Influence and sharp-threshold theorems for monotonic measures

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Abstract

The influence theorem for product measures on the discrete space $\{0,1\}^N$ may be extended to probability measures with the property of monotonicity (which is equivalent to 'strong positive-association'). Corresponding results are valid for probability measures on the cube $[0,1]^N$ that are absolutely continuous with respect to Lebesgue measure. These results lead to a sharp-threshold theorem for measures of random-cluster type, and this may be applied to box-crossings in the two-dimensional random-cluster model.

1 Introduction

Influence and sharp-threshold theorems have proved useful in the study of problems in discrete probability. Reliability theory and random graphs provided early problems of this type, followed by percolation. Important progress has been made since [2, 15] towards a general theory, of which one striking aspect has been the use of discrete Fourier analysis and hypercontractivity. The reader is referred to [10, 11] for a history and bibliography.

Let $\Omega = \{0,1\}^N$ where $N < \infty$, and let μ_p be the product measure on Ω with density p. Vectors in Ω are denoted by $\omega = (\omega(i) : 1 \leq i \leq N)$.

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For any increasing subset A of Ω, and any $i \in \{1, 2, \ldots, N\}$, we define the *conditional influence* $I_A(i)$ by

$$I_A(i) = \mu_p(A \mid X_i = 1) - \mu_p(A \mid X_i = 0), \tag{1.1}$$

where X_i is the indicator function of the event $\{\omega \in \Omega : \omega(i) = 1\}$. It is well known (see [6, 11, 15, 20]) that there exists an absolute positive constant c such that the following holds. For all N, all $p \in (0, 1)$, and all increasing A, there exists $i \in \{1, 2, \ldots, N\}$ such that

$$I_A(i) \geq c \min\{\mu_p(A), 1 - \mu_p(A)\} \frac{\log N}{N}. \tag{1.2}$$

The proof uses discrete Fourier analysis and a technique known as 'hypercontractivity'. Inequality (1.2) is usually stated for the case $p = \frac{1}{2}$, but it holds with the same constant c for all $p \in (0, 1)$.

There is an important application to the theory of sharp thresholds for product measures. Let Π_N be the set of all permutations of the index set $I = \{1, 2, \ldots, N\}$. A subgroup $\mathcal{A}$ of Π_N is said to *act transitively* on I if, for all distinct pairs $j, k \in I$, there exists $\pi \in \mathcal{A}$ with $\pi_j = k$. Any $\pi \in \Pi_N$ acts on Ω by $\pi\omega = (\omega(\pi_i) : 1 \leq i \leq N)$. An event A is called *symmetric* if there exists a subgroup $\mathcal{A}$ of Π_N acting transitively on I such that $A = \pi A$. If A is symmetric, then $I_A(j) = I_A(k)$ for all j, k. By summing (1.2) over i we obtain for symmetric A that

$$\sum_{i=1}^{N} I_A(i) \geq c \min\{\mu_p(A), 1 - \mu_p(A)\} \log N. \tag{1.3}$$

It is standard (see the discussion of Russo's formula in [12]) that

$$\frac{d}{dp}\mu_p(A) = \sum_{i=1}^{N} I_A(i), \tag{1.4}$$

and it follows as in [11] that, for $0 < \epsilon < \frac{1}{2}$, the function $f(p) = \mu_p(A)$ increases from ϵ to $1 - \epsilon$ over an interval of values of p with length smaller in order than $1/\log N$.

We refer to such a statement as a 'sharp-threshold theorem', and we note that such results have wide applications to problems of discrete probability. For example, the observations above have been used recently in [5] to obtain a further proof of the famous theorem of Harris and Kesten that the critical probability p_c of bond percolation on the square lattice satisfies $p_c = \frac{1}{2}$. Using a similar argument in a second paper, [4], they have proved the conjecture that the critical probability of site percolation on a certain Poisson–Voronoi graph in $\mathbb{R}^2$ equals $\frac{1}{2}$ almost surely.

The principal purpose of the current article is to extend the results above to probability measures more general than product measures. We shall prove such results for measures having a certain condition of 'monotonicity', which is equivalent to the FKG lattice condition and is described in the next section. There are many situations in the probabilistic theory of statistical mechanics where such measures are encountered, including the Ising model and the random-cluster model.

We define monotonic probability measures in Section 2, and we note there that monotonicity is equivalent to the FKG lattice condition. This is followed by an influence theorem for monotonic measures.

A monotonic measure μ may be used as the basis of a certain parametric family of measures on Ω indexed by a parameter $p \in (0,1)$. The influence theorem for μ may then be used to obtain a sharp-threshold theorem for this class, as described in Section 3.

The influence theorem on the discrete space Ω was extended in [6] to product measures on the Euclidean cube $[0,1]^N$. Using the methods of Section 2, similar results may be proved for general monotonic measures on $[0,1]^N$. Unlike the discrete case, such an influence theorem does not appear to imply a corresponding sharp-threshold theorem. This is discussed in Section 4.

We turn finally to the random-cluster model, which may be viewed as an extension of percolation and a generalization of the Ising/Potts models for ferromagnetism, see [13, 14]. The random-cluster measure is defined in Section 5, and the sharp-threshold theorem is applied to the existence of box-crossings in two dimensions.

2 Influence for monotonic measures

We begin this section with a classification, further details of which may be found in [14]. Let $1 \le N < \infty$, and write $I = \{1, 2, \dots, N\}$ and $\Omega = \{0,1\}^N$. The set of all subsets of Ω is denoted by $\mathcal{F}$. A probability measure μ on $(\Omega, \mathcal{F})$ is said to be *positive* if $\mu(\omega) > 0$ for all $\omega \in \Omega$. It is said to satisfy the *FKG lattice condition* if

$$\mu(\omega_1 \vee \omega_2)\mu(\omega_1 \wedge \omega_2) \ge \mu(\omega_1)\mu(\omega_2) \qquad \text{for all } \omega_1, \omega_2 \in \Omega, \tag{2.1}$$

where $\omega_1 \vee \omega_2$ and $\omega_1 \wedge \omega_2$ are given by

$$\begin{aligned} \omega_1 \vee \omega_2(i) &= \max\{\omega_1(i), \omega_2(i)\}, && i \in I, \\ \omega_1 \wedge \omega_2(i) &= \min\{\omega_1(i), \omega_2(i)\}, && i \in I. \end{aligned}$$

See [9, 14].

The set Ω is a partially ordered set with the partial order: $\omega \geq \omega'$ if $\omega(i) \geq \omega'(i)$ for all $i \in I$. A non-empty event $A \in \mathcal{F}$ is called *increasing* if: $\omega \in A$ whenever there exists ω' with $\omega \geq \omega'$ and $\omega' \in A$. It is called *decreasing* if its complement is increasing. For probability measures μ_1, μ_2 on $(\Omega, \mathcal{F})$, we write $\mu_1 \leq_{\rm st} \mu_2$, and say that μ_1 is dominated stochastically by μ_2, if

$$\mu_1(A) \leq \mu_2(A) \qquad \text{for all increasing events } A.$$

The indicator function of an event A is denoted by 1_A. For $i \in I$, we write X_i for the indicator function of the event $\{\omega \in \Omega : \omega(i) = 1\}$.

A probability measure μ on Ω is said to be *positively associated* if

$$\mu(A \cap B) \geq \mu(A)\mu(B) \qquad \text{for all increasing events } A,\ B.$$

The famous FKG inequality of [9] asserts that a positive probability measure μ is positively associated if it satisfies the FKG lattice condition. It is well known that the FKG lattice condition is not necessary for positive association, and we explore this next.

We shall for simplicity restrict ourselves henceforth to positive measures. The FKG lattice condition is equivalent to a stronger property termed 'strong positive association'. For $J \subseteq I$ and $\xi \in \Omega$, let $\Omega_J = \{0,1\}^J$ and

$$\Omega_J^\xi = \{\omega \in \Omega : \omega(j) = \xi(j) \text{ for } j \in I \setminus J\}. \tag{2.2}$$

The set of all subsets of Ω_J is denoted by $\mathcal{F}_J$. Let μ be a positive probability measure on $(\Omega, \mathcal{F})$, and define the conditional probability measure μ_J^ξ on $(\Omega_J, \mathcal{F}_J)$ by

$$\mu_J^\xi(\omega_J) = \mu\bigl(X_j = \omega_J(j) \text{ for } j \in J \bigm| X_i = \xi(i) \text{ for } i \in I \setminus J\bigr), \qquad \omega_J \in \Omega_J. \tag{2.3}$$

We say that μ is *strongly positively-associated* if: for all $J \subseteq I$ and all $\xi \in \Omega$, the measure μ_J^ξ is positively associated.

We call μ *monotonic* if: for all $J \subseteq I$, all increasing subsets A of Ω_J, and all $\xi, \zeta \in \Omega$,

$$\mu_J^\xi(A) \leq \mu_J^\zeta(A) \qquad \text{whenever } \xi \leq \zeta. \tag{2.4}$$

That is, μ is monotonic if, for all $J \subseteq I$,

$$\mu_J^\xi \leq_{\rm st} \mu_J^\zeta \qquad \text{whenever } \xi \leq \zeta. \tag{2.5}$$

We call μ 1-*monotonic* if (2.5) holds for all singleton sets J. That is, μ is 1-monotonic if and only if, for all $j \in I$,

$$\mu\bigl(X_j = 1 \bigm| X_i = \xi(i) \text{ for all } i \in I \setminus \{j\}\bigr) \tag{2.6}$$

is non-decreasing in ξ.

The following theorem is fairly standard, and the proof may be found in [14].

Theorem 2.7. *Let μ be a positive probability measure on $(\Omega, \mathcal{F})$. The following are equivalent.*

(i) *μ is strongly positively-associated.*

(ii) *μ satisfies the FKG lattice condition.*

(iii) *μ is monotonic.*

(iv) *μ is 1-monotonic.*

Our principal influence theorem is as follows. For a positive probability measure μ and an increasing event A, the *conditional influence* of the index i $(\in I)$ is given as in (1.1) by

$$I_A(i) = \mu(A \mid X_i = 1) - \mu(A \mid X_i = 0). \tag{2.8}$$

For a product measure μ_p, the influence of the index i was defined in [2, 15] as $\mu_p(\omega^i \in A,\ \omega_i \notin A)$, where ω^i (respectively, ω_i) denotes the configuration obtained from ω by setting $\omega(i)$ equal to 1 (respectively, 0). We refer to the latter quantity as the *absolute influence* of index i. The absolute and conditional influences are equal for product measures, but one should note that

$$I_A(i) \neq \mu(\omega^i \in A,\ \omega_i \notin A) \tag{2.9}$$

for general probability measures μ. Further discussion of this point is provided after the next theorem.

Theorem 2.10 (Influence). *There exists a constant c satisfying $c \in (0, \infty)$ such that the following holds. Let $N \geq 1$ and let A be an increasing subset of $\Omega = \{0, 1\}^N$. Let μ be a positive probability measure on $(\Omega, \mathcal{F})$ that is monotonic. There exists $i \in I$ such that*

$$I_A(i) \geq c \min\{\mu(A), 1 - \mu(A)\} \frac{\log N}{N}. \tag{2.11}$$

Since product measures are monotonic, this extends the influence theorem of [15]. In the proof of Theorem 2.10, we shall encode the measure μ in terms of Lebesgue measure on $[0, 1]^N$, and we shall appeal to the influence theorem of [6]. Thus, we shall require no further arguments of discrete Fourier analysis than those already present in [6, 15].

We return briefly to the discussion of absolute and conditional influences. Suppose, for illustration, that P is chosen at random with $\mathbb{P}(P = \frac{1}{3}) =$

$\mathbb{P}(P=\frac{2}{3})=\frac{1}{2}$ and that, conditional on the value of P, we are provided with independent Bernoulli random variables $X_1, X_2, \dots, X_N$ with parameter P. Consider the increasing event $A=\{S_N>\frac{1}{2}N\}$, where $S_N=X_1+X_2+\dots+X_N$. By symmetry, the conditional influence of each index is the same, as is the absolute influence of each index. It is an easy calculation that

$$I_A(1)=\tfrac{1}{3}+\mathrm{o}(1) \qquad \text{as } N\to\infty.$$

On the other hand,

$$\begin{aligned}\mathbb{P}(\omega^1\in A,\ \omega_1\notin A)&=\mathbb{P}\left(\tfrac{1}{2}N-1<\sum_{i=2}^{N}X_i\le\tfrac{1}{2}N\right)\\ &=\mathrm{o}(e^{-\gamma N}) \qquad \text{as } N\to\infty,\end{aligned}$$

for some $\gamma>0$. This example indicates not only that the absolute and conditional influences can be very different, but also that the conclusion of Theorem 2.10 would be false if re-stated for absolute influences.

In the proof of Theorem 2.10 following, we see that monotonicity has the effect of increasing the influence of each coordinate in I.

Proof of Theorem 2.10. Let $A\in\mathcal{F}$ be an increasing event, and let μ be positive and monotonic. Let λ denote Lebesgue measure on the cube $[0,1]^N$. We propose to construct an increasing subset B of $[0,1]^N$ with the property that $\lambda(B)=\mu(A)$, to apply the influence theorem of [6] to the set B, and to deduce the claim. This will be done via a certain function $f:[0,1]^N\to\{0,1\}^N$ that we construct next.

Let $\mathbf{x}=(x_i:1\le i\le N)\in[0,1]^N$, and let $f(\mathbf{x})=(f_i(\mathbf{x}):1\le i\le N)$ be given recursively as follows. The first coordinate $f_1(\mathbf{x})$ is defined by:

$$\text{with} \quad a_1=\mu(X_1=1), \quad \text{set} \quad f_1(\mathbf{x})=\begin{cases}1 & \text{if } x_1>1-a_1,\\ 0 & \text{otherwise.}\end{cases} \tag{2.12}$$

Suppose we know $f_i(\mathbf{x})$ for $1\le i<k$. Let

$$a_k=\mu(X_k=1\mid X_i=f_i(\mathbf{x}) \text{ for } 1\le i<k), \tag{2.13}$$

and define

$$f_k(\mathbf{x})=\begin{cases}1 & \text{if } x_k>1-a_k,\\ 0 & \text{otherwise.}\end{cases} \tag{2.14}$$

Suppose that $\mathbf{x}\le\mathbf{x}'$, and write a_k and a_k' for the corresponding values in (2.12)–(2.13). Clearly $a_1=a_1'$, so that $f_1(\mathbf{x})\le f_1(\mathbf{x}')$. Since μ is monotonic,

$a_2 \le a_2'$, so that $f_2(\mathbf{x}) \le f_2(\mathbf{x}')$. Continuing inductively, we find that $f_k(\mathbf{x}) \le f_k(\mathbf{x}')$ for all k, which is to say that $f(\mathbf{x}) \le f(\mathbf{x}')$. Therefore, f is non-decreasing on $[0,1]^N$. Let B be the increasing subset of $[0,1]^N$ given by $B = f^{-1}(A)$.

We make four notes concerning the definition of f.

(1) Each a_k depends only on $x_1, x_2, \dots, x_{k-1}$.

(2) Since μ is positive, the a_k satisfy $0 < a_k < 1$ for all $\mathbf{x} \in [0,1]^N$ and $k \in I$.

(3) For any $\mathbf{x} \in [0,1]^N$ and $k \in I$, the values $f_k(\mathbf{x}), f_{k+1}(\mathbf{x}), \dots, f_N(\mathbf{x})$ depend on $x_1, x_2, \dots, x_{k-1}$ only through the values $f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_{k-1}(\mathbf{x})$.

(4) The function f and the event B depend on the ordering of the set I.

Let $U = (U_i : 1 \le i \le N)$ be the identity function on $[0,1]^N$, and note that U has law λ. By the method of construction of the function f, $f(U)$ has law μ. In particular,

$$\mu(A) = \lambda(f(U) \in A) = \lambda(U \in f^{-1}(A)) = \lambda(B). \tag{2.15}$$

Let

$$J_B(i) = \lambda(B \mid U_i = 1) - \lambda(B \mid U_i = 0),$$

where the conditional probabilities are to be interpreted as

$$\lambda(B \mid U_i = u) = \lim_{\epsilon \downarrow 0} \left\{ \frac{1}{\epsilon} \lambda(B \mid U_i \in (u-\epsilon, u+\epsilon)) \right\}, \qquad u = 0, 1.$$

Since B is an event with a certain simple structure, this is the same as $\lambda_{N-1}(B_i^u)$ for $u = 0, 1$, where λ_{N-1} is $(N-1)$-dimensional Lebesgue measure and B_i^u is the set of all $(N-1)$-vectors $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$ such that $(x_1, \dots, x_{i-1}, u, x_{i+1}, \dots, x_N) \in B$.

By Theorem 1 of [6], we may find a constant $c > 0$, independent of the choice of N and A, such that: there exists $i \in I$ with

$$J_B(i) \ge c \min\{\lambda(B), 1 - \lambda(B)\} \frac{\log N}{N}. \tag{2.16}$$

We choose i accordingly.

We claim that

$$I_A(j) \ge J_B(j) \qquad \text{for } j \in I. \tag{2.17}$$

Once (2.17) is shown, the claim follows from (2.15) and (2.16). We prove next that

$$I_A(1) \ge J_B(1). \tag{2.18}$$

We have that

$$\begin{aligned} I_A(1) &= \mu(A \mid X_1 = 1) - \mu(A \mid X_1 = 0) \\ &= \lambda(B \mid f_1(U) = 1) - \lambda(B \mid f_1(U) = 0) \\ &= \lambda(B \mid U_1 > 1 - a_1) - \lambda(B \mid U_1 \le 1 - a_1) \\ &= \lambda(B \mid U_1 = 1) - \lambda(B \mid U_1 = 0) \\ &= J_B(1), \end{aligned} \tag{2.19}$$

where we have used notes (2) and (3) above. This implies (2.18).

We turn our attention to (2.17) with $j \ge 2$. We re-order the set I to bring the index j to the front. That is, we let K be the re-ordered index set $K = (k_1, k_2, \dots, k_N) = (j, 1, 2, \dots, j-1, j+1, \dots, N)$. We write $g = (g_{k_i} : 1 \le i \le N)$ for the associated function given by (2.12)–(2.14) subject to the new ordering, and $C = g^{-1}(A)$. Thinking of (2.12)–(2.14) as an algorithm for constructing f, we are applying the same algorithm to the re-ordered set K.

We claim that

$$J_C(k_1) \ge J_B(j). \tag{2.20}$$

By (2.19) with I replaced by K, $J_C(k_1) = I_A(j)$, and (2.17) follows. It remains to prove (2.20), and we shall use monotonicity again for this.

It suffices for (2.20) to prove that

$$\lambda(C \mid U_j = 1) \ge \lambda(B \mid U_j = 1), \tag{2.21}$$

together with the reversed inequality given $U_j = 0$. The conditioning on the left-hand side of (2.21) refers to the first coordinate encountered by the algorithm (2.12)–(2.14) when applied to the re-ordered set K. Let

$$\overline{U} = (U_1, U_2, \dots, U_{j-1}, 1, U_{j+1}, \dots, U_N). \tag{2.22}$$

The 0/1-vector $f(\overline{U}) = (f_i(\overline{U}) : 1 \le i \le N)$ is constructed sequentially (as above) by considering the indices $1, 2, \dots, N$ in turn. At stage k, we declare $f_k(\overline{U})$ to equal 1 if U_k exceeds a certain function a_k of the variables $f_i(\overline{U})$, $1 \le i < k$. By the monotonicity of μ, this function is non-increasing in these variables. The index j plays a special role in that: (i) $f_j(\overline{U}) = 1$, and (ii) given this fact, it is more likely than before that the variables $f_k(\overline{U})$, $j < k \le N$, will take the value 1. The values $f_k(\overline{U})$, $1 \le k < j$ are unaffected by the value of U_j.

Consider now the 0/1-vector $g(\overline{U}) = (g_{k_r}(\overline{U}) : 1 \le r \le N)$, constructed in the same manner as above but with the new ordering K of the index set I. First we examine index k_1 $(= j)$, and we automatically declare $g_{k_1}(\overline{U}) = 1$

(since $U_j = 1$). We then construct $g_{k_r}(\overline{U})$, $2 \le r \le N$, in sequence. Since the a_k are non-decreasing in the variables constructed so far, we have that

$$g_{k_r}(\overline{U}) \ge f_{k_r}(\overline{U}), \qquad r = 2, 3, \dots, N. \tag{2.23}$$

Therefore, $g(\overline{U}) \ge f(\overline{U})$, implying as required that

$$\lambda(C \mid U_j = 1) = \lambda(g(\overline{U}) \in A) \ge \lambda(f(\overline{U}) \in A) = \lambda(B \mid U_j = 1). \tag{2.24}$$

Inequality (2.21) follows. The same argument implies the reversed inequality obtained from (2.21) by reversing the conditioning to $U_j = 0$. This implies (2.20).

A formal proof of (2.23) follows. Suppose that r is such that $g_{k_s}(\overline{U}) \ge f_{k_s}(\overline{U})$ for $2 \le s < r$. By (2.14), for $r \le j$,

$$\begin{aligned} f_{k_r}(\overline{U}) = 1 &\quad \text{if} \quad U_{k_r} > \mu(X_{k_r} = 0 \mid X_{k_s} = f_{k_s}(\overline{U}) \text{ for } 2 \le s < r), \\ g_{k_r}(\overline{U}) = 1 &\quad \text{if} \quad U_{k_r} > \mu(X_{k_r} = 0 \mid X_{k_s} = g_{k_s}(\overline{U}) \text{ for } 1 \le s < r). \end{aligned}$$

Now $g_{k_1}(\overline{U}) = 1$ and, by the induction hypothesis and monotonicity,

$$\begin{aligned} &\mu(X_{k_r} = 0 \mid X_{k_s} = f_{k_s}(\overline{U}) \text{ for } 2 \le s < r) \\ &\qquad\qquad \ge \mu(X_{k_r} = 0 \mid X_{k_s} = g_{k_s}(\overline{U}) \text{ for } 1 \le s < r), \end{aligned}$$

whence $g_{k_r}(\overline{U}) \ge f_{k_r}(\overline{U})$ as required.

Consider finally the case $j < r \le N$. Then

$$\begin{aligned} f_{k_r}(\overline{U}) = 1 &\quad \text{if} \quad U_{k_r} > \mu(X_{k_r} = 0 \mid X_{k_s} = f_{k_s}(\overline{U}) \text{ for } 1 \le s < r), \\ g_{k_r}(\overline{U}) = 1 &\quad \text{if} \quad U_{k_r} > \mu(X_{k_r} = 0 \mid X_{k_s} = g_{k_s}(\overline{U}) \text{ for } 1 \le s < r), \end{aligned}$$

and the conclusion follows as before. □

3 Sharp-threshold theorem

We consider in this section a family of probability measures indexed by a parameter $p \in (0, 1)$, and we prove a sharp-threshold theorem subject to a hypothesis of monotonicity. The motivating example is the random-cluster model, to which we return in the next section.

Let $1 \le N < \infty$, $I = \{1, 2, \dots, N\}$, and let $\Omega = \{0, 1\}^N$ and $\mathcal{F}$ be given as before. Let μ be a positive probability measure on $(\Omega, \mathcal{F})$. For $p \in (0, 1)$, we define the probability measure μ_p by

$$\mu_p(\omega) = \frac{1}{Z_p} \mu(\omega) \left\{ \prod_{i \in I} p^{\omega(i)} (1 - p)^{1 - \omega(i)} \right\}, \qquad \omega \in \Omega, \tag{3.1}$$

where Z_p is the normalizing constant

$$Z_p = \sum_{\omega\in\Omega} \mu(\omega) \left\{ \prod_{i\in I} p^{\omega(i)}(1-p)^{1-\omega(i)} \right\}. \tag{3.2}$$

It is immediate that μ_p is positive and that $\mu = \mu_{\frac{1}{2}}$. It is easy to check that μ_p satisfies the FKG lattice condition (2.1) if and only if μ satisfies this condition, and it follows that μ is monotonic if and only if, for all $p \in (0,1)$, μ_p is monotonic. In order to prove a sharp-threshold theorem for the family μ_p, we present first a Russo-type formula.

Theorem 3.3 ([3]). *For any event $A \in \mathcal{F}$,*

$$\frac{d}{dp}\mu_p(A) = \frac{1}{p(1-p)} \sum_{i\in I} \mathrm{cov}_p(X_i, 1_A), \tag{3.4}$$

where cov_p denotes covariance with respect to the measure μ_p.

Proof. This may be obtained exactly as in [3], Proposition 4, see also Section 2.4 of [14]. The details are omitted. □

Let $\mathcal{A}$ be a subgroup of the permutation group Π_N. A probability measure ϕ on $(\Omega, \mathcal{F})$ is called $\mathcal{A}$-*invariant* if $\phi(\omega) = \phi(\alpha\omega)$ for all $\alpha \in \mathcal{A}$. An event $A \in \mathcal{F}$ is called $\mathcal{A}$-*invariant* if $A = \alpha A$ for all $\alpha \in \mathcal{A}$. It is easily seen that, for any subgroup $\mathcal{A}$, μ is $\mathcal{A}$-invariant if and only if each μ_p is $\mathcal{A}$-invariant.

Theorem 3.5 (Sharp threshold). *There exists a constant c satisfying $c \in (0,\infty)$ such that the following holds. Let $N \ge 1$ and let $A \in \mathcal{F}$ be an increasing event. Let μ be a positive probability measure on $(\Omega, \mathcal{F})$ which is monotonic. If there exists a subgroup $\mathcal{A}$ of Π_N acting transitively on I such that μ and A are $\mathcal{A}$-invariant, then*

$$\frac{d}{dp}\mu_p(A) \ge \frac{c\xi_p}{p(1-p)} \min\{\mu_p(A), 1-\mu_p(A)\} \log N, \qquad p \in (0,1), \tag{3.6}$$

where $\xi_p = \min\{\mu_p(X_i)(1-\mu_p(X_i)) : i \in I\}$.

We precede the proof with a lemma. Let

$$I_{p,A}(i) = \mu_p(A \mid X_i = 1) - \mu_p(A \mid X_i = 0).$$

Lemma 3.7. *Let $A \in \mathcal{F}$. Suppose there exists a subgroup $\mathcal{A}$ of Π_N acting transitively on I such that μ and A are $\mathcal{A}$-invariant. Then $I_{p,A}(i) = I_{p,A}(j)$ for all $i, j \in I$ and all $p \in (0,1)$.*

Proof of Lemma 3.7. Since μ is $\mathcal{A}$-invariant, so is μ_p for every p. Let $i, j \in I$, and find $\alpha \in \mathcal{A}$ such that $\alpha_i = j$. Under the given conditions,

$$\begin{aligned}\mu_p(A,\, X_j = 1) &= \sum_{\omega \in A} \mu_p(\omega) X_j(\omega) = \sum_{\omega \in A} \mu_p(\alpha\omega) X_i(\alpha\omega) \\ &= \sum_{\omega' \in A} \mu_p(\omega') X_i(\omega') = \mu_p(A,\, X_i = 1).\end{aligned}$$

Applying this with $A = \Omega$, we find that $\mu_p(X_j = 1) = \mu_p(X_i = 1)$. By dividing, we deduce that $\mu_p(A \mid X_j = 1) = \mu_p(A \mid X_i = 1)$. A similar equality holds with 1 replaced by 0, and the claim follows. □

Proof of Theorem 3.5. By Lemma 3.7, every index has the same influence. Since A is increasing,

$$\begin{aligned}\mathrm{cov}_p(X_i, 1_A) &= \mu_p(X_i 1_A) - \mu_p(X_i)\mu_p(A) \\ &= \mu_p(X_i)(1 - \mu_p(X_i)) I_{p,A}(i) \\ &\geq \xi_p I_{p,A}(i).\end{aligned}$$

Summing over the index set I as in (3.4), we deduce (3.6) by Theorem 2.10 applied to the monotonic measure μ_p. □

4 Probability measures on the Euclidean cube

We have so far considered probability measures on the discrete cube $\{0,1\}^N$ only. The method of proof of the influence theorem, Theorem 2.10, may be applied also to probability measures on the Euclidean cube $[0,1]^N$ that are absolutely continuous with respect to Lebesgue measure. Any such measure μ has a density function ρ, which is to say that

$$\mu(A) = \int_A \rho(\mathbf{x})\, \lambda(d\mathbf{x}),$$

for (Lebesgue) measurable subsets A of $[0,1]^N$, with λ denoting Lebesgue measure. Since the density function ρ is non-unique, we shall phrase the results of this section in terms of ρ rather than the associated measure μ. Some may regard this as not entirely satisfactory, arguing that results for *measures* should be based on hypotheses for these measures, rather than for particular versions of their density functions. One may rewrite the conclusions of this section thus, but at the expense of greater measure-theoretic detail which obscures the basic argument.

Let $N \geq 1$, and write $\Omega = [0,1]^N$. Let $\rho : \Omega \to [0,\infty)$ be (Lebesgue) measurable. We call ρ a *density function* if

$$\int_\Omega \rho(\mathbf{x})\,\lambda(d\mathbf{x}) = 1,$$

and in this case we denote by μ_ρ the corresponding probability measure,

$$\mu_\rho(A) = \int_A \rho(\mathbf{x})\,\lambda(d\mathbf{x}).$$

We call ρ *positive* if it is a strictly positive function on Ω, and we say it satisfies the *(continuous) FKG lattice condition* if

$$\rho(\mathbf{x} \vee \mathbf{y})\rho(\mathbf{x} \wedge \mathbf{y}) \geq \rho(\mathbf{x})\rho(\mathbf{y}) \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \Omega, \tag{4.1}$$

where the operations $\vee$, $\wedge$ are defined as the coordinate-wise maximum and minimum, respectively.

Let ρ be a density function. We call μ_ρ *positively associated* if

$$\mu_\rho(A \cap B) \geq \mu_\rho(A)\mu_\rho(B),$$

for all increasing subsets of Ω. [It is presumably well known that increasing subsets of Ω are Lebesgue-measurable but need not be Borel-measurable; see the notes at the end of this section.]

Let $I = \{1, 2, \dots, N\}$. For $J \subseteq I$, let $\Omega_J = [0,1]^J$ and

$$\Omega_J^\xi = \{\mathbf{x} \in \Omega : x_j = \xi_j \text{ for } j \in I \setminus J\}, \qquad \xi \in \Omega. \tag{4.2}$$

The Lebesgue σ-algebra of Ω_J is denoted by $\mathcal{F}_J$. Let ρ be a positive density function. We define the conditional probability measure $\mu_{\rho,J}^\xi$ on $(\Omega_J, \mathcal{F}_J)$ by

$$\mu_{\rho,J}^\xi(E) = \int_E \rho_J^\xi(\mathbf{x})\,\lambda(d(x_j : j \in J)), \qquad E \in \mathcal{F}_J, \tag{4.3}$$

where ρ_J^ξ is the conditional density function

$$\rho_J^\xi(\mathbf{x}) = \frac{1}{Z_J^\xi}\rho(\mathbf{x})1_{\Omega_J^\xi}(\mathbf{x}), \quad Z_J^\xi = \int_{\Omega_J^\xi} \rho(\mathbf{x})\,\lambda(d(x_j : j \in J)).$$

We sometimes write $\mu_\rho\bigl(E \mid (\xi_j : j \in I \setminus J)\bigr)$ for $\mu_{\rho,J}^\xi(E)$, and we recall the standard fact that $\mu_\rho\bigl(\cdot \mid (\xi_j : j \in I \setminus J)\bigr)$ is a version of the conditional expectation given the σ-field $\mathcal{F}_{I \setminus J}$.

We say that ρ is *strongly positively-associated* if: for all $J \subseteq I$ and all $\xi \in \Omega$, the measure $\mu_{\rho,J}^{\xi}$ is positively associated. We call ρ *monotonic* if: for all $J \subseteq I$, all increasing subsets A of Ω_J, and all $\xi, \zeta \in \Omega$,

$$\mu_{\rho,J}^{\xi}(A) \leq \mu_{\rho,J}^{\zeta}(A) \qquad \text{whenever } \xi \leq \zeta. \tag{4.4}$$

That is, ρ is monotonic if, for all $J \subseteq I$,

$$\mu_{\rho,J}^{\xi} \leq_{\text{st}} \mu_{\rho,J}^{\zeta} \qquad \text{whenever } \xi \leq \zeta. \tag{4.5}$$

Here is a basic result concerning stochastic ordering.

Theorem 4.6 ([1, 16]). *Let $N \geq 1$, and let f and g be density functions on $\Omega = [0,1]^N$. If*

$$g(\mathbf{x} \vee \mathbf{y}) f(\mathbf{x} \wedge \mathbf{y}) \geq g(\mathbf{x}) f(\mathbf{y}) \qquad \textit{for all } \mathbf{x}, \mathbf{y} \in [0,1]^N,$$

then $\mu_f \leq_{\text{st}} \mu_g$.

If ρ satisfies the FKG lattice condition and A is an increasing event, then

$$1_A(\mathbf{x} \vee \mathbf{y}) \rho(\mathbf{x} \vee \mathbf{y}) \rho(\mathbf{x} \wedge \mathbf{y}) \geq 1_A(\mathbf{x}) \rho(\mathbf{x}) \rho(\mathbf{y}),$$

whence, by Theorem 4.6,

$$\mu_\rho(A) \mu_\rho(B) \leq \mu_\rho(A \cap B)$$

for all increasing A, B. Therefore, μ_ρ is positively associated.

Henceforth we restrict ourselves to *positive* density functions. Arguments similar to the above are valid with ρ (assumed positive) replaced by the conditional density function ρ_J^{ξ}, and one arrives thus at the following.

Theorem 4.7. *Let $N \geq 1$, and let ρ be a positive density function on $\Omega = [0,1]^N$ satisfying the FKG lattice condition (4.1). Then ρ is strongly positively-associated and monotonic.*

We turn now to a 'continuous' version of Theorem 2.10. Let $N \geq 1$, and let ρ be a monotonic positive density function on $\Omega = [0,1]^N$. Let $U = (U_1, U_2, \ldots, U_N)$ be the identity function on $[0,1]^N$. For an increasing subset A of Ω, we define the *conditional influences* by

$$I_A(i) = \mu_\rho(A \mid U_i = 1) - \mu_\rho(A \mid U_i = 0), \qquad i \in I. \tag{4.8}$$

Theorem 4.9 (Influence). *There exists a constant c satisfying $c \in (0, \infty)$ such that the following holds. Let $N \geq 1$ and let A be an increasing subset of $\Omega = [0,1]^N$. Let ρ be a positive density function on $[0,1]^N$ that is monotonic. There exists $i \in I$ such that*

$$I_A(i) \geq c \min\{\mu(A), 1 - \mu(A)\} \frac{\log N}{N}. \tag{4.10}$$

Proof. The proof is very similar to that of Theorem 2.10. We propose first to construct an increasing event B such that $\lambda(B) = \mu(A)$, by way of a function $f : [0,1]^N \to [0,1]^N$. Let $\mathbf{x} = (x_i : 1 \leq i \leq N) \in [0,1]^N$, and write $f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_N(\mathbf{x}))$. The first coordinate $f_1(\mathbf{x})$ depends on x_1 only and is defined by:

$$\mu_\rho(U_1 > f_1(\mathbf{x})) = 1 - x_1.$$

Since the density function ρ is strictly positive, $f_1(\mathbf{x})$ is a continuous and strictly increasing function of x_1. It is an elementary exercise to check that the law of $f_1(U)$ under λ is the same as that of U_1 under μ_ρ.

Having defined $f_1(\mathbf{x})$, we define $f_2(\mathbf{x})$ in terms of x_1, x_2 only by:

$$\mu_\rho\big(U_2 > f_2(\mathbf{x}) \,\big|\, U_1 = f_1(\mathbf{x})\big) = 1 - x_2.$$

The left-hand side is defined according to (4.3). It is a standard fact that $\mu_\rho(\cdot \mid U_1 = f_1)$ is a version of the conditional expectation $\mu_\rho(\cdot \mid \sigma(U_1))$, where $\sigma(U_1)$ denotes the σ-field generated by U_1, and it is an exercise to check that the pair $(f_1(U), f_2(U))$ has the same law under λ as does the pair (U_1, U_2) under μ_ρ. For each given $x_1 \in (0,1)$, $f(\mathbf{x})$ is a continuous and strictly increasing function of x_2. [We use the assumptions that ρ is positive and monotonic, respectively, here.]

We continue inductively. Suppose we know $f_i(\mathbf{x})$ for $1 \leq i < k$. Then $f_k(\mathbf{x})$ depends on $x_1, x_2, \ldots, x_k$ and is given by:

$$\mu_\rho\big(U_k > f_k(\mathbf{x}) \,\big|\, U_i = f_i(\mathbf{x}) \text{ for } 1 \leq i < k\big) = 1 - x_k.$$

As above, f is strictly increasing (using the assumption of monotonicity), and the law of $f(U)$ under λ is the same as the law of U under μ_ρ. We set $B = f^{-1}(A)$.

Let

$$J_B(i) = \lambda(B \mid U_i = 1) - \lambda(B \mid U_i = 0), \qquad i \in I.$$

Since f_1 is continuous and strictly increasing,

$$\mu_\rho(A \mid U_1 = b) = \lambda(B \mid f_1(U_1) = b) = \lambda(B \mid U_1 = b), \qquad b = 0, 1,$$

implying that $I_A(1) = J_B(1)$. It remains to show that $I_A(j) \geq J_B(j)$ for $j \in I$. Let $j \in I$, $j \neq 1$. We re-order the coordinate set as $K = \{j, 1, 2, \dots, j-1, j+1, \dots, N\}$, and we construct a continuous increasing function g as above but subject to the new ordering. Rather than re-work the details from the proof of Theorem 2.10, we prove only part of that necessary. We sketch a proof that $\mu_\rho(A \mid U_j = 1) \geq \lambda(B \mid U_j = 1)$, a similar argument being valid with 1 replaced by 0 and the inequality reversed. The main step is to show that $f \leq g$ under the assumption that $U_j = 1$. Suppose that $1 \leq r < j$, and assume it has already been proved that $f_i(\mathbf{x}) \leq g_i(\mathbf{x})$ for $\mathbf{x} \in \Omega$ and $1 \leq i < r$. Let $\mathbf{x} \in \Omega$. We claim that

$$\begin{aligned} &\mu_\rho(U_r > \xi \mid U_i = f_i(\mathbf{x}) \text{ for } 1 \leq i < r) \\ &\qquad \leq \mu_\rho(U_r > \xi \mid U_j = 1,\ U_i = g_i(\mathbf{x}) \text{ for } 1 \leq i < r), \qquad \xi \in [0,1]. \end{aligned} \tag{4.11}$$

By monotonicity,

$$\begin{aligned} &\mu_{\rho,J}(\cdot \mid U_j = u,\ U_i = f_i(\mathbf{x}) \text{ for } 1 \leq i < r) \\ &\qquad \leq_{\text{st}} \mu_{\rho,J}(\cdot \mid U_j = 1,\ U_i = g_i(\mathbf{x}) \text{ for } 1 \leq i < r), \qquad u \in [0,1]. \end{aligned} \tag{4.12}$$

The left-hand side of (4.12) is a version of the conditional expectation of the conditional measure $\mu_{\rho,J}(\cdot \mid U_i = f_i(\mathbf{x}) \text{ for } 1 \leq i < r)$ given $\sigma(U_j)$. By averaging over the value of u in (4.12), we obtain (4.11). The other steps are proved similarly. □

Unlike the discrete setting of Section 3, Theorem 4.9 does not imply a sharp-threshold theorem. Any density function ρ on $[0,1]^N$ may be used to generate a parametric family $(\rho_p : 0 < p < 1)$ of densities given by

$$\rho_p(\mathbf{x}) = \frac{1}{Z_{\rho,p}} \rho(\mathbf{x}) \prod_{i=1}^N p^{x_i}(1-p)^{1-x_i}, \qquad \mathbf{x} = (x_1, x_2, \dots, x_N) \in [0,1]^N,$$

and we write $\mu_p = \mu_{\rho_p}$. Let A be an increasing subset of $[0,1]^N$. The proof of Theorem 3.3 may be adapted to this setting to obtain that

$$\frac{d}{dp}\mu_p(A) = \frac{1}{p(1-p)} \sum_{i=1}^N \operatorname{cov}_p(U_i, 1_A),$$

where $U = (U_1, U_2, \dots, U_N)$ is the identity function on $[0,1]^N$, and cov_p denotes covariance with respect to μ_p.

Let ρ be the constant function, so that μ_ρ is Lebesgue measure. As above, let $p \in (0,1)$ and let $Y_1, Y_2, \dots, Y_N$ be independent random variables taking values in $[0,1]$ with common density function

$$\rho_p(x) = \begin{cases} \dfrac{\log[p/(1-p)]}{2p-1} p^x (1-p)^{1-x} & \text{if } p \neq \frac{1}{2},\ x \in (0,1), \\ 1 & \text{if } p = \frac{1}{2},\ x \in (0,1). \end{cases}$$

It is easily checked that the joint density function

$$\rho_p(\mathbf{x}) = \prod_{i=1}^{N} \rho_p(x_i), \qquad \mathbf{x} = (x_1, x_2, \ldots, x_N) \in [0,1]^N,$$

satisfies the FKG lattice condition, and is therefore monotonic.

We now choose A by $A = (N^{-1}, 1]^N$. It is an easy calculation that

$$\mu_p(A) = \begin{cases} \left(1 - \dfrac{\pi^{1/N} - 1}{\pi - 1}\right)^N & \text{if } p \neq \frac{1}{2}, \\ \left(1 - \dfrac{1}{N}\right)^N & \text{if } p = \frac{1}{2}, \end{cases}$$

where $\pi = p/(1-p)$. Therefore, as $N \to \infty$,

$$\mu_p(A) \to \begin{cases} \pi^{-1/(\pi-1)} & \text{if } p \neq \frac{1}{2}, \\ e^{-1} & \text{if } p = \frac{1}{2}. \end{cases}$$

In addition,

$$\mathrm{cov}_{\frac{1}{2}}(U_i, 1_A) = \frac{1}{N}\left(1 - \frac{1}{N}\right)^{N-1} \sim \frac{e^{-1}}{N}.$$

The influence theorem, Theorem 4.9, may be applied to the event A, but there is no sharp threshold for $\mu_p(A)$. This situation diverges from that of the discrete setting at the point where a lower bound for the conditional influence $I_A(i)$ is used to calculate a lower bound for the covariance $\mathrm{cov}_p(U_i, 1_A)$.

We return briefly to the measurability of an increasing subset of $[0,1]^N$.

Theorem 4.13. *Let $N \geq 2$. Every increasing subset of $[0,1]^N$ is Lebesgue-measurable.*

Increasing subsets need not be Borel-measurable, as the following example indicates. Let M be a non-Borel-measurable subset of $[0,1]$. Consider the increasing subset A of $[0,1]^2$ given by

$$A = \{(x,y) \in [0,1]^2 : x + y > 1\} \cup \{(x, 1-x) : x \in M\}.$$

The function $h : x \mapsto (x, 1-x)$ is a continuous, and hence Borel-measurable, function from $\mathbb{R}$ to $\mathbb{R}^2$. If A were Borel-measurable, then so would be

$$A' = A \cap \{(x, 1-x) : x \in \mathbb{R}\} = \{(x, 1-x) : x \in M\}.$$

This would imply that $h^{-1}(A') = M$ is Borel-measurable, a contradiction.

Proof of Theorem 4.13. The statement is trivially true when $N = 1$, and we prove the general case by induction on N. Suppose n is such that the result holds for $N = n$. Let A be an increasing subset of $[0,1]^{n+1}$, and let $g : [0,1]^n \to [0,1] \cup \{\infty\}$ be defined by

$$g(\mathbf{x}) = \inf\{y : (\mathbf{x}, y) \in A\}, \qquad \mathbf{x} \in [0,1]^n.$$

The function g is decreasing on $[0,1]^n$, and hence, for all $c \in \mathbb{R}$, the subset $H_c = \{\mathbf{x} : g(\mathbf{x}) < c\}$ is increasing. By the induction hypothesis, each H_c is Lebesgue-measurable in $[0,1]^n$, and therefore g is a measurable function. Its graph $G = \{(\mathbf{x}, g(\mathbf{x})) : \mathbf{x} \in [0,1]^n\}$ is (by an approximation by simple functions, or otherwise) a Lebesgue-measurable set and is also (by Fubini's Theorem) a null subset of $[0,1]^{n+1}$. Furthermore, the set

$$\overline{A} = \{(\mathbf{x}, y) \in [0,1]^{n+1} : y > g(\mathbf{x})\}$$

is Lebesgue-measurable. Now A differs from $\overline{A}$ only on a subset of the null set G, and the claim follows. □

5 The random-cluster model

The sharp-threshold theorem of Section 3 may be applied as follows to the random-cluster measure. Let $G = (V, E)$ be a finite graph, assumed for simplicity to have neither loops nor multiple edges. We take as configuration space the set $\Omega = \{0,1\}^E$, and write $\mathcal{F}$ for the set of its subsets. For $\omega \in \Omega$, we call an edge e *open* (in ω) if $\omega(e) = 1$, and *closed* otherwise. Let $\eta(\omega) = \{e \in E : \omega(e) = 1\}$ be the set of open edges, and consider the open graph $G_\omega = (V, \eta(\omega))$. The connected components of G_ω are termed *open clusters*, and $k(\omega)$ denotes the number of such clusters (including any isolated vertices).

Let $q \in (0, \infty)$, and let μ be the probability measure on $(\Omega, \mathcal{F})$ given by

$$\mu(\omega) = \frac{1}{Z(q)} q^{k(\omega)}, \qquad \omega \in \Omega, \tag{5.1}$$

where $Z(q)$ is the appropriate normalizing constant. It is clear that μ is positive, and it is easily checked that μ satisfies the FKG lattice condition if $q \geq 1$. See [8, 14]. (The FKG lattice condition does not hold when $q < 1$ and G contains a circuit.) *We assume henceforth that* $q \geq 1$. By Theorem 2.7, μ is monotonic.

The random-cluster measure $\phi_{p,q}$ on the graph G with parameters $p \in (0,1)$ and $q \in [1,\infty)$ is given as in (3.1) by

$$\phi_{p,q}(\omega) = \frac{1}{Z(p,q)} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)}, \qquad \omega \in \Omega. \tag{5.2}$$

It is well known (see [8, 14]) that

$$\frac{p}{p+q(1-p)} \leq \phi_{p,q}(X_e = 1) \leq p, \qquad e \in E. \tag{5.3}$$

We call G $\mathcal{A}$-*transitive* if its automorphism group possesses a subgroup $\mathcal{A}$ acting transitively on E. We may apply Theorem 3.5 to obtain the following. There exists an absolute constant $c > 0$ such that, for all $\mathcal{A}$-transitive graphs G, all p, q, and any increasing $\mathcal{A}$-invariant event $A \in \mathcal{F}$,

$$\frac{d}{dp}\phi_{p,q}(A) \geq c \min\left\{ \frac{q}{\{p+q(1-p)\}^2}, 1 \right\} \min\{\phi_{p,q}(A), 1-\phi_{p,q}(A)\} \log N,$$

whence

$$\frac{d}{dp}\phi_{p,q}(A) \geq \frac{c}{q} \min\{\phi_{p,q}(A), 1-\phi_{p,q}(A)\} \log N. \tag{5.4}$$

The differential inequality (5.4) takes the usual simpler form when $q = 1$, and it may be integrated exactly for general $q \geq 1$. Here is an illustration of (5.4) when integrated. Let $p_1 \in (0,1)$ be chosen such that $\phi_{p_1,q}(A) \geq \frac{1}{2}$, and let $p_1 < p_2 < 1$. We note that $\phi_{p,q}(A) \geq \frac{1}{2}$ for $p \in (p_1, p_2)$. We integrate (5.4) over this interval to obtain that

$$\phi_{p_2,q}(A) \geq 1 - \tfrac{1}{2} N^{-c(p_2-p_1)/q}. \tag{5.5}$$

Bollobás and Riordan have shown in [4, 5] how to apply the sharp-threshold theorem for product measure to percolation in two dimensions, thereby obtaining a further proof of the famous theorem of Harris and Kesten that the critical probability of bond percolation equals $\frac{1}{2}$. Their key step is the proof that there exists a sharp threshold for the event that a large square is traversed by an open path. One obtains similarly the following for the random-cluster model on the square lattice $\mathbb{L}^2$.

Let $\mathbb{Z} = \{\dots, -1, 0, -1, \dots\}$ be the integers, and $\mathbb{Z}^2$ the set of all 2-vectors $x = (x_1, x_2)$ of integers. We turn $\mathbb{Z}^2$ into a graph by placing an edge between any two vertices x, y with $|x-y| = 1$, where

$$|z| = |z_1| + |z_2|, \qquad z \in \mathbb{Z}^2.$$

We write $\mathbb{E}^2$ for the set of such edges, and $\mathbb{L}^2 = (\mathbb{Z}^2, \mathbb{E}^2)$ for the ensuing graph. We shall work on a finite torus of $\mathbb{L}^2$. Let $n \geq 1$. Consider the square $S_n = [0, n]^2$ (this is a convenient abbreviation for $\{0, 1, 2, \ldots, n\}^2$) viewed as a subgraph of $\mathbb{L}^2$. We identify certain pairs of vertices on the boundary of S_n in order to make it symmetric. More specifically, we identify any pair of the form $(0, m)$, (n, m) and of the form $(m, 0)$, (m, n), for $0 \leq m \leq n$, and we merge any parallel edges that ensue. Let $T_n = (V_n, E_n)$ denote the resulting toroidal graph. Let $\mathcal{A}_n$ be the automorphism group of the graph T_n, and note that $\mathcal{A}_n$ acts transitively on E_n. The configuration space of the random-cluster model on T_n is denoted $\Omega(n) = \{0, 1\}^{E_n}$.

Let $p \in (0, 1)$ and $q \in [1, \infty)$. Write $\phi_{n,p}$ for the random-cluster measure on T_n with parameters p and q, and note that $\phi_{n,p}$ is $\mathcal{A}_n$-invariant. Let

$$p_{\mathrm{sd}} = p_{\mathrm{sd}}(q) = \frac{\sqrt{q}}{1 + \sqrt{q}},$$

the self-dual point of the random-cluster model on $\mathbb{L}^2$, see [13, 14]. We note that the (Whitney) dual of T_n is isomorphic to T_n, and the random-cluster measure on T_n is self-dual when $p = p_{\mathrm{sd}}$.

Let $\omega \in \Omega(n)$. Any translate in T_n of a rectangle of the form $[0, r] \times [0, s]$ is said to be of size $r \times s$. When $r \neq s$, such a translate is said to be traversed *long-ways* (respectively, traversed *short-ways*) if the two shorter sides (respectively, longer sides) of the rectangle are joined within the rectangle by an open path of ω.

Let $k \geq 2$, $n \geq 1$. Let $R_n = [0, n+1] \times [0, n]$, viewed as a subgraph of T_{kn}, and let LW_n be the event that R_n is traversed long-ways. By a standard duality argument,

$$\phi_{kn,p_{\mathrm{sd}}}(\mathrm{LW}_n) = \tfrac{1}{2}, \qquad k \geq 2,\ n \geq 1. \tag{5.6}$$

Let A_n be the event that there exists in T_{kn} some translate of the square $S_n = [0, n] \times [0, n]$ that possesses either an open top–bottom crossing or an open left–right crossing. The event A_n is $\mathcal{A}_n$-invariant, and

$$\phi_{kn,p_{\mathrm{sd}}}(A_n) \geq \phi_{kn,p_{\mathrm{sd}}}(\mathrm{LW}_n) = \tfrac{1}{2}. \tag{5.7}$$

We apply (5.5) to the event A_n, with $p_1 = p_{\mathrm{sd}}$ and with $N = 2(kn)^2$ being the number of edges in T_{kn}. This yields that

$$\begin{aligned} \phi_{kn,p}(A_n) &\geq 1 - \tfrac{1}{2}[2(kn)^2]^{-c(p-p_{\mathrm{sd}})/q} \\ &\geq 1 - (kn)^{-2c(p-p_{\mathrm{sd}})/q}, \qquad p_{\mathrm{sd}} < p < 1. \end{aligned} \tag{5.8}$$

The event A_n is defined on the whole of the torus. We next use an argument taken from [4, 5] to obtain a more locally defined event. We shall for simplicity of notation treat certain real-valued quantities as if they were integers. Let $1 < \alpha < k$, and let $H_{n,\alpha} = [0, \alpha n] \times [0, n/\alpha]$ and $V_{n,\alpha} = [0, n/\alpha] \times [0, \alpha n]$. Let $h_{n,\alpha}$, $v_{n,\alpha}$ be the sets of vertices in T_{kn} given by

$$\begin{aligned} h_{n,\alpha} &= \{(l_1 n(\alpha - 1), l_2 n(1 - \alpha^{-1})) \in V_{kn} : l_1, l_2 \in \mathbb{Z}\}, \\ v_{n,\alpha} &= \{(l_1 n(1 - \alpha^{-1}), l_2 n(\alpha - 1)) \in V_{kn} : l_1, l_2 \in \mathbb{Z}\}. \end{aligned}$$

Consider the set $\mathcal{H} = H_{n,\alpha} + h_{n,\alpha}$ of translates of $H_{n,\alpha}$ by vectors in $h_{n,\alpha}$, and also the set $\mathcal{V} = V_{n,\alpha} + v_{n,\alpha}$. If A_n occurs, then some rectangle in $\mathcal{H} \cup \mathcal{V}$ is traversed short-ways. By positive association and symmetry,

$$\begin{aligned} \phi_{kn,p}(\overline{A_n}) &\geq \phi_{kn,p}(\text{no member of } \mathcal{H} \cup \mathcal{V} \text{ is traversed short-ways}) \\ &\geq \{1 - \phi_{kn,p}(\mathrm{SW}_{n,\alpha})\}^M, \end{aligned} \tag{5.9}$$

where $\mathrm{SW}_{n,\alpha}$ is the event that H_n is traversed short-ways, and

$$M = |h_{n,\alpha}| + |v_{n,\alpha}|. \tag{5.10}$$

After taking into account the rounding effects above, we find that

$$M \leq 2\left(1 + \frac{k}{\alpha - 1 - n^{-1}}\right)\left(1 + \frac{k}{1 - \alpha^{-1} - n^{-1}}\right), \tag{5.11}$$

so that M is approximately $2k^2\alpha/(\alpha - 1)^2$ when k and n are large.

Combining (5.8)–(5.10), we arrive at the following theorem, where $\mathrm{SW}_{n,\alpha}$ is the event that the rectangle $\big[0, \lfloor n\alpha \rfloor\big] \times \big[0, \lfloor n/\alpha \rfloor\big]$ is crossed short-ways.

Theorem 5.12. *Let $k \geq 2$, $n \geq 1$, and $p_{\mathrm{sd}} < p < 1$. We have that*

$$\phi_{kn,p}(\mathrm{SW}_{n,\alpha}) \geq 1 - e^{-g(p - p_{\mathrm{sd}})} \tag{5.13}$$

where

$$g = g(k, n, \alpha, q) = \frac{2c}{Mq} \log(kn).$$

In particular, for $p > p_{\mathrm{sd}}$, one may make $\phi_{kn,p}(\mathrm{SW}_{n,\alpha})$ large by holding k fixed and sending $n \to \infty$. It does not seem to be easy to deduce an estimate for $\phi_{p,q}(\mathrm{SW}_{n,\alpha})$ for a random-cluster measure $\phi_{p,q}$ on the infinite lattice $\mathbb{L}^2$. Neither do we know how to use the existence of crossings short-ways to build crossings long-ways. This is in contrast to the case of product measure, see [5, 7, 12, 17, 18, 19].

6 The critical point

There is a famous conjecture that the critical point $p_c(q)$ of the random-cluster model on $\mathbb{L}^2$ equals $p_{sd}(q)$. We do not spell out the details necessary to state this conjecture properly, referring the reader instead to [13, 14]. The conjecture is known to be valid for $q = 1$ (percolation), $q = 2$ (a case corresponding to the Ising model), and for sufficiently large q (namely $q \geq 21.61$). The conjecture would follow if one could prove a strengthening of Theorem 5.12 in which short-ways is replaced by long-ways, and with the toroidal measure replaced by the wired measure on the full lattice. We finish by explaining this.

The so-called 'wired random-cluster measure' on $\mathbb{L}^2$ is denoted by $\phi^1_{p,q}$, and the reader is referred to the references above for a definition of $\phi^1_{p,q}$.

Theorem 6.1. *Let $q \geq 1$. Let p_k be the $\phi^1_{p,q}$-probability that a $2^k \times 2^{k+1}$ rectangle is crossed long-ways. Suppose that*

$$\prod_{k=1}^{\infty} p_k > 0, \qquad p > p_{sd}(q). \tag{6.2}$$

Then the critical point of the random-cluster model on $\mathbb{L}^2$ equals $p_{sd}(q)$.

By duality, $1 - p_k = \phi^0_{p',q}(\mathrm{SW}(k))$, where $\mathrm{SW}(k)$ is the event that the rectangle $[0, 2^{k+1} - 1] \times [0, 2^k + 1]$ is traversed short-ways, and p' is the dual value of p,

$$\frac{p'}{1-p'} = \frac{q(1-p)}{p}.$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty}(1-p_k) &\leq \sum_{k=1}^{\infty} 2^{k+1}\phi^0_{p',q}(\mathrm{rad}(C) \geq 2^k+1)\\ &\leq 4\sum_{n=1}^{\infty}\phi^0_{p',q}(\mathrm{rad}(C) \geq n)\\ &= 4\phi^0_{p',q}(\mathrm{rad}(C)),\end{aligned}$$

where $\mathrm{rad}(C)$ is radius of the open cluster C at the origin, that is, the maximum value of n such that 0 is joined by an open path to the boundary of the box $[-n, n]^2$. It follows that

$$\phi^0_{p',q}(\mathrm{rad}(C)) < \infty, \qquad p < p_{sd}(q),$$

is sufficient for $p_c(q) = p_{sd}(q)$.

Proof. We use a construction given in [7], which was known earlier to one of the current authors and to Paul Seymour. For odd k, let A_k be the event that $[0, 2^k] \times [0, 2^{k+1}]$ is traversed long-ways. For even k, let A_k be the event that $[0, 2^{k+1}] \times [0, 2^k]$ is traversed long-ways. By the positive-associativity and automorphism-invariance of $\phi^1_{p,q}$, under (6.2),

$$\phi^1_{p,q}\left(\bigcap_k A_k\right) \geq \prod_{k=1}^{\infty} \phi^1_{p,q}(A_k) > 0, \qquad p > p_{\mathrm{sd}}(q).$$

On the intersection of the A_k, there exists an infinite open cluster, and therefore $p_{\mathrm{c}}(q) \leq p_{\mathrm{sd}}(q)$. It is standard (see [13, 14]) that $p_{\mathrm{sd}}(q) \leq p_{\mathrm{c}}(q)$, and therefore equality holds as claimed. □

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