} H^{n-k-i+1}(\widehat{Y}, P_y^*). \tag{20}$$

By the choice of y, we see that for any v tangent to S, the homology of the complex in (20) is still of dimension $h^{n-i}(\widehat{Y}, P_y^*)$. Hence the map

$$H^{n-k-i}(\widehat{Y}, P_y^*) \xrightarrow{\cup b_Y^* v} H^{n-k-i+1}(\widehat{Y}, P_y^*) \tag{21}$$

is zero.

Considering the diagram

$$\begin{tikzcd} \widehat{Y} \arrow[r, "b_Y"] \arrow[dr, "h"'] & \mathrm{Alb}(Y) \arrow[d, "pr"] \\ & \widehat{S} \end{tikzcd} .$$

We may rewrite (20) as the map

$$H^0(\widehat{Y}, \Omega_{\widehat{Y}}^{n-k-i} \otimes P_y) \xrightarrow{\wedge h^* \eta} H^0(\widehat{Y}, \Omega_{\widehat{Y}}^{n-k-i+1} \otimes P_y)$$

is zero for any $\eta \in H^0(\widehat{S}, \Omega_{\widehat{S}})$.

If $\mathrm{codim}_{\mathrm{Pic}^0(Y)} S = i$, the dimension of $h(\widehat{S})$ is $\leq n-k-i$. Therefore the dimension of a connected fiber h_s of h is of dimension $\geq i$. Since g_Y is generically finite, $g_Y(h_s)$ is of dimension i and thus is just a fiber of pr which is an abelian variety. Therefore $g_Y(\widehat{Y})$ will be ruled by tori which is a contradiction. Hence we finished the proof of the claim.

$\square$

1.7 A remark to a theorem of Hacon and Pardini

1.7.1 Proof of Theorems 1.15 and 1.17

We begin by a general lemma.

Lemma 1.64 *Suppose that $f : X \to Y$ is a surjective generically finite morphism between smooth projective varieties with $\kappa(Y) \geq 0$. Then for any $j \geq 2$,*

$$f_*(\mathscr{O}_X(jK_X) \otimes \mathscr{J}(||(j-1)K_X||)) \supset \mathscr{O}_Y(jK_Y) \otimes \mathscr{J}(||(j-1)K_Y||).$$

Proof. Take $N > 0$, and let $\tau_Y : Y' \to Y$ be a log resolution such that

$$\tau_Y^*|N(j-1)K_Y| = |L_1| + E_1,$$

where $|L_1|$ is base-point-free and E_1 is the fixed divisor. Then we take a log resolution $\tau_X : X' \to X$ such that we have a commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \tau_X \downarrow & & \tau_Y \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

and $\tau_X^*|N(j-1)K_X| = |L_2| + E_2$ where $|L_2|$ is base-point-free and E_2 is the fixed divisor. Let $D \in |(j-1)K_{X/Y}|$. Then $f'^*E_1 + N\tau_X^*D \succeq E_2$. Hence

$$\begin{aligned} & \mathscr{O}_{X'}(K_{X'/X} + j\tau_X^*K_X - \left\lfloor \frac{1}{N}E_2 \right\rfloor) \\ \supset\ & \mathscr{O}_{X'}(K_{X'/X} + j\tau_X^*K_X - \tau_X^*D - \left\lfloor \frac{1}{N}f'^*E_1 \right\rfloor) \\ =\ & \mathscr{O}_{X'}(K_{X'/X} + \tau_X^*K_X + (j-1)\tau_X^*f^*K_Y - \left\lfloor \frac{1}{N}f'^*E_1 \right\rfloor) \\ =\ & \mathscr{O}_{X'}(K_{X'/Y'} - \left\lfloor \frac{1}{N}f'^*E_1 \right\rfloor + f'^*\left\lfloor \frac{1}{N}E_1 \right\rfloor + f'^*(K_{Y'/Y} + j\tau_Y^*K_Y - \left\lfloor \frac{1}{N}E_1 \right\rfloor)). \end{aligned}$$

We may assume N sufficiently large and divisible. Then

$$\tau_{X*}(\mathscr{O}_{X'}(K_{X'/X} + j\tau_X^*K_X - \lfloor \frac{1}{N}E_2 \rfloor)) = \mathscr{O}_X(jK_X) \otimes \mathscr{J}(||(j-1)K_X||).$$

By step 2 in Hacon and Pardini's proof of Theorem 1 (see Theorem 3.2 [HP]), we know that $K_{X'/Y'} - \lfloor \frac{1}{N}f'^*E_1 \rfloor + f'^*\lfloor \frac{1}{N}E_1 \rfloor$ is an effective divisor, hence

$$\begin{aligned} & \tau_{Y*}f'_*(\mathscr{O}_{X'}(K_{X'/Y'} - \lfloor \frac{1}{N}f'^*E_1 \rfloor + f'^*\lfloor \frac{1}{N}E_1 \rfloor + f'^*(K_{Y'/Y} + j\tau_Y^*K_Y - \lfloor \frac{1}{N}E_1 \rfloor))) \\ & \supseteq \tau_{Y*}(\mathscr{O}_{Y'}(K_{Y'/Y} + j\tau_Y^*K_Y - \lfloor \frac{1}{N}E_1 \rfloor)) = \mathscr{O}_Y(jK_Y) \otimes \mathscr{J}(||(j-1)K_Y||). \end{aligned}$$

This proves the lemma. $\square$

Proof of Theorem 1.15 and Theorem 1.17. As in Proposition 2.1 in [HP], we may assume that we have the following commutative diagram:

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h_X"] & Y \arrow[r, "a_Y"] \arrow[d, "h_Y"] & A \arrow[d, "\pi"] \\
V \arrow[r, "g"] & W \arrow[r, "a_W"] & A/K
\end{tikzcd} \tag{22}
$$

where $\pi : A \to A/K$ is a quotient of abelian varieties, and a_Y and a_W are the Albanese morphisms of Y and W respectively. We set

$$\mathscr{H}_Y = h_{Y*}(\mathscr{O}_Y(jK_Y) \otimes \mathscr{J}(||(j-1)K_Y||)$$

and

$$\mathscr{H}_X = h_{X*}(\mathscr{O}_X(jK_X) \otimes \mathscr{J}(||(j-1)K_X||).$$

By Lemma 1.64, we have

$$a_{W*}\mathscr{H}_Y \subset a_{W*}g_*\mathscr{H}_X. \tag{23}$$

We denote by $\mathscr{F}_Y \subset \mathscr{F}_X$ the two sheaves in (23) respectively, and by $\mathscr{Q}$ the quotient sheaf. Hence we have a short exact sequence:

$$0 \longrightarrow \mathscr{F}_Y \longrightarrow \mathscr{F}_X \longrightarrow \mathscr{Q} \longrightarrow 0$$

of sheaves on A/K. Let $P_j(X) = P_j(Y) = M > 0$. By Theorem 11.1.8 and Proposition 11.2.10 in [La], we have

$$
\begin{aligned}
M = P_j(Y) &= h^0(Y, \mathscr{O}_Y(jK_Y) \otimes \mathscr{J}(||jK_Y||)) \\
&= h^0(Y, \mathscr{O}_Y(jK_Y) \otimes \mathscr{J}(||(j-1)K_Y||)) \\
&= h^0(W, \mathscr{H}_Y) \\
&= h^0(A/K, \mathscr{F}_Y). \qquad (24)
\end{aligned}
$$

Similarly to (24), we can also prove that:

$$M = P_j(X) = h^0(V, \mathscr{H}_X) = h^0(A/K, \mathscr{F}_X).$$

Thus $\mathscr{H}_Y \subset h_{Y*}(\mathscr{O}(jK_Y))$ is a nonzero torsion-free sheaf. Since h_Y is a model of the Iitaka fibration of Y whose general fibers are birational to abelian varieties (see Proposition 2.1 in [HP]), the latter sheaf has rank 1. So the rank of $\mathscr{H}_Y$ is also 1. We have the same situation for h_X, hence the rank of $\mathscr{H}_X$ is again 1.

On the other hand, we have the following claim.

Claim: $\mathscr{Q} = 0$, hence $\mathscr{F}_Y = \mathscr{F}_X$.

In order to prove the Claim, we want to apply Proposition 2.3 in [HP], namely, we just need to prove that $h^j(A/K, \mathscr{F}_Y \otimes P) = h^j(A/K, \mathscr{F}_X \otimes P)$, for all $j \geq 0$ and all $P \in \mathrm{Pic}^0(A/K)$.

We will first prove that when $j \geq 1$. By Lemma 1.32, we have

$$H^i(W, \mathscr{H}_Y \otimes a_W^* P) = H^i(W, g_* \mathscr{H}_X \otimes a_W^* P) = 0, \tag{25}$$

for all $P \in \mathrm{Pic}^0(A/K)$ and all $i \geq 1$.

As in Lemma 1.37, we can prove that

$$R^j a_{W*} \mathscr{H}_Y = R^j a_{W*}(g_* \mathscr{H}_X) = 0, \tag{26}$$

for all $j \geq 1$, as follows.

First we take a very ample line bundle H on A/K such that, for all $k \geq 1$ and $j \geq 0$,

$$H^k(A/K, R^j a_{W*} \mathscr{F}_Y \otimes H) = H^k(A/K, R^j a_{W*}(g_* \mathscr{F}_X) \otimes H) = 0 \tag{27}$$

and $R^j a_{W*} \mathscr{F}_Y \otimes H$ and $R^j a_{W*}(g_* \mathscr{F}_X) \otimes H$ are globally generated. Again by Lemma 1.32,

$$H^j(W, \mathscr{H}_Y \otimes a_W^* H) = H^j(W, g_* \mathscr{H}_X \otimes a_W^* H) = 0,$$

for all $j \geq 1$. Therefore, by Leray's spectral sequence and (27), we conclude that

$$H^0(A/K, R^j a_{W*} \mathscr{H}_Y \otimes H) = H^0(A/K, R^j a_{W*}(g_* \mathscr{H}_X) \otimes H) = 0,$$

for $j \geq 1$. Since $R^j a_{W*} \mathscr{H}_Y \otimes H$ and $R^j a_{W*}(g_* \mathscr{H}_X) \otimes H$ are globally generated, we deduce that $R^j a_{W*} \mathscr{H}_Y = R^j a_{W*}(g_* \mathscr{H}_X) = 0$, for all $j \geq 1$.

Applying the Leray spectral sequence to (25), by (26), we get that, for all $i \geq 1$ and $P \in \mathrm{Pic}^0(A/K)$,

$$H^i(A/K, \mathscr{F}_Y \otimes P) = H^i(W, \mathscr{H}_Y \otimes g^*P) = 0,$$

and

$$H^i(A/K, \mathscr{F}_X \otimes P) = H^i(W, g_*\mathscr{H}_X \otimes g^*P) = 0.$$

Finally, for all $P \in \mathrm{Pic}^0(A/K)$,

$$\begin{aligned} h^0(A/K, \mathscr{F}_Y \otimes P) &= \chi(A/K, \mathscr{F}_Y \otimes P) = \chi(A/K, \mathscr{F}_Y) \\ &= h^0(A/K, \mathscr{F}_Y) = M, \end{aligned}$$

and similarly,

$$h^0(A/K, \mathscr{F}_X \otimes P) = h^0(A/K, \mathscr{F}_X) = M.$$

We have finished the proof of the Claim.

Set $Z = a_W(W)$. Since a_W is generically finite onto its image Z and the rank of $\mathscr{H}_Y$ is 1, the rank of $\mathscr{F}_X = \mathscr{F}_Y = a_{W*}\mathscr{H}_1$ on Z is $\deg(a_W)$.

Now we start to prove Theorem 1.15. We take the Stein factorization of g:

$$V \xrightarrow{p} U \xrightarrow{q} W,$$

where p is an algebraic fiber space and q is surjective and finite. Because $h^0(U, p_*\mathscr{H}_2) = h^0(V, \mathscr{H}_2) = M > 0$, we have that $p_*\mathscr{H}_2$ is a nonzero torsion-free sheaf of rank ≥ 1. We can write

$$\mathscr{F}_X = a_{W*}g_*\mathscr{H}_X = a_{W*}q_*(p_*\mathscr{H}_X),$$

and conclude that the rank of $\mathscr{F}_X$ on Z is $\geq \deg(q) \cdot \deg(a_W)$. This implies $\deg(q) = 1$ hence g has connected fibers. This is essentially Hacon and Pardini's proof of Theorem 1.15.

Similarly, we are going to prove Theorem 1.17. We first assume that g is not birational. Now by Theorem 1, we know that $g \circ h_X$ is an algebraic fiber space. We denote by X_w a general fiber of $g \circ h_X$. The main ingredient is the following lemma.

Lemma 1.65 *In the above situation, the sheaf*

$$g_*\mathscr{H}_X = (g \circ h_X)_*(\mathscr{O}_X(jK_X) \otimes \mathscr{J}(||(j-1)K_X||))$$

has rank $P_j(X_w) > 0$.

This lemma will be proved later. We first use it to finish the proof of Theorem 1.17.

Since $g : V \to W$ is not birational and g is an algebraic fiber space, $\dim W < \dim V$. Hence by the easy addition formula (see Corollary 1.7 in [Mo]), we have $\dim V = \kappa(X) \leq \kappa(X_w) + \dim W$. We get $\kappa(X_w) \geq 1$. Since X is of maximal Albanese dimension, X_w is also of maximal Albanese dimension, hence $P_j(X_w) \geq 2$ by Chen and Hacon's characterization of abelian varieties ([CH1], Theorem 3.2). Then, by Lemma 1.65, the rank of $\mathscr{H}_X$ on Z is $\deg(a_W) \cdot P_j(X_w)$ ($\geq 2\deg(a_W)$), which is a contradiction. This concludes the proof of Theorem 2. $\square$

In order to prove Lemma 1.65, first we need an easy lemma:

Lemma 1.66 *Let X be a smooth projective variety, D_1 a divisor on X with non-negative Iitaka dimension, and D_2 an effective divisor on X. We have the inclusion:*

$$\mathscr{J}(||D_1 + D_2||) \supset \mathscr{J}(||D_1||) \otimes \mathscr{O}_X(-D_2).$$

Proof. Take $M > 0$ such that $|MD_1| \neq \emptyset$. Choose a log resolution

$$\mu : X' \to X$$

for MD_1, MD_2, and $M(D_1 + D_2)$. Write

$$\begin{aligned} \mu^*(|MD_1|) &= |W_1| + E_1 \\ \mu^*(|MD_2|) &= |W_2| + E_2 \\ \mu^*(|M(D_1 + D_2)|) &= |W_3| + E_3, \end{aligned}$$

where E_1, E_2, and E_3 are the fixed divisors and $|W_1|$, $|W_2|$, and $|W_3|$ are free linear series. We have

$$M\mu^*D_2 \succeq E_2 \ \text{ and } \ E_1 + E_2 \succeq E_3,$$

hence

$$\begin{aligned}
\mu_*(K_{X'/X} - \left\lfloor \frac{1}{M} E_3 \right\rfloor) &\supset \mu_*(K_{X'/X} - \left\lfloor \frac{1}{M}(E_1 + E_2) \right\rfloor) \\
&\supset \mu_*(K_{X'/X} - \left\lfloor \frac{1}{M}(E_1 + M\mu^* D_2) \right\rfloor) \\
&= \mu_*(K_{X'/X} - \left\lfloor \frac{1}{M} E_1 \right\rfloor) \otimes \mathscr{O}_X(-D_2).
\end{aligned}$$

By the definition of asymptotic multiplier ideal sheaves, this proves the lemma. □

Proof of Lemma 4. We will reduce Lemma 1.65 to Proposition 1.48. Since Y is of maximal Albanese dimension and h_Y is a model of the Iitaka fibration of Y, by a theorem of Kawamata (see Theorem 1.43), there exists an étale cover $\pi_Y : \widetilde{Y} \to Y$ induced by an étale cover of A and a commutative diagram:

$$\begin{tikzcd}
\widetilde{Y} \arrow[r, "\pi_Y"] \arrow[d, dotted, "h_{\widetilde{Y}}"'] & Y \arrow[d, "h_Y"] \\
\widehat{W} \arrow[r, "b_{\widehat{W}}"] & W,
\end{tikzcd}$$

where $\widehat{W}$ is a smooth projective variety of general type, the rational map $h_{\widetilde{Y}}$ is a model of the Iitaka fibration of $\widetilde{Y}$, and $b_{\widehat{W}}$ is generically finite and surjective.

Let $\widetilde{X}$ be a connected component of $X \times_Y \widetilde{Y}$, denote by $\pi_{\widetilde{X}}$ the induced morphism $\widetilde{X} \to X$, and denote by $f_{\widetilde{X}}$ the induced morphism $\widetilde{X} \to \widetilde{Y}$. Denote by k and $k_{\widetilde{X}}$ respectively the morphism $g \circ h_X = h_Y \circ f$ and the map $h_{\widetilde{Y}} \circ f_{\widetilde{X}}$. After birational modifications of $\widetilde{X}$, we may suppose that $k_{\widetilde{X}}$ is a morphism such that $k_{\widetilde{X}}(E)$ is a proper subvariety of $\widehat{W}$, where E is the $\pi_{\widetilde{X}}$-exceptional divisor. In all, we have the commutative diagram:

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\pi_{\widetilde{X}}"] \arrow[d, "f_{\widetilde{X}}"] \arrow[dd, bend right=60, "k_{\widetilde{X}}"'] & X \arrow[d, "f"] \arrow[dd, bend left=60, "k"] \\
\widetilde{Y} \arrow[r, "\pi_Y"] \arrow[d, dotted, "h_{\widetilde{Y}}"] & Y \arrow[d, "h_Y"] \\
\widehat{W} \arrow[r, "b_{\widehat{W}}"] & W.
\end{tikzcd}$$

We then take the Stein factorization of $k_{\widetilde{X}}$:

$$\widetilde{X} \xrightarrow{k_1} W_1 \xrightarrow{b_{W_1}} \widehat{W}.$$

The important point is that W_1 is still of general type. Again by birational modifications of $\widetilde{X}$ and W_1, we may assume that $k_1 : \widetilde{X} \to W_1$ is an algebraic fiber space between smooth projective varieties. According to the following diagram:

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\pi_{\widetilde{X}}"] \arrow[d, "k_1"] & X \arrow[d, "k"] \\
W_1 \arrow[r, "b_{\widehat{W}} \circ b_{W_1}"'] & W,
\end{tikzcd}$$

By Proposition 1.48, the sheaf

$$k_*(\mathscr{O}_X(jK_X) \otimes \mathscr{J}(||(j-1)K_{X/W} + k^*K_W||)) \otimes \mathscr{O}_W(-(j-2)K_W)$$

has rank $P_j(X_w)$. By Lemma 1.45, the line bundle $(j-1)K_{X/W} + k^*K_W$ has non-negative Iitaka dimension. By Lemma 1.66,

$$\mathscr{J}(||(j-1)K_X||) \supset \mathscr{J}(||(j-1)K_{X/W} + k^*K_W||) \otimes \mathscr{O}_X(-(j-2)k^*K_W).$$

Therefore,

$$\begin{aligned}
& k_*(\mathscr{O}_X(jK_X)) \\
\supset \;& k_*(\mathscr{O}_X(jK_X) \otimes \mathscr{J}(||(j-1)K_X||)) \\
\supset \;& k_*(\mathscr{O}_X(jK_X) \otimes \mathscr{J}(||(j-1)K_{X/W} + k^*K_W||)) \otimes \mathscr{O}_W(-(j-2)K_W).
\end{aligned}$$

Since the first and the third sheaves both have rank $P_j(X_w)$, so has the second. □

1.7.2 An example

Now under the assumptions of Theorem 1.15, it is natural to expect that $f : X \to Y$ might be birational to an étale morphism. However the example below (see also [CH2]) shows that it is not true in general.

Example 1.67 Set $G = \mathbb{Z}_{rs}$ and let $G_2 = s\mathbb{Z}_{rs}$ be the subgroup of G generated by s, with $s \geq 2$ and $r \geq 2$. Let $G_1 = G/G_2 \simeq \mathbb{Z}_s$. Consider an elliptic curve E, let B_1 and B_2 be two points on E, and let L be a line bundle of degree 1 such that $B = (rs - a)B_1 + aB_2 \in |tmL|$ with $1 \leq a \leq m - 2$, $(a, rs) = 1$. Taking the normalization of the (rs)-th root of B, we get a smooth curve C and a Galois cover $\pi : C \to E$ with Galois group G. By construction, π ramifies at two points, B_1 and B_2. Following [Be1] §VI.12, we have $\dim H^0(C, \omega_C^2)^G = 2$.

Let $L^{(i)}$ be $L^i(-\lfloor \frac{iB}{rs} \rfloor)$ and denote $(L^{(i)})^{-1}$ by $L^{-(i)}$. Then, by Proposition 9.8 in [Kol2],

$$\pi_* \mathscr{O}_C = \bigoplus_{i=0}^{rs-1} L^{-(i)}.$$

Let C_1 be the curve

$$\underline{Spec}(\bigoplus_{i=0}^{s-1} L^{-(ri)}),$$

where $\bigoplus_{i=0}^{s-1} L^{-(ri)}$ has the subalgebra structure of $\pi_* \mathscr{O}_C$. We have a factorization of π:

$$C \xrightarrow{g} C_1 \xrightarrow{\pi_1} E.$$

Then $C_1 = C/G_2$, and π_1 is a Galois cover with Galois group G_1 which also ramifies only at B_1 and B_2. Hence we again have

$$\dim H^0(C_1, \omega_{C_1}^2)^{G_1} = 2.$$

Finally we take an abelian variety K such that G acts freely on K by translations and set $K_1 = K/G_2$. Let $\widetilde{X} = C \times K$, $\widetilde{Y} = C_1 \times K$ and $X = \widetilde{X}/G = (C \times K)/G$, $Y = \widetilde{Y}/G = (C_1 \times K_1)/G_1$, where G and G_1 act diagonally. Hence $\widetilde{X}$ and $\widetilde{Y}$ are étale covers of X and Y respectively. There is a natural finite dominant morphism $f : X \to Y$ of degree r. Since its lift $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is not étale, f is not étale.

Since

$$H^0(X, \omega_X^2) \simeq H^0(\widetilde{X}, \omega_{\widetilde{X}}^2)^G = H^0(C, \omega_C^2)^G$$

and

$$H^0(Y,\omega_Y^2) \simeq H^0(\widetilde{Y},\omega_{\widetilde{Y}}^2)^{G_1} = H^0(C_1,\omega_{C_1}^2)^{G_1},$$

we have

$$h^0(X,\omega_X^2) = \dim H^0(C,\omega_C^2)^G = \dim H^0(C_1,\omega_{C_1}^2)^{G_1} = h^0(Y,\omega_Y^2) = 2.$$

1.7.3 A complete description of $f : X \to Y$

Indeed the above example is not a coincidence. By combining Theorem 1.17 and Kawamata's Theorem 13 in [Ka] (see also Theorem 1.43), we obtain:

Theorem 1.68 *Let $f : X \to Y$ be a surjective morphism of smooth n-dimensional projective varieties, with Y of maximal Albanese dimension, and let $X \to V$, $Y \to W$ be the Iitaka fibrations of X, respectively Y.*

If $P_j(X) = P_j(Y)$ for some $j \geq 2$, there exist normal projective varietes V_X and V_Y which are of general type, abelian varieties A_X and A_Y, a finite abelian group G which act faithfully on V_X and on A_X by translations, and a subgroup G_2 of G with $G_1 = G/G_2$ such that $V_Y = V_X/G_2$, $A_Y = A_X/G_2$, and X and Y are birational to $(A_X \times V_X)/G$ and $(A_Y \times V_Y)/G_1$ respectively, where G and G_1 act on $A_X \times V_X$ and $A_Y \times V_Y$ diagonally, respectively. Moreover, f is birational to the quotient morphism $(A_X \times V_X)/G \to (A_Y \times V_Y)/G_1$.

Proof. In the diagram (22), we already know that $g : V \to W$ is birational so we may assume that $V = W$ and g is the identity. We then consider the diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h_X"] & Y \arrow[r, "a_Y"] \arrow[d, "h_Y"] & A \arrow[d] \\ V \arrow[r, equal] & V \arrow[r, "a_V"] & A/K \end{tikzcd} \tag{28}$$

Taking the Stein factorizations for f and a_Y, we may assume that X and Y are normal and f and a_Y are finite. Similarly we take the Stein factorization for $Y \xrightarrow{a_Y} A \to A/K$ and may assume that V is normal and a_V is finite.

By Poincaré reducibility, there exists an isogeny $B \to A/K$ such that $A \times_{A/K} B \simeq K \times B$. We denote by H the kernel of this isogeny. Apply the étale base change $B \to A/K$ to diagram (28) and get

$$\begin{tikzcd} \overline{X} \arrow[r, "\overline{f}"] \arrow[d, "h_{\overline{X}}"] & \overline{Y} \arrow[r, "a_{\overline{Y}}"] \arrow[d, "h_{\overline{Y}}"] & K \times B \arrow[d] \\ \overline{V} \arrow[r, equal] & \overline{V} \arrow[r, "a_V"] & B \end{tikzcd} \tag{29}$$

where $\overline{V} = V \times_{A/K} B$, $\overline{Y} = Y \times_V \overline{V}$, which are connected because a_Y and a_V are the Albanese maps, and $\overline{X} = X \times_Y \overline{Y}$, which is also connected because $\overline{X} = X \times_Y \overline{Y} = X \times_Y (Y \times_V \overline{V}) = X \times_V \overline{V}$ and $h_X : X \to V$ is an algebraic fiber space.

Let A_X and A_Y be the general fibers of $h_{\overline{X}}$ and $h_{\overline{Y}}$ respectively. We have the induced diagram from (29):

$$\begin{tikzcd} A_X \arrow[r, "\beta"'] \arrow[rr, bend left, "\alpha_X"] & A_Y \arrow[r, "\alpha_Y"'] & K \end{tikzcd}$$

By Proposition 2.1 in [HP], A_X and A_Y are birational to abelian varieties. Hence the morphisms α_X and α_Y are birationally equivalent to étale covers. Since $a_{\overline{Y}}$ and $\alpha_{\overline{Y}} \circ \overline{f}$ are finite, α_X and α_Y are also finite. Thus α_X and α_Y are isogenies of abelian varieties by Zariski's Main Theorem. We denote by $\widetilde{G}$, $\widetilde{G}_1$, and $\widetilde{G}_2$ the abelian groups A_X/K, A_Y/K, and A_X/A_Y respectively. Then $\widetilde{G}_1 = \widetilde{G}/\widetilde{G}_2$ and $A_Y = A_X/\widetilde{G}_2$. Let $k \in K$ be a general point, let V_Y be the normal variety $a_{\overline{Y}}^{-1}(k \times B)$, and V_X be the normal variety $\overline{f}^{-1} a_{\overline{Y}}^{-1}(k \times B)$.

We know that A_X and A_Y act on $\overline{X}$, $\overline{Y}$ respectively in such a way that $\overline{f}$ is equivariant for the A_X-action on $\overline{X}$ and the A_Y-action on $\overline{Y}$ (see diagram (6)). Furthermore, the actions induce a faithful $\overline{G}$-action on V_X and a faithful $\widetilde{G}_1$-action on V_Y, and we have an A_X-equivariant isomorphism $\overline{X} \simeq (A_X \times V_X)/\widetilde{G}$ and an A_Y-equivariant isomorphism $\overline{Y} \simeq (A_Y \times V_Y)/\widetilde{G}_1$, where $\widetilde{G}$ acts on $A_X \times V_X$ diagonally and $\widetilde{G}$ acts on $A_Y \times V_Y$ diagonally.

Note that $\overline{f}$ is equivariant for the A_X-action on $\overline{X}$ and the A_Y-action on $\overline{Y}$. Then we consider the induced morphism

$$\begin{tikzcd} V_X \arrow[r, "\overline{f}"] \arrow[d, "h_{\overline{X}}"] & V_Y \arrow[d, "h_{\overline{Y}}"] \\ V \arrow[r, equal] & V. \end{tikzcd}$$

Then $\overline{f}$ is equivariant for the $\widetilde{G}$-action on V_X and the $\widetilde{G}_1$ action on V_Y. Thus $V_Y = V_X/\widetilde{G}_2$ and $\overline{f}|_{V_X}$ is the quotient morphism.

Thus we conclude that $A_Y = A_X/\widetilde{G_2}$, $V_Y = V_X/\widetilde{G_2}$, and $\overline{f} : \overline{X} \to \overline{Y}$ is the quotient morphism $(A_X \times V_X)/\widetilde{G} \to (A_Y \times V_Y)/\widetilde{G}_1$, so

$$\overline{f} : \overline{X} = (A_X \times V_X)/\widetilde{G} \to \overline{Y} = (A_Y \times V_Y)/\widetilde{G}_1$$

is also the quotient morphism.

Let $G = (A_X \times B)/A$ and $G_1 = (A_Y \times B)/A$. We have the short exact sequences of groups

$$1 \to \widetilde{G} \to G \to H \to 1 \quad \text{and} \quad 1 \to \widetilde{\mathrm{G}}_1 \to \mathrm{G}_1 \to \mathrm{H} \to 1.$$

Then $X = (A_X \times V_X)/G$ and $Y = (A_Y \times V_Y)/G_1$ and f is the quotient map. This proves Theorem 7. $\square$

1.7.4 A generalization on surfaces

Let $f : X \to Y$ be a non-birational morphism between smooth projective varieties of general type of dimension n. From Theorem 1.15 and Remark 1.16, we would like to see what is the difference between $P_m(X)$ and $P_m(Y)$ for $m \geq 2$ in general. If $n \geq 3$, this seems to be an extremely hard problem. We have the following examples.

Example 1.69 (Compare with Proposition 8.6.1 in [Kol1]) In this example a hypersurface of degree d in $\mathbb{P}(a_0^{s_0}, \ldots, a_k^{s_k})$ stands for a general hypersurface of degree d in the weighted projective space where we have s_j coordinates with weight a_i. For any integer $k \geq 3$, denote by P_X the weighted projective space $\mathbb{P}(1, (2k)^{4k+5}, (2k+1)^{4k-3})$ with coordinates x_i and denote by P_Y the weighted projective space $\mathbb{P}(2, (2k)^{4k+5}, (2k+1)^{4k-3})$ with coordinates y_i. Like the proof of Proposition 8.6.1, we can check that both P_X and P_Y have canonical singularities. There is a natural degree 2 morphism $\varepsilon : P_X \to P_Y$ by sending $y_0 = x_0^2$ and $y_i = x_i$ for $i \geq 1$. Let $Y^{'}$ be a hypersurface of degree $d = 16k^2 + 8k$ in P_Y and let $X^{'}$ be the pullback by ε of $Y^{'}$. Since $2k(2k+1)|d$ and Y is general, we have that X is also general and both X and Y have canonical singularities. We take suitable resolutions X and Y of $X^{'}$ and $Y^{'}$ respectively. Then we have a degree 2 morphism $f : X \to Y$ induced by ε. The canonical sheaves are $\omega_{X^{'}} = \mathscr{O}_{X^{'}}(2)$ and

$\omega_{Y'} = \mathscr{O}_{Y'}(1)$. Since both $X^{'}$ and $Y^{'}$ have canonical singularities, we have that for any integer $t \geq 0$,

$$P_t(X) = h^0(X^{'}, \mathscr{O}_{X'}(2t)), \text{ and } P_t(Y) = h^0(Y^{'}, \mathscr{O}_{Y'}(t)).$$

Therefore, taking $t = 2m < k$, we have $P_t(X) = P_t(Y) = 1$. We notice that $\dim X = \dim Y = 8k + 2$.

In the case of surface, the above question is much more accessible.

Theorem 1.70 *Let $f : S_1 \to S_2$ be a non-birational morphism between smooth projective surfaces of general type. Assume moreover that $P_m(S_1) = P_m(S_2)$ for some $m \geq 2$. Then $m = 2$ and one of the following cases occurs:*

1) *S_1 is birational to $C_1 \times C_2$, where C_1 and C_2 are smooth projective genus 2 curves, and f is birationally equivalent to the quotient of the diagonal involution.*

2) *S_1 is birational to the principal theta divisor Θ of the Jacobian of a smooth projective genus 3 curve C and f is birationally equivalent to the bicanonical map of S_1.*

3) *S_1 is birational to a double cover of an abelian surface A, branched along a divisor $B \in |2\Theta|$, having at most double points and f is birationally equivalent to the bicanonical map of S_1.*

Proof. Without loss of generality, we may assume that S_2 is a minimal surface of general type (K_{S_2} is big and nef), and we have the following diagram:

$$\begin{array}{ccc} S_1 & \xrightarrow{f} & S_2 \\ \big\downarrow{\scriptstyle g} & & \\ S & & \end{array}$$

where g is the contraction of all the (-2)-curves on S_1, namely S is the minimal model of S_1. Then by Riemann-Roch and Kawamata-Viehweg's vanishing Theorem, we have for $m \geq 2$:

$$\begin{aligned} P_m(S_2) &= \chi(\mathscr{O}_{S_2}) + \frac{m(m-1)}{2} K_{S_2}^2 \\ P_m(S_1) = P_m(S) &= \chi(\mathscr{O}_S) + \frac{m(m-1)}{2} K_S^2. \end{aligned} \qquad (30)$$

Since K_S is big and nef, we have the base-point-free theorem, namely for some $N \geq 0$, $|NK_S|$ is base-point-free. Thus we may write $|NK_{S_1}| = |Ng^*K_S| + E$, where E is exceptional for g and is the fixed divisor of $|NK_{S_1}|$. Since $K_{S_1} \succeq f^*K_{S_2}$ and we may also assume that $|NK_{S_2}|$ is base-point-free, we may write $g^*K_S = f^*K_{S_1} + D$, where D is an effective $\mathbb{Q}$-divisor on S. Therefore,

$$\begin{aligned} K_S^2 &= (g^*K_S \cdot g^*K_S) \geq (g^*K_S \cdot f^*K_{S_2}) \geq (f^*K_{S_2} \cdot f^*K_{S_2}) \\ &= \deg(f) K_{S_2}^2. \end{aligned} \tag{31}$$

Claim 1: $q(S) > 0$ and $q(S_2) = 0$.

If $q(S) = 0$, we have $\chi(\mathscr{O}_S) = 1 + P_1(S) \geq 1 + P_1(S_2) \geq \chi(\mathscr{O}_{S_2})$. Since $K_S^2 \geq \deg(f)K_{S_2}^2$ and $K_{S_2}^2 \geq 1$, we conclude by (30) that $P_m(S_1) > P_m(S_2)$ which is a contradiction.

If $q(S_2) > 0$, we consider the Albanese morphism of S_2, $a : S_2 \to \mathrm{Alb}(S_2)$. By Theorem 1.15, we know that the image of a is a curve. Hence we have a surjective morphism $g : S_2 \to C$ where C is a smooth projective curve of genus $g \geq 1$. Since f is non-birational, $\mathrm{rank}((g \circ f)_*\omega_{S_1}^m) > \mathrm{rank}(f_*\omega_{S_2}^m)$. We know that both $(g \circ f)_*\omega_{S_1/C}^m$ and $f_*\omega_{S_2/C}^m$ are nef ([V1, Corollary 3.6]). If $g(C) \geq 2$, we conclude by Riemann-Roch that

$$P_m(S_1) = h^0((g \circ f)_*\omega_{S_1}^m) > P_m(S_2) = h^0(f_*\omega_{S_2}^m),$$

which is a contradiction. If $g = 1$, we see in Lemma 1.51 that both $(g \circ f)_*\omega_{S_1}^m$ and $f_*\omega_{S_2}^m$ are ample. Again we conclude $P_m(S_1) > P_m(S_2)$ by Riemann-Roch. Hence we have proved Claim 1.

Claim 2: $m = 2$ and $\deg(f) = 2$.

By Claim 1, we have $\chi(\mathscr{O}_{S_2}) = 1 + P_1(S_2)$. By assumption that $P_m(S) = P_m(S_2)$, we have

$$\begin{aligned} 1 + P_1(S_2) + \frac{m(m-1)}{2}K_{S_2}^2 &= \chi(\mathscr{O}_S) + \frac{m(m-1)}{2}K_S^2 \\ &\geq \chi(\mathscr{O}_S) + \frac{\deg(f)m(m-1)}{2}K_{S_2}^2, \end{aligned}$$

where the second inequality holds because of (31). We have

$$\frac{(\deg(f)-1)m(m-1)}{2}K_{S_2}^2 \leq P_1(S_2) + 1 - \chi(\mathscr{O}_S). \tag{32}$$

On the other hand, we have the Noether inequality $P_1(S_2) \leq \frac{1}{2}K_{S_2}^2 + 2$. Thus by (32) and the fact that $\chi(\mathscr{O}_S) \geq 1$, we conclude that $m = 2$ and $\deg(f) = 2$ or $m = 2$ and $\deg(f) = 3$. In the latter case, we have $K_{S_2}^2 = 1$, $\chi(\mathscr{O}_S) = 1$. Therefore by Noether's inequality, $P_1(S_2) \leq 2$. Then by (30), we have

$$1 + K_S^2 = P_2(S) = P_2(S_2) = 2 + P_1(S_2).$$

Since $K_S^2 \geq 3$ and $P_1(S_2) \leq 2$, we conclude that $K_S^2 = 3$ and $P_1(S) \geq P_1(S_2) = 2$. Since $q(S) > 0$ by Claim 1, we have Debarre's inequality (see [D1], [D2]) that $K_S^2 \geq 2P_1(S)$. We then deduce a contradiction. Hence we have $m = \deg(f) = 2$ and

$$K_S^2 + \chi(\mathscr{O}_S) = 1 + P_1(S_2) + K_{S_2}^2. \tag{33}$$

Now by (33) and the fact $\chi(\mathscr{O}_S) \geq 1$, $K_{S_2}^2 \leq \frac{1}{2}K_S^2$, and $P_1(S_2) \leq P_1(S)$, we have $\frac{1}{2}K_S^2 \leq P_1(S)$ where the equality holds only if $K_S^2 = 2K_{S_1}^2$, $\chi(\mathscr{O}_S) = 1$, and $P_1(S_2) = P_1(S) = K_{S_2}^2$. However, since $q(S) > 0$, the equality should hold by Debarre's inequality.

Then by Noether's inequality, $K_{S_2}^2 = P_1(S_2) \leq \frac{1}{2}K_{S_2}^2 + 2$. Hence $K_{S_2}^2 \leq 4$. Then the family of S_2 is bounded and we may treat it case by case using the classification results in [BCP].

Case 1: $K_{S_2}^2 = 1$.

Then $K_S^2 = 2$, $P_1(S) = q(S) = P_1(S_2) = 1$ and $P_2(S) = P_2(S_2) = 3$. Hence by [BCP, Theorem 13], S is a genus 2-fibration over its Albanese variety, $a : S \to E$. Since $\deg(f) = 2$, there exists an involution $\sigma : \dashrightarrow S$ such that f is birationally equivalent to the quotient $S_1 \dashrightarrow S_1/\sigma$. Since S_2 is minimal and of general type, σ is biregular. We get the following commutative diagram by the university o the Albanese morphism,

$$\begin{array}{ccc} S & \xrightarrow{\sigma} & S \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} \\ E & \xrightarrow{h} & E. \end{array}$$

Since σ is an involution, h is the identity, or an involution. But S_2 is of general type and $q(S_2) = 0$, hence h is an involution with fixed points. Hence we have the following commutative diagram:

$$\begin{array}{ccc} S & \longrightarrow & S/\sigma \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle g} \\ E & \longrightarrow & \mathbb{P}^1. \end{array}$$

Hence the fixed locus of σ are smooth curves and thus $S_2 \simeq S/\sigma$. Then $K_S = K_S = f^*(K_{S_2} + g^*\mathscr{O}_{\mathbb{P}^1}(2))$. Since $(K_{S_2} \cdot \mathscr{O}_{\mathbb{P}^1}(1)) = 2$, we conclude that $K_S^2 = 2K_{S_2}^2 + 4(K_{S_2} \cdot \mathscr{O}_{\mathbb{P}^1}(1)) = 10$, which is impossible.

Case 2: $K_{S_2}^2 = 2$.

Then $P_1(S_2) = P_1(S) = 2$ and $K_S^2 = 4$.

We first assume that S_1 does not present the "standard case" (see [BCP, Definition 5]). Namely, there does not exist a dominate rational map onto a curve $g : S_1 \dashrightarrow B$ whose general fiber is irreducible of genus 2. Then by [BCP, Theorem 10], the Albanese morphism of S, $a_S : S \to \mathrm{Alb}(S)$ is branched along a divisor $B \in |2\Theta|$, having at most double points. Then the bicanonical morphism $\phi : S \to \mathbb{P}^4$ is of degree 2 and S_2 is just a smooth model of the image of $\phi(S)$.

If S presents the "standard case", we may assume that there exists an algebraic fiber space $g : S_1 \to B$ whose general fiber is of genus 2. Considering the Albanese morphism a_{S_1} of S_1.

If $g(B) \geq 1$, S_1 hence S has an irrational pencil. If the Albanese morphism of f is not surjective, Zucconi [Z] showed that the Albanese morphism is an étale bundle over a smooth genus 2 curve B. By the same argument in **Case 1**, the involution σ induced by f is induced from an involution on the image of the Albanese morphism and a similar calculation of $K_{S_1}^2$ shows that it is impossible. If the Albanese morphism is surjective. Then Zucconi [Z] showed that B can only be an elliptic curve and the only possibilities are either S is the minimal resolution of $(C_1 \times C_2)/G$, where both C_1 and C_2 are genus 2 curves, $G = \mathbb{Z}_2$ acting diagonally and both C_1/G and C_2/G are elliptic curves or S is the minimal resolution of $(C_1 \times C_2)/G$, where both C_1 and C_2 are genus 3 curves, $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, and both C_1/G and C_2/G are elliptic curves, and $C_i \to C_i/G$ is a Galois covering branched over two points for $i = 1, 2$. In the first case, $K_S^2 > 4$. In the second cases, $K_S^2 = 4$. However, the bicanonical morphism is a 2-cover to $\mathbb{P}^1 \times \mathbb{P}^1$. Hence S_2 should be birational to $\mathbb{P}^1 \times \mathbb{P}^1$ which is again impossible.

If $g(B) = 0$, then $g_*\omega_{S_1} = \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(b)$ for some integers $a \geq b$. Since $P_1(S_1) = q(S_1) = 2$, we conclude that $h^0(g_*\omega_{S_1}) = h^1(g_*\omega_{S_1}) = 2$. Hence $a = 1$ and $b = -3$. Then $\mathscr{O}_{\mathbb{P}^1}(-1)$ is a direct summand of $g_*\omega_{S_1/\mathbb{P}^1}$ which contradicts the fact that $g_*\omega_{S_1/\mathbb{P}^1}$ is nef.

Case 3: $K_{S_2}^2 = 3$.

Then $P_1(S) = q(S) = 3$ and $K_S^2 = 6$. By [BCP, Theorem 9], S is the symmetric square S^2C of a genus 3 curve C hence is then the theta divisor Θ of JC. The bicanonical map of S is a degree 2 map whose image is a surface of general type. Since $P_2(S_1) = P_2(S_2)$, we conclude that f is birationally equivalent to the bicanonical map of S.

Case 4: $K_{S_2}^2 = 4$.

Then $P_1(S) = K_{S_2}^2 = 4$ and $\chi(\mathscr{O}_S) = 1$, therefore by Beauville's inequality [Be2], $S = C_1 \times C_2$, where C_1 and C_2 are two genus 2 smooth projective curves. Let i_1 and i_2 be respectively the involutions on C_1 and C_2. Then ω_X^2 gives a degree 4-morphism from $S = C_1 \times C_2$ to the quotient $C_1/i_1 \times C_2/i_2 = \mathbb{P}^1 \times \mathbb{P}^1$. Since S_2 is of general type, it should be the minimal desingularization of $S/(i_1, i_2)$. This is exactly the case in Remark 1.16. $\square$

2 On the nef cone of symmetric product of curves

2.1 Introduction

Suppose C is a smooth projective curve of genus $g \geq 2$. It is well known that if C is of very general moduli, the Néron-Severi group $\mathrm{N}^1(C \times C)$ is generated by x_1, x_2 and Δ where x_1 and x_2 are the classes of pull-backs of line bundles of degree 1 over each factor and Δ is the class of the diagonal. We have $\Delta^2 = 2 - 2g < 0$. Kollár, see [La], Remark 1.5.11, asked whether Δ is the only irreducible curve with negative self-intersection.

A closely related question is the structure of the nef cone of the symmetric product S^2C of C. Let σ be the following involution of $C \times C$:

$$\begin{gathered}\sigma : C \times C \to C \times C \\ (c_1, c_2) \mapsto (c_2, c_1).\end{gathered}$$

Then S^2C is defined to be $C \times C/\langle\sigma\rangle$ and we denote by $\pi : C \times C \to S^2C$ the quotient morphism. Note that the classes $x_1 + x_2$ and Δ are invariant under the action of σ. There exist classes x and δ' on S^2C such that

$$\pi^* x = x_1 + x_2, \quad \text{and} \quad \pi^* \delta' = \Delta - x_1 - x_2.$$

The intersection numbers are as follows:

$$(x \cdot x) = 1, \quad (x \cdot \delta') = 0, \quad \text{and} \quad (\delta' \cdot \delta') = -\mathrm{g}.$$

Similarly, if C is very general, x and δ' generate the Néron-Severi group $\mathrm{N}^1(S^2C)$ of S^2C.

We then consider the cones of S^2C,

$$\mathrm{Nef}(S^2C) \subset \mathrm{NE}(S^2C) \subset \mathrm{N}^1(S^2C)_{\mathbb{R}},$$

where $\mathrm{NE}(S^2C)$ is the cone spanned by the classes of effective curves and $\mathrm{Nef}(S^2C)$ is the cone spanned by the classes of nef curves (divisors). It is well known that $\mathrm{Nef}(S^2C)$ and $\overline{\mathrm{NE}}(S^2C)$ are dual cones.

Since $2(x + \delta')$ is the class of the push-forward of the diagonal curve of $C \times C$ and

$$\big((x + \delta') \cdot (x + \delta')\big) = 1 - g < 0,$$

we conclude that $x+\delta'$ generates an extremal ray of $\mathrm{NE}(S^2C)$. By duality, $gx+\delta'$ generates an extremal ray of $\mathrm{Nef}(S^2C)$. Thus one side of the effective cone is closed. In order to determine the nef cone, we now focus on the following classes

$$e_t = tx - \delta'.$$

We define $t(C)$ to be the minimal number t such that e_t is nef. As $(e_t \cdot e_t) = t^2 - g$, we see that $t(C) \geq \sqrt{g}$. If there does not exist any irreducible curve with negative self-intersection on $C \times C$ except the diagonal, there does not exist any irreducible curve on S^2C with negative self-intersection except the push-forward of the diagonal on $C \times C$, namely the other side of the effective cone of S^2C is open (which is expected for g big enough). By duality, e_t would be nef for all $t \geq \sqrt{g}$. Thus we would completely describe the nef cone of S^2C.

We collect some known results for the nef cone of S^2C.

1) If $g = 2$, C is an hyperelliptic curve then we take $L \in W_2^1(C)$ then $L \boxtimes L \otimes \mathscr{O}(-\Delta)$ is effective and represented by the curve $D = (x, \tau(x))$ where τ is the involution associated to L and $D^2 = -2$. Hence the nef cone of S^2C is spanned by $2x + \delta'$ and $2x - \delta'$ and the effective cone is closed. Similarly, for a hyperelliptic curve C of genus g, $gx + \delta'$ and $gx - \delta'$ span the nef cone and the effective cone is closed.

2) If $g = 3$, Kouvidakis [Ko, Theorem 2] (see also [D3, Proposition 8]) shows that there exists an irreducible curve with class $20x - 12\delta'$. Hence the nef cone of S^2C is spanned by $3x + \delta'$ and $9x - 5\delta'$. The effective cone is closed.

3) If $g = 4$, Debarre [D3, Proposition 8] showed that $2x - \delta'$ is nef and the effective cone is closed.

4) If $g \geq 9$ is a perfect square, Ciliberto and Kouvidakis [CK] proved that the effective cone is open on the other side. Hence, by duality, $t(C) = \sqrt{g}$.

5) There is a nice theorem of Kouvidakis (see [Ko] or [La, Theorem 1.5.8]) which gives a bound for $t(C)$.

Theorem 2.1 *Assume that C admits a simple branched covering*

$$\pi : C \to \mathbb{P}^1$$

of degree $d \leq [\sqrt{g}]+1$. Then $t(C) = \frac{g}{d-1}$. As a corollary, if C is a very general curve of genus g,

$$\sqrt{g} \leq t(C) \leq \frac{g}{[\sqrt{g}]}.$$

The purpose of this part is to improve on Theorem 2.1. This part was written around 2007. When it was almost finished, I heard from Prof. Debarre that J. Ross had just published in 2007 a paper [Ro] with similar results. The difference is that Ross used degenerations of self-products of curves whereas I use degeneration of symmetric product of curves. Ross used this method to deal the case $g(C) = 5$ while I obtain a general bound for $t(C)$ (Theorem 2.5 and Theorem 2.8). Bastianelli studied this problem in the 2009 article [B1] and the 2010 eprint [B2]. He obtained better bounds for $t(C)$ when $g(C) = 5, 6, 7, 8$. For small genera, we compare the results of Kouvidakis in [Ko], the results of Ross in [Ro], the results of Bastianelli in [B1] and [B2] and my results.

$t(C)$	Kouvidakis' bound	Ross' bound	my bound	Bastianelli's bound	g
$\leq$	$\frac{5}{2}$	$\frac{16}{7}$	$\frac{9}{4}$	$\frac{9}{4}$	5
$\leq$	3	–	$\frac{5}{2}$	$\frac{32}{13}$	6
$\leq$	$\frac{7}{2}$	–	$\frac{8}{3}$	$\frac{77}{29}$	7
$\leq$	4	–	3	$\frac{17}{6}$	8
$\leq$	3	–	3	–	9
$\leq$	$\frac{10}{3}$	–	$\frac{13}{4}$	–	10

2.2 A degeneration method

In this section we describe a degeneration method due to Ciliberto and Kouvidakis [CK]. By this method they are able to prove that the effective cone of the symmetric product of a very general curve whose genus $g \geq 9$ is a square is open on one side and the same statement is true for $g \geq 9$ if we

assume the famous Nagata conjecture. Ciliberto and Kouvidakis considered the Chow variety of a family of curves and we prefer the Hilbert scheme in this paper, although they are of the same nature.
We will always assume $\pi : X \to B$ a flat family of projective curve of genus g over a smooth curve B, $s \in B$ is a closed point, X_b is smooth for all closed point $b \neq s$ and X_s is a nodal curve. Let B^* denote the open subscheme $B \setminus s$ and $X^* \to B^*$ the induced family. Let $\mathrm{Hilb}^2(X/B) : \{\textit{schemes over } B\} \to \{\textit{sets}\}$ denote the following functor:

$$\mathrm{Hilb}^2(X/B)(Y)=\{\textit{0-dimensional subschemes } Z \subset X \times_B Y \\ \textit{such that } Z \textit{ is proper and flat over } Y \textit{ and the colength of } Z \textit{ is } 2\}$$

According to the theory of Hilbert schemes [Gr], there exist a projective scheme $\mathrm{Hilb}^2(X/B)$ over B and a universal family $\mathrm{Univ}(X/B) \subset X \times_B \mathrm{Hilb}^2(X/B)$ represents the functor $\mathrm{Hilb}^2(X/B)$. By the universal property, $\mathrm{Hilb}^2(X/B)_t$ is isomorphic to $\mathrm{Hilb}^2 X_t$ for any point $t \in B$. It is well known that $\mathrm{Hilb}^2(C)$ is just S^2C for C a smooth curve.

We would like to find a line bundle $\mathscr{L}_B$ over $\mathrm{Hilb}^2(X/B)$ such that the class of $\mathscr{L}_t$ is in the form $ax - b\delta$. Then $\mathscr{L}_s$ will give us some interesting information about $ax - b\delta$.

First we give a concrete example about the Hilbert scheme of a nodal curve.

Example 2.2 Let X be the affine curve Spec $k[x,y]/(xy)$. We will denote by o the node. The ideas of the subschemes Z of length 2 contained in X are of the following types:

1. The support of Z doesn't contain o: $(x^2 - (a+b)x + ab, y), (x, (y-a)(y-b)), (x^2 - ax, y^2 - by, xy, ay + bx)$, $ab \neq 0$;
2. The subscheme Z consists of two reduced points, one of which is o: $(x,y)(x-a,y), (x,y)(x,y-b)$, $ab \neq 0$;
3. The support of Z is o: $(ax + by, x^2, y^2)$, $a \neq 0$, or $b \neq 0$.

We now compute the Zariski tangent space of Z in $\mathrm{Hilb}^2(X)_s$, $T_Z \mathrm{Hilb}^2(X)_s$ is naturally isomorphic to $Hom_{\mathscr{O}_Z}(I_Z/I_Z^2, \mathscr{O}_Z)$. Hence $\dim T_Z \mathrm{Hilb}^2(X)_s = 2$ except the cases $I_Z = (x^2 - ax, y), (x, y^2 - by)$. In all the exceptional cases $\dim T_Z \mathrm{Hilb}^2(X)_s = 3$.
We now construct the following families of subschemes of X:

1. A family over the blow up of the origin o of $\mathbf{A^2}$: $\mathscr{Z}^1 \subset \mathrm{Bl}_o\,\mathbf{A^2} \times X$. Let p be the point $(\alpha t, \beta t) \times (\alpha : \beta)$, then the ideal of $\mathscr{Z}^1_p$ is $(x^2 + \alpha t x, y^2 + \beta t y, \beta x + \alpha y + \alpha\beta t)$, we denote the two lines $(t, 0) \times (1 : 0)$ and $(0, t) \times (0 : 1)$ by e_1 and e_2;

2. Two families over $\mathbf{A^2}$: $\mathscr{Z}^1 \subset \mathbf{A^2}_1 \times X$, $\mathscr{Z}^2 \subset \mathbf{A^2}_2 \times X$. Let p be the point (a, b). The ideal of $\mathscr{Z}^1_p$ is $(x^2 - ax + b, y)$ and $\mathscr{Z}^2_p$ is $(x, y^2 - ay + b)$. We take the line $b = 0$ contained in $\mathbf{A^2}_1$ and $\mathbf{A^2}_2$ to be l_1, l_2.

These families glue to the universal family over $\mathrm{Hilb}^2(X)$ by identifying e_1 and l_1, e_2 and l_2, hence $\mathrm{Hilb}^2(X)$ is isomorphic to $\mathbf{A^2}_1\,{}_{e_1 \sim l_1} \sqcup Bl_o\mathbf{A^2}\,{}_{e_2 \sim l_2} \sqcup \mathbf{A^2}_2$. The normalization of $\mathrm{Hilb}^2(X)$ is isomorphic to $\mathbf{A^2}_1 \sqcup Bl_o\mathbf{A^2} \sqcup \mathbf{A^2}_2$.

In general, from the above computation, we obtain (see also [Ra])

Theorem 2.3 *Suppose that C_0 is a connected projective curve with at most ordinary double points $q_1, \ldots, q_\delta$ and that $\epsilon : C \to C_0$ is the normalization of C_0, with p_i^1, p_i^2 the inverse images of q_i. Take $\nu : \mathscr{M} \to \mathrm{Hilb}^2(C_0)$ to be the normalization. Then there is a natural morphism $\rho : \mathscr{M} \to \mathrm{Hilb}^2(C)$ realizing $\mathscr{M}$ as the blow-up of $\mathrm{Hilb}^2(C)$ at $p_i^1 + p_i^2$ for $i = 1, \ldots, \delta$.*

We consider the following subscheme $\Delta_2 \hookrightarrow X^* \times_{B^*} X^*$ defined by the square of the idea $\mathscr{I}_{\Delta^*_B}$ of the diagonal Δ_{B^*}. Since Δ_2 is finite over B^* of length 2, there is morphism $\mu : X^* \to \mathrm{Hilb}^2(X^*/B^*)$ and the pull back of the universal family is Δ_2. Since $\mathrm{Hilb}^2(X^*/B^*)_{B^*}$ is smooth, the Weil divisor $\mu(X^*)$ gives us a line bundle $\mathscr{O}(\Delta)_{B^*}$ over $\mathrm{Hilb}^2(X^*/B^*)_{B^*}$. Since $\mathrm{Hilb}^2(X/B)_B$ is smooth, we could extend it to a line bundle $\mathscr{O}(\Delta)_B$ over the whole family $\mathrm{Hilb}^2(X/B)_B$. It is clear that $\mathscr{O}(\Delta)_B|_{\mathrm{Hilb}^2(X_s)}$ is just $2X_s \hookrightarrow S^2X_s = \mathrm{Hilb}^2 X_s$ for $s \neq o$. Moreover,

Theorem 2.4 *Under the notation in Theorem 2.3, $\nu^*\mathscr{O}(\Delta)_s$ is just $\rho^*(\mathscr{O}(\Delta)_C) \otimes \mathscr{O}(2E)$ where E denotes the sum of the exceptional divisors.*

2.3 New bounds for $t(C)$

In this section we improve the general bound $t(C)$ for C a very general curve of genus g. Let $\tau(g) \in \frac{1}{2}\mathbb{Z}$ be the minimal element larger than $\sqrt{g}$.

Theorem 2.5 *If C is a very general curve of genus $g \geq 5$,*

$$\sqrt{g} \leq t(C) \leq \min\{\frac{g}{\lfloor \sqrt{g} \rfloor}\ ,\ \tau(g)\},$$

in particular, $t(C) \leq \lceil \sqrt{g} \rceil$.

Remark 2.6 The inequality $\sqrt{g} \leq t(C) \leq \frac{g}{\lfloor \sqrt{g} \rfloor}$ is just Kouvidakis' Theorem 2.1. We just need to prove $t(C) \leq \tau(g)$ when g is not a perfect square.

First we fix a very general smooth projective curve C_0 of genus $g-1$. By induction we know that $\lceil \sqrt{g} \rceil x - \delta'$ is ample on $\mathrm{Hilb}^2(C_0)$. We need the following proposition due to Ein and Lazarsfeld [EL2] :

Proposition 2.7 (Ein-Lazarsfeld) *Let X be a smooth projective surface. Let L be an ample divisor on X. For all except perhaps countably many points $p \in X$, the divisor $\pi^*L - E$ is nef on $\mathrm{Bl}_p X$, where $\pi : \mathrm{Bl}_p X \to X$ is the blow-up at p and E is the exceptional divisor.*

Hence $\lceil \sqrt{g} \rceil x - \delta' - E$ is nef on $\mathrm{Bl}_{t_1+t_2} S^2C_0$, for t_1 and t_2 very general points on C_0. We chose a family of curves $\pi : X \to B$ of genus g, with only singular fiber $X_s \simeq C_0/_{t_1 \sim t_2}$. By the above proposition we know that the divisor $\lceil \sqrt{g} \rceil x - \delta' - E$ is nef on $\mathrm{Bl}_{t_1+t_2} S^2C_0$. Then we deduce from Theorem 2.3 and Theorem 2.4 that $\lceil \sqrt{g} \rceil x - \delta'$ is nef on $\mathrm{Hilb}^2 X_t$, $t \in B$ very general.

The proof that $t(C) \leq \tau(g)$ follows the same idea of the above argument. In the following we will assume that $g \geq 6$ since $t(5) = \frac{5}{2} = \frac{5}{\lfloor \sqrt{5} \rfloor}$.
We just need to prove the following claim:

Claim: Suppose that $z = t_1 + t_2$ is a very general point of S^2C_0, then
$2\tau(g)x - 2\delta' - 2E$ is nef on $\mathrm{Bl}_z S^2C_0$.

If the class $2\tau(g)x - 2\delta' - 2E$ is not nef, there exists an integral curve $\widetilde{\Gamma}$ on $\mathrm{Bl}_z S^2C_0$ such that

$$(\widetilde{\Gamma} \cdot (2\tau(g)x - 2\delta' - 2E)) < 0,$$

namely, there exists a reduced irreducible curve Γ on S^2C_0 passes through z with $2\mathrm{mult_z}(\Gamma) > \Gamma \cdot (2\tau(g)x - 2\delta')$.

We consider the following families, just as Ein and Lazarsfeld did in order to prove Proposition 2.7,

> $\{(\Gamma, z) \mid \Gamma \subset S^2C_0$ is an integral curve, $z \in \Gamma$ and $2\mathrm{mult_z}(\Gamma) > \Gamma \cdot (2\tau(g)x - 2\delta')\}$.

If each family is discrete then we are done since there are at most countable many such families and we just need to take $z \in S^2C_0$ outside the curves in these countable families, which is possible by Baire's argument.

Suppose to the contrary that (Γ_t, z_t) is a non-trivial one-parameter family, parameterized by a smooth curve T, of reduced and irreducible curves $\Gamma_t \subseteq \mathrm{Hilb}^2 C_0$ and points $z_t \in \Gamma_t$ with $2m = 2\mathrm{mult_{z_t}(\Gamma_t)} > \Gamma_t \cdot (2\tau(\mathrm{g})\mathrm{x} - 2\delta^{'})$. We first fix (Γ_0, z_0). The deformation (Γ_t, z_t) determines a section of $H^0(\Gamma_0, \mathscr{O}_{\Gamma_0}(\Gamma_0))$ which vanishes to order at least $m-1$ at z_0, therefore

$$(\Gamma_0 \cdot \Gamma_0) \geq m(m-1). \tag{34}$$

By the Hodge index theorem, we have

$$(2m-1)^2 \geq (\Gamma_0 \cdot (2\tau(g)x - 2\delta^{'}))^2 \geq (\Gamma_0^2)(2\tau(g)x - 2\delta^{'})^2 \geq 5(\Gamma_0^2),$$

the last inequality holds because

$$((2\tau(g)x - 2\delta^{'})^2) = 4(\tau(g)^2 - g + 1) \geq 5$$

(g is not a perfect square). We get that $5m(m-1) \leq (2m-1)^2$. The only possible case is $m = 1$ and $(\Gamma_0^2) = 0$. We may write the class of Γ_0 in the form $\alpha x + \beta\delta^{'}$ with $\alpha^2 - (g-1)\beta^2 = 0$. Hence $g-1$ is a perfect square, but it is already known from the work of Cilibeto and Kouvidakis [CK] that there are no such reduced irreducible Γ_0 with self-intersection 0. This finishes the proof of Theorem 2.5.

A better bound can be obtained. Define $\mu(g)$ to be the minimal rational number of the form $\frac{a}{b}$ where a and b are positive integers such that $a^2 \geq gb^2$ and $1 \leq b \leq 4$.

Theorem 2.8 *Let $g \geq 5$ be an integer. Then for a very general curve C of genus g, we have $t(C) \leq \mu(g)$.*

The proof is the same as the proof of Theorem 2.5. We may assume that $\mu(g) = \frac{a}{b}$. Assume that $b\mu(g)x - b\delta^{'}$ is not nef on C. Then $b\mu(g)x - b\delta^{'} - bE$ is not nef on $\mathrm{Bl}_z\, S^2C_0$ for any $z \in S^2C_0$. We consider the family

> $\{(\Gamma, z) \mid \Gamma \subset S^2C_0$ is an integral curve, $z \in \Gamma$ and $b\mathrm{mult_z}(\Gamma) > \Gamma \cdot (b\mu(g)x - b\delta^{'})\}$.

Again by Baire's argument, this family should be of at least 2-dimensional. Hence the deformation (Γ_t, z_t) determines 2 sections of $H^0(\Gamma_0, \mathscr{O}_{\Gamma_0}(\Gamma_0))$ which vanishes to order at least $m-1$ at z_0. Since the geometric genus of Γ_0 is > 0, we have a better estimation:

$$(\Gamma_0 \cdot \Gamma_0) \geq m(m-1) + 2. \tag{35}$$

We then easily deduce the theorem.

Remark 2.9 The above inequality (35) is further improved in Bastianelli's papers [B1] and [B2]. Hence he is able to give better bounds of $t(C)$. For instance, Bastianelli proved in [B1] that we have

$$(\Gamma_0 \cdot \Gamma_0) \geq m(m-1) + 3$$

in the above situation. With his result, we can show that $t(C) \geq \nu(g)$, where $\nu(g)$ to be the minimal rational number of the form $\frac{a}{b}$ where a and b are positive integers such that $a^2 \geq gb^2$ and $1 \leq b \leq 5$ when C is a very general curve of genus $g \geq 5$.

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