

# Borel type bounds for the self-avoiding walk connective constant

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**Abstract.** Let  $\mu$  be the self-avoiding walk connective constant on  $\mathbb{Z}^d$ . We show that the asymptotic expansion for  $\beta_c = 1/\mu$  in powers of  $1/(2d)$  satisfies Borel type bounds. This supports the conjecture that the expansion is Borel summable.

**Keywords** Self-avoiding walk, lace expansion, asymptotic series, Borel summability.

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## 1. Introduction

Let  $\mathbb{Z}^d$  denote the hypercubic lattice, with nearest neighbour edges. A self-avoiding walk of length  $n$  is a sequence of points  $\omega_0, \omega_1, \dots, \omega_n$  in  $\mathbb{Z}^d$  such that  $|\omega_i - \omega_{i+1}| = 1$  and for  $i \neq j$ ,  $\omega_i \neq \omega_j$ . Let  $c_n$  denote the number of self-avoiding walks, up to translation invariance, of length  $n$  on  $\mathbb{Z}^d$ . It is well known that the limit  $\mu(d) = \lim_{n \rightarrow \infty} c_n^{1/n}$  exists [1]; the limit is called the connective constant. Fisher and Gaunt calculated that [2]

$$\mu = 2d - 1 - 1/(2d) - 3/(2d)^2 - 16/(2d)^3 - 102/(2d)^4 - \dots$$

However, their calculation is somewhat mysterious. Firstly, they leave open the question of whether or not the expansion can be continued to higher orders of  $1/d$ . Secondly, even though the error term ‘...’ is left uncontrolled, numerical extrapolation techniques yield surprisingly accurate estimates for  $\mu$ .

Expansions in powers of  $1/d$  have been developed for many other models in statistical physics, such as the Ising model [2], percolation [3], lattice animals [4] and the  $n$ -vector model [5]. Finding the coefficients of the expansion is normally computationally intensive. It is often even more difficult to determine the basic properties of the expansion. What is the radius of convergence? Is it an asymptotic expansion? Can the expansion be interpreted as a Borel sum?

The self-avoiding walk is most easily understood in high dimensions. As  $d \rightarrow \infty$ , paths in  $\mathbb{Z}^d$  with large loops become relatively rare. It is therefore useful to consider a walk with only local self-avoidance. Say that  $\omega_0, \dots, \omega_n$  is a memory- $\tau$  self-avoiding walk if  $\omega(i) \neq \omega(j)$  for  $0 < |i - j| \leq \tau$ . Let  $c_n^{(\tau)}$  denote the number of  $n$ -step memory- $\tau$  walks, up to translation invariance, and let  $\mu_\tau(d) = \lim_{n \rightarrow \infty} (c_n^{(\tau)})^{1/n}$ . Using memory-4 self-avoiding walks as a starting point (taking into account loops of size 2 and 4) Kesten showed that [6]

$$\mu(d) = 2d - 1 - 1/(2d) + O(1/d^2).$$

By considering finite-memory self-avoiding walks with longer memories, the order of the error bound can be improved. However, not only is this method extremely computationally taxing, it provides no guarantee that the resulting expansion will only contain integer powers of  $d$ .

The series expansion for  $\mu$  was put on a much firmer footing by Hara and Slade using the lace expansion [7]. The lace expansion is a powerful technique for exploring the properties of the self-avoiding walk in dimensions  $d > 4$ ; we refer the reader to [8] for a recent introduction. Hara and Slade showed that the connective constant  $\mu$  has an asymptotic expansion in integer powers of  $1/(2d)$  to all orders, with all the coefficients taking integer values.

We will actually phrase their result in terms of the series expansion for the reciprocal of  $\mu(d)$ . The quantity  $\beta_c = 1/\mu(d)$  is the radius of convergence of the self-avoiding walk susceptibility  $\chi(z) = \sum c_n z^n$ . For convenience, set  $s = 1/(2d)$ . Hara and Slade showed

that there are constants  $(\alpha_n)$  such that for  $M = 1, 2, \dots$ , [7]

$$\beta_c(s) = \sum_{n=1}^{M-1} \alpha_n s^n + O(s^M). \quad (1.1)$$

They also verified rigorously that the first six terms in the expansion match the exact calculation of Fisher and Gaunt.

The lace expansion can be used to automate the process of calculating the coefficients of the asymptotic expansion for  $\beta_c$  and  $\mu$ . The computational complexity of the process is reduced using a combinatorial trick known as the two-step method. Using a supercomputer to implement the two-step method, the first 13 coefficients of  $\beta_c$  have been found [9],

$$1, 1, 2, 6, 27, 157, 1065, 7865, 59665, 422421, 1991163, -16122550, -805887918. \quad (1.2)$$

It is not known, but it is widely believed, that the radius of convergence of the expansion for  $\beta_c$  is zero. We will show that the partial sums satisfy Borel type bounds. Borel summability raises the prospect of calculating  $\mu$  from the series expansion even if the radius of convergence is zero.

**Theorem 1.3.** *There exist a constant  $C_1$  such that for all  $d$ ,*

$$\left| \beta_c(s) - \sum_{n=1}^{M-1} \alpha_n s^n \right| \leq C_1^M s^M M!, \quad M = 1, 2, \dots \quad (1.4)$$

The motivation for Theorem 1.3 is discussed in Section 2. In Section 4, we use the lace expansion to derive a formula for the  $\alpha_n$ . This formula is used in Section 5 to control the growth of the  $\alpha_n$  as  $n \rightarrow \infty$ . In Section 6, we consider the diagrammatic estimates for the lace expansion. Finally, in Section 7, we prove Theorem 1.3.

## 2. Borel summability and the spherical model

In light of Theorem 1.3, it is natural to ask if  $\beta_c$  can be recovered from the  $\alpha_n$  by means of a Borel sum. Let  $B$  denote the Borel transform of the asymptotic expansion for  $\beta_c$ ;  $B$  is well defined (see Lemma 5.1) in a neighbourhood of zero by

$$B(t) = \sum_{n=1}^{\infty} \alpha_n t^n / n!$$

We conjecture that  $B$  can be extended analytically to a neighbourhood of the positive real axis, and that  $\beta_c(s)$  is equal to the Borel sum

$$\sum_{\text{Borel}} \alpha_n s^n := \frac{1}{s} \int_0^{\infty} e^{-t/s} B(t) dt.$$

There are two reasons for making this conjecture. Firstly, with  $R > 0$ , let  $C_R := \{z \in \mathbb{C} : \operatorname{Re} z^{-1} > R^{-1}\}$  denote the open disc in  $\mathbb{C}$  with centre  $R/2$  and diameter  $R$ . Suppose that  $\beta_c$  can be extended to an analytic function on  $C_R$  such that (1.4) holds for all  $s \in C_R$ . Under this assumption, the Borel sum is well defined in  $C_R$  and equal to

$\beta_c$  [10]. Unfortunately, it is not clear how to extend  $\beta_c$  to an analytic function on  $C_R$ . Interpreting the Borel sum remains an open problem.

Secondly, there is the case of the spherical model, which is a spin system defined on  $\mathbb{Z}^d$ . There is a surprising connection between the spherical model and self-avoiding walk; both are identified with limits of the  $n$ -vector models (also known as the  $O(n)$  model). The  $n$ -vector model is defined for positive integer values of  $n$ ; for example, the Ising model corresponds to  $n = 1$ . The model has been studied extensively by scientists; many aspects of the models have been solved ‘exactly’ [11]. De Gennes ([12] and [1, Section 2.3]) showed that in an abstract sense, the self-avoiding walk can be viewed as the ‘limit’ of the  $n$ -vector model as  $n \rightarrow 0$ . Stanley showed that as  $n \rightarrow \infty$ , the free energy of the  $n$ -vector model approaches the spherical model free energy [13]. The spherical model is thus said to be the limit as  $n \rightarrow \infty$  of the  $n$ -vector model.

There is an exact solution  $K_c(d)$  for the critical point of the spherical model. Gerber and Fisher show that  $K_c(d)$  can be written as a  $1/d$  expansion [5]; it is a rare example of a  $1/d$  expansion about which a great deal is known. They prove that while the radius of convergence of the expansion is zero, the expansion can be interpreted as a Borel sum [5, (2.14)]:

$$K_c(d) = \sum_{\text{Borel}} \frac{K_n}{(2d)^n} \quad \text{with Borel transform} \quad \sum_{n=1}^{\infty} \frac{K_n x^n}{n!}.$$

Note that the signs of the coefficients ( $K_n$ ) oscillate. The first 12 coefficients are positive, the next 8 are negative, the next 9 are positive; the pattern of signs goes

12, 8, 9, 9, 9, 9, 9, 9, 9, 10, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 10, 9, 9, 9, 9, 9, ...

This oscillation is related to the fact that the Borel transform has no poles on the positive real axis. We saw in (1.2) that the coefficients  $\alpha_n$  for the self-avoiding walk also show a change of sign. The first 11 are positive;  $\alpha_{12}$  and  $\alpha_{13}$  are negative.

### 3. Notation

Given a generating function  $\phi(\beta)$ , we will write  $[\beta^n]\phi(\beta)$  to denote the coefficient of  $\beta^n$ .

We will refer to the fact that  $(n/e)^n \leq n! \leq n^n$  for  $n = 0, 1, 2, \dots$  as Stirling’s approximation.

### 4. From lace expansions to asymptotic expansions

The lace expansion can be thought of as a sum of inclusion/exclusion terms. For a derivation of the lace expansion, see [8, Section 3.2]. The finite memory self-avoiding walks will play a vital role in the proof of Theorem 1.3. For a derivation of the lace expansion for memory- $\tau$  self-avoiding walk, see [14].

A *lace* of type  $N$  and length  $a$  is a sequence of open intervals  $(s_1, t_1), \dots, (s_N, t_N)$  such that

- (i)  $s_i$  and  $t_i$  are integers with  $0 < t_i - s_i \leq \tau$ ,

- (ii)  $s_1 = 0$  and  $t_N = a$ ,
- (iii) for  $i = 1, \dots, N-1$ ,  $(s_i, t_i)$  intersects  $(s_{i+1}, t_{i+1})$ , and
- (iv) if  $|i - j| > 1$ ,  $(s_i, t_i)$  and  $(s_j, t_j)$  are disjoint.

For example,  $(0, 4), (3, 5), (4, 6)$  is a lace if  $\tau \geq 4$ .

A simple walk  $\omega(0), \omega(1), \dots, \omega(a)$  starting from 0 is said to be *compatible* with the lace  $(s_1, t_1), \dots, (s_N, t_N)$  if each interval corresponds to a loop,

$$\omega(s_i) = \omega(t_i) \quad \text{for} \quad i = 1, \dots, N,$$

and if certain self-avoidance constraints are satisfied:

- (i)  $\omega(0), \dots, \omega(t_1 - 1)$  is a memory- $\tau$  self-avoiding walk,
- (ii)  $\omega(1), \dots, \omega(t_2 - 1)$  is a memory- $\tau$  self-avoiding walk (for  $N \geq 2$ ), and
- (iii)  $\omega(t_{i-2}), \dots, \omega(t_i - 1)$  is a memory- $\tau$  self avoiding walk (for  $3 \leq i \leq N$ ).

Let  $\pi_a^{(N)}(x; \tau)$  count the number of  $a$ -step simple walks from 0 to  $x$  that are compatible with a memory- $\tau$  type- $N$  lace. The lace expansion is defined

$$\Pi_\beta(x; \tau) = \sum_{N=1}^{\infty} (-1)^N \Pi_\beta^{(N)}(x; \tau) \quad \text{where} \quad \Pi_\beta^{(N)}(x; \tau) = \sum_{a=N+1}^{\infty} \pi_a^{(N)}(x; \tau) \beta^a.$$

The Fourier transform for functions  $f : \mathbb{Z}^d \rightarrow \mathbb{R}$  is given by

$$\hat{f}(k) = \sum_x f(x) e^{-ik \cdot x}, \quad k \in [-\pi, \pi]^d.$$

We will write  $\hat{\pi}_a^{(N)}(k; \tau)$ ,  $\hat{\Pi}_\beta^{(N)}(k; \tau)$  and  $\hat{\Pi}_\beta(k; \tau)$  for the Fourier transforms of  $\pi_a^{(N)}(x; \tau)$ ,  $\Pi_\beta^{(N)}(x; \tau)$  and  $\Pi_\beta(x; \tau)$ , respectively.

The starting point in our analysis will be [7, (2.2)]. Let  $\beta_\tau = 1/\mu_\tau$ , and take  $\beta_\infty = \beta_c$ . When  $d$  is sufficiently large, for  $\tau$  finite and  $\tau = \infty$ ,

$$\beta_\tau = s(1 - \hat{\Pi}_{\beta_\tau}(0; \tau)). \quad (4.1)$$

In this section, we will use this formula to derive series expansions for  $\beta_\tau$ .

The definition of the lace expansion respects the symmetries of the underlying lattice. There are  $2^d d!$  ways of choosing an ordered orthonormal basis for  $\mathbb{R}^d$  from the set  $\mathbb{Z}^d$ . Each simple walk in  $\mathbb{Z}^d$  with dimensionality  $D$  is equivalent to  $2d(2d-2) \dots (2d-2D+2)$  other walks under the action of this group of symmetries.

Let  $f_\tau(a, N, D)$  count the number of equivalence classes of the set of simple walks in  $\mathbb{Z}^D$  that have dimensionality  $D$ , length  $a$ , and that are compatible with memory- $\tau$  laces of type  $N$ . If  $a < 2D$ ,  $f_\tau(a, N, D) = 0$ . Therefore, we can write the number of walks compatible with memory- $\tau$  laces of length  $a$  and type  $N$  in  $\mathbb{Z}^d$  as a polynomial in powers of  $s^{-1} = 2d$ ,

$$\sum_{D=1}^{\lfloor a/2 \rfloor} f_\tau(a, N, D) 2d(2d-2) \dots (2d-2D+2) = \sum_{b=\lceil a/2 \rceil}^{a-1} c_{a,b,N} s^{b-a}.$$

Let  $I = \{(a, b) : b = 1, 2, \dots; a = b + 1, \dots, 2b\}$  and set

$$c_{a,b} = \sum_{N=1}^{\infty} (-1)^{N+1} c_{a,b,N}, \quad (a, b) \in I.$$

The  $c_{a,b}$  depend implicitly on  $\tau$ , but  $c_{a,b}$  is fixed once  $\tau \geq a$ . Using this notation to rewrite (4.1) yields a formal power series,

$$\begin{aligned}\beta_\tau &= s \left[ 1 + \sum_{(a,b) \in I} \beta_\tau^a c_{a,b} s^{b-a} \right] \\ &= s [1 + \beta_\tau^2 c_{2,1} s^{-1} + \beta_\tau^3 c_{3,2} s^{-1} + \beta_\tau^4 (c_{4,3} s^{-1} + c_{4,2} s^{-2}) + \dots]\end{aligned}\tag{4.2}$$

Plugging “ $\beta_\tau = 0$ ” into the right hand side gives “ $\beta_\tau = s$ ”. Taking “ $\beta_\tau = s$ ” and plugging it back into the right hand side then gives “ $\beta_\tau = s + c_{2,1} s^2 + (c_{3,2} + c_{4,2}) s^3 + \dots$ ”; iterating in this way yields a series expansion for  $\beta_\tau$ :

$$\beta_\tau = \sum_{n=1}^{\infty} \alpha_{n,\tau} s^n = s + c_{2,1} s^2 + (2c_{2,1}^2 + c_{3,2} + c_{4,2}) s^3 + \dots$$

When  $\tau = \infty$ , the  $\alpha_{n,\tau}$  are exactly the  $\alpha_n$  that appear in (1.1). Note that the formulae generated for the  $\alpha_{n,\tau}$ ,

$$\alpha_{1,\tau} = 1, \quad \alpha_{2,\tau} = c_{2,1}, \quad \alpha_{3,\tau} = 2c_{2,1}^2 + c_{3,2} + c_{4,2}, \quad \dots$$

only depend on  $\tau$  through the values of the  $c_{a,b}$ .

**Lemma 4.3.** *With  $S_n := \{(n_{a,b}) \in \mathbb{N}^I : \sum_I b n_{a,b} = n - 1\}$ ,*

$$\alpha_{n,\tau} = \sum_{(n_{a,b}) \in S_n} \frac{[\sum_I a n_{a,b}]!}{[\prod_I n_{a,b}]! [1 + \sum_I (a-1) n_{a,b}]!} \prod_I c_{a,b}^{n_{a,b}}.$$

The big- $\Sigma$  in the formula for  $\alpha_{n,\tau}$  is a sum indexed by the elements of the finite set  $S_n$ ; each element  $(n_{a,b})$  of  $S_n$  is a sequence indexed by  $I$ .

It is a corollary of Lemma 4.3 that  $\alpha_{n,\tau} = \alpha_n$  if the memory  $\tau \geq 2n - 2$ . This follows from the definition of  $S_n$ . The formula for  $\alpha_{n,\tau}$  only depends on  $c_{a,b}$  with  $b \leq n - 1$ . If  $(a,b) \in I$  and  $b \leq n - 1$ , then  $a \leq 2n - 2$ . Recall that  $c_{a,b}$  is defined in terms of laces (and the corresponding compatible walks) of length  $a$ .

*Proof of Lemma 4.3.* Let  $\phi(\beta) = 1 + \sum_I c_{a,b} \beta^a s^{b-a}$ . Setting  $\beta = \sum_{n=1}^{\infty} \alpha_{n,\tau} s^n$ , (4.2) becomes

$$\frac{\beta}{\phi(\beta)} = s.$$

The Lagrange–Bürmann series reversion formula [15, Theorem 1.2.4] states that,

$$\beta = \sum_{k=1}^{\infty} \frac{s^k}{k} [\beta^{k-1}] \phi(\beta)^k.$$

Applying the formula yields,

$$\sum_{n=1}^{\infty} \alpha_{n,\tau} s^n = \sum_{k=1}^{\infty} \frac{s^k}{k} [\beta^{k-1}] \left( 1 + \sum_I c_{a,b} \beta^a s^{b-a} \right)^k.\tag{4.4}$$

Let  $T_k = \{(n_{a,b}) \in \mathbb{N}^I : \sum_I n_{a,b} \leq k\}$ ; by the multinomial theorem,

$$\left( 1 + \sum_I c_{a,b} \beta^a s^{b-a} \right)^k = \sum_{(n_{a,b}) \in T_k} \frac{k!}{[\prod_I n_{a,b}]! (k - \sum_I n_{a,b})!} \prod_I [c_{a,b} \beta^a s^{b-a}]^{n_{a,b}}.$$

Extracting the coefficient of  $\beta^{k-1}$  from the right hand side above leaves only the terms corresponding to  $(n_{a,b})$  in  $U_k := \{(n_{a,b}) \in T_k : k-1 = \sum_I a n_{a,b}\}$ . From (4.4) we get,

$$\sum_{n=1}^{\infty} \alpha_{n,\tau} s^n = \sum_{k=1}^{\infty} \frac{s^k}{k} \sum_{(n_{a,b}) \in U_k} \frac{k!}{[\prod_I n_{a,b}!](k - \sum_I n_{a,b})!} \prod_I [c_{a,b} s^{b-a}]^{n_{a,b}}$$

and so

$$\alpha_{n,\tau} = [s^n] \sum_{k=1}^{\infty} \sum_{(n_{a,b}) \in U_k} \frac{s^k (k-1)!}{[\prod_I n_{a,b}!](k - \sum_I n_{a,b})!} \prod_I [c_{a,b} s^{b-a}]^{n_{a,b}}.$$

Extracting the coefficient of  $s^n$  on the right hand side leaves only the terms with  $n = k + \sum_I (b-a)n_{a,b}$ . By the definition of  $U_k$ , these are the terms with  $(n_{a,b}) \in S_n$ .  $\square$

## 5. Factorial bounds on $(\alpha_{n,\tau})$

We can use Lemma 4.3 to bound the coefficients  $(\alpha_{n,\tau})$  of the asymptotic expansions.

**Lemma 5.1.** *There is a constant  $C_2$  such that  $|\alpha_{n,\tau}| \leq C_2^n n!$*

This is achieved by bounding  $|c_{a,b}|$  in terms of  $b$ .

**Lemma 5.2.** *Let  $c_b = \sum_{a=b+1}^{2b} |c_{a,b}|$ . There is a constant  $C_3$  such that  $c_b \leq C_3^b b!$*

*Proof.* The numbers  $(c_{a,b})$  are defined in terms of laces with length  $a$ ,

$$|c_{a,b}| \leq \sum_{D=a-b}^{\lfloor a/2 \rfloor} \sum_{N=1}^{\infty} f_{\tau}(a, N, D) \times \left| [s^{b-a}] s^{-1} (s^{-1} - 2) \dots (s^{-1} - 2D + 2) \right|.$$

The number of walks of length  $a$  in  $\mathbb{Z}^D$  is  $(2D)^a$ , so

$$\sum_{N=1}^{\infty} f_{\tau}(a, N, D) \leq \frac{(2D)^a}{2^D D!}.$$

The absolute value of  $[s^{b-a}] s^{-1} (s^{-1} - 2) \dots (s^{-1} - 2D + 2)$  is at most

$$[s^{b-a}] (s^{-1} + 2D)^D = (2D)^{D+b-a} \binom{D}{a-b}, \quad D = a-b, \dots, \lfloor a/2 \rfloor.$$

Therefore (as  $D \leq \lfloor a/2 \rfloor \leq b$ )

$$\begin{aligned} |c_{a,b}| &\leq \sum_{D=a-b}^{\lfloor a/2 \rfloor} \frac{(2D)^a}{2^D D!} (2D)^{D+b-a} \binom{D}{a-b} = \sum_{D=a-b}^{\lfloor a/2 \rfloor} \frac{2^b D^{D+b}}{(a-b)!(D-a+b)!} \\ &\leq (1 + \lfloor a/2 \rfloor - (a-b)) \frac{2^b b^{2b}}{(a-b)!(2b-a)!}. \end{aligned}$$

By Stirling's approximation,

$$\frac{b^{2b}}{(a-b)!(2b-a)!} \leq \frac{(b!e^b)^2}{(a-b)!(2b-a)!} \leq \binom{b}{a-b} e^{2b} b! \leq 2^b e^{2b} b!$$

Hence for some constant  $C_3$ ,

$$c_b = \sum_{a=b+1}^{2b} |c_{a,b}| \leq \sum_{a=b+1}^{2b} (1 + \lfloor a/2 \rfloor - (a-b)) \cdot 2^b \cdot 2^b e^{2b} b! \leq C_3^b b!$$

□

Before we can prove Lemma 5.1, we need a bound on how a power series with factorial coefficients behaves under exponentiation.

**Lemma 5.3.** *Let  $\phi(\beta) \equiv \sum_{k=0}^{\infty} k! \beta^k$ . Then*

$$[\beta^k] \phi(\beta)^n \leq k! \prod_{j=1}^k (1 + (n-1)/j^2) \leq 6^n k!$$

Before proving Lemma 5.3, we will state a corollary that will be needed in Section 7.

**Corollary 5.4.** *With  $C$  a positive constant, let  $\psi(\beta) \equiv \sum_{k=1}^{\infty} C^k \beta^k k!$ . For  $k \geq n$ ,*

$$[\beta^k] \psi(\beta)^n \leq (6C)^k (k-n)!$$

*Proof of Corollary 5.4.* For all  $m$ ,  $(m+1)! \leq 2^m m!$  and so

$$[\beta^k] \psi(\beta)^n \leq [\beta^k] (C\beta\phi(2C\beta))^n = C^k 2^{k-n} [\beta^{k-n}] \phi(\beta)^n \leq C^k 2^{k-n} 6^n (k-n)!$$

□

*Proof of Lemma 5.3.* Let  $l_1, \dots, l_n$  denote non-negative integers. The first inequality is equivalent to,

$$\sum_{l_1 + \dots + l_n = k} \prod_{i=1}^n l_i! \leq k! \prod_{j=1}^k (1 + (n-1)/j^2). \quad (5.5)$$

We will show this by induction in  $k$ . For convenience, (5.5) can be written in terms of a multinomial random variable  $X^k \equiv (X_1^k, \dots, X_n^k) \sim \text{Multinomial}(k; 1/n, \dots, 1/n)$ :

$$n^k \mathbb{E} \left[ \binom{k}{X^k}^{-2} \right] \leq \prod_{j=1}^k (1 + (n-1)/j^2), \quad \binom{k}{X^k} = \frac{k!}{X_1^k! \dots X_n^k!}.$$

For the inductive step, we construct  $X^{k+1}$  from  $X^k$  by adding 1 to one of  $X_1^k, \dots, X_n^k$  uniformly at random. Let  $e_1 = (1, 0, 0, \dots)$ ,  $e_2 = (0, 1, 0, \dots)$ , and so on. The inductive step is then,

$$\begin{aligned} n \mathbb{E} \left[ \binom{k+1}{X^{k+1}}^{-2} \right] &= \mathbb{E} \sum_{i=1}^n \binom{k+1}{X^k + e_i}^{-2} \\ &= \mathbb{E} \sum_{i=1}^n \left( \frac{k+1}{X_i^k + 1} \right)^{-2} \binom{k}{X^k}^{-2} \\ &\leq \frac{(k+1)^2 + (n-1)}{(k+1)^2} \mathbb{E} \left[ \binom{k}{X^k}^{-2} \right]. \end{aligned}$$

The inequality is the result of replacing  $\sum_{i=1}^n \left(\frac{k+1}{X_i^k+1}\right)^{-2}$  with its supremum over the range of  $X^k$ .

The second inequality in the statement of Lemma 5.3 follows from a well known result of Euler:  $\sum_{j=1}^{\infty} 1/j^2 = \pi^2/6$ , and so

$$\log \prod_{j=1}^k (1 + (n-1)/j^2) \leq \sum_{j=1}^k \frac{n-1}{j^2} < (n-1)\pi^2/6 < \log(6^n).$$

□

*Proof of Lemma 5.1.* For  $(a, b) \in I$ ,  $a \leq 2b$ . By Lemma 4.3,

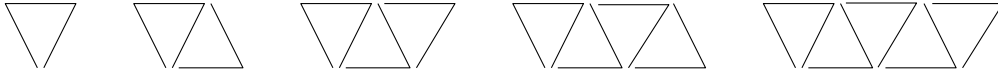
$$\begin{aligned} |\alpha_{n,\tau}| &\leq \sum_{(n_a,b) \in S_n} \frac{(\sum a n_{a,b})!}{\prod n_{a,b}!(1 + \sum (a-1)n_{a,b})!} \prod |c_{a,b}|^{n_{a,b}} \\ &\leq \sum_{(n_a,b) \in S_n} \frac{(2n-2)!}{\prod n_{a,b}!(2n-1 - \sum n_{a,b})!} \prod |c_{a,b}|^{n_{a,b}} \\ &\leq \frac{1}{2n-1} [\beta^{n-1}] \left(1 + \sum_{b=1}^{\infty} c_b \beta^b\right)^{2n-1}. \end{aligned}$$

By Lemma 5.2 and Lemma 5.3,  $|\alpha_n| \leq 6^{2n} C_3^n n!$

□

## 6. Diagrammatic estimates

The walks compatible with type- $N$  laces can be represented by diagrams containing  $N$  segments, with each segment building a new loop. For  $N = 1, 2, 3, 4$  and  $5$ :



Such pictures have inspired a number of simple yet effective bounds on the lace expansion. In particular, there is a number  $C_{\text{HS}}$  such that for  $d$  sufficiently large, for all  $\tau$ , [7]

$$\hat{\Pi}_{\beta\tau}^{(N)}(0; \tau) \leq (sC_{\text{HS}})^N. \quad (6.1)$$

The method of diagrammatic estimates can be used to bound  $\hat{\pi}_a^{(N)}(0; \tau)$ , the number of walks of length  $a$  compatible with type- $N$  laces.

**Lemma 6.2.** *There is a constant  $C_4$  such that,*

$$\hat{\pi}_a^{(N)}(0; \tau) \leq [\beta^a] \left( \sum_{n=1}^{\tau/2} C_4^n s^{-n} n! \beta^{2n} (1 + s/\beta) \right)^N.$$

*Proof.* Let  $f(n, x) = n c_n^{(0)}(x)$ , where  $c_n^{(0)}(x)$  denotes the number of simple walks from 0 to  $x$  of length  $n$ . Let  $g(n) = \sup_x f(n, x)$ . The loops corresponding to memory- $\tau$  laces have length at most  $\tau$ . Consider the function

$$G^\tau(\beta) = \sum_{n=1}^{\tau} \beta^n g(n).$$

We will show, by the method of diagrammatic estimates, that

$$\hat{\pi}_a^{(N)}(0; \tau) \leq [\beta^a] (G^\tau(\beta))^N. \quad (6.3)$$

Let  $(s_1, t_1), \dots, (s_N, t_N)$  represent a typical lace of type  $N$  and length  $a$ . Let  $t_0 = 0$ . Note that

- (i)  $t_1 - t_0, t_2 - t_1, \dots, t_N - t_{N-1} \in \{1, 2, \dots, \tau\}$ , and
- (ii)  $0 \leq s_{i+1} - t_{i-1} < t_i - t_{i-1}$  for  $i = 1, \dots, N - 1$ .

Let  $\omega = (\omega(0), \omega(1), \dots, \omega(a))$  represent a typical simple walk from 0 of length  $a$ . Then

$$\hat{\pi}_a^{(N)}(0; \tau) = \sum_{(t_i)} \sum_{(s_i)} |\{\omega : \omega \text{ is compatible with } (s_1, t_1), \dots, (s_N, t_N)\}|.$$

The first sum is over values of  $t_1, \dots, t_N$  compatible with (i). The second sum is over values of  $s_1, \dots, s_N$  compatible with (ii). Take  $t_1, \dots, t_N$  to be fixed. Suppose that for some  $k = 1, \dots, N$ , we have fixed  $s_1, \dots, s_k$  and  $\omega(1), \dots, \omega(t_{k-1})$ . How many ways are there to pick  $s_{k+1}$  and  $\omega(t_{k-1} + 1), \dots, \omega(t_k)$ ?

The choice of  $\omega(t_{k-1} + 1), \omega(t_{k-1} + 2), \dots, \omega(t_k)$  is constrained by the requirement that  $\omega(t_k) = \omega(s_k)$ . The number of choices for the value of  $s_{k+1}$  is  $t_k - t_{k-1}$ . The total number of choices is at most  $g(t_k - t_{k-1})$ . Therefore

$$\hat{\pi}_a^{(N)}(0; \tau) \leq \sum_{(t_i)} g(t_1 - t_0)g(t_2 - t_1) \dots g(t_N - t_{N-1})$$

and (6.3) follows.

Let  $C_4 = 1000$ . The lemma follows from (6.3) when we show that for  $n = 1, \dots, \tau$ ,

$$g(n) = \sup_x f(n, x) \leq [\beta^n] \sum_{n=1}^{\tau/2} C_4^n s^{-n} n! \beta^{2n} (1 + s/\beta) = C_4^{\lceil n/2 \rceil} s^{-\lfloor n/2 \rfloor} \lceil n/2 \rceil!$$

First consider  $n = 2m \leq 2d$ . A walk from 0 to 0 in  $\mathbb{Z}^d$  of length  $2m$  has dimensionality at most  $m$ . The number of ways to pick an  $m$ -dimensional subspace of  $\mathbb{Z}^d$  is at most  $d^m/m!$  Using Stirling's formula,

$$f(2m, 0) \leq (2m) \frac{d^m}{m!} (2m)^{2m} \leq C_4^m s^{-m} m!$$

For  $x \neq 0$ , let  $i \geq 1$  denote the dimensionality of  $x$ . The number  $j$  of extra dimensions a walk from 0 to  $x$  of length  $2m \leq 2d$  can explore is at most  $m - 1$ , and the total dimensionality of the walk is at most  $i + j \leq 2m$ :

$$f(2m, x) \leq (2m) \frac{d^{m-1}}{(m-1)!} (2 \cdot 2m)^{2m} \leq C_4^m s^{1-m} m!$$

Now consider  $n = 2m + 1$  for  $1 \leq m < d$ . Summing over the neighbours of  $x$ ,  $c_{2m+1}^{(0)}(x) = \sum_{y \sim x} c_{2m}^{(0)}(y)$ . At most 1 of the  $2d$  neighbours of  $x$  can be 0, hence

$$\begin{aligned} f(2m + 1, x) &\leq \left( \frac{2m + 1}{2m} \right) [C_4^m s^{-m} m! + (2d - 1) \cdot C_4^m s^{1-m} m!] \\ &\leq C_4^{m+1} s^{-m} (m + 1)! \end{aligned}$$

Lastly, if  $n \geq 2d$ ,  $c_n^{(0)}(x) \leq (2d)^n$ . Again by Stirling's formula,

$$f(n, x) \leq n(2d)^n \leq C_4^{\lceil n/2 \rceil} s^{-\lfloor n/2 \rfloor} \lceil n/2 \rceil!$$

□

### 7. Proof of Theorem 1.3

It is well known that  $d \leq \mu \leq 2d$ , and hence that  $s \leq \beta_c \leq 2s$ . Let  $C_5 \in [0, C_2^{-1}]$  and suppose that  $C_1 \geq 10C_2/C_5$ . Then for  $s \geq C_5/M$ , inequality (1.4) holds simply by Lemma 5.1. We will show by induction that for some positive constants  $C_5$  and  $C_6$ , for  $k = 0, 1, 2, \dots$  and  $s \leq C_5/k$ ,

$$\beta_c = \sum_{n=1}^k \alpha_n s^n + E_{k+1} s^{k+1} \quad \text{with} \quad |E_{k+1}| \leq C_6^{k+1} (k+1)! \quad (7.1)$$

Theorem 1.3 follows from (7.1) by taking  $C_1 = \max\{C_6, 10C_2/C_5\}$ .

To begin the induction process, notice that  $E_1 = \beta_c/s \in [0, 2]$ , so (7.1) holds for  $k = 0$  if  $C_6 \geq 2$ . As the proof progresses, we will impose a number of conditions on the pair of constants  $(C_5, C_6)$ . The reader will see that all these conditions can be satisfied by first taking  $C_6 \gg 1$  and then taking  $C_5 \ll C_6^{-3}$ .

Assume inductively that (7.1) holds for  $k = 1, \dots, M-1$ . Fix  $s \leq C_5/M$ . We need to show that

$$|E_{M+1}| \leq C_6^{M+1} (M+1)!$$

We will define  $(A_i)_{i=1}^4$  such that

$$E_{M+1} s^M = \sum_{i=1}^4 A_i, \quad |A_i| \leq \frac{1}{4} s^M C_6^{M+1} (M+1)! \quad (7.2)$$

We will now use the lace expansion. We have assumed that  $s = 1/(2d) \leq C_5/M$ . If  $C_5$  is sufficiently small, (4.1) holds for all  $\tau$ .

A type- $N$  memory- $\tau$  lace consists of  $N$  overlapping intervals. Each interval has length at most  $\tau$ , so the total length is at most  $N\tau$ . It is therefore easier to use the finite memory version of the lace expansion. Recall that if  $\tau \geq 2M-2$ , the first  $M$  coefficients of the series expansions for  $\beta_c$  and  $\beta_\tau$  agree: for  $n = 1, \dots, M$ ,  $\alpha_n = \alpha_{n,\tau}$ . Let  $\tau = 2M$ . By [6, Theorem 1] there is a constant  $C_K$  such that,

$$\forall s \leq 1/(52M), \quad 0 \leq \beta_c - \beta_\tau \leq s^{M+2} C_K^M M! \quad (7.3)$$

If  $C_6 \geq C_K$  then  $A_1 := \beta_c - \beta_\tau$  satisfies (7.2). Let  $E_M^\tau = E_M - A_1$ . Then

$$\beta_\tau = \sum_{n=1}^{M-1} \alpha_n s^n + E_M^\tau s^M. \quad (7.4)$$

By (4.1), we must now choose  $A_2, A_3, A_4$  such that

$$A_2 + A_3 + A_4 + \sum_{n=1}^M \alpha_n s^{n-1} = 1 - \hat{\Pi}_{\beta_\tau}(0; \tau). \quad (7.5)$$

With reference to the diagrammatic estimate (6.1), let

$$A_2 = - \sum_{N=M+1}^{\infty} (-1)^N \hat{\Pi}_{\beta_\tau}^{(N)}(0; \tau).$$

If  $C_5 \leq 1/(2C_{\text{HS}})$  and  $C_6 \geq 2C_{\text{HS}}$ , then  $A_2$  satisfies (7.2).

Let  $A_3$  match the terms generated on the right hand side of (7.5) by laces of length  $a \geq 2M$  and type  $N \leq M$ ,

$$A_3 = - \sum_{N=1}^M (-1)^N \sum_{a=2M}^{2MN} \hat{\pi}_a^{(N)}(0; \tau) \beta_\tau^a.$$

By Lemma 6.2,

$$\begin{aligned} |A_3| &\leq \sum_{N=1}^M \sum_{a=2M}^{2MN} \beta_\tau^a [\beta^a] \left( \sum_{n=1}^M C_4 s^{-n} n! \beta^{2n} (1 + s/\beta) \right)^N \\ &\leq \sum_{N=1}^M (1 + s/\beta_\tau)^N \sum_{a=2M}^{2MN} \beta_\tau^a [\beta^a] \left( \sum_{n=1}^M C_4 s^{-n} n! \beta^{2n} \right)^N. \end{aligned}$$

Setting  $x = \beta^2$ , the right hand side is equal to

$$\sum_{N=1}^M (1 + s/\beta_\tau)^N \sum_{a=M}^{MN} (C_4 M \beta_\tau^2 s^{-1})^a [x^a] \left( \sum_{n=1}^M n! x^n / M^n \right)^N.$$

If  $C_5$  is small and  $C_6$  is large then  $(1 + s/\beta_\tau) C_4 M \beta_\tau^2 s^{-1} \leq C_6 e^{-1} M s \leq 1$ , and

$$\begin{aligned} |A_3| &\leq \sum_{N=1}^M \sum_{a=M}^{MN} (C_6 e^{-1} M s)^a [x^a] \left( \sum_{n=1}^M x^n n! / M^n \right)^N \\ &\leq (C_6 e^{-1} M s)^M \sum_{N=1}^M \left( \sum_{n=1}^M n! / M^n \right)^N \leq \frac{1}{4} s^M C_6^{M+1} (M+1)! \end{aligned}$$

By the process of elimination,  $A_4$  is now defined by

$$A_4 + \sum_{n=1}^M \alpha_n s^{n-1} = 1 - \sum_{N=1}^M (-1)^N \sum_{a=2}^{2M-1} \hat{\pi}_a^{(N)}(0; \tau) \beta_\tau^a. \quad (7.6)$$

Substitute (7.4) into the right hand side of (7.6); by Lemma 4.3, we can cancel the powers of  $s$  below  $s^M$ :

$$A_4 = - \sum_{n=M}^{\infty} s^n [s^n] \sum_{N=1}^M (-1)^N \sum_{a=2}^{2M-1} \hat{\pi}_a^{(N)}(0; \tau) \left( \sum_{n=1}^{M-1} \alpha_n s^n + E_M^\tau s^M \right)^a.$$

Recall that  $c_b := \sum_{a=b+1}^{2b} |c_{a,b}|$ . As  $\alpha_1 = 1$  and  $a \leq 2b$ ,

$$\begin{aligned} |A_4| &\leq \sum_{n=M}^{\infty} s^n [s^n] \sum_{a=2}^{2M-1} \sum_{b=\lceil a/2 \rceil}^{a-1} |c_{a,b}| s^{b-a} \left( \sum_{k=1}^{M-1} |\alpha_k| s^k + |E_M^\tau| s^M \right)^a \\ &\leq \sum_{n=M}^{\infty} s^n [s^n] \sum_{b=1}^{2M-2} c_b s^{-b} \left( \sum_{k=1}^{M-1} |\alpha_k| s^k + |E_M^\tau| s^M \right)^{2b}. \end{aligned} \quad (7.7)$$

Expand the  $(\cdot)^{2b}$  terms on line (7.7) and then carry out the sum from  $n = M$  to  $\infty$ : we will split the resulting terms into three groups.

- (i) The terms  $s^n c_b |\alpha_{i_1}| |\alpha_{i_2}| \cdots |\alpha_{i_{2b}}|$  with  $M \leq n \leq 2M - 2$ .
- (ii) The terms  $s^n c_b \dots$  with  $M \leq n \leq 2M - 2$  that are not in group (i) because they contain an  $|E_M^\tau|$  term.
- (iii) The terms  $s^n c_b \dots$  with  $n \geq 2M - 1$ .

Thinking of (4.2) as a recursive formula, Lemma 5.1 is equivalent to the bound:

$$|\alpha_{n+1}| \leq \left| [s^n] \sum_{b=1}^{\infty} c_b s^{-b} \left( \sum_{k=1}^{\infty} |\alpha_k| s^k \right)^{2b} \right| \leq C_2^{n+1} (n+1)!$$

The contribution from the first group is therefore less than

$$\sum_{n=M}^{2M-2} s^n C_2^{n+1} (n+1)! \quad (7.8)$$

The contribution of the second group is

$$\sum_{n=M}^{2M-2} s^n [s^n] \sum_{b=1}^{n+1-M} c_b s^{-b} \times (2b) \left( \sum_{k=1}^{M-1} |\alpha_k| s^k \right)^{2b-1} |E_M^\tau| s^M. \quad (7.9)$$

For  $k = 1, \dots, M$ , define  $\hat{\alpha}_k$  by

$$\hat{\alpha}_k s^k = |\alpha_k| s^k + \dots + |\alpha_{M-1}| s^{M-1} + |E_M^\tau| s^M.$$

We claim that the contribution of the third group is at most

$$s^{2M-1} [x^{2M-1}] \sum_{b=1}^{2M-2} c_b x^{-b} \left( \sum_{k=1}^M \hat{\alpha}_k x^k \right)^{2b}. \quad (7.10)$$

By having the  $\hat{\alpha}_k x^k$  in the  $(\cdot)^{2b}$  term, we catch all the terms in the third group while extracting only the coefficient of  $x^{2M-1}$ . [For example if  $M = 10$ , the term  $s^{28} c_2 \alpha_6 \alpha_7 \alpha_8 \alpha_9$  is accounted for by the term  $c_2 s^{-2} (\hat{\alpha}_6 s^6) (\hat{\alpha}_7 s^7) (\hat{\alpha}_8 s^8) (\hat{\alpha}_9 s^9) =$

$$c_2 s^{-2} (|\alpha_6| s^6 + \dots) (|\alpha_7| s^7 + \dots) (\dots + |\alpha_8| s^8 + \dots) (\dots + |\alpha_9| s^9 + \dots)$$

generated by (7.10).]

The absolute value of  $A_4$  is now bounded by the sum of three intimidating expressions, (7.8)–(7.10). However, the  $\alpha_n$  are controlled by Lemma 5.1, and the  $c_b$  are controlled by Lemma 5.2. By the inductive assumption and (7.3), if  $C_6 \geq C_K$  then  $|E_M^\tau| \leq 2C_6^M M!$  If  $C_6 \geq C_2$ , we have  $\hat{\alpha}_k \leq (2C_6)^k k!$  Substituting in these bounds and applying Corollary 5.4 gives

$$\begin{aligned} |A_4| &\leq \sum_{n=M}^{2M-2} s^n C_2^{n+1} (n+1)! \\ &+ \sum_{n=M}^{2M-2} s^n \sum_{b=1}^{n+1-M} (C_3^b b!) (2b) (6 \cdot C_2)^{n+b-M} (n+1-b-M)! (2C_6^M M!) \\ &+ s^{2M-1} \sum_{b=1}^{2M-2} (C_3^b b!) (6 \cdot 2C_6)^{2M+b-1} (2M-b-1)! \end{aligned} \quad (7.11)$$

Let  $k = n - M$ . On the first line of (7.11), the  $(n + 1)!$  is less than  $(M + 1)!(2M)^k$ ; this turns the summand into a geometric series. On the second line, the summand (of the sum over  $b$ ) is maximized by  $b = k + 1$ , and the sum contains  $k + 1$  terms. On the third line, the summand is maximized by  $b = 2M - 2$ , and the sum contains  $2M - 2$  terms. It follows that

$$\begin{aligned} |A_4| &\leq s^M C_2^{M+1} (M + 1)! \sum_{k=0}^{M-2} (s \cdot C_2 \cdot 2M)^k \\ &\quad + 2s^M C_6^M M! \sum_{k=0}^{M-2} s^k (k + 1) (C_3^{k+1} (k + 1)!)(2k + 2)(6C_2)^{2k+1} \\ &\quad + s^{2M-1} (2M - 2) (C_3^{2M-2} (2M - 2)!)(12C_6)^{4M-3}. \end{aligned}$$

To complete the proof of Theorem 1.3, we simply need to check that with  $s \leq C_5/M$  and  $M \geq 1$ ,  $|A_4| \leq \frac{1}{4} s^M C_6^{M+1} (M + 1)!$

If  $C_5$  is small, the two sums are dominated by their first terms; in particular, we can assume that each sum amounts to no more than twice the value of the summand when  $k = 0$ . As  $(2M - 2) \cdot (2M - 2)! \leq (M + 1)!(2M)^{M-2}$ ,

$$\frac{|A_4|}{s^M C_6^{M+1} (M + 1)!} \leq 2 \left( \frac{C_2}{C_6} \right)^{M+1} + \frac{48C_2 C_3}{C_6 (M + 1)} + \frac{6(12^4 \cdot 2M s C_3^2 C_6^3)^{M-1}}{C_6 M}.$$

The right hand side is smaller than  $1/4$  if  $C_6 \gg C_2 C_3$  and  $C_5 \ll C_3^{-2} C_6^{-3}$ .

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