

Metastability in the dilute Ising model

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Abstract

Consider Glauber dynamics for the Ising model on the hypercubic lattice with a positive magnetic field. Starting from the minus configuration, the system initially settles into a metastable state with negative magnetization. Slowly the system relaxes to a stable state with positive magnetization. Schonmann and Shlosman showed that in the two dimensional case the relaxation time is a simple function of the energy required to create a critical Wulff droplet.

The dilute Ising model is obtained from the regular Ising model by deleting a fraction of the edges of the underlying graph. In this paper we show that even an arbitrarily small dilution can dramatically reduce the relaxation time. This is because of a catalyst effect—rare regions of high dilution speed up the transition from minus phase to plus phase.

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1 Introduction

1.1 Metastability in the Ising model

Consider Glauber dynamics for the supercritical Ising model on the hypercubic lattice ($d \geq 2$) started in the minus configuration but with a positive external magnetic field h . Aizenman and Lebowitz predicted that the model initially settles in a metastable minus phase, eventually relaxing to the plus phase on a time scale that grows exponentially with $1/h^{d-1}$ [1].

To be more precise, let β denote the inverse-temperature and let β_c denote the critical inverse-temperature. Suppose $\beta > \beta_c$. Let μ^+, μ^- denote the plus and minus phases of the equilibrium Ising model. Start the Glauber dynamics at time 0 with all vertices initially taking minus spin. Let $\sigma_t^{0,-}$ denote the state of the Glauber dynamics at time t . With β fixed, let $h \rightarrow 0$ with $t = \exp(\lambda/h^{d-1})$. A heuristic argument suggests that if λ_1 is sufficiently small and λ_2 is sufficiently large then for every local observable f :

- (i) $\mathbb{E}[f(\sigma_t^{0,-})] \rightarrow \mu^-(f)$ if $\lambda < \lambda_1$.
- (ii) $\mathbb{E}[f(\sigma_t^{0,-})] \rightarrow \mu^+(f)$ if $\lambda > \lambda_2$.

Part (i) is a lower bound on the relaxation time and part (ii) is an upper bound. Schonmann proved this behavior in dimensions $d \geq 2$ [15]. However, his proof left open the question of whether or not $\lambda_1 = \lambda_2$. Schonmann and Shlosman

settled this question in dimension two, proving that the above holds with $\lambda_1 = \lambda_2$; the transition is sharp in a logarithmic sense [16]. Their proof refines the heuristic argument and shows that the critical value of λ is a simple function of the surface tension of the Wulff shape. The proof takes advantage of specific features of the two dimensional Ising model such as duality. When considering disordered models, in two and higher dimensions, these simplifying features no longer exist. New arguments from the L_1 -theory of phase coexistence have to be used instead.

The focus of this paper will be the dilute Ising model. For the purpose of comparison, we note that the proof of our main result (Theorem 1.2.3 below) implies that the upper bound of [16] extends to higher dimensions. We believe that our method of proof is valid for all $\beta > \beta_c$ but we did not make verifying this a priority. To avoid certain technicalities we assume that $\beta > \beta_0$ and $\beta \notin \mathcal{N}$ (see Section 1.2).

Let μ^h denote the equilibrium, undiluted Ising measure with a magnetic field $h > 0$. With reference to (4.2.3), the cost of creating a critical droplet under μ^h is $E_c^{2\pi}/h^{d-1}$ and $E_c^{2\pi} = O(\beta)$. Define a *local observable* to be a function that only depends on the spins in the region $[-1/h, 1/h]^d$.

Theorem 1.1.1. *Consider the value*

$$\lambda_2 = \frac{E_c^{2\pi}}{d+1}.$$

Let $\lambda > \lambda_2$. For any positive number C_0 there is a constant $C > 0$ such that for every local observable f and any $h > 0$,

$$\left| \mathbb{E} \left(f \left(\sigma_{\exp(\lambda/h^{d-1})}^{0,-} \right) \right) - \mu^h(f) \right| \leq C \|f\|_\infty \exp(-C_0/h).$$

This is an improvement on the upper bound in [15] and corresponds to the upper bound predicted by the heuristic of [16]. Proving rigorously the lower bound suggested by [16] in dimensions three and higher requires the development of new arguments which we postpone to a future work.

The dilute Ising model is a variant of the Ising model that is obtained by randomizing the Ising model edge coupling strengths. The impact of dilution on the relaxation of Glauber dynamics has been studied in [8, 13]. In the Griffiths phase, which corresponds to the sub-critical regime, the disorder is proven to lead to a slowdown of the dynamics. In the phase transition regime, the metastability has been investigated for the random field Curie-Weiss model [3].

We will consider the Ising model on \mathbb{Z}^d diluted in the simplest way possible. Independently, delete each edge with probability $1 - p$. When p is sufficiently large, the remaining edges form a supercritical percolation cluster. From this point of view, the Ising model is a special case of the dilute Ising model corresponding to $p = 1$. It is natural to ask how the relaxation time depends on p . In this paper we show that even a small dilution can greatly reduce the relaxation time.

1.2 The dilute Ising model

Let \mathbb{Z}^d represent the hypercubic lattice. The Ising model assigns each site of \mathbb{Z}^d a spin of ± 1 . Let $\Sigma = \{\pm 1\}^{\mathbb{Z}^d}$ denote the set of Ising configurations.

Let $E = \{\{x, y\} : \|x - y\|_1 = 1\}$ denote the set of nearest neighbor edges of \mathbb{Z}^d . The equilibrium Ising measure with local coupling strengths $J = (J(e) : e \in E)$ and external magnetic field h is defined using the formal Hamiltonian

$$-\frac{1}{2} \sum_{e=\{x,y\} \in E} J(e)\sigma(x)\sigma(y) - \frac{1}{2} \sum_{x \in \mathbb{Z}^d} h\sigma(x), \quad \sigma \in \Sigma. \quad (1.2.1)$$

We will consider local coupling strengths with the Bernoulli distribution. Let \mathbb{Q} denote the product measure such that for each edge e , $\mathbb{Q}(J(e) = 1) = p$ and $\mathbb{Q}(J(e) = 0) = 1 - p$.

It is well known that when $h \neq 0$, the Ising measure $\mu^{J,h}$ is well defined by the Gibbs formalism for any inverse-temperature $\beta > 0$ and local coupling strengths $J \geq 0$. Consider the spontaneous magnetization of the Ising measure,

$$m^* = \lim_{h \rightarrow 0^+} \mathbb{Q}[\mu^{J,h}(\sigma(0))]. \quad (1.2.2)$$

When $m^* > 0$ there is said to be phase coexistence. For such β there are two different Gibbs measures at $h = 0$, corresponding to the limits $h \rightarrow 0^+$ and $h \rightarrow 0^-$.

It is shown in [9] that if the J -positive edges percolate then there is phase coexistence in the dilute Ising model at low temperatures. In our settings, this means that the critical inverse-temperature

$$\beta_c = \inf \{\beta > 0 : m^* > 0\}$$

is finite if and only if $p > p_c$, where p_c is the threshold for bond percolation on (\mathbb{Z}^d, E) .

As well as defining the equilibrium Ising model, the formal Hamiltonian defines a dynamic model. Let $(\sigma_t^{0,-})_{t \geq 0}$ denote a Markov chain on the set Σ of Ising configurations, starting at time 0 with minus spins everywhere, and evolving with time according to Glauber dynamics. Given a set of coupling strengths J , let \mathbb{E}_J denote expectation with respect to the Glauber dynamics. Our results extend to some other dynamics such as the Metropolis dynamics (see Section 2.5).

A quantity denoted C_{dil} is defined in Section 6 that satisfies $C_{\text{dil}} = O(\log \frac{1}{1-p})$ as $p \rightarrow 1$. For the rest of the paper consider p to be fixed in the range $(p_c, 1)$.

Let β_0 denote the minimum value such that for all $\beta > \beta_0$ the assumptions of slab percolation (see Section 2.4) and spatial mixing (see Section 5.4) hold. Let $\mathcal{N} \subset (0, \infty)$ denote the set of zero measure defined by (2.2.2). For the rest of the paper the inverse-temperature β should be assumed to be greater than β_0 and not in \mathcal{N} .

For $\theta \in (0, \pi)$ let \mathbf{E}_c^θ denote the cost, up to a factor of h^{d-1} , of creating a critical plus droplet in a cone with angle θ . $\mathbf{E}_c^\theta = O(\beta\theta^{d-1})$ and is defined in Section 4.2.

Theorem 1.2.3. *For $\theta \in (0, \pi)$ consider the value*

$$\lambda_2^\theta = \frac{\mathbf{E}_c^\theta + C_{\text{dil}}\theta^{-1}}{d+1}.$$

(i) Let $\lambda > \lambda_2^\theta$. For any positive number C_0 , there are constants $C, c > 0$ such that for any $h > 0$, for every local observable f ,

$$\begin{aligned} \mathbb{Q} \left[\left| \mathbb{E}_J \left(f \left(\sigma_{\exp(\lambda/h^{d-1})}^{0,-} \right) \right) - \mu^{J,h}(f) \right| \leq C \|f\|_\infty \exp(-C_0/h) \right] \\ \geq 1 - C \exp(-c/\sqrt{h}). \end{aligned}$$

(ii) However close p is to one, at low temperatures the diluted Ising model relaxes much more quickly than the corresponding undiluted Ising model; with reference to Theorem 1.1.1, as $\beta \rightarrow \infty$,

$$\frac{1}{\lambda_2} \inf_{\theta \in (0, \pi)} \lambda_2^\theta \rightarrow 0.$$

We expect, based on the undiluted Ising model [4], that the slab percolation threshold is equal to β_c . Further study of slab percolation and spatial-mixing properties for the dilute Ising model would likely extend the domain of validity of Theorem 1.2.3 down to the critical point.

1.3 Heuristic

The metastability phenomenon for the undiluted Ising model [16] is related to the rate of nucleation of plus droplets with linear size order $1/h$. Consider a small neighborhood of the origin. Initially all the spins are minuses. Small clusters of plus spins quickly form and then disappear. After a short time the system looks like it has reached equilibrium with minus spins in the majority. However, if we look at a much larger region we will be in for a surprise. A small number of larger droplets of plus spin will have formed and started to spread. They will eventually merge and cover the whole region, leaving the majority of spins in the plus state.

The rate at which droplets of plus phase form, and what happens to the droplets once they have formed, depends on their energy. Let $\mathcal{V} \subset \mathbb{R}^d$ with unit volume. For $b > 0$ let $E^\mathcal{V}(b)$ denote, up to a factor of h^{d-1} , the energy of a plus droplet with the shape $(b/h)\mathcal{V}$. $E^\mathcal{V}(b)$ can be estimated as a balance between the surface tension at the phase boundary and the effect of the magnetic field h ,

$$E^\mathcal{V}(b)/h^{d-1} = (b/h)^{d-1} \mathcal{F}(\mathcal{V}) - hm^*(b/h)^d. \quad (1.3.1)$$

Here $\mathcal{F}(\mathcal{V})$ is the surface tension of \mathcal{V} (see Section 3.3) and m^* is the mean magnetization in the plus phase (1.2.2).

Let $B_c^\mathcal{V} = \frac{(d-1)\mathcal{F}(\mathcal{V})}{dm^*}$. The energy function $E^\mathcal{V}(b)$ is increasing on the interval $(0, B_c^\mathcal{V})$ and decreasing beyond $B_c^\mathcal{V}$. Droplet with $b < B_c^\mathcal{V}$ are unstable and tend to be eroded by the surrounding minus spins. Droplets with $b > B_c^\mathcal{V}$ are expected to spread. The nucleation of a droplet with $b > B_c^\mathcal{V}$ requires that the system overcomes an energy barrier $E_c^\mathcal{V}/h^{d-1}$, where $E_c^\mathcal{V} := E^\mathcal{V}(B_c^\mathcal{V})$. Given the inverse-temperature β there is a unique shape \mathcal{W} known as the Wulff shape with minimal surface tension; see the definition of $\mathcal{W}_{2\pi}$ in Section 4.1. Setting $\mathcal{V} = \mathcal{W}$ minimizes $E_c^\mathcal{V}$. The critical droplet shape is $(B_c^\mathcal{W}/h)\mathcal{W}$.

In any small neighborhood the rate at which copies of the critical droplet form is approximately $\exp(-E_c^\mathcal{W}/h^{d-1})$. Droplets larger than the critical droplet spread out with roughly uniform speed and eventually invade the whole space.

The space-time cone of points from which one can reach the origin by time t (when growing at a fixed speed) has size $O(t^{d+1})$. If $t = \exp(\lambda/h^{d-1})$ with

$$\lambda > \lambda_c = \frac{E_c^W}{d+1} \quad (1.3.2)$$

then we should expect to see a critical droplet form, and then spread to cover the origin, by time t . This heuristic picture has been turned into a rigorous proof for the two dimensional Ising model [16].

The dilute Ising model is self averaging so the quenched magnetization and the quenched surface tension can unambiguously be defined almost surely with respect to the dilution measure \mathbb{Q} . It is tempting to try to adapt the previous heuristic to the case of the dilute Ising model using the quenched surface tension in (1.3.1) to describe the typical cost of phase coexistence. However, we must be careful. In much simpler models, such as random walk in random environment, it is well known that a small amount of randomness can change the asymptotic behavior.

Dilution seems to be capable of slowing down the dynamics. Consider an expanding droplet of plus phase. If it encounters an area of high dilution it may get blocked and have to seep around the obstruction, slowing down its progress.

However, dilution can also speed up the dynamics. The limiting factor in the undiluted Ising model is the rate at which plus droplets nucleate. Nucleation of plus droplets is infrequent due to the high cost of phase coexistence on their boundaries. The dilution creates atypical regions, which we will call *catalysts*, where the surface tension is unusually low and so the rate of nucleation is unusually high.

The natural human response to catalysts is to try and classify them. Some catalyst do not seem to have much effect on the relaxation time. Consider (when $d = 2$) a circle where all the edges crossing its perimeter have been diluted. If a plus droplet forms inside the circle, there is no way for it to spread outwards.

We therefore want to focus on catalysts that create a sheltered region to help plus droplets nucleate, but are not so closed off they prevent plus droplets from escaping. There seem to be two competing factors. Large catalysts will be relatively rare and so the droplets they help to nucleate will take a long time to reach the origin. Conversely, small catalysts cannot do a great deal to increase the rate at which critical droplets nucleate.

We conjecture that there is an optimal catalyst shape that determines the relaxation time of the system. However, we do not know how to calculate the optimal shape. In this paper we look at a restricted class of catalysts: surfaces of diluted edges that form open-bottom cones. We control the nucleation rate in the cones, and the subsequent growth of the droplet to regions of more typical dilution. This approach leads to an upper bound on the relaxation time that is much smaller than the time predicted by the formula (1.3.2) with quenched surface tension. Indeed, part (ii) of Theorem 1.2.3 shows that asymptotically in β the values of λ_2 differ greatly.

We do not address the issue of the lower bound for the metastable time for disordered models. We believe that the more important point is to show the existence of the catalyst effect of the disorder.

1.4 Outline of the paper

In Section 2 we define the dilute Ising model and recall some of its basic features. The random-cluster representation is used to state a coarse graining property.

In Sections 3 and 4 we look at the Ising model without a magnetic field. In Section 3 we describe the L_1 -theory of phase coexistence. The theory can describe both the typical cost of phase coexistence and the cost of phase coexistence in the neighborhood of catalysts. To combine the two cases we consider the cone $\mathcal{A}_\theta := \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq \|\mathbf{x}\|_2 \cos(\theta/2)\}$ where either $\theta \in (0, \pi)$ or $\theta = 2\pi$. In Section 4 we look at generalizations of the Wulff shape to \mathcal{A}_θ . The Wulff shape is the shape with minimal surface tension given its volume. The Wulff shape can be used to quantify the large deviations of the equilibrium Ising model.

In Section 5 we reintroduce the magnetic field. We justify the energy function featured in the heuristic. We prove regularity results concerning cluster boundaries. The motivation for this is to study the spectral gap of the dilute Ising model in finite regions with various boundary conditions.

Finally in Section 6 we use the accumulated results to prove Theorem 1.2.3. We do this by proving that the cone shaped regions act as catalysts. To show that the clusters of plus phase formed in the catalysts grow we consider another type of cone: space-time cones that are Wulff shaped spatially and growing in size with time.

1.5 Notation

Throughout the paper $C, c, c_{\text{stb}}, c_{\text{hs}}$, etc, will be used to refer to positive numbers that may depend on p, β and θ but not on h . We will recycle C and c to refer to various less important positive constants; the values they represent will change from appearance to appearance.

Let \mathcal{S}^{d-1} denote the set of unit vectors in \mathbb{R}^d . Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ denote the canonical basis vectors. We will use bold to differentiate continuous variables $\mathbf{x} \in \mathbb{R}^d$ from lattice points $x \in \mathbb{Z}^d$.

Consider $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^d$, $\mathbf{x} \in \mathbb{R}^d$ and $c \geq 0$. Let $\mathcal{A} + \mathcal{B} = \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in \mathcal{A}, \mathbf{b} \in \mathcal{B}\}$ denote the sum of the two sets. Let $\mathcal{A} + \mathbf{x}$ denote the translation of \mathcal{A} by \mathbf{x} . Let $c\mathcal{A}$ denote $\{c\mathbf{a} : \mathbf{a} \in \mathcal{A}\}$, the set \mathcal{A} scaled by a factor of c .

2 Properties of the dilute Ising model

2.1 Definition of $\mu_\Lambda^{J, \zeta, h}$

Let $J = (J(e) : e \in E)$ be a given realization of the coupling strengths. We will now define formally the Ising measure $\mu_\Lambda^{J, \zeta, h}$ with a magnetic field $h \in \mathbb{R}$ and boundary conditions $\zeta \in \Sigma$ on a finite domain $\Lambda \subset \mathbb{Z}^d$ at inverse-temperature $\beta > 0$.

Define the *external vertex boundary* $\partial\Lambda$ of Λ :

$$\begin{aligned} \partial\Lambda &= \partial^+\Lambda \cup \partial^-\Lambda \quad \text{where} \\ \partial^\pm\Lambda &= \{x \notin \Lambda : \exists y \in \Lambda, \{x, y\} \in E \text{ and } \zeta(x) = \pm 1\}. \end{aligned}$$

Taking w to stand for *wired*, define edge sets for Λ :

$$\begin{aligned} E(\Lambda) &= \{\{x, y\} \in E : x, y \in \Lambda\}, \\ E^\pm(\Lambda) &= \{\{x, y\} \in E : x \in \Lambda \text{ and } y \in \partial^\pm \Lambda\}, \\ E^w(\Lambda) &= E(\Lambda) \cup E^\pm(\Lambda). \end{aligned}$$

The set of spin configurations compatible with ζ outside Λ is

$$\Sigma_\Lambda^\zeta := \{\sigma \in \Sigma : \forall x \notin \Lambda, \sigma(x) = \zeta(x)\}.$$

Changing a Hamiltonian by an additive constant does not change the resulting measure; with reference to (1.2.1) the Ising Hamiltonian $H_\Lambda^{J, \zeta, h} : \Sigma_\Lambda^\zeta \rightarrow \mathbb{R}$ can be defined by

$$H_\Lambda^{J, \zeta, h}(\sigma) = \sum_{e=\{x, y\} \in E^w(\Lambda)} J(e) 1_{\{\sigma(x) \neq \sigma(y)\}} + \sum_{x \in \Lambda} h 1_{\{\sigma(x) = -1\}}.$$

The dilute Ising measure $\mu_\Lambda^{J, \zeta, h}$ at inverse-temperature β is defined by

$$\mu_\Lambda^{J, \zeta, h}(\{\sigma\}) = \frac{1}{Z_\Lambda^{J, \zeta, h}} \exp\left(-\beta H_\Lambda^{J, \zeta, h}(\sigma)\right)$$

where $Z_\Lambda^{J, \zeta, h}$ is a normalizing constant, the partition function, defined by

$$Z_\Lambda^{J, \zeta, h} = \sum_{\sigma \in \Sigma_\Lambda^\zeta} \exp\left(-\beta H_\Lambda^{J, \zeta, h}(\sigma)\right). \quad (2.1.1)$$

We have used σ above to index summations over Σ_Λ^ζ . It has also been used as a random variable—the mean spin at the origin is written $\mu_\Lambda^{J, \zeta, h}(\sigma(0))$. Furthermore, given a set $V \subset \mathbb{Z}^d$ we will write $\sigma(V)$ to denote the average spin in V ,

$$\sigma(V) = \frac{1}{|V|} \sum_{x \in V} \sigma(x) \in [-1, +1]. \quad (2.1.2)$$

2.2 The random-cluster representation for $\mu_\Lambda^{J, \zeta, h}$

The spin-spin correlations in the Ising model can be described by the $q = 2$ case of the random-cluster model [10]. We have to be extra careful because of the general boundary conditions $\zeta \in \Sigma$, dilute coupling strengths ($J(e)$), and the magnetic field $h \geq 0$. In this section we will describe a random-cluster representation $\phi_\Lambda^{J, \zeta, h}$ for the Ising model $\mu_\Lambda^{J, \zeta, h}$ and a joint measure $\varphi_\Lambda^{J, \zeta, h}$.

The Ising measure $\mu_\Lambda^{J, \zeta, h}$ was defined using the graph $(\Lambda, E^w(\Lambda))$. Add to this graph a ghost vertex \mathfrak{g} through which the magnetic field will act, and a set of ghost edges $E^\mathfrak{g}(\Lambda) = \{\{\mathfrak{g}, x\}, x \in \Lambda\}$.

When defining the random-cluster model on a given graph, for each edge e there is an interaction-strength parameter $p_e \in [0, 1]$. There is also another parameter, $q > 0$, that influences the number of clusters that are formed. In order to describe the correlations of the dilute Ising model we will fix

$$q = 2 \quad \text{and} \quad p_e = \begin{cases} 1 - \exp(-\beta J(e)), & e \in E^w(\Lambda), \\ 1 - \exp(-\beta h), & e \in E^\mathfrak{g}(\Lambda). \end{cases}$$

The state space of $\phi_\Lambda^{J,\zeta,h}$ is $\Omega_\Lambda = \{0, 1\}^{E^w(\Lambda) \cup E^g(\Lambda)}$. With $\omega \in \Omega_\Lambda$, an edge e is *open* if $\omega(e) = 1$ and *closed* if $\omega(e) = 0$. Two vertices of Λ are *connected* if they are joined by paths of open edges either

- (i) to each other, or
- (ii) both to $\partial^+ \Lambda \cup \{\mathbf{g}\}$, or
- (iii) both to $\partial^- \Lambda$.

A cluster is a maximal collection of connected vertices. Let V_+ , V_- denote the clusters connected to $\partial^+ \Lambda \cup \{\mathbf{g}\}$, $\partial^- \Lambda$, respectively. Let $n = n(\Lambda, \omega)$ count the number of other clusters in Λ . Label these clusters V_1, \dots, V_n .

We will define the random-cluster probability measure $\phi_\Lambda^{J,\zeta,h}$ using a coupling probability measure $\varphi_\Lambda^{J,\zeta,h}$ defined on $\Sigma_\Lambda^\zeta \times \Omega_\Lambda$. The marginal distribution of $\varphi_\Lambda^{J,\zeta,h}$ on Σ_Λ^ζ will be the Ising measure $\mu_\Lambda^{J,\zeta,h}$. The marginal distribution of $\varphi_\Lambda^{J,\zeta,h}$ on Ω_Λ defines the random-cluster measure $\phi_\Lambda^{J,\zeta,h}$.

With reference to [10, Section 1.4] define the coupled probability measure $\varphi_\Lambda^{J,\zeta,h}$ as follows. Let $(\sigma, \omega) \in \Sigma_\Lambda^\zeta \times \Omega_\Lambda$. Recall the notation (2.1.2) for average spins. Note that $\sigma(V) = \pm 1$ if and only if $\sigma(x) = \sigma(y)$ for all $x, y \in V$. We will say that σ is an ω -admissible configuration if

$$\sigma(V_+) = +1, \quad \sigma(V_-) = -1 \quad \text{and} \quad \sigma(V_i) = \pm 1, \quad i = 1, \dots, n.$$

If σ is ω -admissible, with reference to (2.1.1) let

$$\varphi_\Lambda^{J,\zeta,h}(\{(\sigma, \omega)\}) = \frac{1}{Z_\Lambda^{J,\zeta,h}} \prod_e p_e^{\omega(e)} (1 - p_e)^{1-\omega(e)},$$

otherwise let $\varphi_\Lambda^{J,\zeta,h}(\{(\sigma, \omega)\}) = 0$.

For configurations $\omega \in \Omega_\Lambda$ under which $\partial^+ \Lambda$ is connected to $\partial^- \Lambda$, there are no ω -admissible configurations. Let $D_\Lambda^\zeta \subset \Omega_\Lambda$ represent the set of configurations such that the vertices in $\partial^+ \Lambda$ are *not* connected to the vertices of $\partial^- \Lambda$; D_Λ^ζ is the support of $\phi_\Lambda^{J,\zeta,h}$. For $\omega \in D_\Lambda^\zeta$, there are $2^{n(\Lambda, \omega)}$ ω -admissible configurations and

$$\phi_\Lambda^{J,\zeta,h}(\{\omega\}) = \frac{2^{n(\Lambda, \omega)}}{Z_\Lambda^{J,\zeta,h}} \prod_e p_e^{\omega(e)} (1 - p_e)^{1-\omega(e)}.$$

The coupling $\varphi_\Lambda^{J,\zeta,h}$ has a probabilistic interpretation. To sample an Ising configuration $\sigma \sim \mu_\Lambda^{J,\zeta,h}$ given a sample $\omega \sim \phi_\Lambda^{J,\zeta,h}$, set $\sigma(V_+) = +1$, set $\sigma(V_-) = -1$, and independently for $i = 1, \dots, n$ set

$$\sigma(V_i) = \begin{cases} +1 & \text{with probability } 1/2, \\ -1 & \text{otherwise.} \end{cases}$$

It is sometimes easier to ignore the ghost edges. Let $r(\omega)$ denote the edge configuration obtained by closing all the ghost edges. We will say that $V \subset \Lambda$ is a *real* cluster if V is an $r(\omega)$ -cluster. If V is a real cluster under $\phi_\Lambda^{J,\zeta,h}$, but not connected to $\partial^\pm \Lambda$, then

$$\sigma(V) = \begin{cases} +1 & \text{with probability } e^{\beta h|V|}/(1 + e^{\beta h|V|}), \\ -1 & \text{otherwise.} \end{cases} \quad (2.2.1)$$

Let $x \leftrightarrow y$ denote the event that x and y are in the same real cluster, and let $A \leftrightarrow B$ denote the event that $a \leftrightarrow b$ for some $a \in A$ and $b \in B$. Note that when $h = 0$, all the clusters are real clusters.

There are two special cases of the $h = 0$ random-cluster model, wired and free boundary conditions. Wired boundary conditions refers to either all-plus or all-minus boundary conditions. Under wired boundary conditions $D_\Lambda^\zeta = \Omega_\Lambda$. The limit $\phi^{J,w} = \lim_{\Lambda \rightarrow \mathbb{Z}^d} \phi_\Lambda^{J,+,0}$ is called the wired random-cluster measure on \mathbb{Z}^d . Free boundary conditions refers to pretending that $\partial\Lambda = \emptyset$ whilst defining $\phi_\Lambda^{J,\zeta,h}$. The resulting measure $\phi_\Lambda^{J,f,0}$ depends only on $(J(e) : e \in E(\Lambda))$. The measure $\phi^{J,f}$ obtained by taking the limit of $\phi_\Lambda^{J,f,0}$ as $\Lambda \rightarrow \mathbb{Z}^d$ is called the free random-cluster measure on \mathbb{Z}^d .

Let $0 \leftrightarrow \infty$ denote the event that the origin is in an infinite real-cluster. The set

$$\mathcal{N} := \{ \beta : \mathbb{Q} [\mu^{J,f}(0 \leftrightarrow \infty)] < \mathbb{Q} [\mu^{J,w}(0 \leftrightarrow \infty)] \} \quad (2.2.2)$$

is at most countable [17]. It is conjectured that $\mathcal{N} = \emptyset$.

2.3 Stochastic orderings

Given two measures μ_1, μ_2 on a set \mathbb{R}^Λ , we will write $\mu_1 \leq_{\text{st}} \mu_2$ if there is a coupling (σ_1, σ_2) on $\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ such that

- (i) $\sigma_1 \sim \mu_1$,
- (ii) $\sigma_2 \sim \mu_2$, and
- (iii) $\sigma \leq \sigma'$ with probability one.

Holley's inequality [10, Theorem 2.1] can be used to prove stochastic orderings for the ferromagnetic Ising model. The Ising model on a fixed graph Λ is stochastically increasing with respect to the magnetic field and the boundary conditions: for $h_1 \leq h_2$ and any $\zeta_1 \leq \zeta_2$,

$$\mu_\Lambda^{J,\zeta_1,h_1} \leq_{\text{st}} \mu_\Lambda^{J,\zeta_2,h_2}.$$

The effect of expanding the region depends on the boundary conditions. With $\Delta \subset \Lambda$,

$$\mu_\Delta^{J,+,h} \geq_{\text{st}} \mu_\Lambda^{J,+,h} \quad \text{but} \quad \mu_\Delta^{J,-,h} \leq_{\text{st}} \mu_\Lambda^{J,-,h}.$$

Under plus boundary conditions, the random-cluster representation increases with $h \in [0, \infty)$ and J ,

$$\phi_\Lambda^{J_1,+,h_1} \leq_{\text{st}} \phi_\Lambda^{J_2,+,h_2} \quad \text{if} \quad h_2 \geq h_1 \geq 0 \quad \text{and} \quad \forall e, J_2(e) \geq J_1(e) \geq 0.$$

Note that sending $J(e)$ to zero or infinity on the boundary allows us to compare free and wired boundary conditions.

2.4 Coarse graining

Coarse graining is an important technique in the study of percolation and the random-cluster model. The open edges of the random-cluster model percolate for $\beta > \beta_c$. Slab percolation is a stronger property than percolation [17]. In three and higher dimensions, slab percolation refers to percolation in a *slab*

$\mathbb{Z}^{d-1} \times \{1, \dots, n\}$. In two dimensions it refers to the existence of spanning clusters in rectangles with arbitrarily high aspect ratios. The slab-percolation threshold is defined

$$\hat{\beta}_c = \inf\{\beta > 0 : \text{slab percolation occurs under } \mathbb{Q}[\mu^{J,\dagger}]\} \geq \beta_c.$$

It is conjectured that $\hat{\beta}_c = \beta_c$.

With K a positive integer, let

$$\mathbb{B}_K = [-K/2, K/2)^d \cap \mathbb{Z}^d.$$

For $i \in \mathbb{Z}^d$, let $\mathbb{B}_K(i) := \mathbb{B}_K + Ki$ denote a copy of \mathbb{B}_K centered at Ki .

Let $\Lambda \subset \mathbb{Z}^d$. To allow unusually high dilution on the edge boundary of Λ , let $J \sim \mathbb{Q}$ and let $J' \in \{0, 1\}^E$ denote a set of coupling strengths that agrees with J in $E(\Lambda)$. Let $\omega \in \Omega_\Lambda$ denote an edge configuration for Λ .

If $\mathbb{B}_K(i) \subset \Lambda$ and, looking at the restriction of ω to $\mathbb{B}_K(i)$, if there is a unique real-cluster $A \subset \mathbb{B}_K(i)$ connecting the $2d$ faces of $\mathbb{B}_K(i)$, let $\mathbb{B}_K^\dagger(i) = A$; let $\mathbb{B}_K^\ddagger(i)$ denote the real Λ -cluster containing $\mathbb{B}_K^\dagger(i)$. Otherwise, let $\mathbb{B}_K^\dagger(i) = \mathbb{B}_K^\ddagger(i) = \emptyset$.

Let $\varepsilon_{\text{cg}} > 0$. Recall the definition (1.2.2) of the spontaneous magnetization m^* .

Definition 2.4.1. A box $\mathbb{B}_K(i) \subset \Lambda$ is ε_{cg} -good if:

- (i) $\mathbb{B}_K^\dagger(i)$ is connected (by paths of length one) to each $\mathbb{B}_K^\dagger(j)$ such that $\mathbb{B}_K(j) \subset \Lambda$ and $\|i - j\|_1 = 1$.
- (ii) The diameters of the real-clusters of $\Lambda \setminus (\cup_j \mathbb{B}_K^\dagger(j))$ intersecting $\mathbb{B}_K(i)$ are at most $K/2$.
- (iii) $\mathbb{B}_K^\ddagger(i) \cap \mathbb{B}_K(i)$ contains between $K^d m^*(1 - \varepsilon_{\text{cg}})$ and $K^d m^*(1 + \varepsilon_{\text{cg}})$ vertices.

Otherwise $\mathbb{B}_K(i)$ is ε_{cg} -bad.

Note that any box not entirely contained in Λ is automatically ε_{cg} -bad. If a box is 1-bad then it is also ε_{cg} -bad for all $\varepsilon_{\text{cg}} \in (0, 1)$. Recall that we have fixed $\beta > \beta_0 \geq \hat{\beta}_c$. The supremum below is over $J' \in \{0, 1\}^E$ that agree with J on $E(\Lambda)$. Combining [17, Theorem 2.1 and Proposition 2.2] yields:

Proposition 2.4.2. There are constants $c_{\text{cg}} = c_{\text{cg}}(\varepsilon_{\text{cg}}) > 0$ and $K_0 = K_0(\varepsilon_{\text{cg}}) \in \mathbb{N}$ such that for $K \geq K_0$ and any $\mathbb{B}_K(i_1), \dots, \mathbb{B}_K(i_n) \subset \Lambda$, with \mathbb{Q} -probability $1 - \exp(-c_{\text{cg}}Kn)$,

$$\sup_{J'} \varphi_\Lambda^{J', +, 0}(\mathbb{B}_K(i_1), \dots, \mathbb{B}_K(i_n) \text{ are } \varepsilon_{\text{cg}}\text{-bad}) \leq \exp(-c_{\text{cg}}Kn).$$

Coarse graining gives a crude measure of the cost of phase coexistence. Consider a path of neighboring boxes $\mathbb{B}_K(i_1), \dots, \mathbb{B}_K(i_j)$ in Λ . If the first box and last box are not connected by a path of open edges, then there must be a 1-bad box somewhere along the path of boxes. Moreover, there must be a surface of at least $\lfloor K/K_0(1) \rfloor^{d-1}$ 1-bad $\mathbb{B}_{K_0(1)}$ -boxes separating $\mathbb{B}_K(i_1)$ from $\mathbb{B}_K(i_j)$ [5, cf. Lemma 4.2]. We obtain the following corollary to Proposition 2.4.2. Let c'_{cg} denote a positive constant, independent of ε_{cg} .

Corollary 2.4.3. *Let $K \geq K_0(1)$. For $k = 1, \dots, n$, let $\mathbb{B}_K(i_1^k), \dots, \mathbb{B}_K(i_{j_k}^k)$ denote a simple path of neighboring boxes in Λ with length $j_k \leq \exp(\sqrt{K})$. Assume the n chains are disjoint. Let A denote the event that for each k , $\mathbb{B}_K(i_1^k)$ is not connected to $\mathbb{B}_K(i_{j_k}^k)$ in $\cup_{l=1}^{j_k} \mathbb{B}_K(i_l^k)$. With \mathbb{Q} -probability $1 - \exp(-c'_{\text{cg}} K^{d-1} n)$,*

$$\sup_{J'} \varphi_\Lambda^{J', +, 0}(A) \leq \exp(-c'_{\text{cg}} K^{d-1} n).$$

We can quantify the extent to which a magnetic field and mixed boundary conditions affect the coarse graining property. Recall that $E^\pm(\Lambda)$ denotes the set of edges connecting Λ to the external vertex boundary $\partial^\pm \Lambda$.

Lemma 2.4.4. *Let $\zeta \in \Sigma$. For any $\mathbb{B}_K(i_1), \dots, \mathbb{B}_K(i_n) \subset \Lambda$, with \mathbb{Q} -probability $1 - \exp(-c_{\text{cg}} K n)$,*

$$\begin{aligned} \sup_{J'} \varphi_\Lambda^{J', \zeta, h}(\mathbb{B}_K(i_1), \dots, \mathbb{B}_K(i_n) \text{ are } \varepsilon_{\text{cg}}\text{-bad}) \\ \leq \exp(\beta |E^\pm(\Lambda)| + \beta h |\Lambda| - c_{\text{cg}} K n). \end{aligned}$$

Proof. Let (σ, ω) be an element of the support of $\varphi_\Lambda^{J', +, 0}$; under ω no ghost edges are open. Let B denote the set of configurations that agree with (σ, ω) as far as the vertices and the real edges are concerned,

$$\frac{\varphi_\Lambda^{J', \zeta, h}(B)}{\varphi_\Lambda^{J', +, 0}(\{(\sigma, \omega)\})} \leq \frac{Z_\Lambda^{J', +, 0}}{Z_\Lambda^{J', \zeta, h}}.$$

The right-hand side is bounded above by $\exp(\beta |E^\pm(\Lambda)| + \beta h |\Lambda|)$. The lemma follows by Proposition 2.4.2. \square

2.5 The graphical construction of the Glauber dynamics

For $\xi \in \Sigma_\Lambda^\zeta$, let $(\sigma_t^{s, \xi})_{t \geq s}$ denote the dynamic Ising model, started at time s in state ξ and evolving according to the Glauber dynamics (also known as heat-bath dynamics). The Glauber dynamics can be described by the following graphical construction. Let $\sigma_s^{s, \xi} = \xi$. Place a rate-one Poisson process at each vertex. Label the points of the Poisson process (x_i, t_i) with $t_1 < t_2 < \dots$; to each point (x_i, t_i) attach a uniform $[0, 1]$ random variable U_i . Let $\sigma_{t-}^{s, \xi}$ denote the Ising configuration immediately before time t . For each point of the Poisson process, resample the spin at x_i from the Ising measure conditional on the state of the neighboring spins. If the probability $\sigma(x_i) = +1$ conditional on $\{\sigma(y) = \sigma_{t-}^{s, \xi}(y) : y \sim x\}$ is q , set $\sigma_{t_i}^{s, \xi}(x_i) = +1$ if $U_i > 1 - q$, and -1 otherwise. The dynamics are:

- (i) Monotonic, if $\xi < \eta$ then $\sigma_t^{s, \xi} \leq \sigma_t^{s, \eta}$.
- (ii) Finite range, to update site x only requires knowledge of the neighbors.
- (iii) Bounded with respect to the transition rates.

Our results are also valid for other dynamics, such as the Metropolis dynamics, that share these properties.

3 L_1 -theory

3.1 Microscopic and mesoscopic scales

Recall that we have fixed $p \in (p_c, 1)$ and $\beta > \beta_0$ with $\beta \notin \mathcal{N}$.

To take advantage of the coarse graining result, we introduce some notation. With reference to Theorem 1.2.3, let $h > 0$. Define

$$K = \lfloor h^{-1/(2d)} \rfloor, \quad N = K \lfloor h^{-1}/K \rfloor \approx h^{-1}. \quad (3.1.1)$$

We will call N the macroscopic scale. This is the scale at which nucleation of plus droplets occurs. The number K denotes a mesoscopic scale. We will consider regions with size order N composed of boxes $\mathbb{B}_K(i)$.

Let $\mathcal{D} \subset \mathbb{R}^d$ denote a connected region. Let $\mathbb{D}(\mathcal{D}, N, K)$ denote a discretized version of \mathcal{D} composed of mesoscopic boxes,

$$\mathbb{D}(\mathcal{D}, N, K) = \bigcup_{i \in I} \mathbb{B}_K(i), \quad I = \left\{ x \in \mathbb{Z}^d : x + \left[-\frac{K}{2N}, \frac{K}{2N} \right]^d \subset \mathcal{D} \right\}. \quad (3.1.2)$$

We have used $h > 0$ to define a set $\Lambda = \mathbb{D}(\mathcal{D}, N, K)$ with size $O(N) = O(h^{-1})$ on which we wish to study the Ising model $\mu_\Lambda^{J, \zeta, h}$. Given Λ , we will also want to consider the Ising measure $\mu_\Lambda^{J, \zeta, 0}$. From now on, h will always determine the scale N but will not always indicate the strength of the magnetic field. The coarse graining implies that under $\mu_\Lambda^{J, \zeta, 0}$, $\sigma(\mathbb{B}_K(i))$ is close to either $\pm m^*$ with high probability. This motivates the definition below of the magnetization profile $\mathbb{M}_K^\zeta : \mathbb{R}^d \rightarrow \mathbb{R}$ associated with the Ising configuration σ .

Let Λ denote an arbitrary finite subset of \mathbb{Z}^d ; we will mostly be interested in sets of the form $\mathbb{D}(\mathcal{D}, N, K)$ but we will also consider sets that differ from $\mathbb{D}(\mathcal{D}, N, K)$ around the boundary. Let $\zeta \in \Sigma$ denote a boundary condition. Recall the notation (2.1.2) and that under $\mu_\Lambda^{J, \zeta, h}$, $\sigma(x) = \zeta(x)$ for $x \notin \Lambda$.

For $\sigma \in \Sigma_\Lambda^\zeta$, define the profile $\mathbb{M}_K^\zeta : \mathbb{R}^d \rightarrow \mathbb{R}$ as follows. For $\mathbf{x} \in \mathbb{R}^d$, choose i such that $N\mathbf{x} \in [-K/2, K/2]^d + Ki$. Let

$$\mathbb{M}_K^\zeta(\mathbf{x}) = \frac{1}{2} \times \begin{cases} 1 + \sigma(\mathbb{B}_K(i))/m^*, & \mathbb{B}_K(i) \subset \Lambda, \\ 1 + \sigma(\mathbb{B}_K(i)), & \text{otherwise.} \end{cases} \quad (3.1.3)$$

A value of 1 indicates plus phase, 0 indicates minus phase. The idea behind L_1 -theory is that \mathbb{M}_K^ζ can be approximated by the class of bounded variation profiles. The large deviations of \mathbb{M}_K^ζ can be described in terms of surface tension.

3.2 Surface tension in a parallelepiped

In statistical physics, surface tension is the excess free energy per unit area due to the presence of an interface. The definitions of surface tension in [16] and [18] differ by a factor of β ; we have chosen to follow [18].

Let $(\mathbf{n}, \mathbf{u}_2, \dots, \mathbf{u}_d)$ denote an orthonormal basis for \mathbb{R}^d and let \mathcal{R} denote the rectangular parallelepiped

$$\mathcal{R}_{L,H}(\mathbf{n}, \mathbf{u}_2, \dots, \mathbf{u}_d) := \left\{ t_1 \mathbf{n} + \sum_{k=2}^d t_k \mathbf{u}_k : \mathbf{t} \in \left[-\frac{H}{2}, \frac{H}{2} \right] \times \left[-\frac{L}{2}, \frac{L}{2} \right]^{d-1} \right\}.$$

\mathcal{R} is centered at the origin, has height H in the direction \mathbf{n} and extension L in the other directions.

Let $\Lambda = \mathbb{D}(\mathcal{R}, N, 1)$ denote a discrete version of Λ (3.1.2). The box Λ has sides of length NL in the directions $\mathbf{u}_2, \dots, \mathbf{u}_d$. The surface tension can be written in terms of either the Ising model partition function (2.1.1) or the random-cluster representation.

Definition 3.2.1. Let ζ denote the configuration in Σ given by $\zeta(y) = +1$ if $y \cdot \mathbf{n} \geq 0$ and $\zeta(y) = -1$ otherwise. The surface tension τ_Λ^J is defined by

$$\tau_\Lambda^J = \frac{1}{(NL)^{d-1}} \log \frac{Z_\Lambda^{J,+0}}{Z_\Lambda^{J,\zeta,0}} = \frac{1}{(NL)^{d-1}} \log \frac{1}{\phi_\Lambda^{J,+0}(\mathbb{D}_\Lambda^\zeta)}.$$

Let $J \sim \mathbb{Q}$. Surface tension converges in probability as $N \rightarrow \infty$ [18, Theorem 1.3]:

Proposition 3.2.2. For $\beta > 0$ and $\mathbf{n} \in \mathcal{S}^{d-1}$, there exists $\tau(\mathbf{n}) \geq 0$, the surface tension perpendicular to \mathbf{n} , such that for all parallelepipeds $\mathcal{R} = \mathcal{R}_{L,H}(\mathbf{n}, \mathbf{u}_2, \dots, \mathbf{u}_d)$,

$$\tau_\Lambda^J \xrightarrow[\mathbb{Q}\text{-probability}]{} \tau(\mathbf{n}) \text{ as } N \rightarrow \infty. \quad (3.2.3)$$

Surface tension is strictly positive at temperatures below the threshold for slab percolation [18, Proposition 2.11]:

Proposition 3.2.4. There are constants $C, c > 0$ such that for $\mathbf{n} \in \mathcal{S}^{d-1}$ and $\beta > \hat{\beta}_c$, $c\tau(\mathbf{e}_1) \leq \tau(\mathbf{n}) \leq C\tau(\mathbf{e}_1) \leq C\beta$.

We note for completeness that we are discussing *quenched* surface tension. *Annealed* surface tension, which will not be used in this paper, describes the cost of phase coexistence under the averaged measure $\mathbb{Q}^{\mu^{J,\zeta,h}}$. When studying large deviations under the annealed measure, the environment J changes to reduce the surface tension. Although we are interested in the large deviations of J , we prefer to control them ‘by hand’ using the \mathbb{Q}_θ notation defined in (4.3.1). This is less efficient in terms of the size of the large deviation needed. However, it is much simpler.

3.3 Surface tension in cones

L_1 -theory describes the Ising model at equilibrium [6, 7, 18]. We will restrict our attention to certain subsets of \mathbb{R}^d with zero surface tension at the boundary. At the microscopic scale, this corresponds to the sampling $J \sim \mathbb{Q}$ but conditioned on the existence of a surface of edges with $J(e) = 0$.

For $\theta \in (0, 2\pi]$ define a linear cone \mathcal{A}_θ ,

$$\mathcal{A}_\theta := \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq \|\mathbf{x}\|_2 \cos(\theta/2)\}.$$

For $\theta = 2\pi$, \mathcal{A}_θ is simply the whole of \mathbb{R}^d . Let \mathcal{L}^d denote the Lebesgue measure on \mathcal{A}_θ and let $\mathcal{V}(u, \varepsilon)$ denote the \mathcal{L}^d -ball of radius ε about $u : \mathcal{A}_\theta \rightarrow \mathbb{R}$.

The *perimeter* $\mathcal{P}(U)$ of a Borel subset $U \subset \mathcal{A}_\theta$ can be written in terms of functions of bounded variation (see [2, Chapter 3] and [18, Section 3.1]). The set of bounded variation profiles BV is given by

$$\text{BV} := \{U : U \subset \mathcal{A}_\theta \text{ is a Borel set and } \mathcal{P}(U) < \infty\}.$$

Bounded variation profiles $U \in \text{BV}$ have a *reduced boundary* ∂^*U and an outer normal $\mathbf{n}^U : \partial^*U \rightarrow \mathcal{S}^{d-1}$. Let \mathcal{H}^{d-1} denote the $d - 1$ dimensional Hausdorff measure on \mathcal{A}_θ ; $\mathcal{H}^{d-1}(\partial^*U) = \mathcal{P}(U)$. We will write ∂U to refer to the reduced boundary of U excluding (when $\theta < 2\pi$) the boundary of \mathcal{A}_θ ,

$$\partial U := \partial^*U \setminus \partial^*\mathcal{A}_\theta. \quad (3.3.1)$$

The outer normal \mathbf{n}^U defined on ∂U is Borel measurable. With reference to (3.2.3), this allows us to define the *surface tension* and *energy* of bounded variation profiles for the dilute Ising model. Define the surface tension \mathcal{F} by

$$\mathcal{F}(U) = \int_{\partial U} \tau(\mathbf{n}^U(\mathbf{x})) d\mathcal{H}^{d-1}(\mathbf{x}), \quad U \in \text{BV}. \quad (3.3.2)$$

Define the energy \mathcal{E} by

$$\mathcal{E}(U) = \mathcal{F}(U) - \beta m^* \mathcal{L}^d(U), \quad U \in \text{BV}. \quad (3.3.3)$$

The motivation for these quantities is that they measure, in the following sense, the cost of phase coexistence associated with the Ising model. Sample J from \mathbb{Q} conditional on $J(e) = 0$ for all $e = \{x, y\}$ such that x but not y is in $\mathbb{D}(\mathcal{A}_\theta, N, K)$ (3.1.2). This makes the surface tension on $\partial^*\mathcal{A}_\theta$ zero. Let \mathcal{D} denote a compact subset of \mathcal{A}_θ . For a profile of bounded variation $U \subset \mathcal{D}$, the surface of $\mathbb{D}(U, N, K)$ has size $O(1/h^{d-1})$ and $\mathbb{D}(U, N, K)$ has volume $O(1/h^d)$ (3.1.1). Heuristically, we expect that the probability of seeing the plus phase in $\mathbb{D}(U, N, K)$ and the minus phase in $\mathbb{D}(\mathcal{D} \setminus U, N, K)$ to be approximately

$$\begin{aligned} & \exp\left(-\frac{\mathcal{F}(U)}{h^{d-1}}\right) \quad \text{under } \mu_{\mathbb{D}(\mathcal{D}, N, K)}^{J, -, 0}, \quad \text{and} \\ & \exp\left(\frac{1}{h^{d-1}} \left[\inf_{U'} \mathcal{E}(U') - \mathcal{E}(U) \right]\right) \quad \text{under } \mu_{\mathbb{D}(\mathcal{D}, N, K)}^{J, -, h}. \end{aligned}$$

The infimum is over profiles $U' \in \text{BV}$ compatible with the boundary conditions. For general \mathcal{D} , it is difficult to evaluate the infimum—the conflicting contributions of the positive field and negative boundary conditions may lead to a complicated equilibrium magnetization profile under $\mu_{\mathbb{D}(\mathcal{D}, N, K)}^{J, -, h}$. The problem is simpler if \mathcal{D} is the Wulff shape.

4 The Wulff shape in \mathcal{A}_θ

4.1 Wulff, Winterbottom and Summertop shapes

Let $U \subset \mathcal{A}_\theta$ denote a set of bounded variation. Consider the problem of minimizing $\mathcal{F}(U)$ given that U has volume b^d .

Proposition 4.1.1. *Let $\theta \in (0, \pi] \cup \{2\pi\}$. The problem of finding a set of bounded variation $U \subset \mathcal{A}_\theta$ with volume b^d and minimal surface tension has a unique solution when $\theta < \pi$; for $\theta = \pi$ and $\theta = 2\pi$ the solution is unique up to translations. There is a scaling constant w_θ such that the solution is the convex shape*

$$\mathcal{W}_\theta(b) = w_\theta b \{ \mathbf{x} \in \mathcal{A}_\theta : \forall \mathbf{n} \in \mathcal{S}^{d-1}, \mathbf{x} \cdot \mathbf{n} \leq \tau(\mathbf{n}) \}. \quad (4.1.2)$$

If we omit the b , take $b = w_\theta^{-1}$ so that $\mathcal{W}_\theta \equiv \mathcal{W}_\theta(w_\theta^{-1})$.

Special cases of \mathcal{W}_θ are known by a variety of names. When $\theta = 2\pi$, \mathcal{W}_θ is the Wulff shape. When $\theta = \pi$, \mathcal{W}_θ is the Winterbottom shape. When $\theta \in (0, \pi)$ and $d = 2$, \mathcal{W}_θ is the Summertop shape [19]. We will refer to \mathcal{W}_θ as the Wulff shape in \mathcal{A}_θ .

Proof of Proposition 4.1.1. Let U denote a compact subset of \mathcal{A}_θ . With reference to [11, (G)], the formula (3.3.2) that defines the surface tension $\mathcal{F}(U)$ is equivalent to

$$\mathcal{F}(U) = \lim_{\varepsilon \rightarrow 0} \frac{|(U + \varepsilon \mathcal{W}_{2\pi}) \cap \mathcal{A}_\theta| - |U|}{\varepsilon}.$$

We can use (4.1.2) to define a second measure of surface tension. Let

$$\hat{\mathcal{F}}(U) = \lim_{\varepsilon \rightarrow 0} \frac{|U + \varepsilon \mathcal{W}_\theta| - |U|}{\varepsilon}.$$

Observe that:

- (i) For any $\mathbf{x} \in \mathcal{A}_\theta$ and $\varepsilon > 0$,

$$\mathbf{x} + \varepsilon \mathcal{W}_\theta \subset (\mathbf{x} + \varepsilon \mathcal{W}_{2\pi}) \cap \mathcal{A}_\theta.$$

Thus for $U \in \text{BV}$, $\hat{\mathcal{F}}(U) \leq \mathcal{F}(U)$.

- (ii) By the Brunn–Minkowski theorem, $\mathcal{W}_\theta(b)$ is the unique shape in \mathcal{A}_θ (up to translations) with volume b^d and minimal $\hat{\mathcal{F}}$ -surface tension.
(iii) By the convexity of $\mathcal{W}_{2\pi}$, when $U = \mathcal{W}_\theta$,

$$(U + \varepsilon \mathcal{W}_{2\pi}) \cap \mathcal{A}_\theta = U + \varepsilon \mathcal{W}_\theta$$

and so $\hat{\mathcal{F}}(\mathcal{W}_\theta(b)) = \mathcal{F}(\mathcal{W}_\theta(b))$.

Therefore any shape with volume b^d in \mathcal{A}_θ with minimal \mathcal{F} -surface tension must take the shape $\mathcal{W}_\theta(b)$. The claim of uniqueness when $\theta < \pi$ follows from the fact that for any $\mathbf{x} \in \mathcal{A}_\theta \setminus \{0\}$, $\mathcal{F}(\mathbf{x} + \mathcal{W}_\theta(b)) > \mathcal{F}(\mathcal{W}_\theta(b))$. \square

4.2 Critical droplets

Let $\mathbf{E}^\theta(b)$ account for the cost of filling $\mathcal{W}_\theta(b)$ with the plus phase,

$$\mathbf{E}^\theta(b) := \mathcal{E}(\mathcal{W}_\theta(b)) = b^{d-1} \mathcal{F}(\mathcal{W}_\theta(1)) - b^d \beta m^*. \quad (4.2.1)$$

The positive term represents the cost of phase coexistence, the negative term represents the benefit of conforming to the magnetic field. Let B_c^θ denote the maximizer of \mathbf{E}^θ , and let B_{root}^θ denote the positive root of \mathbf{E}^θ ,

$$B_c^\theta = \frac{d-1}{d} \frac{\mathcal{F}(\mathcal{W}_\theta(1))}{\beta m^*}, \quad B_{\text{root}}^\theta = \frac{\mathcal{F}(\mathcal{W}_\theta(1))}{\beta m^*}. \quad (4.2.2)$$

The significance of B_{root}^θ is that the Ising measure with minus boundary conditions and magnetic field h on $\mathbb{D}(\mathcal{W}_\theta(b), N, K)$ favors the minus phase if $b < B_{\text{root}}^\theta$, whereas it favors the plus phase if $b > B_{\text{root}}^\theta$. Let

$$\mathbf{E}_c^\theta := \mathbf{E}^\theta(B_c^\theta) = \left(\frac{\mathcal{F}(\mathcal{W}_\theta(1))}{d} \right)^d \left(\frac{d-1}{\beta m^*} \right)^{d-1}. \quad (4.2.3)$$

The maximum E_c^θ of E^θ characterizes the energy needed to create arbitrarily large plus droplets in the cone \mathcal{A}_θ , starting from the minus phase.

As the $J(e)$ are independent we can find regions that resemble, due to high local dilution, $\mathbb{D}(\mathcal{W}_\theta(b), N, K)$ for any $\theta \in (0, \pi)$ and any $b > 0$. However, to maximize the number of catalysts we do not want to take b any larger than we have to.

Proposition 4.2.4. *The diameter of the critical droplet is bounded uniformly over $\beta > \hat{\beta}_c$ and $\theta \in (0, \pi) \cup \{2\pi\}$. As $\theta \rightarrow 0$, $E_c^\theta = O(\beta\theta^{d-1})$.*

Proof. For $U \in \text{BV}$, $\mathcal{F}(U)$ has order $\beta\mathcal{H}^{d-1}(\partial U)$ by Proposition 3.2.4. Choose u_θ such that the cone

$$\mathcal{U}_\theta := u_\theta\{\mathbf{x} \in \mathcal{A}_\theta : x_1 \leq 1\}$$

has unit volume. As $\theta \rightarrow 0$, both \mathcal{U}_θ and $\mathcal{W}_\theta(1)$ have length of order $\theta^{-(d-1)/d}$ in the x_1 -direction. By the optimality of the Wulff shape, $\mathcal{F}(\mathcal{W}_\theta(1)) \leq \mathcal{F}(\mathcal{U}_\theta)$ which has order $\beta(\theta u_\theta)^{d-1}$. Substitute this approximation into (4.2.2) and (4.2.3). \square

4.3 Notation for Wulff shapes

Consider the discrete analogue \mathbb{A}_θ of \mathcal{A}_θ at microscopic scale N and mesoscopic scale K (3.1.2),

$$\mathbb{A}_\theta := \mathbb{D}(\mathcal{A}_\theta, N, K),$$

and the discrete analogues of the Wulff shape,

$$\mathbb{W}_\theta(b) := \mathbb{D}(\mathcal{W}_\theta(b), N, K), \quad b \geq 0.$$

For $\theta \neq 2\pi$, the edge boundary of \mathbb{A}_θ is infinite, thus the probability of finding a pattern of dilution that carves out a translation of the \mathbb{A}_θ anywhere in \mathbb{Z}^d is zero. Instead let $B_{\max}^\theta > 0$ denote a fixed, but as yet unknown, quantity. We will limit our attention to the region $\mathbb{W}_\theta(B_{\max}^\theta)$. To impose free boundary conditions on the portion of the boundary of $\mathbb{W}_\theta(B_{\max}^\theta)$ corresponding to $\partial^* \mathcal{A}_\theta$, let \mathbb{Q}_θ denote the dilution measure \mathbb{Q} conditioned appropriately,

$$\mathbb{Q}_\theta := \mathbb{Q}[\cdot \mid \forall x \sim y \text{ such that } x \in \mathbb{W}_\theta(B_{\max}^\theta) \text{ but } y \notin \mathbb{A}_\theta, J(\{x, y\}) = 0]. \quad (4.3.1)$$

The probability of seeing such a pattern of dilution is simply $(1 - p)$ raised to the power of the number of edges that are conditioned to be closed in the definition of \mathbb{Q}_θ . $\mathcal{W}_\theta(b)$ has size order $b\theta^{-(d-1)/d}$ along the x_1 -direction and order $b\theta^{1/d}$ in the directions x_2, \dots, x_d . The number of edges that need to be diluted is therefore order $(B_{\max}^\theta \theta^{-(d-1)/d}) \times (B_{\max}^\theta \theta^{1/d})^{d-2}$. The \mathbb{Q} -probability of the event conditioned on in (4.3.1) is thus

$$\exp\left(- (B_{\max}^\theta)^{d-1} \theta^{-1/d} O\left(\log \frac{1}{1-p}\right) / h^{d-1}\right). \quad (4.3.2)$$

As well as Wulff shaped regions, we also need to consider Wulff ‘annuli’: the difference between two Wulff shapes. With $0 \leq b_1 \leq b_2 \leq B_{\max}^\theta$, let

$$\mathcal{W}_\theta(b_1, b_2) := \mathcal{W}_\theta(b_2) \setminus \mathcal{W}_\theta(b_1) \quad \text{and} \quad \mathbb{W}_\theta(b_1, b_2) := \mathbb{W}_\theta(b_2) \setminus \mathbb{W}_\theta(b_1).$$

When $\theta = 2\pi$, $\mathcal{W}_\theta(b_1, b_2)$ is annular; otherwise it is simply-connected. There can be three parts to the boundary of $\mathcal{W}_\theta(b_1, b_2)$:

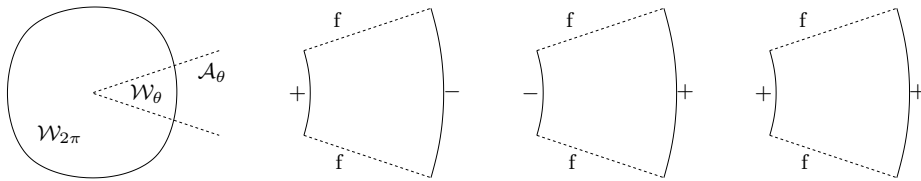


Figure 1: From left to right: \mathcal{W}_θ is the intersection of \mathcal{A}_θ with $\mathbb{W}_{2\pi}$. The set $\mathbb{W}_\theta(b_1, b_2)$ with $(+, -)$, $(-, +)$ and $(+, +)$ boundary conditions. The dotted lines indicate free boundary conditions.

- (i) the inner boundary $\partial\mathcal{W}_\theta(b_1)$,
- (ii) the outer boundary $\partial\mathcal{W}_\theta(b_2)$, and
- (iii) the free part of the boundary $\partial^*\mathcal{A}_\theta \cap \partial^*\mathcal{W}_\theta(b_1, b_2)$.

If $b_1 = 0$ then there is no inner boundary and $\mathbb{W}_\theta(b_1, b_2) = \mathbb{W}_\theta(b_2)$. If $\theta = 2\pi$ then the free part of the boundary is empty.

Let $(+, -)$ denote an Ising configuration that is equal to $+1$ on $\mathbb{W}_\theta(b_1)$, and equal to -1 on $\mathbb{A}_\theta \setminus \mathbb{W}_\theta(b_2)$; see Figure 1. We will show in Section 4.6 that for $b \in [b_1, b_2]$, the probability that $\mathbb{M}_K^{(+, -)}$ is close to $\mathcal{W}_\theta(b)$ under $\mu_\Lambda^{J, (+, -), 0}$ is approximately

$$\exp\left(\frac{\mathcal{F}(\mathcal{W}_\theta(b_1)) - \mathcal{F}(\mathcal{W}_\theta(b))}{h^{d-1}}\right).$$

As well as $(+, -)$ boundary conditions, we will also consider boundary conditions of $(-, +)$, $(+, +)$ and $(-, -)$. We may simplify $(+, +)$ to $+$ and $(-, -)$ to $-$.

We will also need to consider some sets that only differ from $\mathbb{W}_\theta(b)$ and $\mathbb{W}_\theta(b_1, b_2)$ at the mesoscopic scale. The sets $\mathbb{W}_\theta(b)$ are subsets of the discrete cone \mathbb{A}_θ . Let $\{x_1, x_2, \dots\}$ denote an ordering of \mathbb{A}_θ and let

$$\Delta_\theta^n = \{x_1, \dots, x_n\}, \quad n \geq 0. \quad (4.3.3)$$

As a function of b , $|\mathbb{W}_\theta(b)|$ is non-decreasing. $\mathbb{W}_\theta(b)$ is composed of boxes of K^d vertices, so $|\mathbb{W}_\theta(\cdot)|$ is a step function (i.e. piece-wise constant and cadlag). We can assume that the ordering has been chosen so that for $b \geq 0$, $\mathbb{W}_\theta(b) = \Delta_\theta^{|\mathbb{W}_\theta(b)|}$. Let $\mathbb{W}'_\theta(\cdot)$ denote a second family of increasing subsets of \mathbb{A}_θ such that

- (i) $\mathbb{W}_\theta(b) = \mathbb{W}'_\theta(b)$ at the points b of discontinuity of $|\mathbb{W}_\theta(b)|$,
- (ii) $\mathbb{W}_\theta(b) \subset \mathbb{W}'_\theta(b)$,
- (iii) for each n , for some $b_\theta(n)$, $\mathbb{W}'_\theta(b_\theta(n)) = \Delta_\theta^n$.

Let $\mathbb{W}'_\theta(b_1, b_2) = \mathbb{W}'_\theta(b_2) \setminus \mathbb{W}_\theta(b_1)$.

4.4 High and uniformly high \mathbb{Q}_θ -probability

We will say that an event occurs with *high \mathbb{Q}_θ -probability* if under \mathbb{Q}_θ it occurs with probability at least $1 - C \exp(-c/\sqrt{h})$. Note that by taking C large, we only have to consider $h \in (0, h_0)$ where h_0 can be arbitrarily small.

Given $B_{\max}^\theta > 0$, suppose that

$$0 \leq b_1 \leq b \leq b_2 \leq B_{\max}^\theta \quad \text{and} \quad \Lambda = \mathbb{W}'_\theta(b_1, b_2). \quad (4.4.1)$$

Given a class of events defined in terms of b, b_1 and b_2 we will say that they occur with *uniformly high* \mathbb{Q}_θ -probability if each event occurs with probability at least $1 - C \exp(-c/\sqrt{h})$, uniformly over (4.4.1). Abusing this notation, we may place some additional restriction on b_1 and b_2 ; for example fixing $b_1 = 0$. In that case interpret (4.4.1) with the additional restriction in place.

4.5 L_1 -theory under $(w, -)$ boundary conditions

The L_1 -theory developed in [18] describes the dilute Ising model in cubes $\Lambda = \{1, \dots, N\}^d$ under the measure $\mu_\Lambda^{J, -, 0}$, $J \sim \mathbb{Q}$. The proofs in [18] are easily adapted to sets of the form $\mathbb{W}'_\theta(b_1, b_2)$ with $J \sim \mathbb{Q}_\theta$. Moreover, the methodology accommodates the $(w, -)$ boundary conditions described below.

Consider Λ as in (4.4.1). Let $(w, -)$ denote wired boundary conditions on the inner boundary of Λ , and minus boundary conditions on the outer boundary of Λ . If $b_1 = 0$, $(w, -)$ simply means minus boundary conditions.

This is equivalent to starting from $(+, -)$ or $(-, -)$ boundary conditions, and then replacing the inner boundary of Λ with a single Ising spin variable,

$$\mu_\Lambda^{J, (w, -), 0}(\{\sigma\}) = \frac{1}{\sum_{s=\pm 1} Z_\Lambda^{J, (s, -), 0}} \cdot \begin{cases} \exp\left(-\beta H_\Lambda^{J, (+, -), 0}(\sigma)\right), & \sigma \in \Sigma_\Lambda^{(+, -)}, \\ \exp\left(-\beta H_\Lambda^{J, (-, -), 0}(\sigma)\right), & \sigma \in \Sigma_\Lambda^{(-, -)}. \end{cases}$$

This is a measure on $\Sigma_\Lambda^{(+, -)} \cup \Sigma_\Lambda^{(-, -)}$. Let $\partial^w \Lambda$ denote the wired inner-boundary of Λ . Let $\sigma(\partial^w \Lambda) = \pm 1$ according to whether $\sigma \in \Sigma_\Lambda^{(+, -)}$ or $\sigma \in \Sigma_\Lambda^{(-, -)}$. Define $\mathbb{M}_K^{(w, -)}$ to be either $\mathbb{M}_K^{(+, -)}$ or $\mathbb{M}_K^{(-, -)}$ according to $\sigma(\partial^w \Lambda)$.

The $(w, -)$ measure with $h = 0$ has a natural random-cluster representation with wired-inner and wired-outer boundary conditions. The corresponding coupled measure is

$$\varphi_\Lambda^{J, (w, -), 0}(\{(\sigma, \omega)\}) = \frac{\sum_{s=\pm 1} Z_\Lambda^{J, (s, -), 0} \varphi_\Lambda^{J, (s, -), 0}(\{(\sigma, \omega)\})}{\sum_{s=\pm 1} Z_\Lambda^{J, (s, -), 0}}. \quad (4.5.1)$$

Note that only the $s = \sigma(\partial^w \Lambda)$ term in the numerator of (4.5.1) is positive.

Let $D_\Lambda^{(w, -)}$ refer to the event that the inner boundary $\partial^w \Lambda$ and the outer boundary $\partial^- \Lambda$ are not connected in the random-cluster representation. For $\varepsilon > 0$, let $D_{b, \varepsilon}$ denote the event that the inner-boundary takes the plus spin and that the phase profile is in a neighborhood of $\mathcal{W}_\theta(b)$,

$$D_{b, \varepsilon} := D_\Lambda^{(w, -)} \cap \{\sigma(\partial^w \Lambda) = 1\} \cap \left\{ \mathbb{M}_K^{(w, -)} \in \mathcal{V}(\mathcal{W}_\theta(b), \varepsilon) \right\}.$$

Parts (i) and (ii) below follow from the proofs in [18] of Proposition 3.10 and Theorem 1.11 respectively.

Proposition 4.5.2. *Let $b \in [b_1, b_2]$ and $\varepsilon_{\text{wm}} > 0$. With high \mathbb{Q}_θ -probability:*

(i) *The probability of phase coexistence is bounded below,*

$$h^{d-1} \log \varphi_\Lambda^{J, (w, -), 0}(D_{b, \varepsilon_{\text{wm}}}) \geq -\mathcal{F}(\mathcal{W}_\theta(b)) - \varepsilon_{\text{wm}}.$$

(ii) The probability of phase coexistence is bounded above,

$$h^{d-1} \log \mu_\Lambda^{J,(\mathbf{w},-),0} \left(\int \mathbb{M}_K^{(\mathbf{w},-)} d\mathcal{L}^d \geq b^d \right) \leq -\mathcal{F}(\mathcal{W}_\theta(b)) + \varepsilon_{\text{wm}}.$$

4.6 Large deviations under $(+, -)$ boundary conditions

With b and Λ as in (4.4.1), we will give upper and lower bounds for the cost of phase coexistence under mixed boundary conditions in the absence of an external magnetic field.

Under $(+, -)$ boundary conditions, the Ising measure favors the minus phase because the minus boundary is bigger. Let $\varepsilon_{\text{pm}} > 0$. Large deviations of the magnetization $\mathbb{M}_K^{(+,-)}$ defined in (3.1.3) away from the minus phase are controlled as follows.

Proposition 4.6.1 (Upper bound). *With high \mathbb{Q}_θ -probability*

$$h^{d-1} \log \mu_\Lambda^{J,(+,-),0} \left(\int \mathbb{M}_K^{(+,-)} d\mathcal{L}^d \geq b^d \right) \leq \mathcal{F}(\mathcal{W}_\theta(b_1)) - \mathcal{F}(\mathcal{W}_\theta(b)) + \varepsilon_{\text{pm}}.$$

Proposition 4.6.2 (Lower bound). *With high \mathbb{Q}_θ -probability*

$$h^{d-1} \log \mu_\Lambda^{J,(+,-),0} \left(\int \mathbb{M}_K^{(+,-)} d\mathcal{L}^d \geq b^d \right) \geq \mathcal{F}(\mathcal{W}_\theta(b_1)) - \mathcal{F}(\mathcal{W}_\theta(b)) - \varepsilon_{\text{pm}}.$$

Proof of Proposition 4.6.1. By the definition of the $(\mathbf{w}, -)$ measure,

$$\begin{aligned} & \mu_\Lambda^{J,(+,-),0} \left(\int \mathbb{M}_K^{(+,-)} d\mathcal{L}^d \geq b^d \right) \\ & \leq \mu_\Lambda^{J,(\mathbf{w},-),0} \left(\int \mathbb{M}_K^{(\mathbf{w},-)} d\mathcal{L}^d \geq b^d \right) \cdot \left(\frac{Z_\Lambda^{J,(+,-),0}}{\sum_{s=\pm 1} Z_\Lambda^{J,(s,-),0}} \right)^{-1}. \end{aligned} \quad (4.6.3)$$

Proposition 4.5.2 part (ii) provides an upper bound on the first term on the right-hand side of (4.6.3),

$$\mu_\Lambda^{J,(\mathbf{w},-),0} \left(\int \mathbb{M}_K^{(\mathbf{w},-)} d\mathcal{L}^d \geq b^d \right) \leq \exp \left(-\frac{\mathcal{F}(\mathcal{W}_\theta(b)) - \varepsilon_{\text{wm}}}{h^{d-1}} \right).$$

For $\omega \in \mathbb{D}_\Lambda^{(\mathbf{w},-)}$, the $s = +1$ and $s = -1$ terms in the numerator of the right-hand side of (4.5.1) are equal, so

$$\phi_\Lambda^{J,(\mathbf{w},-),0}(\omega) = \frac{2Z_\Lambda^{J,(+,-),0} \phi_\Lambda^{J,(+,-),0}(\omega)}{\sum_{s=\pm 1} Z_\Lambda^{J,(s,-),0}}.$$

Summing over ω ,

$$\phi_\Lambda^{J,(\mathbf{w},-),0}(\mathbb{D}_\Lambda^{(\mathbf{w},-)}) = \frac{2Z_\Lambda^{J,(+,-),0}}{\sum_{s=\pm 1} Z_\Lambda^{J,(s,-),0}}. \quad (4.6.4)$$

By Proposition 4.5.2 part (i) with $b = b_1$,

$$\phi_\Lambda^{J,(\mathbf{w},-),0}(\mathbb{D}_\Lambda^{(\mathbf{w},-)}) \geq \varphi_\Lambda^{J,(\mathbf{w},-),0}(\mathbb{D}_{b_1, \varepsilon_{\text{wm}}}) \geq \exp \left(-\frac{\mathcal{F}(\mathcal{W}_\theta(b_1)) + \varepsilon_{\text{wm}}}{h^{d-1}} \right). \quad (4.6.5)$$

Combining inequalities (4.6.4) and (4.6.5) produces an upper bound for the second term on the right-hand side of (4.6.3). The proposition follows by taking $\varepsilon_{\text{wm}} = \varepsilon_{\text{pm}}/3$. \square

Proof of Proposition 4.6.2. With $\varepsilon_{\text{wm}} > 0$ let $b' = \sqrt[d]{b^d + \varepsilon_{\text{wm}}}$ so that $D_{b', \varepsilon_{\text{wm}}}$ implies $\int \mathbb{M}_K^{(+, -)} d\mathcal{L}^d \geq b^d$. By equation (4.6.4) and the definition of the $(w, -)$ measure,

$$\varphi_\Lambda^{J, (+, -), 0}(D_{b', \varepsilon_{\text{wm}}}) = \frac{2\varphi_\Lambda^{J, (w, -), 0}(D_{b', \varepsilon_{\text{wm}}})}{\phi_\Lambda^{J, (w, -), 0}(D_\Lambda^{(w, -)})}.$$

Proposition 4.5.2 part (i) gives a lower bound for the numerator on the right-hand side. Proposition 4.5.2 part (ii) with $b = b_1$ gives an upper bound on the denominator on the right-hand side. Take ε_{wm} sufficiently small. \square

Proposition 4.6.6. *With reference to (4.4.1), for fixed ε_{pm} both Proposition 4.6.1 and Proposition 4.6.2 hold with uniformly high \mathbb{Q}_θ -probability.*

Proof. Consider Proposition 4.6.1; the case of Proposition 4.6.2 follows similarly. Controlling uniformly the \mathbb{Q}_θ -probability can be reduced to the control of a finite number of events. Let $M(b_1, b_2, b)$ denote the increasing event $\left\{ \sigma : \int \mathbb{M}_K^{(+, -)} d\mathcal{L}^d \geq b^d \right\}$ with $\mathbb{M}_K^{(+, -)}$ defined with respect to $\mathbb{W}'_\theta(b_1, b_2)$.

The measure $\mu_\Lambda^{J, (+, -), h}$ is stochastically increasing with b_1 and b_2 . Let $C, \varepsilon > 0$ and let

$$b'_1 = \varepsilon \lceil b_1 / \varepsilon \rceil, \quad b'_2 = \varepsilon \lceil b_2 / \varepsilon \rceil, \quad b' = \varepsilon \left\lceil \varepsilon^{-1} \sqrt[d]{b^d - C\varepsilon(B_{\max}^\theta)^{d-1}} \right\rceil.$$

For triples (b_1, b_2, b) satisfying (4.4.1), the triple (b'_1, b'_2, b') takes a finite number of values. With high \mathbb{Q}_θ -probability,

$$h^{d-1} \log \mu_{\mathbb{W}'_\theta(b'_1, b'_2)}^{J, (+, -), 0}(M(b'_1, b'_2, b')) \leq \mathcal{F}(\mathcal{W}_\theta(b'_1)) - \mathcal{F}(\mathcal{W}_\theta(b')) + \frac{\varepsilon_{\text{pm}}}{2}.$$

C can be chosen such that if $\mathcal{F}(\mathcal{W}_\theta(b_1)) - \mathcal{F}(\mathcal{W}_\theta(b)) + \varepsilon_{\text{pm}} < 0$ and ε is sufficiently small then $M(b_1, b_2, b) \implies M(b'_1, b'_2, b')$ and

$$\mathcal{F}(\mathcal{W}_\theta(b'_1)) - \mathcal{F}(\mathcal{W}_\theta(b')) + \frac{\varepsilon_{\text{pm}}}{2} \leq \mathcal{F}(\mathcal{W}_\theta(b_1)) - \mathcal{F}(\mathcal{W}_\theta(b)) + \varepsilon_{\text{pm}}. \quad \square$$

5 Spatial and Markov chain mixing

We will describe the dilute Ising model at equilibrium under mixed boundary conditions and a magnetic field. These results will be used in the next section to show that sufficiently large plus droplets spread in a predictable way.

5.1 Stability of minimum energy profiles

Consider the Ising measure with a magnetic field h and $(+, -)$ boundary conditions. Recall the definitions of the energy functions \mathcal{E} (3.3.3) and \mathbf{E}^θ (4.2.1). With reference to (4.4.1), consider the case $\mathbf{E}^\theta(b_1) > \mathbf{E}^\theta(b_2)$. The minimum value of $\mathbf{E}^\theta(b)$ is attained when $b = b_2$. Geometrically, this means that the profile U with $\mathcal{W}_\theta(b_1) \subset U \subset \mathcal{W}_\theta(b_2)$ that minimizes $\mathcal{E}(U)$ is $\mathcal{W}_\theta(b_2)$. The minimizer is unique and stable.

Proposition 5.1.1. *Let $\varepsilon_{\text{stb}} \in (0, 1)$ such that $b_1 < b_2(1 - \varepsilon_{\text{stb}})$ and $\mathbf{E}^\theta(b_1) > \mathbf{E}^\theta(b_2(1 - \varepsilon_{\text{stb}}))$. There is a constant $c_{\text{stb}} = c_{\text{stb}}(\varepsilon_{\text{stb}}) > 0$, independent of B_{max}^θ , such that for profiles $U \in \text{BV}$,*

$$\mathcal{L}^d(U) \in [b_1^d, b_2^d(1 - \varepsilon_{\text{stb}})^d] \implies \mathcal{E}(U) \geq \mathcal{E}(\mathcal{W}_\theta(b_2)) + 2b_2c_{\text{stb}}. \quad (5.1.2)$$

Given ε_{stb} , with uniformly high \mathbb{Q}_θ -probability,

$$\mu_\Lambda^{J, (+, -), h} \left(\int \mathbb{M}_K^{(+, -)} d\mathcal{L}^d \leq b_2^d(1 - \varepsilon_{\text{stb}})^d \right) \leq \exp \left(-\frac{b_2c_{\text{stb}}}{h^{d-1}} \right). \quad (5.1.3)$$

Proof. We will first show inequality (5.1.2) with

$$c_{\text{stb}} := \frac{(d-1)\mathcal{F}(\mathcal{W}_\theta(1)) (B_c^\theta)^{d-2} \varepsilon^2}{4d^2 \sqrt[d]{1-\varepsilon}} \quad \text{where} \quad \varepsilon := 1 - (1 - \varepsilon_{\text{stb}})^d.$$

By the optimality of the Wulff shape, Proposition 4.1.1, we can assume that $U = \mathcal{W}_\theta(b)$ for some b . The unique maximum of \mathbf{E}^θ occurs at B_c^θ . As $\mathbf{E}^\theta(b_1) > \mathbf{E}^\theta(b_2(1 - \varepsilon_{\text{stb}}))$ the function \mathbf{E}^θ must be decreasing in the region $[b_2(1 - \varepsilon_{\text{stb}}), b_2]$; it is optimal to consider

$$b = b_2(1 - \varepsilon_{\text{stb}}) \geq B_c^\theta. \quad (5.1.4)$$

With b as above,

$$\begin{aligned} \mathbf{E}^\theta(b) - \mathbf{E}^\theta(b_2) &= \mathcal{F}(\mathcal{W}_\theta(1))(b^{d-1} - b_2^{d-1}) - \beta m^*(b^d - b_2^d) \\ &= \mathcal{F}(\mathcal{W}_\theta(1))b_2^{d-1}((1 - \varepsilon)^{(d-1)/d} - 1) + \beta m^*\varepsilon b_2^d. \end{aligned} \quad (5.1.5)$$

By (4.2.2) and (5.1.4),

$$\beta m^*\varepsilon b_2^d \geq \beta m^*\varepsilon \frac{b_2^{d-1} B_c^\theta}{\sqrt[d]{1-\varepsilon}} = \mathcal{F}(\mathcal{W}_\theta(1))b_2^{d-1} \frac{\varepsilon}{\sqrt[d]{1-\varepsilon}} \frac{d-1}{d}. \quad (5.1.6)$$

Let $f(\varepsilon) = (1 - \varepsilon) - \sqrt[d]{1 - \varepsilon}$. For $\varepsilon \in (0, 1)$, $f'''(\varepsilon) > 0$ so

$$f(\varepsilon) - f(0) - \varepsilon f'(0) \geq f''(0)\varepsilon^2/2, \quad \varepsilon \in [0, 1],$$

or more explicitly

$$(1 - \varepsilon) - \sqrt[d]{1 - \varepsilon} + \varepsilon \frac{d-1}{d} \geq \varepsilon^2 \cdot \frac{d-1}{2d^2}, \quad \varepsilon \in [0, 1]. \quad (5.1.7)$$

Substituting (5.1.6) into (5.1.5) and then using (5.1.7) gives

$$\mathbf{E}^\theta(b) - \mathbf{E}^\theta(b_2) \geq \mathcal{F}(\mathcal{W}_\theta(1))b_2^{d-1} \cdot \frac{\varepsilon^2}{\sqrt[d]{1-\varepsilon}} \cdot \frac{d-1}{2d^2} \geq 2b_2c_{\text{stb}}$$

as required.

We will now show inequality (5.1.3). Let S count, up to an additive constant, the number of plus spins in Λ ,

$$S := \sum_{x \in \Lambda} \frac{\sigma(x) + m^*}{2}.$$

By the definition of Λ , $\mathbb{M}_K^{(+,-)}$ and S ,

$$\left| m^* b_1^d + h^d S - m^* \int \mathbb{M}_K^{(+,-)} d\mathcal{L}^d \right| = O(K/N).$$

The magnetic field corresponds to a Radon-Nikodym derivative controlled by S . With $Z := \mu_\Lambda^{J,(+,-),0} [\exp(\beta h S)]$,

$$\frac{\mu_\Lambda^{J,(+,-),h}(\{\sigma\})}{\mu_\Lambda^{J,(+,-),0}(\{\sigma\})} = \frac{1}{Z} \exp(\beta h S).$$

We will find a lower bound on Z . When h is sufficiently small,

$$\mathbb{M}_K^{(+,-)} \in \mathcal{V}(\mathcal{W}_\theta(b), \varepsilon_{\text{pm}}) \implies |h^d S - m^*(b^d - b_1^d)| = O(\varepsilon_{\text{pm}}).$$

Applying Proposition 4.6.2 with $b = b_2$,

$$\begin{aligned} h^{d-1} \log Z &\geq h^{d-1} \log \mu_\Lambda^{J,(+,-),0}(\mathbb{M}_K^{(+,-)} \in \mathcal{V}(\mathcal{W}_\theta(b_2), \varepsilon_{\text{pm}})) + \beta m^*(b_2^d - b_1^d) - O(\varepsilon_{\text{pm}}) \\ &\geq \mathcal{F}(\mathcal{W}_\theta(b_1)) - \mathcal{F}(\mathcal{W}_\theta(b_2)) + \beta m^*(b_2^d - b_1^d) - O(\varepsilon_{\text{pm}}) \\ &\geq \mathcal{E}(\mathcal{W}_\theta(b_1)) - \mathcal{E}(\mathcal{W}_\theta(b_2)) - O(\varepsilon_{\text{pm}}). \end{aligned}$$

Let $n = \lfloor (b_2(1 - \varepsilon_{\text{stb}}) - b_1)/\varepsilon_{\text{pm}} \rfloor$. We can write

$$\begin{aligned} \left\{ \sigma : \int \mathbb{M}_K^{(+,-)} d\mathcal{L}^d \leq b_2^d(1 - \varepsilon_{\text{stb}})^d \right\} \subset \\ \bigcup_{b \in \{b_1, b_1 + \varepsilon_{\text{pm}}, \dots, b_1 + \varepsilon_{\text{pm}} n\}} \left\{ \sigma : \int \mathbb{M}_K^{(+,-)} d\mathcal{L}^d \in [b^d, (b + \varepsilon_{\text{pm}})^d] \right\}. \end{aligned}$$

By Proposition 4.6.1, for $b \in \{b_1, b_1 + \varepsilon_{\text{pm}}, \dots, b_1 + \varepsilon_{\text{pm}} n\}$,

$$\begin{aligned} &\mu_\Lambda^{J,(+,-),0} \left[\exp(\beta h S); \int \mathbb{M}_K^{(+,-)} d\mathcal{L}^d \in [b^d, (b + \varepsilon_{\text{pm}})^d] \right] \\ &\leq \exp(h^{1-d} [\mathcal{F}(\mathcal{W}_\theta(b_1)) - \mathcal{F}(\mathcal{W}_\theta(b)) + \beta m^*[(b + \varepsilon_{\text{pm}})^d - b_1^d] + O(\varepsilon_{\text{pm}})]) \\ &\leq \exp(h^{1-d} [\mathcal{E}(\mathcal{W}_\theta(b_1)) - \mathcal{E}(\mathcal{W}_\theta(b)) + O(\varepsilon_{\text{pm}})]). \end{aligned}$$

The left-hand side of (5.1.3) is therefore at most

$$\sum_{b \in \{b_1, b_1 + \varepsilon_{\text{pm}}, \dots, b_1 + \varepsilon_{\text{pm}} n\}} \exp(h^{1-d} [\mathcal{E}(\mathcal{W}_\theta(b_2)) - \mathcal{E}(\mathcal{W}_\theta(b)) + O(\varepsilon_{\text{pm}})]).$$

Taking ε_{pm} small with respect to ε_{stb} , inequality (5.1.3) follows by (5.1.2). \square

Consider now the case $E^\theta(b_1) < E^\theta(b_2)$. The optimum profile matching $(+, -)$ boundary conditions is $\mathcal{W}_\theta(b_1)$ so the minus phase is dominant.

Proposition 5.1.8. *Let $\varepsilon_{\text{stb}} > 0$. Suppose that $E^\theta(b_1 + \varepsilon_{\text{stb}}) < E^\theta(b_2)$. There is a constant $c'_{\text{stb}} = c'_{\text{stb}}(\varepsilon_{\text{stb}}) > 0$ such that with uniformly high \mathbb{Q}_θ -probability*

$$\mu_\Lambda^{J,(+,-),h} \left(\int \mathbb{M}_K^{(+,-)} d\mathcal{L}^d \geq (b_1 + \varepsilon_{\text{stb}})^d \right) \leq \exp\left(-\frac{c'_{\text{stb}}}{h^{d-1}}\right).$$

We will omit the proof of Proposition 5.1.8 as it is similar to the proof of Proposition 5.1.1.

Now consider the case $b_1 = 0$ and $b_2 > B_c^\theta$ under minus boundary conditions. In order to get a stability property that does not depend on the sign of $\mathbf{E}^\theta(b_2)$ we will condition on seeing a large region of the plus-phase. With $\varepsilon_C > 0$ let

$$\hat{\mu}_\Lambda^{J,-,h} := \mu_\Lambda^{J,-,h}(\cdot \mid \mathcal{C}) \quad \text{where} \quad (5.1.9)$$

$$\mathcal{C} = \mathcal{C}(\varepsilon_C) := \left\{ \sigma \in \Sigma_\Lambda^- : \int_{\mathcal{W}_\theta(B_c^\theta + \varepsilon_C)} \mathbb{M}_K^- d\mathcal{L}^d \geq (B_c^\theta)^d \right\}.$$

Proposition 5.1.10. *Let $\varepsilon_{\text{stb}}, \varepsilon_C > 0$. Suppose $b_1 = 0$ and $b_2(1 - \varepsilon_{\text{stb}}) > B_c^\theta + \varepsilon_C$. Recall c_{stb} from Proposition 5.1.1. Given ε_{stb} and ε_C , with uniformly high \mathbb{Q}_θ -probability,*

$$\hat{\mu}_\Lambda^{J,-,h} \left(\int \mathbb{M}_K^- d\mathcal{L}^d \leq b_2^d(1 - \varepsilon_{\text{stb}})^d \right) \leq \exp \left(-\frac{b_2 c_{\text{stb}}}{h^{d-1}} \right).$$

Proof. The proof of (5.1.2) (with B_c^θ playing the role of b_1) implies that for all $U \in \text{BV}$,

$$\mathcal{L}^d(U) \in [(B_c^\theta)^d, b_2^d(1 - \varepsilon_{\text{stb}})^d] \implies \mathcal{E}(U) \geq \mathcal{E}(\mathcal{W}_\theta(b_2)) + 2b_2 c_{\text{stb}}.$$

Therefore $\mathbf{E}^\theta(b_2(1 - \varepsilon_{\text{stb}})) - \mathbf{E}^\theta(b_2) \geq 2b_2 c_{\text{stb}}$.

Returning to the context of Proposition 5.1.10, we have $b_1 = 0$. Let $a = \min\{0, \mathbf{E}^\theta(b_2)\}$ denote the minimum of $\mathbf{E}^\theta(b)$ for $b \in [b_1, b_2]$. Let b denote the minimum value in the range $[b_2(1 - \varepsilon_{\text{stb}}), b_2]$ such that

$$b^d - b^d \leq (B_c^\theta + \varepsilon_C)^d - (B_c^\theta)^d.$$

Treating the magnetic field as a Radon-Nikodym derivative as in the proof of Proposition 5.1.1,

$$\mu_\Lambda^{J,-,h} \left(\int \mathbb{M}_K^- d\mathcal{L}^d \in [(B_c^\theta)^d, b_2^d(1 - \varepsilon_{\text{stb}})^d] \right) \leq \exp \left(\frac{a - \mathbf{E}^\theta(b_2) - 3b_2 c_{\text{stb}}/2}{h^{d-1}} \right)$$

and

$$\mu_\Lambda^{J,-,h}(\mathcal{C}) \geq \mu_\Lambda^{J,-,h} \left(\int \mathbb{M}_K^- d\mathcal{L}^d \in [b^d, b_2^d] \right) \geq \exp \left(\frac{a - \mathbf{E}^\theta(b_2) - b_2 c_{\text{stb}}/2}{h^{d-1}} \right). \quad \square$$

We will now consider two different boundary conditions. By Proposition 4.6.1, plus/minus symmetry when $h = 0$, and monotonicity, the plus phase is dominant under $\mu_\Lambda^{J,(-,+),h}$.

Proposition 5.1.11. *Let $\varepsilon_{\text{stb}} > 0$. There is a constant $c''_{\text{stb}} = c''_{\text{stb}}(\varepsilon_{\text{stb}}) > 0$ such that with uniformly high \mathbb{Q}_θ -probability,*

$$\mu_\Lambda^{J,(-,+),h} \left(\int_{\mathcal{W}_\theta(b_1, b_2)} \mathbb{M}_K^{(-,+)} d\mathcal{L}^d \leq b_2^d - (b_1 + \varepsilon_{\text{stb}})^d \right) \leq \exp \left(-\frac{c''_{\text{stb}}}{h^{d-1}} \right).$$

Finally, consider boundary conditions of plus on the inner boundary and free on the outer boundary.

Proposition 5.1.12. *Let $\varepsilon_{\text{stb}} > 0$ and suppose $b_2^d \geq 2\varepsilon_{\text{stb}}$. With uniformly high \mathbb{Q}_θ -probability,*

$$\mu_\Lambda^{J, (+, f), h} \left(\int_{\mathcal{W}_\theta(b_1, b_2)} \mathbb{M}_K^+ d\mathcal{L}^d \leq b_2^d - \varepsilon_{\text{stb}} \right) \leq \exp \left(-\frac{\varepsilon_{\text{stb}}}{h^{d-1}} \right).$$

We omit the proof as it is similar to the others in this section.

5.2 Phase labels in \mathbb{A}_θ

Lemma 2.4.4 provides a simple measure of the cost of phase coexistence in a region conditional on the boundary. Using phase labels—defined in terms of the coarse graining—to describe the boundary conditions allows for a sharper bound.

Take Λ as in (4.4.1). Let $\omega \in \Omega_\Lambda$, and let $\sigma \in \Sigma_\Lambda^\zeta$ denote an ω -admissible spin configuration. Define phase labels in terms of the coarse graining with $\varepsilon_{\text{cg}} = 1$:

$$\Psi(i) = \begin{cases} +1, & \mathbb{B}_K(i) \text{ is 1-good and } \sigma(\mathbb{B}_K^\dagger(i)) = +1, \\ -1, & \mathbb{B}_K(i) \text{ is 1-good and } \sigma(\mathbb{B}_K^\dagger(i)) = -1, \\ 0, & \text{otherwise.} \end{cases} \quad (5.2.1)$$

Let $\Gamma = \cup_{i \in I} \mathbb{B}_K(i)$ denote a subset of Λ composed of whole mesoscopic boxes. Let ∂I denote the set of points $i \in \mathbb{Z}^d$ at L_∞ -distance 2 from I such that $\mathbb{B}_K(i) \subset \mathbb{A}_\theta$. Let $\psi : I \cup \partial I \rightarrow \{\pm 1, 0\}$ denote a label configuration assigning labels to the boxes $\mathbb{B}_K(i)$. Let Int_ψ (respectively Ext_ψ) denote the event that $\Psi(i) = \psi(i)$ for $i \in I$ (respectively ∂I). Given ψ , let f_I^0 count the number of 0 labels in I . Let $f_{\partial I}^+, f_{\partial I}^-, f_{\partial I}^0$ count the number of boxes in ∂I with phase labels $+1$, -1 and 0. Let f_I^{\leftrightarrow} count the maximal number of disjoint paths (the maximal flow) inside I between the $+1$ labels and -1 labels.

We can find constants $C_{\text{cut}}, c_{\text{cut}} > 0$ such that the following hold.

Lemma 5.2.2. *With \mathbb{Q}_θ -probability $1 - \exp(-c_{\text{cut}} f_I^{\leftrightarrow} K^{d-1})$,*

$$\varphi_\Lambda^{J, \zeta, h}(\text{Int}_\psi \cap \text{Ext}_\psi) \leq \exp(C_{\text{cut}}(f_{\partial I}^0 + f_{\partial I}^-)K^{d-1} - c_{\text{cut}} f_I^{\leftrightarrow} K^{d-1}).$$

Lemma 5.2.3. *With \mathbb{Q}_θ -probability $1 - \exp(-c_{\text{cut}}[f_I^0 K + f_I^{\leftrightarrow} K^{d-1}])$,*

$$\varphi_\Lambda^{J, \zeta, h}(\text{Int}_\psi \cap \text{Ext}_\psi) \leq \exp(C_{\text{cut}} \min\{f_{\partial I}^0 + f_{\partial I}^-, f_{\partial I}^0 + f_{\partial I}^+\} K^{d-1} + \beta h |\Gamma| 4^d - c_{\text{cut}}[f_I^0 K + f_I^{\leftrightarrow} K^{d-1}]).$$

Proof of Lemma 5.2.2. Let Γ_k denote the union of the mesoscopic boxes at distance at most k from Γ ,

$$\Gamma_k = \bigcup \{ \mathbb{B}_K(j) : \exists i \in I, \|i - j\|_\infty \leq k \}.$$

Note that $\Gamma \subset \Gamma_1 \subset \Gamma_2$, and for $i \in \partial I$, $\mathbb{B}_K(i) \subset \Gamma_2 \setminus \Gamma_1$. Write $\omega = \omega_{\mathfrak{g}} \oplus \omega_{\text{int}} \oplus \omega_{\text{ext}}$ where $\omega_{\mathfrak{g}}$ is the configuration of the ghost edges and ω_{int} is the configuration of the real edges $E(\Gamma_1)$.

We would like to condition on the event Ext_ψ . However, the resulting measure is complicated because Ext_ψ carries information about not just ω_{ext} and

($\sigma(\mathbb{B}_K^\dagger(i)) : i \in \partial I$), but also about ω_{int} . Let Ext'_ψ denote the event that ω_{ext} , and the states of the vertices in $\Lambda \setminus \Gamma_1$, are compatible with Ext_ψ . Let Int'_ψ denote the event that ω_{int} is compatible with Int_ψ .

The coupled measure $\varphi_\Lambda^{J,\zeta,h}$ has a property related to the finite energy property of the regular random-cluster model. The probability of edge e being closed, conditional on all the other edge states *and* all the spin states, is bounded away from 0:

$$\varphi_\Lambda^{J,\zeta,h}(\omega(e) = 0 \mid \sigma \text{ and } (\omega(f))_{f \neq e}) \geq 1 - p_e. \quad (5.2.4)$$

This tells us the cost of conditioning on edges being closed. Surgically closing edges saves us from having to consider mixed boundary conditions.

Let T denote the event that all edges spanning between a box $\mathbb{B}_K(i)$ with $i \in \partial I$ and $\psi(i) \leq 0$ and a box $\mathbb{B}_K(j) \subset \Gamma_1$ are closed. By (5.2.4), for some constant $C_{\text{cut}} > 0$,

$$\varphi_\Lambda^{J,\zeta,h}(\text{Int}'_\psi \cap \text{Ext}'_\psi) \leq \varphi_\Lambda^{J,\zeta,h}(\text{Int}'_\psi \cap \text{Ext}'_\psi \cap T) \exp(C_{\text{cut}}(f_{\partial I}^0 + f_{\partial I}^-)K^{d-1}). \quad (5.2.5)$$

Suppose that ω_{ext} splits $\Lambda \setminus \Gamma_1$ into ω_{ext} -clusters $W_+, W_-, W_1, \dots, W_n$. We stress that ω_{ext} is a partial edge configuration: the ω_{ext} -clusters in $\Lambda \setminus \Gamma_1$ may be connected by an $(\omega_{\text{g}} \oplus \omega_{\text{int}})$ -path. Assume that the event $\text{Ext}'_\psi \cap T$ holds. If $\mathbb{B}_K^\dagger(i)$, $i \in \partial I$, intersects Γ_1 then $\psi(i) = 1$. Without loss of generality we can assume that

- (i) W_1, \dots, W_l correspond to $\mathbb{B}_K^\dagger(i)$ with $i \in \partial I$ and $\psi(i) = +1$,
- (ii) W_{l+1}, \dots, W_m correspond to clusters with diameter less than $K/2$,
- (iii) W_{m+1}, \dots, W_n (and W_-) are not connected to Γ_1 .

Consider the conditional measure $\varphi_\Lambda^{J,\zeta,h}(\cdot \mid \text{Ext}'_\psi \cap T, \omega_{\text{ext}})$. The clusters W_+ and W_1, \dots, W_l act as plus boundary conditions. The spins of the clusters W_{l+1}, \dots, W_m are unknown so the clusters act as wired boundary conditions. Let $\hat{\phi}^h$ denote the marginal measure on ω_{int} corresponding to $\varphi_\Lambda^{J,\zeta,h}(\cdot \mid \text{Ext}'_\psi \cap T, \omega_{\text{ext}})$,

$$\varphi_\Lambda^{J,\zeta,h}(\text{Int}'_\psi \cap \text{Ext}'_\psi \cap T) \leq \sup_{\omega_{\text{ext}} \in \text{Ext}'_\psi \cap T} \hat{\phi}^h(\text{Int}'_\psi). \quad (5.2.6)$$

Let A denote the decreasing event that there are no ω_{int} -open paths in Γ_1 between \mathbb{B}_K -boxes with $\psi(i) = +1$ and $\psi(i) = -1$; note that $\text{Int}'_\psi \subset A$. By monotonicity, as A is a decreasing event and $\hat{\phi}^h$ is increasing with $h \in [0, \infty)$,

$$\hat{\phi}^h(\text{Int}'_\psi) \leq \hat{\phi}^h(A) \leq \hat{\phi}^0(A). \quad (5.2.7)$$

With reference to Corollary 2.4.3, we can find a collection of f_I^{\leftrightarrow} disjoint chains of boxes such that A implies that for each chain, the first box is not connected to the last box. With \mathbb{Q}_θ -probability $1 - \exp(-c'_{\text{cg}} f_I^{\leftrightarrow} K^{d-1})$,

$$\hat{\phi}^0(A) \leq \exp(-c'_{\text{cg}} f_I^{\leftrightarrow} K^{d-1}). \quad (5.2.8)$$

Collecting (5.2.5)-(5.2.8) gives the lemma. \square

Proof of Lemma 5.2.3. We will first consider the case $f_{\partial I}^+ \geq f_{\partial I}^-$. Let Γ_1 , Int'_ψ , Ext'_ψ , T , W_1, \dots, W_n and $\hat{\phi}^h$ be defined as above.

Recall inequality (5.2.6). With reference to (2.2.1), the sizes of the W_{l+1}, \dots, W_m affect the interaction (under $\hat{\phi}^h$) of open clusters in Γ_1 with the external magnetic field. Recall that the clusters W_{l+1}, \dots, W_m have diameter at most $K/2$:

$$\hat{\phi}^h(\text{Int}'_\psi) / \hat{\phi}^0(\text{Int}'_\psi) \leq \exp(\beta h |\Gamma_1 \cup W_{l+1} \cup \dots \cup W_m|) \leq \exp(\beta h |\Gamma| 4^d). \quad (5.2.9)$$

By Proposition 2.4.2 and Corollary 2.4.3, there is a positive constant $c > 0$ such that with \mathbb{Q}_θ -probability $1 - \exp(-c \max\{f_I^0 K, f_I^{\leftrightarrow} K^{d-1}\})$,

$$\hat{\phi}^0(\text{Int}'_\psi) \leq \exp(-c \max\{f_I^0 K, f_I^{\leftrightarrow} K^{d-1}\}). \quad (5.2.10)$$

Collecting (5.2.5)-(5.2.6) and (5.2.9)-(5.2.10) gives the lemma in the case $f_{\partial I}^+ \geq f_{\partial I}^-$. The proof in the case $f_{\partial I}^- > f_{\partial I}^+$ follows by swapping + and - in the definition of T and $\hat{\phi}^h$. \square

5.3 Hausdorff stability of random-cluster boundaries

Consider the context of Proposition 5.1.1: $E^\theta(b_1) > E^\theta(b_2)$. Under $\mu_\Lambda^{J, (+, -), h}$ the plus phase is dominant so the minus boundary does not affect the bulk of the domain. In this section, we will show that, in a random-cluster sense, the $\partial^- \Lambda$ boundary-cluster is small.

Proposition 5.3.1. *Let $\varepsilon_{\text{hs}} > 0$. Suppose that $E^\theta(b_1) > E^\theta(b_2(1 - \varepsilon_{\text{hs}}))$. There is a constant $c_{\text{hs}} = c_{\text{hs}}(\varepsilon_{\text{hs}}) > 0$, independent of B_{max}^θ , such that with uniformly high \mathbb{Q}_θ -probability*

$$\phi_\Lambda^{J, (+, -), h}(\partial^- \Lambda \leftrightarrow \mathbb{W}_\theta(b_1, b_2(1 - \varepsilon_{\text{hs}}))) \leq \exp(-c_{\text{hs}} b_2 / h).$$

The proof develops the technique of truncation used in [5]. The differences in geometry, and the presence of a magnetic field, pose extra challenges.

Proof of Proposition 5.3.1. Let w denote the minimum L_∞ -distance between $\partial \mathcal{W}_\theta(1)$ and the origin,

$$w = \inf_{\mathbf{x} \in \partial \mathcal{W}_\theta(1)} \|\mathbf{x}\|_\infty. \quad (5.3.2)$$

Let $R = \lfloor \varepsilon_{\text{hs}} b_2 N w / (8K) \rfloor$ and let $S = \lfloor (b_2 - b_1) N w / (2K) \rfloor$. Define mesoscopic layers

$$\mathbb{H}_l = \mathbb{W}_\theta \left(b_2 - \frac{2lK}{wN}, b_2 - \frac{2(l-1)K}{wN} \right), \quad l = 1, \dots, S.$$

The layers divide the annulus $\mathbb{W}_\theta(b_1, b_2)$ into S layers. Let $\mathbb{H}_{j,k} = \cup_{l=j}^k \mathbb{H}_l$. The annulus $\mathbb{W}_\theta(b_2(1 - \varepsilon_{\text{hs}}), b_2)$ corresponds to the first $4R$ layers, $\mathbb{H}_{1,4R}$, with \mathbb{H}_1 the outermost layer. The layers are essentially $d - 1$ dimensional, resembling scalar multiples of $\partial \mathcal{W}_\theta$; Figure 2 shows the case $d = 2$. The layers have been constructed so that:

- (i) The mesoscopic boxes in \mathbb{H}_l form a surface separating \mathbb{H}_{l-1} from \mathbb{H}_{l+1} .

(ii) Each \mathbb{H}_l is between 2 and $2d$ mesoscopic boxes thick.

Define mesoscopic phase labels $\Psi = (\Psi(i) : \mathbb{B}_K(i) \subset \mathbb{H}_{1,S})$ according to (5.2.1). We will show that with high probability, a surface of $+1$ boxes separates \mathbb{H}_1 from \mathbb{H}_{4R} .

Given a phase label $\psi : \{i : \mathbb{B}_K(i) \subset \mathbb{H}_{1,S}\} \rightarrow \{\pm 1, 0\}$, define the profile \vec{f} of ψ as follows. For $s \in \{-1, 0, 1\}$, let f_l^s count the number of s -boxes in \mathbb{H}_l ,

$$f_l^s = \#\{i : \mathbb{B}_K(i) \subset \mathbb{H}_l \text{ and } \psi(i) = s\}.$$

Let $\vec{f} = (f_l^s)_{l=1, \dots, S}^{s=-1, 0, 1}$. We will write $\mathcal{F}(\vec{f})$ to denote the set of phase labels ψ compatible with \vec{f} . The number of configurations of the phase labels ψ compatible with \vec{f} is limited by the definition of the coarse graining. Surfaces of 0-boxes must separate the plus-boxes from the minus-boxes. The surfaces of 0-boxes cannot separate \mathbb{H}_l into more than $f_l^0 + 1$ connected components. Therefore the number of ways of assigning the labels in layer l is bounded by

$$\binom{|\mathbb{H}_l|K^{-d}}{f_l^0} \exp([f_l^0 + 1] \log 2) = \exp(f_l^0 O(\log N)). \quad (5.3.3)$$

We will say that a profile \vec{f} is *spanning* if $f_l^{-1} + f_l^0 > 0$ for $l = 1, \dots, 4R$. Recall from (3.1.1) that $K \approx N^{1/(2d)}$. The number of spanning profiles is less than

$$\prod_{l=1}^S \left(\frac{|\mathbb{H}_l|}{K^d} \right)^3 = O\left((N/K)^{d-1}\right)^{O(N/K)}$$

which grows more slowly than $\exp(cN)$ for every positive constant c . It is therefore sufficient to find a constant $c_{\text{hs}} = c_{\text{hs}}(\varepsilon_{\text{hs}}) > 0$ and a \mathbb{Q}_θ -event \mathcal{J} such that

$$\mathcal{J} \subset \left\{ J : \forall \vec{f} \text{ spanning, } \varphi_\Lambda^{J, (+, -), h}(\Psi \in \mathcal{F}(\vec{f})) \leq \exp(-2c_{\text{hs}} b_2 N) \right\} \quad (5.3.4)$$

and \mathcal{J} has uniformly high \mathbb{Q}_θ -probability. The event \mathcal{J} is defined as follows. Let $\mathcal{J} = \emptyset$ for $h \geq h_0$ (for some $h_0 > 0$) so we can assume that h is arbitrarily small. We will appeal below to Proposition 5.1.1 and Lemmas 2.4.4, 5.2.2 and 5.2.3. For $h < h_0$, let \mathcal{J} denote the intersection of the associated \mathbb{Q}_θ -events.

Let $\alpha, \hat{\alpha} > 0$. Consider three constraints on the label profile:

$$\sum_{l=1}^S f_l^0 \leq N^{d-1}/\sqrt{K}, \quad (5.3.5)$$

$$\sum_{l=1}^S f_l^{-1} \leq \alpha \left(\frac{b_2 N}{K} \right)^d, \quad (5.3.6)$$

$$\max_{l=R, \dots, 3R} f_l^{-1} \leq \hat{\alpha} \left(\frac{b_2 N}{K} \right)^{d-1}. \quad (5.3.7)$$

For $J \in \mathcal{J}$ we will check the inequality in (5.3.4) in four parts. We will show:

- (I) For any $c_{\text{hs}} > 0$, if (5.3.5) fails then the inequality in (5.3.4) holds.
- (II) For any $\alpha > 0$, if (5.3.5) holds but (5.3.6) fails, then the inequality in (5.3.4) holds if c_{hs} is sufficiently small.

- (III) For any $\hat{\alpha} > 0$, if (5.3.5) and (5.3.6) holds but (5.3.7) fails, then the inequality in (5.3.4) holds provided that α and c_{hs} are sufficiently small.
- (IV) If (5.3.5)-(5.3.7) hold, then the inequality in (5.3.4) holds if $\alpha, \hat{\alpha}$ and c_{hs} are sufficiently small.

For part (I), choose \vec{f} such that (5.3.5) fails. Inequality (5.3.5) is an upper bound on the volume of bad boxes. We will apply Lemma 2.4.4 for $\psi \in \mathcal{F}(\vec{f})$ with $\varepsilon_{\text{cg}} = 1$ and $n = \sum_{l=1}^S f_l^0 > N^{d-1}/\sqrt{K}$. The positive terms in the exponential in Lemma 2.4.4 are

$$\beta|\partial^\pm \Lambda| + \beta h|\Lambda| = O(b_2^d N^{d-1}).$$

The absolute value of the negative term is greater than $c_{\text{cg}}K(N^{d-1}/\sqrt{K})$ which is a higher order of N (3.1.1). For h sufficiently small,

$$\varphi_\Lambda^{J, (+, -), h}(\Psi = \psi) \leq \exp(-c_{\text{cg}}nK/2),$$

and so by (5.3.3),

$$\varphi_\Lambda^{J, (+, -), h}(\Psi \in \mathcal{F}(\vec{f})) \leq \exp(nO(\log N) - c_{\text{cg}}nK/2).$$

Whatever the value of the constant $c_{\text{hs}} > 0$, for h sufficiently small the right-hand side above is less than $\exp(-2c_{\text{hs}}b_2N)$.

Now to part (II). Inequality (5.3.6) is an upper bound on the volume of minus phase. We can expect the majority of boxes to have label $+1$ because of Proposition 5.1.1.

Recall the symbol ε_{cg} used in the definition of the coarse graining. Let ψ denote a label configuration. Let n^+ count the number of phase labels $\psi(i) = +1$ such that $\mathbb{B}_K(i)$ is also ε_{cg} -good. Similarly for n^- . Let n^0 count the number of ε_{cg} -bad boxes in $\mathbb{H}_{1,S}$. Note that

$$\sum_{l=1}^S f_l^{-1} \leq n^0 + n^-.$$

Let $M(\varepsilon_{\text{stb}})$ refer to the $\mu_\Lambda^{J, (+, -), h}$ -event in Proposition 5.1.1. Under $M(\varepsilon_{\text{stb}})$,

$$\frac{1}{N^d m^*} \sum_{x \in \Lambda} \sigma(x) \geq b_2^d - b_1^d - 2\varepsilon_{\text{stb}}b_2^d + O(K^{-1}). \quad (5.3.8)$$

For each box $\mathbb{B}_K(i)$ counted by n^- there are at least $(1 - \varepsilon_{\text{cg}})K^d m^*$ vertices in $\mathbb{B}_K^\dagger(i) \cap \mathbb{B}_K(i)$. For each box $\mathbb{B}_K(i)$ counted by n^+ there are at most $(1 + \varepsilon_{\text{cg}})K^d m^*$ vertices in $\mathbb{B}_K^\dagger(i) \cap \mathbb{B}_K(i)$. There are at most $n^0 K^d$ vertices in ε_{cg} -bad boxes. The remaining vertices lie in clusters with diameter less than $K/2$. Small clusters only interact weakly with the magnetic field. If an open cluster has volume of less than $(K/2)^d$ then the odds of it taking plus spin are at most $\exp(\beta h(K/2)^d)$ to 1 (2.2.1). Conditional on $\Psi = \psi$, with probability $1 - \exp(-b_2N)$,

$$\frac{1}{K^d m^*} \sum_{x \in \Lambda} \sigma(x) \leq n^+(1 + \varepsilon_{\text{cg}}) + \frac{n^0}{m^*} - n^-(1 - \varepsilon_{\text{cg}}) + \left(\frac{N}{K}\right)^d O(K^{-1}). \quad (5.3.9)$$

Lemma 5.3.12. *Suppose that $0 < a_1 < a_2$ and $U \subset \mathcal{A}_\theta$. Let $A_i = U \cap \partial\mathcal{W}_\theta(a_i)$, $i = 1, 2$. Let S denote the portion of ∂U contained in the closure of $\mathcal{W}_\theta(a_1, a_2)$. Then*

$$\mathcal{H}^{d-1}(S) \geq d^{-1/2}[\mathcal{H}^{d-1}(A_1) - \mathcal{H}^{d-1}(A_2)].$$

Proof. Let P denote the linear projection $x \rightarrow (a_1/a_2)x$. S must separate $A_1 \setminus PA_2$ from $(P^{-1}A_1) \setminus A_2$. The surface area of $A_1 \setminus PA_2$ is at least $\mathcal{H}^{d-1}(A_1) - \mathcal{H}^{d-1}(A_2)$ as P is a contraction. Let $\mathbf{x} \in \partial\mathcal{W}_\theta$. Recall that $\mathcal{W}_\theta = \mathcal{A}_\theta \cap \mathcal{W}_{2\pi}$; by the symmetry of $\mathcal{W}_{2\pi}$, the angle between the vector \mathbf{x} and the normal vector $\mathbf{n}^{\mathcal{W}_\theta}(\mathbf{x})$ is at most $\cos^{-1}(d^{-1/2})$. \square

Lemma 5.3.12 implies that there is a constant $c \in (0, 1)$ such that

$$f_I^{\leftrightarrow} \geq cf_m^{-1} - c^{-1}(f_l^{-1} + f_l^0).$$

Given $\hat{\alpha}$, if α is sufficiently small then we can choose $k < l < m < n$ such that

$$\begin{aligned} f_k^0 + f_k^{-1} &\leq (b_2N/K)^{d-1} \times \hat{\alpha}c_{\text{cut}}c/(6C_{\text{cut}}), \\ f_l^0 + f_l^{-1} &\leq (b_2N/K)^{d-1} \times \hat{\alpha}c^2/6, \\ f_m^{-1} &\geq (b_2N/K)^{d-1} \times \hat{\alpha}, \\ f_n^0 + f_n^{-1} &\leq (b_2N/K)^{d-1} \times \hat{\alpha}c_{\text{cut}}c/(6C_{\text{cut}}). \end{aligned} \tag{5.3.13}$$

Lemma 5.2.2 gives

$$\begin{aligned} \varphi_\Lambda^{J, (+, -), h}(\text{Int}_\psi \cap \text{Ext}_\psi) &\leq \exp(C_{\text{cut}}(f_k^0 + f_k^{-1} + f_n^0 + f_n^{-1})K^{d-1} - c_{\text{cut}}f_I^{\leftrightarrow}K^{d-1}) \\ &\leq \exp(-\hat{\alpha}c_{\text{cut}}c(b_2N)^{d-1}/2). \end{aligned} \tag{5.3.14}$$

We complete part (III) by checking that the right hand side of (5.3.14), when multiplied by the size of the set $\mathcal{F}(\vec{f})$ [cf. (5.3.3) and (5.3.5)] is less than $\exp(-2c_{\text{hs}}b_2N)$.

Before we start part (IV) of the proof of (5.3.4), we will give an isoperimetric inequality for $\partial\mathcal{W}_\theta$. The surface $\partial\mathcal{W}_\theta$ is $d - 1$ dimensional, so subsets of $\partial\mathcal{W}_\theta$ have $d - 2$ dimensional boundaries.

Lemma 5.3.15. *There is a positive constant $u = u(\theta)$ such that for any $A \subset \partial\mathcal{W}_\theta$,*

$$\frac{\mathcal{L}^{d-1}[A]}{\mathcal{L}^{d-1}[\partial\mathcal{W}_\theta]} \leq \frac{1}{2} \implies \mathcal{H}^{d-2}[\partial A] \geq u (\mathcal{L}^{d-1}[A])^{(d-2)/(d-1)}.$$

Proof. First consider the case $\theta = 2\pi$. Let w denote the minimum L_∞ -distance between $\partial\mathcal{W}_\theta$ and the origin (5.3.2). By convexity, $\partial\mathcal{W}_\theta$ lies inside the L_2 -annulus with inner radius w and outer radius wd .

Consider the projection P of $\partial\mathcal{W}_\theta$ onto the unit sphere \mathcal{S}^{d-1} . Associated with P are two Radon-Nikodym derivatives, one for the $d - 1$ dimensional Lebesgue measures on the domain and codomain, and one for the $d - 2$ dimensional Hausdorff measures on the domain and codomain. By symmetry and convexity (see [14, Theorem 2.2.4]) both Radon-Nikodym derivatives are bounded away from 0 and ∞ . The result follows from Lévy's isoperimetric inequality for the unit sphere.

A similar argument works when $0 < \theta < \pi$. Consider a projection from \mathcal{W}_θ to the $(d - 1)$ -dimensional unit ball. \square

Now for part (IV) of the proof of (5.3.4). Let \vec{f} denote a spanning profile and let $\psi \in \mathcal{F}(\vec{f})$. Choose k and n such that $R \leq k \leq 2R \leq n \leq 3R$. We will apply Lemma 5.2.3 with Γ defined according to (5.3.11).

If (5.3.7) holds with $\hat{\alpha}$ sufficiently small then Lemma 5.3.15 implies that for some constant $c > 0$,

$$f_l^{\leftrightarrow} \geq c \sum_{l=k+1}^{n-1} (f_l^{-1})^{(d-2)/(d-1)}.$$

Let

$$g_l := c_{\text{cut}} f_l^0 K + c_{\text{cut}} c (f_l^{-1})^{(d-2)/(d-1)} K^{d-1}.$$

Lemma 5.2.3 gives

$$\begin{aligned} & \varphi_{\Lambda}^{J(+,-),h}(\text{Int}_{\psi} \cap \text{Ext}_{\psi}) \\ & \leq \exp \left(C_{\text{cut}} (f_k^0 + f_k^{-1} + f_n^0 + f_n^{-1}) K^{d-1} + \beta h |\Gamma| 4^d - \sum_{l=k+1}^{n-1} g_l \right). \end{aligned} \quad (5.3.16)$$

The term corresponding to the magnetic field in (5.3.16) is bounded by (5.3.7). If $\hat{\alpha}$ is sufficiently small,

$$\beta h |\Gamma| 4^d \leq \beta h (12K)^d \sum_{l=k+1}^{n-1} f_l^0 + f_l^{-1} \leq \frac{1}{4} \sum_{l=k+1}^{n-1} g_l.$$

With reference to (5.3.3) and the assumption that h is small, the number of ways of choosing $(\psi(i) : \mathbb{B}_K(i) \subset \mathbb{H}_{k,n})$ is at most

$$\sum_{l=k}^n \exp(f_l^0 \text{O}(\log N)) \leq \exp \left((f_k^0 + f_n^0) \text{O}(\log N) + \frac{1}{4} \sum_{l=k+1}^{n-1} g_l \right).$$

Thus if h is sufficiently small,

$$\varphi_{\Lambda}^{J(+,-),h}(\Psi \in \mathcal{F}(\vec{f})) \leq \exp \left(2C_{\text{cut}} (f_k^0 + f_k^{-1} + f_n^0 + f_n^{-1}) K^{d-1} - \frac{1}{2} \sum_{l=k+1}^{n-1} g_l \right). \quad (5.3.17)$$

In our notation, [5, (4.40)] states that if \vec{f} satisfies (5.3.5)-(5.3.6) with α sufficiently small then there is a positive constant c such that

$$\begin{aligned} & \min_{R < k < 2R} \left\{ 2C_{\text{cut}} (f_k^0 + f_k^{-1}) K^{d-1} - \frac{1}{2} \sum_{l=k+1}^{2R} g_l \right\} \leq -cb_2 N, \text{ and} \\ & \min_{2R < n < 3R} \left\{ 2C_{\text{cut}} (f_n^0 + f_n^{-1}) K^{d-1} - \frac{1}{2} \sum_{l=2R+1}^{n-1} g_l \right\} \leq -cb_2 N. \end{aligned} \quad (5.3.18)$$

Part (IV) of the proof of inequality (5.3.4) follows from (5.3.17) and (5.3.18). This completes the proof of Proposition 5.3.1. \square

We will give three analogous results below. They follow, mutatis mutandis, from the proof of Proposition 5.3.1. Consider first the context of Proposition 5.1.10.

Proposition 5.3.19. *Recall the event \mathcal{C} defined by (5.1.9). Suppose that $b_1 = 0$ and $b_2(1 - \varepsilon_{\text{hs}}) > B_c^\theta + \varepsilon_{\mathcal{C}}$. Given $\varepsilon_{\mathcal{C}}$ and ε_{hs} , with uniformly high \mathbb{Q}_θ -probability*

$$\varphi_\Lambda^{J,-,h}(\partial^- \Lambda \leftrightarrow \mathbb{W}_\theta(b_1, b_2(1 - \varepsilon_{\text{hs}})) \mid \mathcal{C}) \leq \exp(-c_{\text{hs}} b_2/h).$$

Proof. In part (II) of the proof of Proposition 5.3.1 replace Proposition 5.1.1 with Proposition 5.1.10. \square

In the context of Proposition 5.1.11, the plus phase is dominant and so the minus boundary does not affect the bulk of the domain.

Proposition 5.3.20. *There is a constant $c'_{\text{hs}} = c'_{\text{hs}}(\varepsilon_{\text{hs}}, B_{\text{max}}^\theta) > 0$ such that with uniformly high \mathbb{Q}_θ -probability*

$$\phi_\Lambda^{J,(-,+),h}(\partial^- \Lambda \leftrightarrow \mathbb{W}_\theta(b_1 + \varepsilon_{\text{hs}}, b_2)) \leq \exp(-c'_{\text{hs}}/h).$$

Proof. Proposition 5.3.20 differs from Proposition 5.3.1 in that it shows that the inner (rather than the outer) boundary condition has limited influence. Let

$$\begin{aligned} \mathbb{H}_l &= \mathbb{W}_\theta \left(b_1 + \frac{2(4R - l)K}{wN}, b_1 + \frac{2(4R + 1 - l)K}{wN} \right), \quad l = 1, \dots, 4R, \\ \mathbb{H}_l &= \mathbb{W}_\theta \left(b_1 + \frac{2(l - 1)K}{wN}, b_1 + \frac{2lK}{wN} \right), \quad l = 4R + 1, \dots, S, \end{aligned}$$

with $R = \lfloor \varepsilon_{\text{hs}} N w / (8K) \rfloor$ and $S = \lfloor (b_2 - b_1) N w / (2K) \rfloor$. The proof of Proposition 5.3.1, mutandis mutandis, shows that a surface of +1 boxes separates \mathbb{H}_1 from \mathbb{H}_{4R} . In (5.3.6)-(5.3.7), replace $b_2 N / K$ with N / K . Proposition 5.1.11 replaces Proposition 5.1.1 in part (II) of the proof.

Notice that the proof of part (III) has become slightly more flexible; the additional flexibility will be important below in the proof of Proposition 5.3.21. The quantity c'_{hs} is allowed to depend on B_{max}^θ . This means that Lemma 5.2.3 can be used in place of Lemma 5.2.2; the extra term due to the magnetic field can be controlled by taking α , and therefore $|\Gamma|$, sufficiently small. \square

Consider the context of Proposition 5.1.8. The minus phase is dominant so the plus boundary does not affect the bulk of the domain.

Proposition 5.3.21. *Suppose that $E^\theta(b_1 + \varepsilon_{\text{hs}}) < E^\theta(b_2)$. There is a constant $c''_{\text{hs}} = c''_{\text{hs}}(\varepsilon_{\text{hs}}) > 0$ such that with uniformly high \mathbb{Q}_θ -probability*

$$\phi_\Lambda^{J,(+,-),h}(\partial^+ \Lambda \leftrightarrow \mathbb{W}_\theta(b_1 + \varepsilon_{\text{hs}}, b_2)) \leq \exp(-c''_{\text{hs}}/h).$$

Proof. The proof can be obtained from the proof of Proposition 5.3.20 by swapping the roles of plus and minus. The sign of the magnetic field has to stay the same, but, for example, in (5.3.6) replace f_l^{-1} with f_l^{+1} , etc. Proposition 5.1.8 replaces Proposition 5.1.11 in part (II) of the proof. \square

5.4 Hausdorff stability implies spatial mixing

In Proposition 5.3.21 we showed that under $\mathbb{Q}_\theta[\mu_\Lambda^{J,(+,-),h}]$, with high probability, a surface of -1 mesoscopic blocks separates the region $\mathbb{W}'_\theta(b_1 + \varepsilon_{\text{hs}}, b_2)$ from the plus boundary $\partial^+\Lambda$. In two dimensions, by planar duality, this implies that there are no Ising spin-clusters connecting $\partial^+\Lambda$ to $\mathbb{W}'_\theta(b_1 + \varepsilon_{\text{hs}}, b_2)$. By the Ising model's domain Markov property, and monotonicity, we can compare $\mathbb{Q}_\theta[\mu_\Lambda^{J,(+,-),h}]$ to $\mathbb{Q}_\theta[\mu_\Lambda^{J,(-,-),h}]$ in $\mathbb{W}'_\theta(b_1 + \varepsilon_{\text{hs}}, b_2)$.

In contrast in higher dimensions, especially when close to the critical temperature, the Ising spin-cluster associated with $\partial^-\Lambda$ under $\mu_\Lambda^{J,(+,-),h}$ may be much larger than the cluster associated with $\partial^-\Lambda$ under the random-cluster representation $\phi_\Lambda^{J,(+,-),h}$. We cannot make such a comparison. Instead we will appeal to a spatial mixing property, stated below as Proposition 5.4.1.

Much is known about the spatial mixing properties of the Ising model in the absence of a magnetic field. The difficulty here is that the magnetic field is acting to weaken the dominant phase.

We conjecture that Proposition 5.4.1, and the coarse graining property, holds for all $\beta > \beta_c$. If that is the case then Theorem 1.2.3 holds up to the critical point. For simplicity we will reuse the constants c_{hs} , c'_{hs} and c''_{hs} , adjusting their values if necessary.

Proposition 5.4.1. *There is a finite β_0 such that if $\beta > \beta_0$ then for $\varepsilon_{\text{hs}} > 0$, with uniformly high \mathbb{Q}_θ -probability:*

(i) For $x \in \mathbb{W}_\theta(b_1, b_2(1 - 2\varepsilon_{\text{hs}}))$,

$$\begin{aligned} & \varphi_\Lambda^{J,(+,-),h}(\sigma(x) = 1 \mid \partial^-\Lambda \leftrightarrow \mathbb{W}_\theta(b_1, b_2(1 - \varepsilon_{\text{hs}}))) \\ & \geq \varphi_\Lambda^{J,(+,+),h}(\sigma(x) = 1) - \exp(-c_{\text{hs}}b_2/h). \end{aligned}$$

(ii) For $x \in \mathbb{W}'_\theta(b_1 + 2\varepsilon_{\text{hs}}, b_2)$,

$$\begin{aligned} & \varphi_\Lambda^{J,(-,+),h}(\sigma(x) = 1 \mid \partial^-\Lambda \leftrightarrow \mathbb{W}_\theta(b_1 + \varepsilon_{\text{hs}}, b_2)) \\ & \geq \varphi_\Lambda^{J,(+,+),h}(\sigma(x) = 1) - \exp(-c'_{\text{hs}}/h). \end{aligned}$$

(iii) If $\mathbf{E}^\theta(b_1 + 2\varepsilon_{\text{hs}}) < \mathbf{E}^\theta(b_2)$, then for $x \in \mathbb{W}'_\theta(b_1 + 2\varepsilon_{\text{hs}}, b_2)$,

$$\begin{aligned} & \varphi_\Lambda^{J,(+,-),h}(\sigma(x) = 1 \mid \partial^+\Lambda \leftrightarrow \mathbb{W}_\theta(b_1 + \varepsilon_{\text{hs}}, b_2)) \\ & \leq \varphi_\Lambda^{J,(-,-),h}(\sigma(x) = 1) + \exp(-c''_{\text{hs}}/h). \end{aligned}$$

(iv) If $b_1 > B_c^\theta$, for $x \in \mathbb{W}'_\theta(b_1 + \varepsilon_{\text{hs}}b_2, b_2)$,

$$\begin{aligned} & \varphi_\Lambda^{J,(+,-),h}(\sigma(x) = 1) \\ & \leq \varphi_{\mathbb{W}'_\theta(b_2)}^{J,-,h} \left(\sigma(x) = 1 \mid \int_{\mathbb{W}_\theta(b_1)} \mathbb{M}_K^- d\mathcal{L}^d \geq (B_c^\theta)^d \right) + \exp(-c_{\text{hs}}b_2/h). \end{aligned}$$

The statement of Proposition 5.4.1 is fine tuned to suit our needs—we have only considered spatial mixing in Wulff-shaped regions. Also, the restriction in part (iii) is stricter than necessary.

We will prove Proposition 5.4.1 after first showing how it can be used with the results of Section 5.3. By part (i), in the context of Proposition 5.3.1 with uniformly high \mathbb{Q}_θ -probability, for $x \in \mathbb{W}_\theta(b_1, b_2(1 - 2\varepsilon_{\text{hs}}))$,

$$|\mu_\Lambda^{J, (+, -), h}(\sigma(x) = 1) - \mu_\Lambda^{J, +, h}(\sigma(x) = 1)| \leq 2 \exp(-c_{\text{hs}} b_2/h). \quad (5.4.2)$$

By part (ii), in the context of Proposition 5.3.20 with uniformly high \mathbb{Q}_θ -probability, for $x \in \mathbb{W}'_\theta(b_1 + 2\varepsilon_{\text{hs}}, b_2)$,

$$|\mu_\Lambda^{J, (-, +), h}(\sigma(x) = 1) - \mu_\Lambda^{J, +, h}(\sigma(x) = 1)| \leq 2 \exp(-c'_{\text{hs}}/h). \quad (5.4.3)$$

By part (iii), in the context of Proposition 5.3.21 with uniformly high \mathbb{Q}_θ -probability, for $x \in \mathbb{W}'_\theta(b_1 + 2\varepsilon_{\text{hs}}, b_2)$,

$$|\mu_\Lambda^{J, (+, -), h}(\sigma(x) = 1) - \mu_\Lambda^{J, -, h}(\sigma(x) = 1)| \leq 2 \exp(-c''_{\text{hs}}/h). \quad (5.4.4)$$

Suppose that $b_2 > b_1 > B_{\text{root}}^\theta$ and consider $x \in \mathbb{W}'_\theta(b_1 + \varepsilon_{\text{hs}} b_2, b_2)$. By part (iv), and by Proposition 5.1.1 applied to $\mathbb{W}_\theta(b_1)$ with $\varepsilon_{\text{stb}} = 1 - B_c^\theta/b_1$, with uniformly high \mathbb{Q}_θ -probability,

$$\begin{aligned} & |\mu_{\mathbb{W}'_\theta(b_1, b_2)}^{J, (+, -), h}(\sigma(x) = 1) - \mu_{\mathbb{W}'_\theta(0, b_2)}^{J, -, h}(\sigma(x) = 1)| \\ & \leq \exp(-c_{\text{stb}} b_1/h^{d-1}) + \exp(-c_{\text{hs}} b_2/h). \end{aligned} \quad (5.4.5)$$

Proposition 5.4.1 is also relevant to the conditioned measure defined in (5.1.9). With $b_1 = 0$, the total variation distance between $\hat{\mu}_\Lambda^{J, -, h}$ and $\mu_\Lambda^{J, -, h}(\cdot \mid \partial^- \Lambda \leftrightarrow \mathbb{W}_\theta(b_2(1 - \varepsilon_{\text{hs}})))$ is bounded by Proposition 5.1.12, monotonicity, and Proposition 5.3.19. Thus by part (i) of Proposition 5.4.1, for some constant $c > 0$, with uniformly high \mathbb{Q}_θ -probability for $x \in \mathbb{W}'_\theta(b_2(1 - 2\varepsilon_{\text{hs}}))$,

$$|\hat{\mu}_\Lambda^{J, -, h}(\sigma(x) = 1) - \mu_\Lambda^{J, +, h}(\sigma(x) = 1)| \leq \exp(-c/h). \quad (5.4.6)$$

If $b_2 > b_1 \geq B_c^\theta + 2\varepsilon_{\text{C}}$ then part (iv) can be used to compare $\mu_\Lambda^{J, (+, -), h}$ and $\hat{\mu}_{\mathbb{W}'_\theta(0, b_2)}^{J, -, h}$. For $x \in \mathbb{W}'_\theta(b_1 + \varepsilon_{\text{hs}} b_2, b_2)$, with uniformly high \mathbb{Q}_θ -probability for some positive constant $c > 0$,

$$|\mu_\Lambda^{J, (+, -), h}(\sigma(x) = 1) - \hat{\mu}_{\mathbb{W}'_\theta(0, b_2)}^{J, -, h}(\sigma(x) = 1)| \leq \exp(-c/h). \quad (5.4.7)$$

Proof of Proposition 5.4.1. We will prove part (i); the other parts are similar. By monotonicity it is sufficient to show that for $x \in \mathbb{W}_\theta(b_1, b_2(1 - 2\varepsilon_{\text{hs}}))$,

$$\mu_{\mathbb{W}_\theta(b_1, b_2(1 - \varepsilon_{\text{hs}}))}^{J, (+, \text{f}), h}(\sigma(x) = 1) \geq \mu_{\mathbb{W}_\theta(b_1, b_2(1 - \varepsilon_{\text{hs}}))}^{J, (+, +), h}(\sigma(x) = 1) - \exp(-c_{\text{hs}} b_2/h).$$

Let A denote the event that the inner- and outer-boundaries of $\mathbb{W}_\theta(b_2(1 - 2\varepsilon_{\text{hs}}), b_2(1 - \varepsilon_{\text{hs}}))$ are separated by a set of plus spins blocking all paths between the two. Of course, the set of plus spins only needs to block paths composed entirely of edges with $J(e) = 1$. By monotonicity

$$\mu_{\mathbb{W}_\theta(b_1, b_2(1 - \varepsilon_{\text{hs}}))}^{J, (+, \text{f}), h}(\sigma(x) = 1 \mid A) \geq \mu_{\mathbb{W}_\theta(b_1, b_2(1 - \varepsilon_{\text{hs}}))}^{J, (+, +), h}(\sigma(x) = 1)$$

so we need to show that

$$\mu_{\mathbb{W}_\theta(b_1, b_2(1 - \varepsilon_{\text{hs}}))}^{J, (+, \text{f}), h}(A) \geq 1 - \exp(-c_{\text{hs}} b_2/h).$$

We will do this using a stronger coarse-graining property.

Definition 5.4.8. Consider a box $\mathbb{B}_K(i) \subset \Lambda$. If

- (i) $\mathbb{B}_K(i)$ is ε_{cg} -good and
- (ii) the σ -spin clusters composed of vertices with spin $-\sigma(\mathbb{B}_K^\dagger(i))$ intersecting $\mathbb{B}_K(i)$ have diameter at most $K/2$

then say that $\mathbb{B}_K(i)$ is ε_{cg} -Ising-good.

With $p \in (p_c, 1)$ fixed, as $\beta \rightarrow \infty$ the annealed random-cluster measure $\mathbb{Q}[\phi^{J,h}]$ converges weakly to product measure with density p ; the density of edges with $J(e) = 1$ but $\omega(e) = 0$ goes to zero. Taking K_0 large, and then taking β large, we can make the $\mathbb{Q}[\varphi_\Lambda^{J,+0}]$ -probability that $\mathbb{B}_{K_0}(i) \subset \Lambda$ is ε_{cg} -Ising good arbitrarily close to 1. By a standard renormalization argument we can find β_0, K_1 and $c > 0$ such that if $\beta > \beta_0$ and $K > K_1$ then $\mathbb{B}_K(i) \subset \Lambda$ is ε_{cg} -Ising-good with $\mathbb{Q}[\varphi_\Lambda^{J,f,0}]$ -probability $1 - \exp(-cK)$.

The result now follows by adapting the proof of Proposition 5.3.1. Substitute ‘ ε_{cg} -Ising-good’ for ‘ ε_{cg} -good’ in the definition of the phase labels and, because of the free outer boundary conditions, use Proposition 5.1.12 in place of Proposition 5.1.1. If the profile \vec{f} associated with the phase label configuration Ψ is not spanning then the event A holds.

Parts (ii) and (iii) of Proposition 5.4.1 follow from the proofs of Propositions 5.3.20 and 5.3.21, respectively, by substituting ‘ ε_{cg} -Ising-good’ for ‘ ε_{cg} -good’. Part (iv) follows from the proof of Proposition 5.3.19; with high probability there is a surface of plus spins separating the inner- and outer-boundaries of $\mathbb{W}_\theta(b_1, b_1 + \varepsilon_{\text{hs}}b_2)$ under $\mu_\Lambda^{J,-,h}(\cdot \mid \int_{\mathbb{W}_\theta(b_1)} \mathbb{M}_K^- d\mathcal{L}^d \geq (B_c^\theta)^d)$. \square

5.5 Spectral gap of the dynamics

The Glauber dynamics for $\mu_\Lambda^{J,\zeta,h}$ can be studied by introducing a block dynamics. With $\varepsilon_{\text{block}} > 0$, let $n = \lfloor (b_2 - b_1)/\varepsilon_{\text{block}} \rfloor - 1$. Consider a sequence of overlapping annuli that cover Λ ,

$$\begin{aligned} \Delta_j &= \mathbb{W}_\theta(b_1 + (j-1)\varepsilon_{\text{block}}, b_1 + (j+1)\varepsilon_{\text{block}}), & j = 1, 2, \dots, n-1, \\ \Delta_n &= \mathbb{W}'_\theta(b_1 + (n-1)\varepsilon_{\text{block}}, b_2). \end{aligned}$$

Consider a block dynamics for $\mu_\Lambda^{J,\zeta,h}$ with blocks $\Delta_1, \dots, \Delta_n$; update each block Δ_j at rate 1, resampling the block conditional on the configuration restricted to $\Lambda \setminus \Delta_j$.

Lemma 5.5.1. For $\varepsilon > 0$, if $\varepsilon_{\text{block}}$ and h_0 are sufficiently small and $0 < h < h_0$,

$$\text{gap}(\Lambda, \zeta, h) \geq \exp(-\varepsilon/h^{d-1}) \text{gap}(\Lambda, \{\Delta_1, \dots, \Delta_n\}, \zeta, h).$$

Proof. \mathcal{W}_θ is a subset of $\mathcal{W}_{2\pi}$, so we can assume $\theta = 2\pi$ without loss of generality.

Let $y_1^j, \dots, y_{|\Delta_j|}^j$ denote an ordering of the vertices in Δ_j such that the angle between y_i^j and \mathbf{e}_1 is increasing with i . For each vertex y_i^j in block Δ_j consider the edge-boundary between $\{y_1^j, \dots, y_i^j\}$ and $\{y_{i+1}^j, \dots, y_{|\Delta_j|}^j\}$; let L denote the maximum (over $i = 1, \dots, |\Delta_j|$ and $j = 1, \dots, n$) size of the boundary. Given B_{max}^θ , $Lh^{d-1} = O(\varepsilon_{\text{block}})$ as $\varepsilon_{\text{block}} \rightarrow 0$.

As noted in [16], the proof of [12, Theorem 2.1] implies that for some $C, c > 0$,

$$\text{gap}(\Lambda, \zeta, h) \geq \frac{c \exp(-CL)}{|\Lambda|} \text{gap}(\Lambda, \{\Delta_1, \dots, \Delta_n\}, \zeta, h).$$

Choose $\varepsilon_{\text{block}}$ so that $CL < \varepsilon/h^{d-1}$. \square

We are now in a position to extend [16, Propositions 3.5.1–3.5.3] from the Ising model on \mathbb{Z}^2 to the dilute Ising model on \mathbb{Z}^d with $d \geq 2$.

Proposition 5.5.2. *Let $b_1 = 0$ and $\varepsilon > 0$. With uniformly high \mathbb{Q}_θ -probability,*

$$\text{gap}(\Lambda, +, h) \geq \exp(-\varepsilon/h^{d-1}).$$

Proof. Let $(\sigma_t)_{t \geq 0}$ denote a copy of the block dynamics Markov chain. The graphical construction can be extended to the block dynamics by coupling from the past: if block Δ_j is to be updated at time t , use a copy of the regular graphical construction in Δ_j over the time interval $(-\infty, 0]$ with boundary conditions σ_{t-} to produce the new configuration σ_t . By monotonicity, σ_t is an increasing function of the initial configuration σ_0 .

When t is sufficiently large, σ_t is independent of σ_0 ; σ_t then corresponds to a sample from the equilibrium distribution $\mu_\Lambda^{J,+,h}$. Let $(\sigma_t^{\text{eqm}})_{t \geq 0}$ denote a copy of the block dynamics Markov chain started in equilibrium.

We will show that with uniformly high \mathbb{Q}_θ -probability, the probability that $\sigma_1 = \sigma_1^{\text{eqm}}$ is bounded away from zero. This implies that the spectral gap of the block dynamics is bounded away from zero and so the result follows by Lemma 5.5.1.

Say that an update of block Δ_j at time t is *good* if the update maps all configurations that agree with σ_{t-}^{eqm} on $\Lambda \setminus \cup_{i=1}^j \Delta_i$ to configurations that agree with σ_t^{eqm} on the larger set $\Lambda \setminus \cup_{i=1}^{j-1} \Delta_i = \mathbb{W}'_\theta(b_1 + j\varepsilon_{\text{block}}, b_2)$. By monotonicity, if σ_{t-} agrees with $\sigma_{t-}^{\text{eqm}} \sim \mu_\Lambda^{J,+,h}$ on $\Lambda \setminus \cup_{i=1}^j \Delta_i$ then

$$\mu_{\mathbb{W}'_\theta(b_1+(j-1)\varepsilon_{\text{block}}, b_2)}^{J,(-,+,h)} \leq_{\text{st}} \sigma_t \leq_{\text{st}} \mu_{\mathbb{W}'_\theta(b_1+(j-1)\varepsilon_{\text{block}}, b_2)}^{J,(+,+,h)}. \quad (5.5.3)$$

Inequality (5.4.3) used with the sandwich (5.5.3) gives a lower bound on the probability that the update is good; σ_t and σ_t^{eqm} agree on $\mathbb{W}'_\theta(b_1 + j\varepsilon_{\text{block}}, b_2)$ with probability at least $1 - 2|\Lambda| \exp(-c'_{\text{hs}}/h)$.

With probability $\exp(-n)/n!$ there is an uninterrupted sequence of updates on $\Delta_n, \Delta_{n-1}, \dots, \Delta_1$ in the time interval $[0, 1]$. If all the updates are good, which occurs with probability at least $1 - 2n|\Lambda| \exp(-c'_{\text{hs}}N)$, then $\sigma_1 = \sigma_1^{\text{eqm}}$. Note that $n \leq B_{\text{max}}^\theta/\varepsilon_{\text{block}}$ so the probability of seeing such a sequence of updates is bounded away from zero uniformly over $b_2 \in [0, B_{\text{max}}^\theta]$. \square

Proposition 5.5.4. *Let $b_1 = 0$ and $\varepsilon > 0$. One can choose b_2 slightly larger than B_c^θ such that with high \mathbb{Q}_θ -probability,*

$$\text{gap}(\Lambda, -, h) \geq \exp(-\varepsilon/h^{d-1}).$$

Proof. Take $\varepsilon_{\text{block}}$ according to Lemma 5.5.1. Taking $n = \lfloor B_c^\theta/\varepsilon_{\text{block}} \rfloor - 1$, choose $b_2 \in (B_c^\theta, (n+2)\varepsilon_{\text{block}})$ such that $\mathbf{E}^\theta((n-1)\varepsilon_{\text{block}}) < \mathbf{E}^\theta(b_2)$. We can then follow the proof of Proposition 5.5.2. The boundary conditions in (5.5.3) should be changed to $(-, -)$ on the left-side and $(+, -)$ on the right-side. The probability of a block update being good is then bounded below using inequality (5.4.4) in place of (5.4.3) \square

Proposition 5.5.5. *Consider the case $b_1 > B_c^\theta$. Let $\varepsilon > 0$. With uniformly high \mathbb{Q}_θ -probability,*

$$\text{gap}(\Lambda, (+, -), h) \geq \exp(-\varepsilon/h^{d-1}).$$

Proof. Say that an update of block Δ_j at time t is *good* if the update maps all configurations that agree with σ_{t-}^{eqm} on $\mathbb{W}_\theta(b_1 + (j-1)\varepsilon_{\text{block}}) = \Lambda \setminus \cup_{i=j}^n \Delta_i$ to configurations that agree with σ_t^{eqm} on $\Lambda \setminus \cup_{i=j+1}^n \Delta_i$. The boundary conditions in (5.5.3) should be changed to $(+, -)$ on the left-side and $(+, +)$ on the right-side. Inequality (5.4.2) shows that updates are good with high probability. If there is an uninterrupted sequence of good updates in the order $\Delta_1, \dots, \Delta_n$ in the time interval $[0, 1]$ then $\sigma_1 = \sigma_1^{\text{eqm}}$. \square

6 Space-time cones and rescaling

In this section we will turn the heuristic description of plus-cluster nucleation from Section 1.3 into a proof of Theorem 1.2.3. We will apply the results in Section 5 with two values of θ .

We will take $\theta \in (0, \pi)$ to denote the argument of λ_2^θ in the statement of the theorem. We will consider regions with the shape $\mathbb{W}_\theta(b)$ for $b \in [B_{\min}^\theta, B_{\max}^\theta]$. The lower bound B_{\min}^θ will be chosen to maximize the rate of nucleation of plus clusters. We will take B_{\max}^θ to be the minimum value such that a translation of $\mathbb{W}_{2\pi}(1.01B_c^{2\pi})$ fits inside $\mathbb{W}_\theta(B_{\max}^\theta)$; see parts 1 and 2 of Figure 3. By Proposition 4.2.4, $\mathcal{W}_{2\pi}(B_c^{2\pi})$ has diameter of order 1 as $\beta \rightarrow \infty$ and $\theta \rightarrow 0$; $\mathcal{W}_\theta(B_{\max}^\theta)$ must have volume of order θ^{-1} . The Wulff shape $\mathcal{W}_\theta(b)$ has volume b^d so B_{\max}^θ must be of order $\theta^{-1/d}$. By (4.3.2) the probability of the event conditioned on in the definition of \mathbb{Q}_θ is

$$\exp(-C_{\text{dil}}\theta^{-1}/h^{d-1}) \quad \text{with} \quad C_{\text{dil}} = O\left(\log \frac{1}{1-p}\right). \quad (6.0.6)$$

This gives the density of nucleation sites in \mathbb{Z}^d . We will show that at these nucleation sites, droplets of plus phase form at the rate $\exp(-E_c^\theta/h^{d-1})$.

We must then show that the clusters of plus-phase can spread out from the sheltered nucleation sites. We do this by considering the full Wulff shape $\mathcal{W}_{2\pi}$. In areas of typical dilution, sufficiently large Wulff-shaped droplets of plus phase expand with high probability. With reference to (5.4.2) we will take $B_{\max}^{2\pi}$ large so that $\mu_{\mathbb{W}_{2\pi}(B_{\max}^{2\pi})}^{J, -, h}$ provides a good approximation to the equilibrium measure $\mu^{J, h}$ in a neighborhood of the origin.

6.1 The graphical construction in space-time regions

Before we give the proof of Theorem 1.2.3, we need to extend the Ising dynamics to allow the size of the graph to change with time. With $\Gamma_0, \Gamma_1, \dots, \Gamma_n \subset \mathbb{Z}^d$ and $t_0 < t_1 < \dots < t_{n+1}$, consider the space-time region

$$\Gamma = \text{ST}(\Gamma_0, \dots, \Gamma_n; t_0 < \dots < t_{n+1}) := \bigcup_{i=0}^n \Gamma_i \times [t_i, t_{i+1}]. \quad (6.1.1)$$

The graphical construction for the Ising model $\mu_\Lambda^{J, \zeta, h}$ described in Section 2.5 can be extended to Γ .

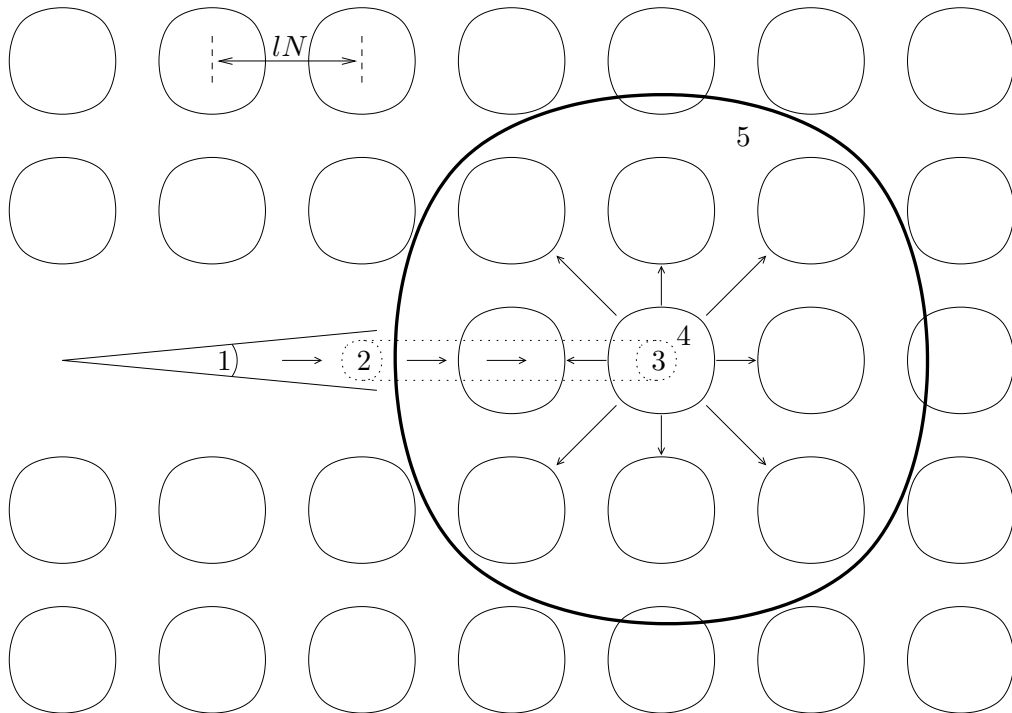


Figure 3: Illustration of a \mathbb{Q} -catalyst. The nucleation event occurs at rate $\exp(-\mathbb{E}_c^\theta/h^{d-1})$ and consists of the following steps. (1) A droplet of plus phase with the shape $\mathbb{W}_\theta(B_{\min}^\theta)$ forms in a region of high dilution that resembles $\mathbb{W}_\theta(B_{\max}^\theta)$ [Proposition 6.2.2]. (2) The droplet expands in the sheltered region to cover a copy of $\mathbb{W}_{2\pi}(1.01B_c^{2\pi})$ [Proposition 6.3.1]. (3) The droplet of plus phase spreads to the right [Proposition 6.4.2] and (4) expands to cover $\mathbb{W}_{2\pi}(B_{\min}^{2\pi})$ [Proposition 6.3.1 with θ taken to be 2π]. There is now a droplet of plus phase at the center of a \mathbb{Q} -conductive site of the rescaled lattice. The droplet expands (5) to cover $\mathbb{W}_{2\pi}(B_{\max}^{2\pi})$ [Proposition 6.5.1] which contains neighboring \mathbb{Q} -conductive sites.

- (i) Let s denote the start time.
- (ii) Let ξ denote an initial configuration compatible with boundary conditions ζ at time s , i.e. if $s \in [t_i, t_{i+1})$ then $\xi \in \Sigma_{\Gamma_i}^\zeta$.
- (iii) Let $\sigma_{\Gamma, \zeta, h; s}^{s, \xi} = \xi$.
- (iv) If a vertex x is added to the dynamics at time t_i (i.e. $x \in \Gamma_i \setminus \Gamma_{i-1}$) then the spin $\sigma_{\Gamma, \zeta, h; t_i}^{s, \xi}(x)$ is taken to be $\xi(x)$ to match the boundary conditions. The spin at x may then change with each arrival of the corresponding Poisson process.
- (v) If x is removed from the dynamics at time t_i (i.e. $x \in \Gamma_{i-1} \setminus \Gamma_i$) then the spin at x is immediately switched to $\xi(x)$ to conform to the boundary conditions.

The graphical construction of \mathbb{P}_J allows us to link together the Ising dynamics run in overlapping space-time regions. This can be used to chain together the different steps involved in the growth of a region of plus-phase.

Remark 6.1.2. *Consider two space-time regions such that the top layer of the first region covers the start of the second region:*

$$\begin{aligned}\Gamma &= \text{ST}(\Gamma_0, \Gamma_1, \dots, \Gamma_m; t_0 < t_1 < \dots < t_{m+1}), \\ \Delta &= \text{ST}(\Delta_0, \Delta_1, \dots, \Delta_n; u_0 < u_1 < \dots < u_{n+1}), \\ \Gamma_m &= \Delta_0 \text{ and } u_0 = t_m < t_{m+1} \leq u_1.\end{aligned}$$

If $\sigma_{\Gamma, -, h; t_{m+1}}^{t_0, \xi} = \sigma_{\Gamma, -, h; t_{m+1}}^{t_m, +}$ and $\sigma_{\Delta, -, h; u_{n+1}}^{u_0, +} = \sigma_{\Delta, -, h; u_{n+1}}^{u_n, +}$ then

$$\sigma_{\Gamma \cup \Delta, -, h; u_{n+1}}^{t_0, \xi} = \sigma_{\Delta, -, h; u_{n+1}}^{u_n, +}.$$

6.2 Droplet creation in a Summertop cone

Let $\theta \in (0, \pi)$ and $\delta > 0$. By Proposition 5.5.4 we can choose $B_{\min}^\theta \in (B_c^\theta, B_{\text{root}}^\theta)$ such that with high \mathbb{Q}_θ -probability,

$$\text{gap}(\mathbb{W}_\theta(B_{\min}^\theta), -, h) \geq \exp(-\delta/(2h^{d-1})). \quad (6.2.1)$$

Let $\Lambda = \mathbb{W}_\theta(B_{\min}^\theta)$. Heuristically, we expect critical droplets to form in Λ at rate $\exp(-E_c^\theta/h^{d-1})$. Let $\hat{\mu}_\Lambda^{J, -, h} = \mu_\Lambda^{J, -, h}(\cdot \mid \mathcal{C})$ denote the conditional measure defined by (5.1.9) with $\varepsilon_{\mathcal{C}} = (B_{\min}^\theta - B_c^\theta)/3$.

Proposition 6.2.2. *With high \mathbb{Q}_θ -probability we can construct a random variable $\hat{\sigma} \sim \hat{\mu}_\Lambda^{J, -, h}$ such that the event $\{\hat{\sigma} = \sigma_{\Lambda, -, h; \exp(\delta/h^{d-1})}^{0, -}\}$ has probability $\exp(-E_c^\theta/h^{d-1})$ and is independent of the value of $\hat{\sigma}$.*

Proof. Taking $a = 0$ in the last inequality in the proof of Proposition 5.1.10, we can assume that $\mu_\Lambda^{J, -, h}(\mathcal{C}) \geq 2 \exp(-E_c^\theta/h^{d-1})$. By (6.2.1) and a Markov chain mixing inequality (i.e. [15, (59)]) the total variation distance between $\sigma_{\Lambda, -, h; \exp(\delta/h^{d-1})}^{0, -}$ and $\mu_\Lambda^{J, -, h}$ is less than $\exp(-E_c^\theta/h^{d-1})$. \square

6.3 Growing in a Summertop cone

In this section we will use the “inverted space-time pyramids” of [16] to show that under \mathbb{Q}_θ , droplets of plus phase tends to expand from $\mathbb{W}_\theta(B_{\min}^\theta)$ to $\mathbb{W}_\theta(B_{\max}^\theta)$ with high probability.

With $\delta > 0$, and with reference to (4.3.3) and (6.1.1), consider the space-time region

$$\begin{aligned} \nabla &= \nabla(B_{\min}^\theta, B_{\max}^\theta, \delta, \theta) := \text{ST}(\Lambda_0, \dots, \Lambda_n; t_0 < \dots < t_{n+1}), \\ \Lambda_i &:= \Delta_\theta^{i+|\mathbb{W}_\theta(B_{\min}^\theta)|}, \quad t_i := i \exp(\delta/h^{d-1}), \quad n := |\mathbb{W}_\theta(B_{\max}^\theta)| - |\mathbb{W}_\theta(B_{\min}^\theta)|. \end{aligned}$$

For each i let $\hat{\mu}_{\Lambda_i}^{J,-,h} = \mu_{\Lambda_i}^{J,-,h}(\cdot | \mathcal{C})$ with $\varepsilon_{\mathcal{C}} = (B_{\min}^\theta - B_{\mathcal{C}}^\theta)/3$.

For $\eta \in \Sigma_{\mathbb{W}_\theta(B_{\min}^\theta)}^-$ consider the event

$$G_\eta := \left\{ \sigma_{\nabla,-,h;t_{n+1}}^{0,\eta} = \sigma_{\nabla,-,h;t_{n+1}}^{t_n,+} \right\}.$$

For $\eta \in \mathcal{C}$, G_η describes the plus phase spreading from $\mathbb{W}_\theta(B_{\min}^\theta)$ to $\mathbb{W}_\theta(B_{\max}^\theta)$ in time t_n . The event G_η depends only on the elements of the graphical construction contained in ∇ . Here is an extension of [16, Proposition 3.2.2] to the dilute Ising model.

Proposition 6.3.1. *There are positive constants δ_0, C, C_1 such that if $0 < \delta \leq \delta_0$ then with high \mathbb{Q}_θ -probability*

$$\int \mathbb{P}_J(G_\eta) d\hat{\mu}_{\Lambda_0}^{J,-,h}(\eta) \geq 1 - C \exp(-C_1/h).$$

Proof of Proposition 6.3.1. We will assume that J belongs to a certain event with high \mathbb{Q}_θ -probability; the set is defined implicitly by our use of results from Section 5.

For $i = 1, \dots, n$ and $\zeta \in \Sigma_{\Lambda_{i-1}}^-$ let $G_\zeta^i = \left\{ \sigma_{\nabla,-,h;t_{i+1}}^{t_{i-1},\zeta} = \sigma_{\nabla,-,h;t_{i+1}}^{t_i,+} \right\}$;

$$G_\eta^1 \cap \left(\bigcap_{i=2}^n G_+^i \right) \implies G_\eta.$$

By monotonicity $G_\zeta^i \subset G_+^i$. It is sufficient to show that for each i ,

$$\int \mathbb{P}_J(G_\zeta^i) d\hat{\mu}_{\Lambda_{i-1}}^{J,-,h}(\zeta) \geq 1 - C \exp(-C_1/h). \quad (6.3.2)$$

With reference to Proposition 5.1.10, if we start the dynamics with initial distribution $\hat{\mu}_{\Lambda_{i-1}}^{J,-,h}$ we expect to stay inside \mathcal{C} for a long time. Let $(\hat{\sigma}_{\nabla,-,h;t}^{s,\zeta})_{t \geq s}$ denote the Markov chain obtained from the graphical construction by suppressing any jumps from \mathcal{C} to \mathcal{C}^c . By introducing a stopping time

$$\tau_i^\zeta = \inf\{t \geq t_i : \sigma_{\nabla,-,h;t}^{t_i,\zeta} \neq \hat{\sigma}_{\nabla,-,h;t}^{t_i,\zeta}\},$$

we will see that the modified dynamics are likely to agree with the regular dynamics over the interval $[t_i, t_{i+1}]$.

Let σ^x denote the configuration obtained from σ by flipping the spin at x , and let

$$\partial\mathcal{C} = \{\sigma \in \mathcal{C} : \exists x, \sigma^x \in \mathcal{C}^c\}.$$

Proposition 5.1.10 gives an upper bound on $\hat{\mu}_{\Lambda_i}^{J,-,h}(\partial\mathcal{C})$. Given B_{\min}^θ we can find $\delta_0 > 0$ such that

$$\hat{\mu}_{\Lambda_i}^{J,-,h}(\partial\mathcal{C}) \leq \exp(-3\delta_0/h^{d-1}). \quad (6.3.3)$$

If the starting state ζ is sampled from $\hat{\mu}_{\Lambda_i}^{J,-,h}$ then the process $(\hat{\sigma}_{\nabla,-,h;t}^{t_i,\zeta})_{t \in [t_i, t_{i+1}]}$ is stationary. By (6.3.3) (cf. [16, (2.12)]), if $\delta \leq \delta_0$ and h is sufficiently small,

$$\int \mathbb{P}_J(\tau_i^\zeta \leq t_{i+1}) d\hat{\mu}_{\Lambda_i}^{J,-,h}(\zeta) \leq \exp(-\delta_0/h^{d-1}). \quad (6.3.4)$$

For some x , $\Lambda_i = \Lambda_{i-1} \cup \{x\}$. We will need a bound on the effect of adding this extra vertex has on the conditional Ising measures $(\hat{\mu}_{\Lambda_i}^{J,-,h})_i$. By the Ising model's finite-energy property, for any $h_0 > 0$,

$$\alpha := \inf_{0 < h < h_0} \inf_{\zeta \in \Sigma} \inf_J \inf_{s = \pm 1} \mu_{\{0\}}^{J,\zeta,h}(\sigma(0) = s) > 0. \quad (6.3.5)$$

By the Ising model's Markov property, for $\zeta \in \mathcal{C} \cap \Sigma_{\Lambda_{i-1}}^-$,

$$\hat{\mu}_{\Lambda_i}^{J,-,h}(\zeta) / \hat{\mu}_{\Lambda_{i-1}}^{J,-,h}(\zeta) = \mu_{\Lambda_i}^{J,-,h}(\sigma(x) = -1 \mid \mathcal{C}) \geq \alpha.$$

Therefore (cf. [16, (3.28)]),

$$\begin{aligned} & \int \mathbb{P}_J((G_\zeta^i)^c) d\hat{\mu}_{\Lambda_{i-1}}^{J,-,h}(\zeta) \\ &= \int \mathbb{P}_J(\sigma_{\nabla,-,h;t_{i+1}}^{t_{i-1},\zeta} \neq \sigma_{\nabla,-,h;t_{i+1}}^{t_i,+}) d\hat{\mu}_{\Lambda_{i-1}}^{J,-,h}(\zeta) \\ &\leq \int \mathbb{P}_J(\sigma_{\nabla,-,h;t_{i+1}}^{t_i,\zeta} \neq \sigma_{\nabla,-,h;t_{i+1}}^{t_i,+}) d\hat{\mu}_{\Lambda_{i-1}}^{J,-,h}(\zeta) + \int \mathbb{P}_J(\tau_{i-1}^\zeta \leq t_i) d\hat{\mu}_{\Lambda_{i-1}}^{J,-,h}(\zeta) \\ &\leq \alpha^{-1} \int \mathbb{P}_J(\sigma_{\nabla,-,h;t_{i+1}}^{t_i,\zeta} \neq \sigma_{\nabla,-,h;t_{i+1}}^{t_i,+}) d\hat{\mu}_{\Lambda_i}^{J,-,h}(\zeta) + \exp(-\delta_0/h^{d-1}). \end{aligned} \quad (6.3.6)$$

Set

$$b_\gamma = (1 - \gamma)B_c^\theta + \gamma B_{\min}^\theta, \quad \gamma \in [0, 1]. \quad (6.3.7)$$

Thus $B_c^\theta = b_0 < b_{1/3} < b_{2/3} < b_1 = B_{\min}^\theta$. Choose b such that $\mathbb{W}'_\theta(b) = \Lambda_i$. By monotonicity and the invariance of the modified dynamics with respect to

$$\begin{aligned}
& \hat{\mu}_{\mathbb{W}'_\theta(b)}^{J,-,h}, \\
& \int \mathbb{P}_J(\sigma_{\nabla,-,h;t_{i+1}}^{t_i,\zeta} \neq \sigma_{\nabla,-,h;t_{i+1}}^{t_i,+}) d\hat{\mu}_{\mathbb{W}'_\theta(b)}^{J,-,h}(\zeta) \\
& \leq \int \mathbb{P}_J(\sigma_{\nabla,-,h;t_{i+1}}^{t_i,+} > \hat{\sigma}_{\nabla,-,h;t_{i+1}}^{t_i,\zeta}) + \hat{\mu}_{\mathbb{W}'_\theta(b)}^{J,-,h}(\tau_i^\zeta \leq t_{i+1}) d\hat{\mu}_{\mathbb{W}'_\theta(b)}^{J,-,h}(\zeta) \\
& \leq \int \sum_{y \in \mathbb{W}'_\theta(b)} \left\{ \mathbb{P}_J(\sigma_{\nabla,-,h;t_{i+1}}^{t_i,+}(y) = 1) - \mathbb{P}_J(\hat{\sigma}_{\nabla,-,h;t_{i+1}}^{t_i,\zeta}(y) = 1) \right\} d\hat{\mu}_{\mathbb{W}'_\theta(b)}^{J,-,h}(\zeta) \\
& \quad + \exp(-\delta_0/h^{d-1}) \\
& \leq \sum_{y \in \mathbb{W}'_\theta(b_{2/3})} \left\{ \mathbb{P}_J(\sigma_{\mathbb{W}'_\theta(b),+,h;\exp(\delta/h^{d-1})}^{0,+}(y) = 1) - \hat{\mu}_{\mathbb{W}'_\theta(b)}^{J,-,h}(\sigma(y) = 1) \right\} \\
& \quad + \sum_{y \in \mathbb{W}'_\theta(b_{2/3},b)} \left\{ \mathbb{P}_J(\sigma_{\mathbb{W}'_\theta(b_{1/3},b),(+,-),h;\exp(\delta/h^{d-1})}^{0,+}(y) = 1) - \hat{\mu}_{\mathbb{W}'_\theta(b)}^{J,-,h}(\sigma(y) = 1) \right\} \\
& \quad + \exp(-\delta_0/h^{d-1}).
\end{aligned}$$

For $y \in \mathbb{W}'_\theta(b_{2/3})$, by Proposition 5.5.2 and Markov chain mixing [15, (59)],

$$\begin{aligned}
& |\mathbb{P}_J(\sigma_{\mathbb{W}'_\theta(b),+,h;\exp(\delta/h^{d-1})}^{0,+}(y) = 1) - \mu_{\mathbb{W}'_\theta(b)}^{J,+h}(\sigma(y) = 1)| \\
& \leq \exp[-\exp(\delta/h^{d-1})\text{gap}(\mathbb{W}'_\theta(b), +, h)] / \mu_{\mathbb{W}'_\theta(b)}^{J,+h}(\sigma = +) \\
& \leq \exp[-\exp(\delta/(2h^{d-1}))].
\end{aligned}$$

Similarly for $y \in \mathbb{W}'_\theta(b_{2/3}, b)$, by Proposition 5.5.5,

$$\begin{aligned}
& |\mathbb{P}_J(\sigma_{\mathbb{W}'_\theta(b_{1/3},b),(+,-),h;\exp(\delta/h^{d-1})}^{0,+}(y) = 1) - \mu_{\mathbb{W}'_\theta(b_{1/3},b)}^{J,(+,-),h}(\sigma(y) = 1)| \\
& \leq \exp[-\exp(\delta/h^{d-1})\text{gap}(\mathbb{W}'_\theta(b_{1/3}, b), (+, -), h)] / \mu_{\mathbb{W}'_\theta(b_{1/3},b)}^{J,+h}(\sigma = +) \\
& \leq \exp[-\exp(\delta/(2h^{d-1}))].
\end{aligned}$$

By the above

$$\begin{aligned}
& \int \mathbb{P}_J(\sigma_{\nabla,-,h;t_{i+1}}^{t_i,\zeta} \neq \sigma_{\nabla,-,h;t_{i+1}}^{t_i,+}) d\hat{\mu}_{\mathbb{W}'_\theta(b)}^{J,-,h}(\zeta) \\
& \leq \sum_{y \in \mathbb{W}'_\theta(b_{2/3})} \mu_{\mathbb{W}'_\theta(b)}^{J,+h}(\sigma(y) = +1) - \hat{\mu}_{\mathbb{W}'_\theta(b)}^{J,-,h}(\sigma(y) = +1) \quad (6.3.8) \\
& \quad + \sum_{y \in \mathbb{W}'_\theta(b_{2/3},b)} \mu_{\mathbb{W}'_\theta(b_{1/3},b)}^{J,(+,-),h}(\sigma(y) = +1) - \hat{\mu}_{\mathbb{W}'_\theta(b)}^{J,-,h}(\sigma(y) = +1) \\
& \quad + \exp(-\delta_0/h^{d-1}) + |\mathbb{W}'_\theta(b)| \exp[-\exp(\delta/(2h^{d-1}))].
\end{aligned}$$

By (5.4.6) and (5.4.7),

$$\begin{aligned}
& \sum_{y \in \mathbb{W}'_\theta(b_{2/3})} \mu_{\mathbb{W}'_\theta(b)}^{J,+h}(\sigma(y) = +1) - \hat{\mu}_{\mathbb{W}'_\theta(b)}^{J,-,h}(\sigma(y) = +1) \quad (6.3.9) \\
& \quad + \sum_{y \in \mathbb{W}'_\theta(b_{2/3},b)} \mu_{\mathbb{W}'_\theta(b_{1/3},b)}^{J,(+,-),h}(\sigma(y) = +1) - \hat{\mu}_{\mathbb{W}'_\theta(b)}^{J,-,h}(\sigma(y) = +1) \\
& \leq |\mathbb{W}'_\theta(b)| \exp(-c/h).
\end{aligned}$$

Inequality (6.3.2) now follows by (6.3.6), (6.3.8) and (6.3.9). \square

6.4 Escaping from Summertop-cones

In Proposition 6.3.1 we considered space-time pyramids. Consider now “space-time parallelepipeds”. From now on we will write 2π in place of θ to make it clear that θ refers to the angle of the catalyst cone. Let a denote a positive constant and let $b = 1.01B_c^{2\pi}$. We can find a sequence of graphs $\Lambda_0, \Lambda_1, \dots, \Lambda_n$ such that

- (i) $\Lambda_0 = \mathbb{W}_{2\pi}(b)$,
- (ii) $\Lambda_n = \mathbb{W}_{2\pi}(b) + aN\mathbf{e}_1$,
- (iii) Λ_{i+1} differs from Λ_i by adding a vertex or removing a vertex,
- (iv) for any i , for some $k_i \in (0, aN)$, Λ_i differs from $\mathbb{W}_\theta(b) + k_i\mathbf{e}_1$ by at most a mesoscopic layer of vertices around the boundary, and
- (v) $n = O(ab^{d-1}/h^d)$,

Let

$$\diamond = \diamond(a, b, \delta) = \text{ST}(\Lambda_0, \dots, \Lambda_n; t_0 < \dots < t_{n+1}), \quad (6.4.1)$$

with $t_i := i \exp(\delta/h^{d-1})$. In Figure 3, the dotted lines indicate the area swept out by a space-time parallelepiped that starts inside the copy of $\mathbb{W}_\theta(B_{\max}^\theta)$.

For $\eta \in \Sigma_{\Lambda_0}^-$ consider the event

$$G_\eta := \left\{ \sigma_{\diamond, -, h; t_{n+1}}^{0, \eta} = \sigma_{\diamond, -, h; t_{n+1}}^{t_n, +} \right\}.$$

Here is an extension of Proposition 6.3.1 to \diamond . Let $\varepsilon_C = (b - B_c^{2\pi})/3$ and let $\hat{\mu}_{\Lambda_i}^{J, -, h} := \mu_{\Lambda_i}^{J, -, h}(\cdot \mid \mathcal{C}_i)$ with

$$\mathcal{C}_i = \left\{ \sigma : \int_{\mathbb{W}_\theta(B_c^\theta + \varepsilon_C) + (k_i/N)\mathbf{e}_1} \mathbb{M}_K^- d\mathcal{L}^d \geq (B_c^\theta)^d \right\}.$$

Proposition 6.4.2. *Let \diamond be defined according to (6.4.1) with $\delta \leq \delta_0$. There are positive constants C, C_1 such that with high \mathbb{Q} -probability*

$$\int \mathbb{P}_J(G_\eta) d\hat{\mu}_{\Lambda_0}^{J, -, h}(\eta) \geq 1 - C \exp(-C_1/h).$$

Proof. We can adapt the proof of Proposition 6.3.1, showing that (6.3.2) holds when, for example, $\Lambda_i = \Lambda_{i-1} \setminus \{x\}$ for some vertex x on the boundary of Λ_{i-1} . Let $(\hat{\sigma}_{\diamond, -, h; t}^{s, \zeta})_{t \geq s}$ denote the Markov chain obtained from the graphical construction by suppressing any jumps from \mathcal{C}_{i-1} to \mathcal{C}_{i-1}^c . The only place where the change is important is in inequality (6.3.6). Recall that the spin of vertices leaving \diamond are set to -1 . Let μ denote the measure obtained by sampling from $\hat{\mu}_{\Lambda_{i-1}}^{J, -, h}$ and then setting the spin at x equal to -1 . Let $\mu' = \mu_{\Lambda_i}^{J, -, h}(\cdot \mid \mathcal{C}_{i-1})$. By the definition of α (6.3.5),

$$\mu(\zeta) \leq \alpha^{-1} \mu'(\zeta), \quad \zeta \in \Sigma_i^-.$$

In place of (6.3.6) we have that

$$\begin{aligned}
& \int \mathbb{P}_J((G_\zeta^i)^c) d\hat{\mu}_{\Lambda_{i-1}}^{J,-,h}(\zeta) \\
&= \int \mathbb{P}_J(\sigma_{\diamond,-,h;t_{i+1}}^{t_{i-1},\zeta} \neq \sigma_{\diamond,-,h;t_{i+1}}^{t_i,+}) d\hat{\mu}_{\Lambda_{i-1}}^{J,-,h}(\zeta) \\
&\leq \int \mathbb{P}_J(\sigma_{\diamond,-,h;t_{i+1}}^{t_i,\zeta} \neq \sigma_{\diamond,-,h;t_{i+1}}^{t_i,+}) d\mu(\zeta) + \int \mathbb{P}_J(\tau_{i-1}^\zeta \leq t_i) d\hat{\mu}_{\Lambda_{i-1}}^{J,-,h}(\zeta) \\
&\leq \alpha^{-1} \int \mathbb{P}_J(\sigma_{\diamond,-,h;t_{i+1}}^{t_i,\zeta} \neq \sigma_{\diamond,-,h;t_{i+1}}^{t_i,+}) d\mu'(\zeta) + \exp(-\delta_0/h^{d-1}).
\end{aligned}$$

The rest of the proof follows mutatis mutandis. \square

6.5 Growth on a rescaled lattice

In Proposition 6.3.1 we require $B_{\min}^\theta > B_c^\theta$. If in addition $B_{\min}^\theta > B_{\text{root}}^\theta$ then we get the following stronger result corresponding to [16, Proposition 3.2.1].

Proposition 6.5.1. *Let $B_{\max}^{2\pi} > B_{\min}^{2\pi} > B_{\text{root}}^{2\pi}$ and $\delta > 0$. Consider $\nabla = \nabla(B_{\min}^{2\pi}, B_{\max}^{2\pi}, \delta, 2\pi)$. There are positive constants C, C_1 such that with high \mathbb{Q} -probability*

$$\int \mathbb{P}_J(G_\eta) d\mu_{\mathbb{W}_{2\pi}(B_{\min}^{2\pi})}^{J,-,h}(\eta) \geq 1 - C \exp(-C_1/h).$$

Moreover, C_1 is a function of $B_{\min}^{2\pi}$ and $C_1 \rightarrow \infty$ as $B_{\min}^{2\pi} \rightarrow \infty$.

Proof of Proposition 6.5.1. Taking $\theta = 2\pi$, the proof of this proposition is very similar to the proof of Proposition 6.3.1. Define $b_\gamma = (1 - \gamma)B_{\text{root}}^{2\pi} + \gamma B_{\min}^{2\pi}$ in place of (6.3.7). We can then simply replace $\hat{\mu}_{\Lambda_i}^{J,-,h}$ with $\mu_{\Lambda_i}^{J,-,h}$. The need for the modified dynamics and the stopping time has disappeared; the $\exp(-\delta_0/h^{d-1})$ terms can be removed from the proof.

Let $\varepsilon_{\text{hs}} = (B_{\min}^{2\pi} - B_{\text{root}}^{2\pi})/(8B_{\min}^{2\pi})$. In (6.3.9), (5.4.2) and (5.4.5) replace (5.4.6) and (5.4.7), respectively. We can therefore replace the term $\exp(-c/h)$ with $2 \exp(-c_{\text{hs}} B_{\min}^{2\pi}/h)$. This yields the claim that $C_1 \rightarrow \infty$ as $B_{\min}^{2\pi} \rightarrow \infty$. \square

Proof of Theorem 1.2.3. Let $\lambda, \lambda_2^\theta, C_0$ and f refer to the corresponding quantities in the statement of the theorem. Let $\delta = (\lambda - \lambda_2^\theta)/3 > 0$.

The idea of a droplet of plus phase growing can be formalized using Proposition 6.5.1. With reference to Proposition 6.5.1, choose $B_{\min}^{2\pi}$ such that $C_1 \geq C_0 + 1$. Let $l = \text{diameter}(\mathbb{W}_{2\pi}(B_{\min}^{2\pi} + 1))$. With reference to (5.4.2), take $B_{\max}^{2\pi}$ to be greater than $3(B_{\min}^{2\pi} + 1)$ and large enough that for some constant C ,

$$|\mu_{\mathbb{W}_{2\pi}(B_{\max}^{2\pi})}^{J,-,h}(f) - \mu^{J,h}(f)| \leq C \|f\|_\infty \exp(-C_0/h).$$

Define a collection of overlapping translations of $\nabla = \nabla(B_{\min}^{2\pi}, B_{\max}^{2\pi}, \delta, 2\pi)$: let

$$\nabla_{x,i} = \nabla + (lNx, iT), \quad (x, i) \in \mathbb{Z}^d \times \mathbb{N},$$

where T denotes the time from the start of the first slice of ∇ to the start of the final slice of ∇ ,

$$T = |\mathbb{W}_{2\pi}(B_{\min}^{2\pi}, B_{\max}^{2\pi})| \exp(\delta/h^{d-1}).$$

Time-wise, the top slice of $\nabla_{x,i}$ overlaps the bottom slice of $\nabla_{x,i+1}$. We have chosen $B_{\min}^{2\pi}, l$ and $B_{\max}^{2\pi}$ so that

$$\begin{aligned} [\mathbb{W}_{2\pi}(B_{\min}^{2\pi}) + lN\mathbf{e}_1] \cap \mathbb{W}_{2\pi}(B_{\min}^{2\pi}) &= \emptyset, \quad \text{and} \\ [\mathbb{W}_{2\pi}(B_{\min}^{2\pi}) + lN\mathbf{e}_1] &\subset \mathbb{W}_{2\pi}(B_{\max}^{2\pi}). \end{aligned}$$

If $\|x - y\|_1 = 1$, then $\nabla_{x,0}$ and $\nabla_{y,0}$ do not intersect at time 0, but they then ‘invade’ each other: at time T , $\nabla_{x,0}$ covers $\nabla_{y,1}$.

Say that $x \in \mathbb{Z}^d$ is \mathbb{Q} -conductive if, translated by lNx , the \mathbb{Q} -event from Proposition 6.5.1 holds; x is \mathbb{Q} -conductive with high \mathbb{Q} -probability. When h is small the \mathbb{Q} -conductive vertices form a supercritical site-percolation type of process on \mathbb{Z}^d . Let $G_{+,x,i}$ denote the translation by (lNx, iT) of the \mathbb{P}_J -event G_+ from Proposition 6.5.1; space-time paths of $G_{+,x,i}$ -events show how clusters of plus phase spread out once they have formed.

We will say $x \in \mathbb{Z}^d$ is a \mathbb{Q} -catalyst if the event conditioned on in the definition of \mathbb{Q}_θ , translated by lNx , occurs. Let D denote the density of \mathbb{Q} -catalysts (6.0.6).

For a \mathbb{Q} -catalyst to be effective, the edges that do not need to be closed should have typical dilution. Choose k minimal such that

$$\mathbb{W}_\theta(B_{\max}^\theta) \cap [\mathbb{W}_{2\pi}(B_{\max}^{2\pi}) + klN\mathbf{e}_1] = \emptyset.$$

With reference to Figure 3, define a \mathbb{P}_J -measurable event corresponding to the nucleation and escape of a plus droplet,

$$\text{Nuc}_x := \left\{ \sigma_{\nabla_{(x+k\mathbf{e}_1,0),-,h;T+\exp(\delta/h^{d-1})}}^{0,-} = \sigma_{\nabla_{(x+k\mathbf{e}_1,0),-,h;T+\exp(\delta/h^{d-1})}}^{T,+} \right\}.$$

Figure 3 illustrates how Nuc_x can be written as the concatenation of the events described in Propositions 6.2.2-6.5.1; in the applications of Propositions 6.2.2-6.4.2 take the value of δ to be $\min\{\delta_0, (\lambda - \lambda_2^\theta)/6\}$.

We will say that a \mathbb{Q} -catalyst x is *good* if

- (i) $x + k\mathbf{e}_1$ is \mathbb{Q} -conductive, and
- (ii) $\mathbb{P}_J(\text{Nuc}_x) \geq \exp(-E_c^\theta/h^{d-1})/2$.

\mathbb{Q} -catalysts are good with high \mathbb{Q} -probability. If x is a good \mathbb{Q} -catalyst, let $\text{Nuc}_{x,i}$ denote Nuc_x translated iT forward in time.

Let $\text{Con}_M(x, y)$ denote the \mathbb{Q} -event that x and y are joined by a simple path of exactly M \mathbb{Q} -conductive vertices, and let

$$A := \{x \in \mathbb{Z}^d : x - k\mathbf{e}_1 \text{ is a good } \mathbb{Q}\text{-catalyst and } \text{Con}_M(x, 0)\},$$

Take M maximal such that $\nabla_{0,3M}$ finishes before time $\exp(\lambda/h^{d-1})$. By a Peierls argument, there is a constant $c > 0$ such that with high \mathbb{Q} -probability $\{|A| \geq cDM^d\}$. Assume that $|A| \geq cDM^d$.

The expected number of $(x, i) \in A \times \{0, 1, \dots, M\}$ such that $\text{Nuc}_{x,i}$ occurs is

$$\exp(-E_c^\theta/h^{d-1})/2 \cdot cDM^d \cdot (M+1) \geq \exp\left(\frac{\delta(d+1)}{h^{d-1}}\right) \gg 1.$$

We can assume that $\text{Nuc}_{x,i}$ does occur for some $(x, i) \in A \times \{0, 1, \dots, M\}$. As $x \in A$, there is path $y_0, y_1, \dots, y_M \in \mathbb{Z}^d$ of \mathbb{Q} -conductive sites from $y_0 = x$ to $y_M = 0$. The time between $\nabla_{x,i}$ and $\nabla_{0,3M}$ is between $2MT$ and $3MT$.

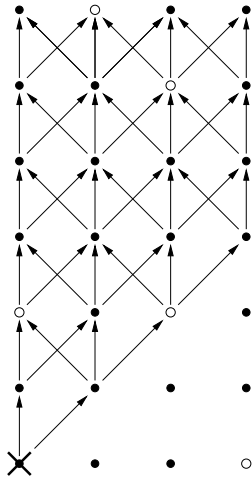


Figure 4: The x marks a nucleation event $\text{Nuc}_{x,i}$. The horizontal axis corresponds to a path of length $M = 3$ from $y_0 = x \in A$ to the origin $y_M = 0$. The vertical axis corresponds to time. The arrows indicate how the region of plus-phase can spread to neighboring points of the rescaled lattice [Proposition 6.5.1]. The black dots indicate points of the rescaled space-time lattice where $G_{+,y,k}$ occurs. The spread of the plus phase is thus bounded below by a supercritical directed-percolation cluster.

The growth of the region of plus-phase along the path y_0, \dots, y_M corresponds to a directed percolation cluster on the graph $\{0, 1, \dots, M\} \times \{i, i+1, \dots, 3M\}$; see Figure 4. By a Peierls argument, with high probability there is a space-time path

$$((y_{j_k}, k) : k = i, \dots, 3M; j_i = 0, j_{3M} = M \text{ and } \forall k, |j_k - j_{k+1}| \leq 1)$$

such that $G_{+,y_{j_k},k}$ occurs for $k = i, i+1, \dots, 3M$. □

This completes the proof of Theorem 1.2.3 part (i). Part (ii) follows by Proposition 4.2.4.

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