

HELFFER-SJÖSTRAND REPRESENTATION FOR CONSERVATIVE DYNAMICS

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ABSTRACT. We consider a Helffer-Sjöstrand representation for the correlations in canonical Gibbs measures with convex interactions under conservative Ginzburg-Landau dynamics. We investigate the rate of relaxation to equilibrium.

Keywords Ginzburg-Landau dynamics, canonical Gibbs measure, random walk representation, monotonicity.

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1. INTRODUCTION

The complicated interactions in particle systems lead to subtle correlations which in some cases can be represented in term of simpler quantities, e.g. random walk crossings [9]. A celebrated representation for the correlations of Gibbs measures has been obtained by Helffer and Sjöstrand by means of the Witten Laplacian [16, 17]. In this representation, the decay of correlations is related to spectral properties and it can be studied by using spectral theory [13, 14, 15]. For effective interface models, this representation triggered a probabilistic reinterpretation of the Witten Laplacian as the generator of a random walk coupled to the evolution of the particle system [25, 8, 12]. For a large class of effective interface models, the correlations between two sites x, y can be understood as the total time spent at y by a random walk (in a random environment) starting from x . The strength of the Helffer-Sjöstrand representation is to relate the equilibrium correlations to the behavior of the dynamics associated to the model.

In this paper, we investigate the Helffer-Sjöstrand representation for the correlations in canonical Gibbs measures, i.e. measures conditioned to have a fixed mean density. This relates the equilibrium correlations in canonical measures to the conservative Ginzburg-Landau dynamics. In contrast to the non-conservative dynamics considered in the previous works [16, 8], fixing the total density leads to a Witten Laplacian with a different structure: the

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gradients are now replaced by gradient differences. In one-dimension, we will show that for a class of Hamiltonian with convex interactions, (2.3), the correlations for the canonical Gibbs measure can be interpreted as the occupation time of a random walk evolving in a random environment coupled to the conservative Ginzburg-Landau dynamics. Furthermore, the space-time correlations of the dynamics are also encoded in the diffusive mechanism of the random walk. For some specific dynamics, like the symmetric simple exclusion process, a similar property for the correlations was obtained by duality [23]. In our model, the random walk is not a consequence of a duality property but it is reminiscent of the stochastic process considered in [8].

We will use the Helffer-Sjöstrand representation to study the relaxation of the one-dimensional conservative Ginzburg-Landau dynamics. Equilibrium fluctuations of the Ginzburg-Landau model and the convergence of the density field to a generalized Ornstein-Uhlenbeck process have been obtained in [24, 28]. The relaxation is also expected to occur at a microscopic scale: for initial data $\eta \in \mathbb{R}^{\mathbb{Z}}$ sampled from the equilibrium measure $\langle \cdot \rangle_\rho$, the space-time correlation is conjectured to obey the following scaling form for large t and $|i|$ ([27] page 177 equation (2.14))

$$\langle \eta_0(0); \eta_i(t) \rangle_\rho \approx \frac{\chi(\rho)}{\sqrt{2\pi\hat{q}(\rho)t}} \exp\left(-\frac{i^2}{2t\hat{q}(\rho)}\right), \quad (1.1)$$

where $\hat{q}(\rho)$ is a diffusion coefficient and $\chi(\rho) = \langle \eta_0; \eta_0 \rangle_\rho$ is the susceptibility. The intuition behind the scaling (1.1) is that an initial fluctuation of the density at the origin will diffuse in the course of time. By combining localization techniques and spectral gap bounds, sharp relaxation estimates of the type (1.1) for functions supported in a neighborhood of the origin were derived (in any dimension) for discrete models in [1, 2, 3, 4, 18] and for continuous variables in [21].

For continuous variable models, the Helffer-Sjöstrand representation provides an alternative way to understand (1.1) as it relates the correlation between the origin and a site i at time t to the probability that the random walk starting at 0 touches the site i at time t . This random walk evolves in a random environment coupled to the evolution of the particle system, therefore its precise limiting properties are difficult to study. By using the general theory of Aronson, De Giorgi, Nash, Moser for uniformly elliptic second order operators, some estimates can be obtained on the kernel of this random walk. This leads to bounds on the correlation functions in (1.1) (see Section 4). In the spirit of [12], more global relaxation estimates can be obtained by using the homogenization theory of Kipnis, Varadhan [19] (see Section 5).

2. GINZBURG-LANDAU DYNAMICS

Let Λ denote the one dimensional torus $\Lambda = (\mathbb{Z}/N\mathbb{Z})$ with nearest neighbor edges. We are going to study the relaxation properties of conservative

Ginzburg-Landau dynamics on \mathbb{R}^Λ

$$\forall i \in \Lambda, \quad d\eta_i(t) = \sum_{j=i\pm 1} \left(\frac{\partial H}{\partial \eta_j}(\eta) - \frac{\partial H}{\partial \eta_i}(\eta) \right) dt + \sqrt{2}(dB_{(i,i+1)}(t) - dB_{(i-1,i)}(t)). \quad (2.1)$$

where $(B_{(i,i+1)}(t))_{i \in \Lambda}$ denote independent standard Brownian motions associated to each edge and we will consider Hamiltonians of the form

$$H(\eta) = \sum_{i \in \Lambda} V_1(\eta_i) + V_2(\eta_i + \eta_{i+1}). \quad (2.2)$$

We will require convexity assumptions on the potentials, i.e. that there are constants C_\pm such that for

$$0 < C_- \leq V_k''(\cdot) \leq C_+, \quad k = 1, 2. \quad (2.3)$$

The dynamics (2.1) can be extended on \mathbb{Z} (see [10, 28]). In Sections 4 and 5, we will investigate the relaxation properties in the infinite volume limit and for this, we will have to restrict to Hamiltonians without interactions, i.e. $V_2 = 0$.

The Gibbs measure on \mathbb{R}^Λ associated to H will be denoted by μ_N and the canonical Gibbs measure with mean density ρ by

$$\mu_{\rho,N}(\cdot) = \mu_N \left(\cdot \mid \sum_i \eta_i = |\Lambda|\rho \right).$$

We will write $\langle \cdot \rangle$ to denote expectation with respect to $\mu_{\rho,N}$.

Let $\mathcal{S}_{\rho,N} = \{\eta \in \mathbb{R}^\Lambda : \sum_i \eta_i = |\Lambda|\rho\}$. In Λ , the dynamics (2.1) conserve the total density $\sum_{i \in \Lambda} \eta_i$. We will see in (2.4) that $\mu_{\rho,N}$ is a (reversible) invariant measure.

The generator of the dynamics (2.1) is given by

$$L_\Lambda = \sum_{i \in \Lambda} - \left(\frac{\partial}{\partial \eta_{i+1}} - \frac{\partial}{\partial \eta_i} \right)^2 + \left(\frac{\partial H}{\partial \eta_{i+1}} - \frac{\partial H}{\partial \eta_i} \right) \left(\frac{\partial}{\partial \eta_{i+1}} - \frac{\partial}{\partial \eta_i} \right).$$

Let $\mathcal{B} = \{(i, i+1)\}_{i \in \Lambda}$ denote the set of oriented nearest neighbor bonds of Λ . It is natural to associate a differential operator with each edge. For $b = (i, j) \in \mathcal{B}$, we will write ∂_b to denote $\partial/\partial \eta_j - \partial/\partial \eta_i$. We will write $\bar{\nabla}$ to denote the vector of all such operators: $\bar{\nabla} = (\partial_b)_{b \in \mathcal{B}}$. The generator L_Λ can now be written:

$$L_\Lambda = -\bar{\nabla} \cdot \bar{\nabla} + \bar{\nabla} H \cdot \bar{\nabla}.$$

Unless otherwise stated, take ρ and N to be fixed quantities. Note that there is an integration by parts formula

$$\langle f \bar{\nabla} g \rangle = \langle -g \bar{\nabla} f \rangle + \langle f g \bar{\nabla} H \rangle, \quad (2.4)$$

where $\langle \cdot \rangle$ denotes the expectation with respect to $\mu_{\rho,N}$. Thus L_Λ is self-adjoint with respect to $\mu_{\rho,N}$ and the dynamics are reversible. The operator L_Λ has a self-adjoint extension with domain included in $L^2(\mathcal{S}_{\rho,N}, \mu_{\rho,N})$.

3. CORRELATIONS FOR THE CANONICAL MEASURE

3.1. Helffer-Sjöstrand representation. We will derive a formula similar to the Helffer-Sjöstrand representation [17, 13, 14, 15] for the correlations under the canonical Gibbs measure $\mu_{\rho,N}$. We follow a formalism similar to the one applied previously for the non-conservative (Langevin) dynamics [16].

The correlation between two functions f, g under the measure $\mu_{\rho,N}$ is defined by

$$\langle f; g \rangle = \langle (f - \langle f \rangle)(g - \langle g \rangle) \rangle .$$

The operator L_Λ has a spectral gap (see [20, 6] or the Appendix for an alternative proof) and thus for any smooth function g in $\mathcal{C}_0^\infty(\mathcal{S}_{\rho,N}, \mathbb{R})$ there exists a unique inverse u in $\mathcal{C}_0^\infty(\mathcal{S}_{\rho,N}, \mathbb{R})$ such that

$$L_\Lambda u = g - \langle g \rangle .$$

Using integration by parts (2.4), one has:

$$\langle f; g \rangle = \langle (f - \langle f \rangle)L_\Lambda u \rangle = \mathcal{E}(f, u) ,$$

where the Dirichlet form \mathcal{E} is defined by

$$\mathcal{E}(f, u) = \langle f L_\Lambda u \rangle = \langle \bar{\nabla} f \cdot \bar{\nabla} u \rangle . \quad (3.1)$$

Let Hess H denote the (bond-wise) Hessian matrix

$$\text{Hess } H = [\partial_b \partial_c H]_{b,c \in \mathcal{B}} .$$

One has

$$\bar{\nabla} g = \bar{\nabla} L_\Lambda u = \mathcal{L}_\Lambda \bar{\nabla} u \quad (3.2)$$

where \mathcal{L}_Λ denotes the ‘Witten-Laplacian’ operator defined on $\mathcal{C}_0^\infty(\mathcal{S}_{\rho,N}, \mathbb{R}^{\mathcal{B}})$ by,

$$\mathcal{L}_\Lambda U(\eta) = L_\Lambda \otimes \text{Id } U(\eta) + (\text{Hess } H) \cdot U(\eta), \quad U : \mathbb{R}^\Lambda \rightarrow \mathbb{R}^{\mathcal{B}} .$$

Here $L_\Lambda \otimes \text{Id}$ denotes the identity matrix with diagonal elements equal to L_Λ . Note that the operator \mathcal{L}_Λ is also self-adjoint in $\mathbb{L}^2(\mathcal{S}_{\rho,N}, \mu_{\rho,N})$.

Combining (3.1) and (3.2), we therefore have for any f, g in $\mathcal{C}_0^\infty(\mathcal{S}_{\rho,N}, \mathbb{R})$

$$\langle f; g \rangle = \langle \bar{\nabla} f \mathcal{L}_\Lambda^{-1} \bar{\nabla} g \rangle . \quad (3.3)$$

We remark that the Witten Laplacian \mathcal{L}_Λ and the identity (3.3) could have been defined for more general Hamiltonians than (2.2), and in any dimension. In the Appendix, we exploit (3.3) to estimate the spectral gap in dimension $d \geq 1$. However the interpretation of \mathcal{L}_Λ as the generator of stochastic dynamics (Section 3.2) requires the assumption (2.3) on the Hamiltonian H and the one-dimensional structure.

3.2. The random walk representation. Let $X(t)$ denote a continuous time random walk on the edges in \mathcal{B} with jump rates determined by the Hessian of H : $X(t)$ steps from b to c at rate $-\partial_b \partial_c H$. Thus if $X(t) = b = (i, i+1)$, the non-zero jump rate to the bond $b+k = (i+k, i+k+1)$ is given by (with $k \in \{-2, -1, 1, 2\}$)

$$\begin{aligned} -\partial_b \partial_{b-2} H &= V_2''(\eta_i + \eta_{i-1}), \\ -\partial_b \partial_{b-1} H &= V_1''(\eta_i), \\ -\partial_b \partial_{b+1} H &= V_1''(\eta_{i+1}), \\ -\partial_b \partial_{b+2} H &= V_2''(\eta_{i+1} + \eta_{i+2}). \end{aligned}$$

Thus \mathcal{L}_Λ can be interpreted as the generator of the joint evolution $(\eta(t), X(t))$ in $\mathcal{S}_{\rho, N} \times \mathcal{B}$. On this space, the representation (3.3) has a probabilistic interpretation.

Proposition 3.4. *Let \mathbb{E}_b^η denote the expectation for the joint process starting at $\eta(0) = \eta$ and $X(0) = b$.*

$$\langle f; g \rangle = \langle \bar{\nabla} f \mathcal{L}_\Lambda^{-1} \bar{\nabla} g \rangle = \int_0^\infty \sum_{b, c \in \mathcal{B}} \langle \partial_b f(\eta) \mathbb{E}_b^\eta(1_{X(t)=c} \partial_c g(\eta(t))) \rangle dt. \quad (3.5)$$

With no extra effort, we can extend the Hamiltonian to include terms such as $V_3(\eta_i + \eta_{i+1} + \eta_{i+2})$, $V_4(\eta_i + \eta_{i+1} + \eta_{i+2} + \eta_{i+3})$, and so on. In this case the random walk would perform jumps from $(i, i+1)$ of length k with intensities $V_k''(\eta_i + \eta_{i-1} + \dots + \eta_{i-k+1})$ and $V_k''(\eta_{i+1} + \eta_{i+2} + \dots + \eta_{i+k})$. For this class of models, the variables η can be reinterpreted as the gradient field of effective interface models and therefore the representation (3.5) is equivalent to the one derived for the non-conservative Ginzburg-Landau dynamics (see for example [8, Proposition 2.2]). However, looking directly at the non-conservative dynamics changes the point of view. Below, we derive new identities for the conservative Ginzburg Landau dynamics.

It does not seem possible to extend this random walk representation to dimensions $d \geq 2$. The matrix $-\text{Hess } H$ cannot be interpreted (in an easy way) as generating a continuous time Markov chain: many of the off-diagonal elements are negative, so they cannot be interpreted as jump rates.

Proof of Proposition 3.4. The evolution under the semi-group generated by \mathcal{L}_Λ can be rewritten as follow. For any g in $\mathcal{C}_0^\infty(\mathcal{S}_{\rho, N}, \mathbb{R})$

$$e^{-t\mathcal{L}_\Lambda} \bar{\nabla} g(\eta) = \left(\mathbb{E}_b^\eta \left[\sum_c 1_{X(t)=c} \partial_c g(\eta_t) \right] \right)_{b \in \mathcal{B}}.$$

Applying \mathcal{L}_Λ yields,

$$\begin{aligned} \mathcal{L}_\Lambda \int_0^T e^{-t\mathcal{L}_\Lambda} \bar{\nabla} g(\eta) dt &= \mathcal{L}_\Lambda \left(\int_0^T \mathbb{E}_b^\eta \left[\sum_c 1_{X(t)=c} \partial_c g(\eta_t) \right] dt \right)_{b \in \mathcal{B}} \\ &= \bar{\nabla} g(\eta) - e^{-T\mathcal{L}_\Lambda} \bar{\nabla} g(\eta). \end{aligned}$$

The space of gradients is invariant under the operator \mathcal{L}_Λ , and therefore under the semi-group. Theorem A.1 implies that

$$\langle \bar{\nabla}g(\eta) e^{-t\mathcal{L}_\Lambda} \bar{\nabla}g(\eta) \rangle \leq \exp\left(-\frac{kC_-}{N^2}t\right) \langle (\bar{\nabla}g(\eta))^2 \rangle.$$

Taking the limit $T \rightarrow \infty$ yields

$$\mathcal{L}_\Lambda \left(\int_0^\infty \mathbb{E}_b^\eta \left[\sum_c 1_{X(t)=c} \partial_c g(\eta_t) \right] dt \right)_{b \in \mathcal{B}} = \bar{\nabla}g.$$

Hence

$$\mathcal{L}_\Lambda^{-1} \bar{\nabla}g = \left(\int_0^\infty \mathbb{E}_b^\eta \left[\sum_c 1_{X(t)=c} \partial_c g(\eta_t) \right] dt \right)_{b \in \mathcal{B}}$$

can be substituted into (3.3). \square

The joint dynamics including the field $\eta(t)$ and the random walk $X(t)$ can be extended to \mathbb{Z} (see e.g. [10]). In Section 4, we will relate the relaxation to equilibrium in \mathbb{Z} to the fluctuation of the random walk $X(t)$.

4. RELAXATION TO EQUILIBRIUM

In this section, we will consider the relaxation of the dynamics on \mathbb{Z} . We focus only on systems with Hamiltonians $H(\eta) = \sum_x V(\eta_x)$ and with a potential V satisfying the convexity assumption (2.3). This is equivalent to setting $V_1 = V$ and $V_2 = 0$ in (2.2). Let $\langle \cdot \rangle_\rho$ denote the corresponding invariant measure on \mathbb{Z} with mean density ρ ; $\langle \cdot \rangle_\rho$ is a product measure and the marginal probability density function can be written

$$Z^{-1} \exp(-V(\eta) + \alpha\eta),$$

where Z and α are constant chosen to give mass one and mean ρ .

The random walk representation provides a clear connection between the relaxation of the particle system and the diffusive mechanism conjectured in (1.1).

Proposition 4.1. *For any i and $t \geq 0$, the following identity holds*

$$\langle V'(\eta_0(0)); \eta_i(t) \rangle_\rho = \left\langle \mathbb{E}_{(0,1)}^\eta \left(1_{X(t)=(i,i+1)} \right) \right\rangle_\rho. \quad (4.2)$$

In the Gaussian case $V(x) = x^2/2$, the jump rates are uniform and the identity (4.2) coincides with the conjecture (1.1)

$$\langle \eta_0(0); \eta_i(t) \rangle_\rho = \mathbb{E}_{(0,1)}(1_{X(t)=(i,i+1)}) \simeq \frac{1}{(4\pi t)^{1/2}} \exp\left(-\frac{i^2}{4t}\right),$$

where X_t is just a simple random walk. A similar relation also holds for the space-time correlations of the symmetric simple exclusion process [23]. For general potentials, (4.2) confirms the conjecture (1.1) as it explicitly relates the relaxation to equilibrium to the relaxation of the random walk.

Relation (4.2) is non-symmetric, however upper and lower bound on the relaxation of the two-point correlation function can be obtained.

Proposition 4.3. *For any i and $t \geq 0$, one has*

$$\frac{1}{C_+} \leq \frac{\langle \eta_0(0); \eta_i(t) \rangle_\rho}{\left\langle \mathbb{E}_{(0,1)}^\eta (1_{X(t)=(i,i+1)}) \right\rangle_\rho} \leq \frac{1}{C_-}, \quad (4.4)$$

where the constants C_\pm were introduced in (2.3).

The previous estimates (4.4) can be turned into quantitative bounds using Aronson estimates for the transition kernel of strictly elliptic operators. Following [12], there exists c_1, c_2 such that for $t > 1$

$$\left\langle \mathbb{E}_{(0,1)}^\eta (1_{X(t)=(i,i+1)}) \right\rangle_\rho \leq \frac{c_1}{\sqrt{t}} \exp\left(-\frac{|i|}{c_1 \sqrt{t}}\right), \quad (4.5)$$

and for $|i| \leq \sqrt{t}$

$$\left\langle \mathbb{E}_{(0,1)}^\eta (1_{X(t)=(i,i+1)}) \right\rangle_\rho \geq \frac{c_2}{1 \vee \sqrt{t}}. \quad (4.6)$$

Proof of Proposition 4.1. First we will prove (4.2) for the dynamics in a finite domain Λ and then we will pass to the limit $\Lambda \rightarrow \mathbb{Z}$ for a fixed time t .

Given two functions f and g , and a time $t \geq 0$, we can apply formula (3.5) to f and $e^{-tL_\Lambda} g = P_t(g)$ which is the semi-group at time t for the Ginzburg-Landau dynamics

$$\begin{aligned} \langle f(\eta); e^{-tL_\Lambda} g(\eta) \rangle &= \langle f(\eta); P_t(g)(\eta) \rangle \\ &= \int_t^\infty \sum_{b,c \in \mathcal{B}} \langle \partial_b f(\eta) \mathbb{E}_b^\eta (1_{X(s)=c} \partial_c g(\eta(s))) \rangle ds, \end{aligned}$$

where $\langle \cdot \rangle$ refers to the finite volume measure $\mu_{\rho,N}$ with the canonical constraint. For $f(\eta) = V'(\eta_0)$ and $g(\eta) = \eta_i$, this gives

$$\bar{\nabla} f = \begin{cases} \partial_b f = V''(\eta_0), & \text{if } b = (-1, 0), \\ \partial_b f = -V''(\eta_0), & \text{if } b = (0, 1), \\ \partial_b f = 0, & \text{otherwise,} \end{cases}$$

and

$$\bar{\nabla} g = \begin{cases} \partial_b g = 1, & \text{if } b = (i-1, i), \\ \partial_b g = -1, & \text{if } b = (i, i+1), \\ \partial_b g = 0, & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned} \langle V'(\eta_0(0)); \eta_i(t) \rangle &= \\ &= \int_t^\infty ds \left\langle V''(\eta_0) \mathbb{E}_{(-1,0)}^\eta (1_{X(s)=(i-1,i)}) \right\rangle + \left\langle V''(\eta_0) \mathbb{E}_{(0,1)}^\eta (1_{X(s)=(i,i+1)}) \right\rangle \\ &\quad - \left\langle V''(\eta_0) \mathbb{E}_{(-1,0)}^\eta (1_{X(s)=(i,i+1)}) \right\rangle - \left\langle V''(\eta_0) \mathbb{E}_{(0,1)}^\eta (1_{X(s)=(i-1,i)}) \right\rangle. \end{aligned}$$

We introduce the function

$$F_s(b, \eta) = \mathbb{E}_b^\eta(1_{X(s)=(i, i+1)})$$

which is the probability that the walk starting at the edge b in an initial field η is located at $(i, i+1)$ at time s .

Using translation invariance,

$$\begin{aligned} \langle V'(\eta_0(0)); \eta_i(t) \rangle &= \int_t^\infty ds \left\langle V''(\eta_1) \left[F_s((0, 1), \eta) - F_s((1, 2), \eta) \right] \right\rangle \\ &\quad + \left\langle V''(\eta_0) \left[F_s((0, 1), \eta) - F_s((-1, 0), \eta) \right] \right\rangle. \end{aligned}$$

As $\langle \cdot \rangle$ is the invariant measure, $\langle L_\Lambda F_s((0, 1), \eta) \rangle = 0$, and so

$$\begin{aligned} \langle V'(\eta_0(0)); \eta_i(t) \rangle &= \int_t^\infty \langle [\mathcal{L}_\Lambda F_s]((0, 1), \eta) \rangle ds = - \int_t^\infty \partial_s \langle F_s((0, 1), \eta) \rangle ds \\ &= \langle F_t((0, 1), \eta) \rangle - \lim_{s \rightarrow \infty} \langle F_s((0, 1), \eta) \rangle. \end{aligned} \quad (4.7)$$

The random walk is uniformly distributed in the limit $s \rightarrow \infty$, and so $\langle F_s((0, 1), \eta) \rangle \rightarrow |\Lambda|^{-1}$ where $|\Lambda|$ is the number of sites in Λ .

For any fixed time $t > 0$, one can take the limit $\Lambda \rightarrow \mathbb{Z}$ in (4.7). This concludes the proof of (4.2). \square

Proof of Proposition 4.3. The dynamics have a monotonicity property.

Lemma 4.8. *Let $f, g : \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$ be two locally defined functions that are non-decreasing with respect to the coordinatewise partial order. For $t \geq 0$, $\langle f(\eta(0)); g(\eta(t)) \rangle_\rho \geq 0$.*

We postpone the proof of the Lemma and first use it to deduce Proposition 4.3. The function $h(\xi) = \frac{V'(\xi)}{C_-} - \xi$ is non-decreasing. By Lemma 4.8,

$$0 \leq \langle h(\eta_0(0)); \eta_i(t) \rangle_\rho.$$

Hence, by the bilinearity of covariances,

$$\langle \eta_0(0); \eta_i(t) \rangle_\rho \leq \frac{1}{C_-} \langle V'(\eta_0(0)); \eta_i(t) \rangle_\rho = \frac{1}{C_-} \left\langle \mathbb{E}_{(0,1)}^\eta(1_{X(t)=(i, i+1)}) \right\rangle_\rho.$$

The lower bound follows in the same way. \square

Monotonicity properties for non-conservative dynamics have been considered in [11]. We provide below an alternative proof tailored for our model.

Proof of Lemma 4.8. By a standard approximation argument, it is sufficient to show that for locally defined, increasing events A and B ,

$$\mu_\rho(\eta(0) \in A \text{ and } \eta(t) \in B) \geq \mu_\rho(A)\mu_\rho(B). \quad (4.9)$$

Let $\eta, \tilde{\eta}$ represent two solutions to the SDE (2.1) such that

- (i) $\tilde{\eta}(0)$ has distribution μ_ρ , and
- (ii) $\eta(0)$ has distribution $\mu_\rho(\cdot | A)$.

We will write $\tilde{\eta}(t) \leq \eta(t)$ if $\tilde{\eta}_i(t) \leq \eta_i(t)$ for all $i \in \mathbb{Z}$. Note that (4.9) holds if with probability one, $\tilde{\eta}(t) \leq \eta(t)$.

The product measure μ_ρ only differs from the conditional measure $\mu_\rho(\cdot | A)$ on the support of A . By Preston's FKG inequality [26, Theorem 3], $\mu_\rho(\cdot | A)$ stochastically dominates μ_ρ . In other words, there is a probability measure under which $\tilde{\eta}(0) \leq \eta(0)$ almost surely. We can then let $\tilde{\eta}$ and η evolve with the same driving noise $(B_b)_{b \in \mathcal{B}}$.

Let $\varphi(t) = \eta(t) - \tilde{\eta}(t)$ so that $\varphi(0) \geq 0$. By (2.1), if $\varphi_i(t) < 0$, then

$$\frac{d}{dt} \varphi_i(t) \geq \sum_{j \sim i} V'(\eta_j(t)) - V'(\tilde{\eta}_j(t)) \geq \sum_{\{j \sim i : \varphi_j(t) < 0\}} C_+ \varphi_j(t). \quad (4.10)$$

By [28, Theorem 2.1], for any $r > 0$, η and $\tilde{\eta}$ lie in the Hilbert space

$$L_r^2 = \left\{ \zeta : \sum_{i \in \mathbb{Z}} \zeta_i^2 \exp(-r|i|) \leq \infty \right\}.$$

We can therefore define

$$A(t) = \sum_{i \in \mathbb{Z}} 2^{-|i|} \varphi_i(t)^2 \mathbf{1}_{\{\varphi_i(t) < 0\}}.$$

Note that $\frac{d}{dt}(t^2 \mathbf{1}_{\{t < 0\}}) = 2t \mathbf{1}_{\{t < 0\}}$; by (4.10) and the chain rule,

$$\frac{d}{dt} A(t) \leq \sum_{\{i \sim j : \varphi_i(t), \varphi_j(t) < 0\}} 2^{1-|i|} C_+ \varphi_i(t) \varphi_j(t).$$

Checking that for any $\varphi_{i-1}, \varphi_i, \varphi_{i+1} \in \mathbb{R}$ that

$$2^{-|i|} \varphi_i(\varphi_{i-1} + \varphi_{i+1}) \leq 2^{-|i-1|} \varphi_{i-1}^2 + 2^{-|i|} \varphi_i^2 + 2^{-|i+1|} \varphi_{i+1}^2,$$

we obtain $dA(t)/dt \leq 6C_+ A(t)$. Applying Gronwall's lemma with

$$A(0) = 0, \quad A(t) \leq \int_0^t 6C_+ A(s) ds,$$

we find $A(t) = 0$ for all $t \geq 0$. \square

5. DIFFUSION COEFFICIENT

In Propositions 4.1 and 4.3, the relaxation to equilibrium was rephrased in terms of the diffusion of the random walk $X(t)$. We compute below a Central Limit Theorem for this random walk.

We consider the Ginzburg-Landau dynamics on \mathbb{Z} starting from the equilibrium measure $\langle \cdot \rangle_\rho$ at density ρ . We will show that after rescaling, the random walk $X(t)$ converges to a Brownian motion. With $\varepsilon > 0$, let $X^\varepsilon(t) = \varepsilon X(\varepsilon^{-2}t)$ for $t \geq 0$. Let $Y(t)$ denote a Brownian motion with $Y(0) = 0$ and variance $q(\rho) > 0$ (defined in (5.2)),

$$\mathbb{E}_0[Y(s)Y(t)] = (s \wedge t) q(\rho).$$

Let $R_1 = V''(\eta_1)$ (respectively, $R_{-1} = V''(\eta_0)$) denote the jump rate of the random walk X from $(0, 1)$ to $(1, 2)$ (respectively, $(-1, 0)$). For $i \in \mathbb{Z}$, let τ_i denote the shift operator that moves vertex i to 0,

$$\tau_i \eta_j = \eta_{i+j}, \quad j \in \mathbb{Z}.$$

Theorem 5.1. *Over any finite time interval $[0, T]$, as $\varepsilon \rightarrow 0$, $X^\varepsilon(t) \rightarrow Y(t)$ weakly in the Skorohod space, with*

$$q(\rho) = 2 \inf_f \left\{ \langle (1 - f(\eta) + f(\tau_1 \eta))^2 R_1 \rangle_\rho + \mathcal{E}(f, f) \right\}. \quad (5.2)$$

The infimum is taken over smooth, bounded local functions f . By abuse of notation, $\mathcal{E}(f, f)$ stands for the Dirichlet form (3.1) extended to \mathbb{Z} .

This formula implies that $2C_- \leq q(\rho) \leq 2C_+$. For the upper bound, set $f = 1$. The lower bound follows by an application of the Cauchy-Schwarz inequality (see [12, (4.14)]).

Combining Proposition 4.1 and Theorem 5.1, we get a weaker formulation of (1.1). Let φ be a smooth function with compact support $[-1, 1]$. For any $x \in \mathbb{R}$ and $t > 0$

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left\langle V'(\eta_0(0)); \varepsilon \sum \varphi(\varepsilon i) \eta_{i+\lfloor x\varepsilon \rfloor} \left(\frac{t}{\varepsilon^2} \right) \right\rangle_\rho \\ = \frac{1}{\sqrt{2\pi q(\rho)} t} \int_{\mathbb{R}} dy \varphi(y - x) \exp\left(-\frac{y^2}{2t q(\rho)}\right), \end{aligned}$$

where $\lfloor \cdot \rfloor$ stands for the integer part.

Proof of Theorem 5.1. We follow the approach of Kipnis and Varadhan (see [12, Section 4]) and consider the process viewed from the position of the random walk $X(t)$. Let $\tilde{\eta}(t)$ denote the configuration $\eta(t)$ as viewed from $X(t)$:

$$\text{if } X(t) = (i, i + 1), \quad \tilde{\eta}(t) = \tau_i \eta(t).$$

Note that the edge $(0, 1)$ in $\tilde{\eta}(t)$ corresponds to the edge $X(t)$ in $\eta(t)$.

We can write the displacement of the random walk $X(t)$ as the sum of a drift term and a martingale,

$$X(t) - X(0) = \int_0^t j(\tilde{\eta}(s)) ds + M(t).$$

Here $j(\tilde{\eta}(s)) = (R_1 - R_{-1})(\tilde{\eta}(s))$ is the drift of the random walk X at time s . The process $M(t)$ is a martingale with respect to the family of σ -algebras generated by $(\eta(s), X(s))_{s \in [0, t]}$; $M(t)$ is cadlag with the same jumps as $X(t)$; $\mathbb{E}(M(t)^2)$ is therefore $t \langle R_{-1} + R_1 \rangle_\rho = 2t \langle R_1 \rangle_\rho$. We will apply Kipnis and Varadhan's [19, Theorem 1.8] to the drift term. To do this, there are two conditions we must check, see Lemmas 5.4 and 5.6. First some notation. Let \tilde{L} denote the generator of $\tilde{\eta}(t)$,

$$\tilde{L}F(\eta) = LF(\eta) + \sum_{k=\pm 1} [F(\eta) - F(\tau_k \eta)] R_k,$$

where L stands for the generator of the Ginzburg-Landau dynamics on \mathbb{Z} .

Define a Dirichlet form $\tilde{\mathcal{E}}$ by

$$\tilde{\mathcal{E}}(f, f) = \left\langle f \tilde{L} f \right\rangle_{\rho}.$$

Note that by translation invariance, $\langle \cdot \rangle_{\rho} = \mu_{\rho}$ is the invariant measure corresponding to both L and \tilde{L} , and

$$\tilde{\mathcal{E}}(f, g) = \mathcal{E}(f, g) + \langle (f(\eta) - f(\tau_1 \eta)) R_1 (g(\eta) - g(\tau_1 \eta)) \rangle_{\rho}. \quad (5.3)$$

Define dual norms $\| \cdot \|_1$ and $\| \cdot \|_{-1}$ by

$$\|f\|_1^2 = \tilde{\mathcal{E}}(f, f), \quad \|g\|_{-1}^2 = \sup_f \left\{ 2 \langle fg \rangle_{\rho} - \|f\|_1^2 \right\}.$$

Note that $\|g\|_{-1} = \infty$ unless $\langle g \rangle = 0$.

Lemma 5.4. *The shifted process $(\tilde{\eta}(s))$ is time ergodic with respect to μ_{ρ} .*

Proof. If $e^{-\tilde{L}t} f = f$ for all $t \geq 0$, then $\tilde{L}f = 0$ and therefore $\tilde{\mathcal{E}}(f, f) = 0$. As $V'' \geq C_-$,

$$0 = \tilde{\mathcal{E}}(f, f) \geq C_- \langle (f(\eta) - f(\tau_1 \eta))^2 \rangle_{\rho}. \quad (5.5)$$

Hence f is translation invariant. The stationary measure is ergodic, so f is constant with probability one. \square

Lemma 5.6. *The drift function is finite in the dual norm: $\|j\|_{-1} < \infty$.*

Proof. A sufficient condition [19, 1.14] for $\|j\|_{-1} < \infty$ is that,

$$\forall f \in \mathcal{D}(\tilde{L}), \quad \langle fj \rangle_{\rho} \leq C \|f\|_1.$$

By (5.5) and translation invariance,

$$\begin{aligned} \langle f(\eta) j(\eta) \rangle_{\rho} &= \sum_{k=\pm 1} \langle f(\eta) k R_k \rangle_{\rho} = \langle R_1 (f(\eta) - f(\tau_1 \eta)) \rangle_{\rho} \\ &\leq \langle R_1^2 \rangle_{\rho}^{1/2} \langle (f(\eta) - f(\tau_1 \eta))^2 \rangle_{\rho}^{1/2} \\ &\leq C_+ \langle (f(\eta) - f(\tau_1 \eta))^2 \rangle_{\rho}^{1/2} \\ &\leq C_+ C_-^{-1/2} \|f\|_1. \end{aligned} \quad \square$$

We now seek to determine the diffusion coefficient $q(\rho)$. The random walk $X(t)$ is antisymmetric [7]; on the time interval $[0, T]$, the law of $(X(t), j(t))$ is equal to the law of $(X(T-t) - X(T), j(T-t))$, but the displacement of the random walk part is in the opposite direction. This symmetry implies that,

$$\mathbb{E} \left[X(t) \int_0^t j(s) ds \right] = 0.$$

This allows us to expand $\frac{1}{t} \mathbb{E}(M_t^2) = \frac{1}{t} \mathbb{E}((X(t) - \int_0^t j(\tilde{\eta}(s)) ds)^2)$:

$$2 \langle R_1 \rangle_{\rho} = \frac{1}{t} \mathbb{E}(X(t)^2) + \frac{1}{t} \mathbb{E} \left[\left(\int_0^t j(\tilde{\eta}(s)) ds \right)^2 \right].$$

By [19, Remark 1.7 and Theorem 1.8],

$$\lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \left[\left(\int_0^t j(\tilde{\eta}(s)) ds \right)^2 \right] = 2 \|j\|_{-1}^2.$$

By (5.3),

$$\begin{aligned} q(\rho) &= 2 \langle R_1 \rangle_\rho - 2 \sup_f \left\{ 2 \langle f j(\eta) \rangle_\rho - \tilde{\mathcal{E}}(f, f) \right\} \\ &= 2 \inf_f \left\{ \langle R_1 \rangle_\rho - 2 \langle f(R_1 - R_{-1}) \rangle_\rho + \langle (f(\eta) - f(\tau_1 \eta))^2 R_1 \rangle_\rho + \mathcal{E}(f, f) \right\} \end{aligned}$$

The variational formula (5.2) now follows as $\langle f R_{-1} \rangle_\rho = \langle f(\tau_1 \eta) R_1 \rangle_\rho$. \square

6. CONCLUSION

In this paper, we derived the Helffer-Sjöstrand representation for the equilibrium correlations in canonical Gibbs measures. For a class of one-dimensional Hamiltonian, this representation can be reinterpreted in terms of a stochastic process describing the joint evolution of the conservative Ginzburg-Landau dynamics and a random walk coupled to this dynamics. Using the random walk analogy, the diffusive relaxation of the Ginzburg-Landau dynamics can be related to the diffusive behavior of this random walk (Proposition 4.1). In this way several bounds on the return to equilibrium for the Ginzburg-Landau dynamics are obtained (4.5), (4.6).

To sharpen the estimates in this paper and to derive the precise relaxation to equilibrium conjectured in (1.1), one would need to prove a local central limit theorem for the random walk in the Helffer-Sjöstrand representation. This seems to be a challenging task. It would be also interesting to obtain a stochastic interpretation of the Witten Laplacian in higher dimensions. This would provide a straightforward approach to derive relaxation bounds for Ginzburg-Landau dynamics in dimension $d \geq 2$.

APPENDIX A. SPECTRAL GAP

Sharp bounds on the spectral gap have been derived for conservative Ginzburg-Landau type dynamics in [20, 6, 5]. In this appendix, we show how to recover these bounds by using the Witten Laplacian formalism when the potential is strictly convex. We will follow the probabilistic approach devised in [22].

In this appendix, we will relax some assumptions on the dynamics and we suppose that it is defined in dimension $d \geq 1$. Let Λ denote the d -dimensional torus $(\mathbb{Z}/N\mathbb{Z})^d$. Let \mathcal{B} denote the set of oriented nearest neighbor edges of Λ ; how the edges are oriented will not be important. With $\bar{\nabla} = (\partial_b)_{b \in \mathcal{B}}$, the definitions of L_Λ and \mathcal{L}_Λ extend naturally to this higher dimensional setting. Consider the Hamiltonian

$$H(\eta) = H_1(\eta) + H_2(\eta) = \sum_x V_1(\eta_x) + \sum_{(x,y) \in \mathcal{B}} V_2(\eta_x + \eta_y).$$

We will assume that V_1 is strictly convex, with $V_1'' \geq C_- > 0$, and that V_2 is convex.

Let $\lambda = \lambda(\rho, N, d)$ denote the spectral gap of the operator L_Λ with respect to $\mu_{\rho, N}$,

$$\lambda = \inf_{f \perp 1} \frac{\mathcal{E}(f, f)}{\langle f; f \rangle}.$$

The Dirichlet form \mathcal{E} was introduced in (3.1). Define also

$$\tilde{\lambda} = \inf_{f \perp 1} \frac{\langle \bar{\nabla} f \cdot \mathcal{L}_\Lambda \bar{\nabla} f \rangle}{\langle \bar{\nabla} f \cdot \bar{\nabla} f \rangle}.$$

Theorem A.1. *There is a constant k such that for all N and ρ ,*

$$\lambda \geq \tilde{\lambda} \geq \frac{kC_-}{N^2}.$$

The scaling N^2 is of the correct order as it characterizes the diffusive behavior of the conservative dynamics. The assumption of strict convexity should only be seen as a limitation in the method of proof.

Proof. We will first show that $\lambda \geq \tilde{\lambda}$. Recall that $P_t = e^{-tL_\Lambda}$ denotes the semi-group associated with the dynamics. As $P_0 f = f$ and $P_\infty f = \langle f \rangle$,

$$\begin{aligned} \langle f; f \rangle &= \langle f[P_0 f - P_\infty f] \rangle = \int_0^\infty \langle f L_\Lambda P_t f \rangle dt \\ &= \int_0^\infty \langle P_{t/2} f L_\Lambda P_{t/2} f \rangle dt = \int_0^\infty \mathcal{E}(P_{t/2} f, P_{t/2} f) dt. \end{aligned}$$

Let $F(t) = \mathcal{E}(P_t f, P_t f)$. Then by (3.2)

$$F'(t) = -2 \langle \bar{\nabla} P_t f \cdot \bar{\nabla} L_\Lambda P_t f \rangle = -2 \langle \bar{\nabla} P_t f \cdot \mathcal{L}_\Lambda \bar{\nabla} P_t f \rangle.$$

For any u ,

$$\langle \bar{\nabla} u \cdot \mathcal{L}_\Lambda \bar{\nabla} u \rangle \geq \tilde{\lambda} \langle \bar{\nabla} u \cdot \bar{\nabla} u \rangle = \tilde{\lambda} \mathcal{E}(u, u).$$

With $u = P_t f$, $F(t) \leq \exp(-2t\tilde{\lambda})F(0)$ and the bound on λ follows:

$$\langle f; f \rangle = \int_0^\infty F(t/2) dt \leq \int_0^\infty \exp(-t\tilde{\lambda}) F(0) dt = \tilde{\lambda}^{-1} \mathcal{E}(f, f).$$

We will now show the lower bound for $\tilde{\lambda}$. With $F = \bar{\nabla} f$ we need,

$$\langle F \cdot \mathcal{L}_\Lambda F \rangle = \langle F \cdot L_\Lambda \otimes \text{Id} F \rangle + \langle F \cdot (\text{Hess } H) F \rangle \geq \frac{kC_-}{N^2} \langle F \cdot F \rangle.$$

The term $\langle F \cdot L_\Lambda \otimes \text{Id} F \rangle$ is equal to $\sum_b \mathcal{E}(\partial_b f, \partial_b f) \geq 0$, thus

$$\langle F \cdot \mathcal{L}_\Lambda F \rangle \geq \langle F \cdot (\text{Hess } H) F \rangle$$

Let S denote either $\{x\}$ (for $x \in \Lambda$) or $\{x, y\}$ (for $(x, y) \in \mathcal{B}$). If $S = \{x\}$, let $W = V_1(\eta_x)$; if $S = \{x, y\}$, let $W = V_2(\eta_x + \eta_y)$. Let $v_S(b) = +1$ if b

points in to S , let $v_S(b) = -1$ if b points out from S , and let $v_S(b) = 0$, if b points neither into or out from S . Note that $\partial_b \partial_c W = v_S(b)v_S(c)W''$ and

$$F \cdot (\text{Hess } W)F \geq C_- \left(\sum_b v_S(b) \partial_b f \right)^2,$$

Hence,

$$F \cdot (\text{Hess } H_1)F \geq C_- \sum_x \left(\sum_{i=1}^d \partial_{(x, x+e_i)} f - \partial_{(x-e_i, x)} f \right)^2$$

and $F \cdot (\text{Hess } H_2)F \geq 0$. It is sufficient now to show that,

$$\sum_x \left(\sum_{i=1}^d \partial_{(x, x+e_i)} f - \partial_{(x-e_i, x)} f \right)^2 \geq \frac{k}{N^2} (F \cdot F). \quad (\text{A.2})$$

Let $\varphi_x = \partial f / \partial \eta_x$, and let

$$g(\varphi) = \sum_x \left(\sum_{i=1}^d -\varphi_{x-e_i} + 2\varphi_x - \varphi_{x+e_i} \right)^2, \quad h(\varphi) = \sum_{(x,y) \in \mathcal{B}} (\varphi_x - \varphi_y)^2.$$

Inequality (A.2) is equivalent to $g(\varphi) \geq (k/N^2)h(\varphi)$. Let $Q = [q_{xy}]$ denote the generator matrix for the rate-1 nearest-neighbor simple random walk on Λ : $q_{xy} = 1$ iff $x \sim y$ and $\sum_y q_{xy} = 0$. Q has $|\Lambda|$ eigenvalues $0 = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_{|\Lambda|}$, and corresponding eigenvectors $\psi_1, \dots, \psi_{|\Lambda|}$. Q has a uniform stationary distribution, so we can assume that the eigenvectors form an orthonormal basis for $\ell_2(\mathbb{R}^\Lambda)$. The spectral gap of the walk $|\lambda_2|$ is known to be at least k/N^2 with k a constant. Write $\varphi = \sum_{i=1}^{|\Lambda|} a_i \psi_i$. Then

$$g(\varphi) = \sum_x (\varphi Q)(x)^2 = \sum_{i=2}^{|\Lambda|} a_i^2 \lambda_i^2,$$

and

$$h(\varphi) = - \sum_x \varphi(x) (\varphi Q)(x) \geq |\lambda_2| \sum_{i=2}^{|\Lambda|} a_i^2,$$

and so $g(\varphi) \geq |\lambda_2| h(\varphi)$. \square

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